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Cahn-Hilliard Equation with a Logarithmic Free  
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by

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# Convergence to Equilibrium for the Cahn-Hilliard Equation with a Logarithmic Free Energy

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## Abstract

In this paper we investigate the asymptotic behavior of the nonlinear Cahn-Hilliard equation with a logarithmic free energy and similar singular free energies. We prove an existence and uniqueness result with the help of monotone operator methods, which differs from the known proofs based on approximation by smooth potentials. Moreover, we apply the Łojasiewicz-Simon inequality to show that each solution converges to a steady state as time tends to infinity.

**Key words:** Cahn-Hilliard equation, monotone operators, logarithmic potential, Łojasiewicz-Simon inequality, convergence to steady states

**AMS-Classification:** 35K55, 35B40, 35Q99, 47H05, 47J35, 80A22

## 1 Introduction and Main Results

The nonlinear Cahn-Hilliard equation

$$\partial_t c = \Delta \mu, \quad \mu = -\Delta c + f'(c), \quad (1.1)$$

has been studied by several authors over the last years. This system describes the dynamics of phase separation of a two-component mixture. Here  $c = c(t, x)$  is proportional to the concentration difference of the two components and is often called the order parameter of the system, e.g. the mass density,  $\mu$  is the chemical potential, which accounts for mass transport in the system and  $f'$  is the derivative of a function  $f$ , representing the physical potential, which characterizes the stable phases

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of the system (e.g. the pure phases of the mixture). To give a derivation of the Cahn-Hilliard equation, we start with a free-energy functional of the form

$$E(c) = \int_{\Omega} \left( \frac{1}{2} |\nabla c(x)|^2 + f(c(x)) \right) dx, \quad (1.2)$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n \leq 3$ , is a bounded domain with  $C^3$ -boundary  $\partial\Omega$ . Moreover, we require that  $c$  is a conserved quantity, i.e.

$$\frac{d}{dt} \int_{\Omega} c(t, x) dx = 0, \quad (1.3)$$

for all  $t \geq 0$ , with the associated conservation law  $\partial_t c + \operatorname{div} \vec{j} = 0$ , where  $\vec{j}$  is the flux of the order parameter  $c$ . According to Visintin [20, Chapter V.6, p. 150] the flux is given by

$$\vec{j} = -\nabla \left( \frac{\delta E}{\delta c} \right),$$

where  $\frac{\delta E}{\delta c}$  stands for the variational derivative of the functional (1.2) with respect to  $c$ . This yields

$$\mu := \frac{\delta E}{\delta c} = -\Delta c + f'(c),$$

hence the equations (1.1). Note that for simplicity all physical constants are chosen to be 1. In the next step we have to equip (1.1) with some boundary and initial conditions. To ensure that  $c$  satisfies (1.3), we require  $\partial_\nu \mu = 0$  on  $\partial\Omega$ , where  $\partial_\nu$  is defined in the sense of traces, i.e.  $\partial_\nu c = \nu \cdot \nabla c|_{\partial\Omega}$ , where  $\nu = \nu(x)$  is the outer normal at  $x \in \partial\Omega$ . Since (1.1) is a fourth order system we will need an additional boundary condition to obtain well-posedness. With a view on variations of  $E(c)$  the natural boundary condition is  $\partial_\nu c = 0$  on  $\partial\Omega$ . Hence we consider the problem

$$\partial_t c = \Delta \mu \quad \text{in } \Omega \times (0, \infty), \quad (1.4)$$

$$\mu = -\Delta c + f'(c) \quad \text{in } \Omega \times (0, \infty), \quad (1.5)$$

$$\partial_\nu c = \partial_\nu \mu = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.6)$$

$$c|_{t=0} = c_0 \quad \text{in } \Omega. \quad (1.7)$$

The function  $f$ , also called homogeneous free energy, is often assumed to be a smooth function in order to simplify the mathematical analysis. A famous example is the so-called *double-well* potential  $f(s) = ((s - a)(s - b))^2$ , which describes two stable phases ( $s = a, b$ ). Once we have a solution  $c = c(t, x)$  to (1.4)-(1.7) we would like to conclude from the equations, that its range lies in the physical reasonable interval  $[a, b]$ . Unfortunately one may not make use of a maximum principle since (1.1) is of fourth order. A way out is the use of potentials satisfying the following assumption:

**Assumption 1.1** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function, which is twice continuously differentiable in  $(a, b)$ , with

$$\lim_{s \rightarrow a} f'(s) = -\infty, \quad \lim_{s \rightarrow b} f'(s) = \infty,$$

and  $f''(s) \geq -d$  for some  $d \geq 0$ .

If  $f$  is defined on an interval  $[a, b]$ , then we extend  $f(x)$  in (1.2) by  $+\infty$  if  $x \notin [a, b]$ . Hence  $E(c) < \infty$  implies  $c(x) \in [a, b]$  for almost every  $x \in \Omega$ .

The latter assumptions are motivated by the so-called regular solution model free energy suggested by Cahn and Hilliard [3]:

$$f(c) = \frac{\theta}{2} ((1+c) \ln(1+c) + (1-c) \ln(1-c)) - \frac{\theta_c}{2} c^2, \quad (1.8)$$

where  $\theta, \theta_c > 0$ ,  $a = -1$ ,  $b = 1$ . Here the logarithmic terms are related to the entropy of the system. We note that  $f$  is convex if and only if  $\theta \geq \theta_c$ . In this case the mixed phase is stable. On the other hand, if  $0 < \theta < \theta_c$ , the mixed phase is unstable and phase separation occurs.

This logarithmic free energy (1.8) and its mathematical properties are the main motivation for the present work. However, in the following analysis general homogeneous free energies  $f$  with the properties stated above can be treated. The main observation is that, although  $f$  is in general non-convex, it can be considered as a perturbation of a convex potential in the following way:

By the assumption we have the decomposition  $f(s) = \phi(s) - \frac{d}{2}c^2$  where  $\phi: [a, b] \rightarrow \mathbb{R}$  is continuous, convex and twice continuously differentiable in  $(a, b)$ . This will be the key point in the following analysis, which is based on a decomposition of the associated operators in a monotone operator plus a Lipschitz perturbation. The condition  $\lim_{c \rightarrow a} \phi'(c) = -\infty$ ,  $\lim_{c \rightarrow b} \phi'(c) = \infty$  will keep the concentration difference  $c$  in the (physical reasonable) interval  $[a, b]$  and ensures that the subgradient of the associated functional is a single valued function with a suitable domain.

In the case that  $f$  is smooth on  $\mathbb{R}$ , Elliott & Zheng [10] proved the existence of global solutions to (1.4) in an  $L_2$ -setting, with  $f(s) = (s^2 - 1)^2$ . Global well-posedness of strong solutions of (1.4) with dynamic boundary conditions in the sense of  $L_p$  has been shown by Prüss, Racke & Zheng [17] and Prüss & Wilke [18], whereas in [18] they considered a more general case, namely the non-isothermal Cahn-Hilliard equation with physical potentials of polynomial growth. In [17] the existence of a uniform attractor has been proven if  $f(s) = (s^2 - 1)^2$ . Results on convergence of solutions to steady states with the help of the Łojasiewicz-Simon inequality in case of smooth potentials  $f$  can be found in Chill et. al. [6], Hoffmann & Rybka [15], Prüss & Wilke [18], and Wu & Zheng [21].

Concerning the regular model free energy, existence and uniqueness of solutions was first proven by Elliot and Luckhaus [9] in the case of a multi-component mixture. An alternative proof in the case of a two-component mixture was given by Debussche & Dettori [7]. Moreover, they proved existence of a global attractor as

$t \rightarrow \infty$  and estimated its dimension in some cases. Further properties of the attractor were studied by Dupaix [8] and Miranville & Zelik [16]. Bonfoh [1] investigated a generalized Cahn-Hilliard equation in an anisotropic medium with a logarithmic potential. He used a variational approach to show existence and uniqueness of weak solutions. Furthermore he proved the existence of finite dimensional attractors for this system. Singular homogeneous free energies together with a degenerate mobility were studied by Elliot & Garcke [11]. Finally, we want to mention the paper of Garcke [12] in which he studied the Cahn-Hilliard equation with elastic misfit and with a logarithmic potential. He proved existence, uniqueness in a special case, and some regularity properties of weak solutions.

Our main result is that every solution  $c$  of (1.4)-(1.7) converges to a solution of the stationary system/a critical point of  $E$  as  $t \rightarrow \infty$ , provided that  $f$  is analytic in  $(a, b)$ . The proof is based on the Łojasiewicz-Simon inequality, which was already applied successfully in the case of analytic potentials  $f: \mathbb{R} \rightarrow \mathbb{R}$  (see the references above). Moreover, we include a proof of existence of unique solutions, which is based on monotone operator theory, and prove all necessary uniform estimates as  $t \rightarrow \infty$ . The first proof of existence by Elliot & Luckhaus [9] and all later approaches, known to the authors, to variants of Cahn-Hilliard equations with a logarithmic potential are based on first considering the system for suitably smoothed potentials and then passing to the limit. In our approach the existence of unique solutions is shown directly by solving an abstract Cauchy problem for a suitable Lipschitz perturbation of a monotone operator, which is the subgradient of the convex part of the energy of  $E(c)$  in  $H_{(0)}^{-1}(\Omega)$ , cf. Section 2.1 below for the definition of  $H_{(0)}^{-1}(\Omega)$ . Once the domain of the subgradient is characterized, which is the most technical part, the existence of unique solutions follows more or less directly from the general theory. We hope that this method will be useful for further application to evolution problems with similar singular potentials.

The definitions of the function spaces in the following are given in Section 2.1 below. The precise results are as follows:

**THEOREM 1.2 (Global Existence, Uniqueness, Regularity)**

*Let  $f$  be as in Assumption 1.1 For every  $c_0 \in H^1(\Omega)$  with  $E(c_0) < \infty$  there is a unique solution  $c \in L^\infty(0, \infty; H^1(\Omega))$  of (1.4)-(1.7) with  $\partial_t c \in L^2(0, \infty; H_{(0)}^{-1}(\Omega))$ ,  $\mu \in L^2(0, \infty; H^1(\Omega))$ , satisfying*

$$E(c(T)) + \int_0^T \|\nabla \mu(t)\|_{L^2(\Omega)}^2 dt = E(c_0) \quad (1.9)$$

*for all  $T > 0$ . Furthermore, it holds that*

$$\begin{aligned} \kappa c &\in L^\infty(0, \infty; H^2(\Omega)), \\ \kappa \phi'(c) &\in L^\infty(0, \infty; L^2(\Omega)), \\ \kappa \mu &\in L^\infty(0, \infty; H^1(\Omega)), \quad \text{and} \\ \kappa \partial_t c &\in L^\infty(0, \infty; H_{(0)}^{-1}(\Omega)) \cap L^2(0, \infty; H^1(\Omega)), \end{aligned}$$

where  $\kappa(t) = \left(\frac{t}{1+t}\right)^{\frac{1}{2}}$ . Finally, let  $Z_m := \{c_0 \in H^1(\Omega) : E(c_0) < \infty, \int_{\Omega} c_0(x) dx = m\}$ ,  $m \in (a, b)$ . Then the mapping  $Z_m \ni c_0 \mapsto c(t) \in Z_m$  is strongly continuous.

**THEOREM 1.3 (Convergence to Equilibrium)**

Let  $f$  be as in Assumption 1.1 and additionally let  $f$  be analytic in  $(a, b)$ . Moreover, let  $c_0 \in H^1(\Omega)$  with  $E(c_0) < \infty$  and let  $c$  be the solution due to Theorem 1.2. Then

$$\lim_{t \rightarrow \infty} c(t) = c_{\infty} \quad \text{in } H^{2r}(\Omega), r \in (0, 1),$$

where  $c_{\infty} \in H^2(\Omega)$  satisfies  $\overline{c_{\infty}(\Omega)} \subset (a, b)$ ,  $\phi'(c_{\infty}) \in L^2(\Omega)$  and  $c_{\infty}$  is a solution of the stationary system

$$-\Delta c_{\infty} + f'(c_{\infty}) = \text{const.} \quad \text{in } \Omega, \quad (1.10)$$

$$\partial_{\nu} c_{\infty}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \quad (1.11)$$

$$\int_{\Omega} c_{\infty}(x) dx = \int_{\Omega} c_0(x) dx. \quad (1.12)$$

This paper is organized as follows. In Section 2 we introduce some basic notations and function spaces. Then in Section 3 we prove a existence of unique solutions for an evolution equation for a monotone operator plus a globally Lipschitz continuous operator. This is done by applying well-known methods in the theory of monotone operators. Section 4 is devoted to the computation of the subgradient of the functional

$$F(c) = \frac{1}{2} \int_{\Omega} |\nabla c(x)|^2 dx + \int_{\Omega} \phi(c(x)) dx$$

and characterize its domain, where  $\phi$  is the function in the decomposition  $f(s) = \phi(s) - \frac{d}{2}s^2$  (see Theorem 4.3 below). Combining the results of Sections 3 and 4 we give the proof of Theorem 1.2 in Section 5. Finally, in Section 6 we show that each solution  $c_{\infty}$  of the stationary system (1.10)-(1.11) with the side condition (1.12) is uniformly bounded with range in an interval  $[a + \varepsilon, b - \varepsilon]$  for some  $\varepsilon > 0$ . Using this and the compactness of the  $\omega$ -limit set, we show that the same is true for the solution of the instationary system for sufficiently large time. Then we may extend  $f$  to a smooth function  $\tilde{f}$  on  $\mathbb{R}$  and apply the Lojasiewicz-Simon inequality for smooth potentials to show the convergence stated in Theorem 1.3, provided that  $f$  is analytic in  $(a, b)$ .

## 2 Preliminaries

For a set  $M$  the power set will be denoted by  $\mathcal{P}(M)$ . Moreover, we denote  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$  and  $\mathbb{R}_+ = \mathbb{R}_+^1$ . If  $X$  is a Banach space and  $X^*$  is its dual, then

$$\langle f, g \rangle \equiv \langle f, g \rangle_{X^*, X} = f(g), \quad f \in X^*, g \in X,$$

denotes the duality product. Moreover, if  $H$  is a Hilbert space  $(\cdot, \cdot)_H$  will denote its inner product. In the following all Hilbert spaces will be real-valued and separable.

## 2.1 Function Spaces

We denote by  $L^p(\Omega)$ ,  $1 \leq p \leq \infty$ , the usual set of  $p$ -integrable/essentially bounded functions  $f: \Omega \rightarrow \mathbb{R}$  and denote by  $\|\cdot\|_p \equiv \|\cdot\|_{L^p(\Omega)}$  its norm. Moreover,  $H^m(\Omega)$ ,  $m \in \mathbb{N}$ , denotes the usual  $L^2$ -Sobolev space of order  $m$  and  $H_0^m(\Omega)$  denotes the closure of  $C_0^\infty(\Omega)$  in  $H^m(\Omega)$ .

Given  $f \in L^1(\Omega)$ , we denote by

$$m(f) = \frac{1}{|\Omega|} \int_{\Omega} f(x) dx$$

its mean value. Moreover, for  $m \in \mathbb{R}$  we set

$$L_{(m)}^2(\Omega) := \{f \in L^2(\Omega) : m(f) = m\}$$

and  $P_0 f := f - m(f)$  denotes the orthogonal projection onto  $L_{(0)}^2(\Omega)$ .

We define

$$H_{(0)}^1 = H_{(0)}^1(\Omega) = \left\{ c \in H^1(\Omega) : \int_{\Omega} c(x) dx = 0 \right\}$$

equipped with the inner product

$$(c, d)_{H_{(0)}^1(\Omega)} = (\nabla c, \nabla d)_{L^2(\Omega)}, \quad c, d \in H_{(0)}^1(\Omega).$$

Then  $H_{(0)}^1(\Omega)$  is a Hilbert space because of Poincaré's inequality. Moreover, let  $H_{(0)}^{-1} \equiv H_{(0)}^{-1}(\Omega) = H_{(0)}^1(\Omega)^*$ . Then the Riesz isomorphism  $\mathcal{R}: H_{(0)}^1(\Omega) \rightarrow H_{(0)}^{-1}(\Omega)$  is given by

$$\langle \mathcal{R}c, d \rangle_{H_{(0)}^{-1}, H_{(0)}^1} = (c, d)_{H_{(0)}^1} = (\nabla c, \nabla d)_{L^2}, \quad c, d \in H_{(0)}^1(\Omega),$$

i.e.,  $\mathcal{R} = -\Delta_N$  is the negative Laplace operator with Neumann boundary conditions. In particular, this means that we equip  $H_{(0)}^{-1}(\Omega)$  with the inner product

$$(f, g)_{H_{(0)}^{-1}} = (\nabla \Delta_N^{-1} f, \nabla \Delta_N^{-1} g)_{L^2} = (\Delta_N^{-1} f, \Delta_N^{-1} g)_{H_{(0)}^1}.$$

This implies the useful interpolation inequality

$$\|f\|_{L^2}^2 = -(\nabla \Delta_N^{-1} f, \nabla f)_{L^2} \leq \|f\|_{H_{(0)}^{-1}} \|f\|_{H_{(0)}^1} \quad \text{for all } f \in H_{(0)}^1(\Omega). \quad (2.1)$$

Moreover, we embed  $H_{(0)}^1(\Omega)$  and  $L_{(0)}^2(\Omega)$  into  $H_{(0)}^{-1}(\Omega)$  in the standard way:

$$\langle c, \varphi \rangle_{H_{(0)}^{-1}, H_{(0)}^1} = \int_{\Omega} c(x) \varphi(x) dx, \quad \varphi \in H_{(0)}^1(\Omega).$$

Finally, we note that, if  $u \in H_{(0)}^1(\Omega)$  solves  $\Delta_N u = f$  for some  $f \in L_{(0)}^2(\Omega)$  and  $\partial\Omega$  is  $C^2$ , then it follows from standard elliptic theory that  $u \in H^2(\Omega)$  and  $u$  solves the Laplace equation with Neumann boundary conditions in the strong sense:

$$\Delta u = f \quad \text{in } \Omega, \quad (2.2)$$

$$\partial_\nu u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \quad (2.3)$$

where the second equation holds in the sense of traces.



### 3 Evolution Equations for Monotone Operators

We refer to Brézis [2] and Showalter [19] for basic results in the theory of monotone operators. In the following we just summarize some basic facts and definitions. Let  $H$  be a real-valued and separable Hilbert space. Recall that  $\mathcal{A}: H \rightarrow \mathcal{P}(H)$  is a monotone operator if

$$(w - z, x - y)_H \geq 0 \quad \text{for all } w \in \mathcal{A}(x), z \in \mathcal{A}(y).$$

Moreover,  $\mathcal{D}(\mathcal{A}) = \{x \in H : \mathcal{A}(x) \neq \emptyset\}$ . Now let  $\varphi: H \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function. Then  $\text{dom}(\varphi) = \{x \in H : \varphi(x) < \infty\}$  and  $\varphi$  is called proper if  $\text{dom}(\varphi) \neq \emptyset$ . Moreover, the subgradient  $\partial\varphi: H \rightarrow \mathcal{P}(H)$  is defined by  $w \in \partial\varphi(x)$  if and only if

$$\varphi(\xi) \geq \varphi(x) + (w, \xi - x)_H \quad \text{for all } \xi \in H.$$

Then  $\partial\varphi$  is a monotone operator and, if additionally  $\varphi$  is lower semi-continuous, then  $\partial\varphi$  is maximal monotone, cf. [2, Exemple 2.3.4].

The proof of Theorem 1.2 is based on the following result for the evolution problem associated to Lipschitz perturbation of monotone operators.

**THEOREM 3.1** *Let  $H_0, H_1$  be real-valued, separable Hilbert spaces such that  $H_1$  is densely embedded into  $H_0$ . Moreover, let  $\varphi: H_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lower semi-continuous functional such that  $\varphi = \varphi_1 + \varphi_2$ , where  $\varphi_2 \geq 0$  is convex and lower semi-continuous,  $\text{dom } \varphi_1 = H_1$ , and  $\varphi_1|_{H_1}$  is a bounded, coercive, quadratic form on  $H_1$  and set  $\mathcal{A} = \partial\varphi$ . Furthermore, assume that  $\mathcal{B}: H_1 \rightarrow H_0$  is a globally Lipschitz continuous function. Then for every  $u_0 \in \mathcal{D}(\mathcal{A})$  and  $f \in L^2(0, T; H_0)$  there is a unique  $u \in W_2^1(0, T; H_0) \cap L^\infty(0, T; H_1)$  with  $u(t) \in \mathcal{D}(\mathcal{A})$  for a.e.  $t > 0$  solving*

$$\frac{du}{dt}(t) + \mathcal{A}(u(t)) \ni \mathcal{B}(u(t)) + f(t) \quad \text{for a.a. } t \in (0, T), \quad (3.1)$$

$$u(0) = u_0. \quad (3.2)$$

Moreover,  $\varphi(u) \in L^\infty(0, T)$ .

**Proof:** First of all, we show that

$$(y_1 - y_2, u_1 - u_2)_{H_0} \geq c \|u_1 - u_2\|_{H_1}^2 \quad \text{for all } y_j \in \mathcal{A}(u_j), j = 1, 2. \quad (3.3)$$

To this end, let  $\tilde{\varphi} = \varphi|_{H_1}$ ,  $\tilde{\varphi}_j = \varphi_j|_{H_1}$ ,  $j = 1, 2$ . Then  $\tilde{\varphi}$  is a convex, proper, and lower semi-continuous functional on  $H_1$  since  $\text{dom } \varphi \subseteq H_1$  and  $H_1 \hookrightarrow H_0$ . Moreover, from the definition of a subgradient it immediately follows that  $\mathcal{D}(\partial\varphi) \subseteq \mathcal{D}(\partial\tilde{\varphi})$  and for every  $y \in \partial\varphi(u)$  there is a unique  $w_y \in \partial\tilde{\varphi}(u)$  such that

$$(y, d)_{H_0} = (w_y, d)_{H_1} \quad \text{for all } d \in H_1.$$

Moreover, since  $\tilde{\varphi}_1$  is a bounded, coercive quadratic form on  $H_1$ ,  $\tilde{\varphi}_1(u) = \frac{1}{2}(Lu, u)_{H_1}$ ,  $u \in H_1$ , and therefore  $\partial\tilde{\varphi}_1(u) = Lu$  and  $\mathcal{D}(\tilde{\varphi}_1) = H_1$ , where  $L \in \mathcal{L}(H)$  is self-adjoint positive operator. Hence by [2, Corollaire 2.11, Chapter II]  $\partial\tilde{\varphi}(u) = \partial\tilde{\varphi}_1(u) + \partial\tilde{\varphi}_2(u) = Lu + \tilde{\varphi}_2(u)$ . Thus, if  $y_j \in \mathcal{A}(u_j)$ ,  $j = 1, 2$ ,

$$\begin{aligned} (y_1 - y_2, u_1 - u_2)_{H_0} &= (w_{y_1} - w_{y_2}, u_1 - u_2)_{H_1} \\ &= (L(u_1 - u_2), u_1 - u_2)_{H_1} + (w'_{y_1} - w'_{y_2}, u_1 - u_2)_{H_1} \\ &\geq c\|u_1 - u_2\|_{H_1}^2, \end{aligned}$$

where  $w'_{y_j} = w_{y_j} - Lu_j \in \partial\tilde{\varphi}_2(u_j)$ ,  $j = 1, 2$ . This proves (3.3).

Now let  $X_T = L^2(0, T; H_1)$ . For a given  $v \in X_T$  we define  $u = F(v)$  as the solution of

$$\begin{aligned} \frac{d}{dt}u(t) + \mathcal{A}(u(t)) &\ni \mathcal{B}(v(t)) + f(t) \quad \text{for almost all } t > 0, \\ u(0) &= u_0, \end{aligned}$$

which exists due to [19, Theorem 4.1A and Theorem 4.3, Chapter IV] and satisfies

$$\varphi \circ u \in L^\infty(0, T), \quad \frac{du}{dt} \in L^2(0, T; H_0). \quad (3.4)$$

Since  $\varphi(u(t)) \geq \varphi_1(u(t)) \geq c\|u(t)\|_{H_1}^2$ , we conclude that  $u \in L^\infty(0, T; H_1)$ . Moreover, if  $u_j = F(v_j)$ ,  $j = 1, 2$ , then

$$\frac{d}{dt}(u_1(t) - u_2(t)) + y_1(t) - y_2(t) = \mathcal{B}(v_1(t)) - \mathcal{B}(v_2(t))$$

where  $y_j(t) \in \partial\varphi(u_j(t))$ . Hence taking the inner product with  $u_1 - u_2$ , integrating w.r.t.  $s \in (0, t)$ , and using (3.3) we obtain

$$\begin{aligned} &\sup_{0 \leq s \leq t} \|u_1(s) - u_2(s)\|_{H_0}^2 + c \int_0^t \|(u_1 - u_2)(s)\|_{H_1}^2 ds \\ &\leq \int_0^t |(\mathcal{B}(v_1(s)) - \mathcal{B}(v_2(s)), u_1(s) - u_2(s))_{H_0}| ds \\ &\leq Mt^{\frac{1}{2}} \|v_1 - v_2\|_{L^2(0, t; H_1)} \|u_1 - u_2\|_{L^\infty(0, t; H_0)}. \end{aligned}$$

Thus we conclude

$$\|u_1 - u_2\|_{X_T}^2 \leq CT \|v_1 - v_2\|_{X_T}^2,$$

where  $C > 0$  is independent of  $T$ ,  $u_0$ , and  $f$ . Hence choosing  $0 < T < C^{-1}$ ,  $F: X_T \rightarrow X_T$  is a contraction and there is a unique solution  $u \in X_T$  of (3.1), which is in  $W_2^1(0, T; H_0) \cap L^\infty(0, T; H_1)$  because of (3.4). Moreover, since  $\varphi \circ u \in L^\infty(0, T)$  and the constant  $C$  above is independent of  $u_0$  and  $f$ , the solution  $u$  can be extended to an arbitrary large interval  $[0, T]$ ,  $T > 0$ .  $\blacksquare$

## 4 Subgradients

In the following  $\phi: [a, b] \rightarrow \mathbb{R}$  denotes a continuous function and we set  $\phi(x) = +\infty$  for  $x \notin [a, b]$ .

In this section we study the subgradient of the functional

$$F(c) = \frac{1}{2} \int_{\Omega} |\nabla c(x)|^2 dx + \int_{\Omega} \phi(c(x)) dx \quad (4.1)$$

first defined on  $L^2_{(m)}(\Omega)$ ,  $m \in (a, b)$ , with

$$\text{dom } F = \{c \in H^1(\Omega) \cap L^2_{(m)}(\Omega) : \phi(c) \in L^1(\Omega)\}.$$

We denote by  $\partial F(c): L^2_{(m)}(\Omega) \rightarrow \mathcal{P}(L^2_{(0)}(\Omega))$  the subgradient of  $F$  at  $c \in \text{dom } F$  in the sense that  $w \in \partial F(c)$  if and only if

$$(w, c' - c)_{L^2} \leq F(c') - F(c) \quad \text{for all } c' \in L^2_{(m)}(\Omega).$$

Note that  $L^2_{(m)}(\Omega)$  is an *affine* subspace of  $L^2(\Omega)$  with tangent space  $L^2_{(0)}(\Omega)$ . Therefore the standard definition of  $\partial F$  for functionals on Hilbert spaces does not apply. But the definition above is the obvious generalization to affine subspaces of Hilbert spaces.

First of all, we have:

**Lemma 4.1** *Let  $\phi: [a, b] \rightarrow \mathbb{R}$  be a continuous and convex function. Then  $F$  defined as in (4.1) is a proper, lower semi-continuous, convex functional.*

**Proof:** It only remains to prove the lower semi-continuity. By adding a suitable constant we can w.l.o.g. assume that  $\phi \geq 0$ . For this let  $c_k \in L^2_{(m)}(\Omega)$  such that  $c_k \rightarrow_{k \rightarrow \infty} c$  in  $L^2(\Omega)$ . It suffices to consider the case  $\liminf_{k \rightarrow \infty} F(c_k) < \infty$ . Note that after passing to a subsequence, we can assume that  $F(c_k) \leq C$  for all  $k \in \mathbb{N}$  and some  $C > 0$ . In particular, this implies that  $c_k \in \text{dom } F$  for all  $k \in \mathbb{N}$ . Because of  $\phi(c) \geq 0$ , there is a subsequence (again denoted by  $c_k$ ) such that  $c_k \rightarrow_{k \rightarrow \infty} c^*$  in  $H^1(\Omega)$  and  $c_k \rightarrow_{k \rightarrow \infty} c^*$  in  $L^2(\Omega)$  and almost everywhere in  $\Omega$ . Since  $c_k \rightarrow_{k \rightarrow \infty} c$  in  $L^2(\Omega)$ ,  $c = c^* \in H^1_{(0)}(\Omega)$ . Therefore the Lemma of Fatou yields

$$\int_{\Omega} \phi(c(x)) dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \phi(c_k(x)) dx \leq C,$$

which proves that  $c \in \text{dom } \varphi$  and

$$F(c) \leq \liminf_{k \rightarrow \infty} F(c_k)$$

since  $\|\nabla c\|_2^2 \leq \liminf_{k \rightarrow \infty} \|\nabla c_k\|_2^2$ . ■

**Corollary 4.2** *Let  $\phi$  and  $F$  be as in Lemma 4.1 and let  $m = 0 \in (a, b)$ . Then  $\partial F$  is a maximal monotone operator on  $H = L^2_{(0)}(\Omega)$ .*

**Proof:** Because of Lemma 4.1, this fact follows from Corollary 1.2 and Lemma 1.3 in [19, Chapter IV].  $\blacksquare$

Now we state our main result on the following characterization of  $\partial F(c)$ :

**THEOREM 4.3** *Let  $\phi: [a, b] \rightarrow \mathbb{R}$  be a continuous and convex function that is twice continuously differentiable in  $(a, b)$  and satisfies  $\lim_{x \rightarrow a} \phi'(x) = -\infty$ ,  $\lim_{x \rightarrow b} \phi'(x) = +\infty$ . Moreover, we set  $\phi'(x) = +\infty$  for  $x \notin (a, b)$  and let  $F$  be defined as in (4.1). Then*

$$\begin{aligned} \mathcal{D}(\partial F) &= \{c \in H^2(\Omega) \cap L^2_{(m)}(\Omega) : \\ &\quad \phi'(c) \in L^2(\Omega), \phi''(c)|\nabla c|^2 \in L^1(\Omega), \partial_\nu c|_{\partial\Omega} = 0\} \end{aligned}$$

and

$$\partial F(c) = -\Delta c + P_0 \phi'(c).$$

Moreover,

$$\|c\|_{H^2(\Omega)}^2 + \|\phi'(c)\|_2^2 + \int_{\Omega} \phi''(c(x))|\nabla c(x)|^2 dx \leq C (\|\partial F(c)\|_2^2 + \|c\|_2^2) \quad (4.2)$$

for some constant  $C > 0$  independent of  $c \in \mathcal{D}(\partial F)$ .

Before proving the lemma, we introduce some technical tools and simplifications. First of all, if we replace  $c(x)$  by  $\bar{c}(x) = c(x) - m$  and  $\phi$  by  $\bar{\phi}(c) = \phi(c + m)$ , we can assume w.l.o.g. that  $m = 0 \in (\bar{a}, \bar{b})$ . Moreover, replacing  $\phi(c)$  by  $\bar{\phi}(c) = \phi(c) + b_1 c(x) + b_2$ ,  $b_j \in \mathbb{R}$ , changes  $F$  only by an affine linear functional, for which the subgradient is trivial. In this way, we may as well assume that  $\phi'(0) = \phi(0) = 0$ . Furthermore, we define  $\phi_+(c) = \phi(c)$  if  $c > 0$ ,  $\phi_+(c) = 0$  if  $c \leq 0$  and  $\phi_-(c) = \phi(c) - \phi_+(c)$ . Then  $\phi_{\pm}: \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  are convex functions, which are continuously differentiable in  $(a, b)$ .

In the following, we would like to evaluate the directional derivative of  $F(c)$  in direction  $\phi'(c)$ . Formally, this implies the estimate of  $\|\phi'(c)\|_2$ . But we cannot do this directly due to the singular behavior of  $\phi$ . Therefore we approximate  $\phi'_+$  (and analogously  $\phi'_-$ ) from below by a sequence  $f_n^+$  of smooth potentials as follows: Since  $\phi'$  is continuous and monotone,  $\phi'(0) = 0$ , and  $\lim_{c \rightarrow b} \phi'(c) = +\infty$ , for every  $n \in \mathbb{N}$  sufficiently large there is some  $c_n \in (\frac{b}{2}, b)$  such that  $\phi'(c_n) = n$ . Therefore we can define

$$f_n^+(c) = \begin{cases} \phi'(c) & \text{for } c \in [\frac{b}{2}, c_n), \\ n + \phi''(c_n)(c - c_n) & \text{for } c \geq c_n, \\ 0 & \text{for } c \leq 0. \end{cases}$$

for  $c \notin (0, \frac{b}{2})$ . Moreover, we can extend  $f_n^+$  to  $\mathbb{R}$  such that  $f_n^+ : \mathbb{R} \rightarrow \mathbb{R}$  are  $C^1$ -functions with  $0 \leq f_n^+ \leq \phi'_+$  and with first derivative bounded by  $M_n := \sup_{0 \leq x \leq c_n} \phi''(x)$ .

Since we have to work in the subspace  $L^2_{(0)}(\Omega)$ , we will use “bump functions” supported in suitable sets to correct the mean value of functions. For this let  $c \in H^1_{(0)}(\Omega)$  be fixed and let  $I \subset [a, b]$  be an interval such that  $|\{c(x) \in I\}| > 0$ . Then we say that  $\varphi$  is a *bump function supported in  $\{c \in I\}$*  if  $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ ,  $\varphi \geq 0$ ,  $\varphi(x) = 0$  if  $c(x) \notin I$  and if  $m(\varphi) = 1$ . Such a function can be constructed as follows: Choose a smooth function  $\psi : \mathbb{R} \rightarrow [0, 1]$  with bounded first derivative such that  $\psi(s) = 0$  if  $s \notin I$  and  $\psi(s) > 0$  else. Then  $\varphi(x) = \frac{\psi(c(x))}{m(\psi(c))}$  has the stated properties. Furthermore, we note that, if  $I = [a, a']$  with  $a' \in (a, b)$ , then we can choose  $\psi$  such that  $\psi'(s) \leq 0$ . This implies that the constructed function  $\varphi$  has the property

$$(\nabla c, \nabla \varphi)_{L^2(\Omega)} = \frac{1}{m(\psi(c))} \int_{\Omega} \psi'(c) |\nabla c|^2 dx \leq 0. \quad (4.3)$$

Given such a bump function  $\varphi$ , we define  $M_\varphi : L^2(\Omega) \rightarrow H^1(\Omega) \cap L^\infty(\Omega)$  by

$$(M_\varphi f)(x) = m(f)\varphi, \quad f \in L^2(\Omega).$$

Then  $f - M_\varphi f \in L^2_{(0)}(\Omega)$  and

$$\|M_\varphi f\|_{H^1} \leq C \left| \int_{\Omega} f(x) dx \right| \quad \text{for all } f \in L^2(\Omega). \quad (4.4)$$

Finally, we note that

$$|\{c(x) - a \geq t\}| \leq \frac{1}{t} \int_{\Omega} (c(x) - a) dx = \frac{|a||\Omega|}{t}$$

for  $t > 0$  since  $c \in L^2_{(0)}(\Omega)$ . This implies that  $|\{c < \frac{b}{2}\}| \geq \frac{b}{b+2|a|}|\Omega| > 0$ . Hence the interval  $I = [a, \frac{b}{2})$  is admissible for the construction of bump function supported in  $\{c \in (a, \frac{b}{2})\}$ .

**Proof of Theorem 4.3:** Let  $c \in \mathcal{D}(\partial F)$ . We define  $\tilde{c}_t(x)$ ,  $0 < t \leq \frac{2}{M_n}$ ,  $x \in \Omega$ , as solution of

$$\tilde{c}_t(x) = c(x) - t f_n^+(\tilde{c}_t(x)), \quad (4.5)$$

which exists by the contraction mapping principle. Then  $\tilde{c}_t(x) = c(x)$  if  $c(x) < 0$  since  $f_n^+(\tilde{c}_t(x)) = 0$  in this case. Moreover,  $0 \leq \tilde{c}_t(x) = c(x) - t f_n^+(\tilde{c}_t(x)) \leq c(x)$  if  $c(x) \geq 0$ . More formally,  $\tilde{c}_t$  can be expressed in the form  $\tilde{c}_t(x) = F_t^n(c(x))$ , where  $F_t^n : [a, b] \rightarrow [a, b]$  is a continuous differentiable mapping with  $F_t^n(x) \rightarrow x$ ,  $(F_t^n)'(x) \rightarrow 1$  as  $t \rightarrow 0+$  uniformly in  $[a, b]$ . Hence  $\tilde{c}_t \in H^1(\Omega)$  and  $\tilde{c}_t \rightarrow_{t \rightarrow 0} c$  in  $H^1(\Omega)$  and almost everywhere.

Since  $\tilde{c}_t(x) \notin L^2_{(0)}(\Omega)$  in general, we set  $c_t = \tilde{c}_t + t M_\varphi(f_n^+(\tilde{c}_t))$ , where  $\varphi$  is a bump function supported in  $\{c(x) < \frac{b}{2}\}$  with the property (4.3). Then  $c_t \in L^2_{(0)}(\Omega)$ .

Furthermore,  $c_t(x) = \tilde{c}_t(x)$  and  $f_n^+(c_t(x)) = f_n^+(\tilde{c}_t(x))$  if  $c(x) > \frac{b}{2}$  and  $c_t(x) = \tilde{c}_t(x) + tM_\varphi(f_n^+(c_t)) \in [a, \frac{3}{4}b]$  if  $c(x) \leq \frac{b}{2}$  and if  $0 < t < \frac{b}{4M'_n}$  where  $M'_n = \sup_{0 \leq t \leq b} f_n^+(t) \|\varphi\|_\infty$ . For short we write  $d_t = M_\varphi(f_n^+(\tilde{c}_t))$ .

Now we assume that  $w \in \partial F(c)$ . Then

$$F(c) - F(c_t) \leq t(w, f_n^+(\tilde{c}_t) - d_t)_{L^2(\Omega)}.$$

Moreover, if  $t > 0$  is sufficiently small, then

$$\begin{aligned} & F(c) - F(c_t) \\ &= \int_{\Omega} (\phi(c(x)) - \phi(c_t(x))) dx + t(\nabla c, \nabla f_n^+(\tilde{c}_t))_{L^2} - tm(f_n^+(\tilde{c}_t))(\nabla c, \nabla \varphi)_{L^2} \\ &\quad - \frac{t^2}{2} \|f_n^+(\tilde{c}_t) - d_t\|_{L^2}^2 \\ &\geq t \int_{\{c(x) > \frac{b}{2}\}} \phi'(c_t(x)) f_n^+(c_t(x)) dx + t \int_{\{c(x) \leq \frac{a}{2}\}} (\phi(c(x)) - \phi(c(x) + td_t)) dx \\ &\quad + \int_{\{\frac{a}{2} \leq c(x) \leq \frac{b}{2}\}} (\phi(c(x)) - \phi(\tilde{c}_t(x) + td_t)) dx \\ &\quad + t(\nabla c, \nabla f_n^+(\tilde{c}_t)) - \frac{t^2}{2} \|f_n^+(\tilde{c}_t) - d_t\|_{L^2}^2 \\ &\geq t \int_{\{c(x) > \frac{b}{2}\}} f_n^+(c_t(x))^2 dx + t(\nabla c, \nabla f_n^+(\tilde{c}_t)) - \frac{t^2}{2} \|f_n^+(\tilde{c}_t) - d_t\|_{L^2}^2 \\ &\quad + \int_{\{\frac{a}{2} \leq c(x) \leq \frac{b}{2}\}} (\phi(c(x)) - \phi(c(x) + td_t)) dx, \end{aligned}$$

where we have used that  $\phi(c(x)) - \phi(c_t(x)) \geq \phi'(c_t(x))(c(x) - c_t(x))$  and  $c_t(x) < c(x)$  if  $c(x) > \frac{b}{2}$ ,  $\phi'(c_t(x)) \geq f_n^+(c_t(x))$  as well as (4.3) and  $\phi(c(x)) - \phi(c(x) + td_t(x)) \geq 0$  if  $c(x) \leq \frac{a}{2}$  and  $t \leq \frac{a}{2M'_n}$ . Hence

$$\begin{aligned} (w, f_n^+(\tilde{c}_t) - d_t)_{L^2(\Omega)} &\geq \int_{\{c(x) > \frac{b}{2}\}} f_n^+(c_t(x))^2 dx + (\nabla c, \nabla f_n^+(\tilde{c}_t)) - \frac{t}{2} \|f_n^+(c_t) - d_t\|_{L^2}^2 \\ &\quad + \int_{\{\frac{a}{2} \leq c(x) \leq \frac{b}{2}\}} \frac{1}{t} (\phi(c(x)) - \phi(\tilde{c}_t(x) + td_t)) dx, \end{aligned}$$

which yields for  $t \rightarrow 0$

$$\begin{aligned} (w, f_n^+(c) - M_\varphi(f_n^+(c)))_{L^2(\Omega)} &\geq \int_{\{c(x) > \frac{b}{2}\}} f_n^+(c(x))^2 dx + (\nabla c, \nabla f_n^+(c)) \\ &\quad + \int_{\{\frac{a}{2} \leq c(x) \leq \frac{b}{2}\}} \phi'(c(x))(f_n^+(c(x)) - M_\varphi(f_n^+(c))) dx \end{aligned}$$

due to  $\lim_{t \rightarrow 0} \tilde{c}_t(x) = c(x)$  in  $H^1(\Omega)$  and almost everywhere and since  $\phi(c)$  is continuously differentiable in  $[\frac{a}{2}, \frac{3}{4}b]$ . Now we observe that

$$(\nabla c, \nabla f_n^+(c)) = \int_{\Omega} (f_n^+)'(c(x)) |\nabla c(x)|^2 dx \geq 0$$

and use that  $\|M_\varphi(f_n^+(c))\|_2 \leq C\|f_n^+(c)\|_2$  due to (4.4). Therefore

$$\begin{aligned} & \|f_n^+(c)\|_{L^2(\Omega)}^2 + \int_{\Omega} (f_n^+)'(c(x))|\nabla c(x)|^2 dx \\ & \leq C \left( \|w\|_{L^2(\Omega)}^2 + \int_{\{\frac{a}{2} \leq c(x) \leq \frac{b}{2}\}} |\phi'(c(x))|^2 dx \right) \\ & \leq C' \left( \|w\|_{L^2(\Omega)}^2 + \int_{\Omega} |c(x)|^2 dx \right) \end{aligned}$$

by Young's inequality. Letting  $n \rightarrow \infty$  we obtain

$$\|\phi'_+(c)\|_{L^2(\Omega)}^2 + \int_{\Omega} \phi''_+(c(x))|\nabla c(x)|^2 dx \leq C \left( \|w\|_{L^2(\Omega)}^2 + \|c\|_{H^1(\Omega)}^2 \right) \quad (4.6)$$

by Fatou's lemma. By symmetry the same is true for  $\phi_-$  instead of  $\phi_+$  and therefore also for  $\phi$ .

In particular,  $\phi'(c) \in L^2(\Omega)$  implies  $c(x) \in (a, b)$  almost everywhere. Thus  $|\{c(x) \in (a + \delta, b - \delta)\}| > 0$  for sufficiently small  $\delta > 0$ . Because of this, we can use a bump function  $\varphi$  supported in  $\{c(x) \in (a + \delta, b - \delta)\}$  for some fixed  $\delta > 0$ . Moreover, let  $\psi_M: \mathbb{R} \rightarrow [0, 1]$ ,  $M \in \mathbb{N}$ , be smooth functions such that  $\psi_M(s) = 0$  if  $|s| \geq M + 1$ ,  $\psi_M(s) = 1$  if  $|s| \leq M$ , and  $|\psi'_M(s)| \leq 2$ . Set  $\chi_M(x) = \psi_M(\phi'(c(x)))$ . Then  $\chi_M \in H^1(\Omega)$  and  $\chi_M(x) = 0$  if  $|\phi'(c(x))| \geq M + 1$ . Moreover,  $\chi_M \rightarrow_{M \rightarrow \infty} 1$  almost everywhere and in  $L^p(\Omega)$ ,  $1 \leq p < \infty$ , and

$$(\nabla c, \nabla(\chi_M \psi))_{L^2(\Omega)} = (\nabla c, \chi_M \nabla \psi)_{L^2(\Omega)} + \int_{\Omega} \phi''(c(x))|\nabla c(x)|^2 \psi(x) \psi'_M(\phi'(c(x))) dx$$

for all  $\psi \in C^\infty(\overline{\Omega})$ . Since  $\phi''(c)|\nabla c|^2 \in L^1(\Omega)$  due to (4.6) and  $\psi'_M(\phi'(c(x))) \rightarrow_{M \rightarrow \infty} 0$  almost everywhere, we conclude

$$\lim_{M \rightarrow \infty} (\nabla c, \nabla(\chi_M \psi))_{L^2(\Omega)} = (\nabla c, \nabla \psi)_{L^2(\Omega)} \quad \text{for all } \psi \in C^\infty(\overline{\Omega}). \quad (4.7)$$

Now we define  $c_t^M = c - t\chi_M \psi + tM_\varphi(\chi_M \psi)$ ,  $\psi \in C^\infty(\overline{\Omega})$ ,  $t > 0$ ,  $M \in \mathbb{N}$ .

Then  $c_t^M \in \text{dom } F$  for sufficiently small  $t > 0$  (depending on  $M$ ) and

$$\begin{aligned} & t(w, \chi_M \psi - M_\varphi(\chi_M \psi)) \geq F(c) - F(c_t^M) \\ & = \int_{\{\phi'(c(x)) \leq M+1\}} (\phi(c(x)) - \phi(c_t^M(x))) dx + t(\nabla c, \nabla(\chi_M \psi - M_\varphi(\chi_M \psi)))_{L^2} \\ & \quad - t^2 \|\nabla(\chi_M \psi - M_\varphi(\chi_M \psi))\|_{L^2}^2. \end{aligned}$$

Dividing by  $t$  and passing to the limit  $t \rightarrow 0$ , we conclude

$$\begin{aligned} & (w, \chi_M \psi - M_\varphi(\chi_M \psi)) \\ & \geq \int_{\Omega} \phi'(c(x))(\chi_M \psi - M_\varphi(\chi_M \psi)) dx + (\nabla c, \nabla(\chi_M \psi - M_\varphi(\chi_M \psi)))_{L^2} \end{aligned}$$

for all  $\psi \in C^\infty(\overline{\Omega})$ . Replacing  $\psi$  by  $-\psi$ , we obtain equality above. Finally, letting  $M \rightarrow \infty$  we obtain

$$(w, \psi)_{L^2(\Omega)} = (\phi'(c), \psi)_{L^2(\Omega)} + (\nabla c, \nabla \psi)_{L^2(\Omega)} \quad \text{for all } \psi \in C^\infty(\overline{\Omega}), m(\psi) = 0,$$

where we have used (4.7), (4.4), and

$$\lim_{M \rightarrow \infty} \int_{\Omega} \chi_M \psi \, dx = \lim_{M \rightarrow \infty} \int_{\Omega} (\chi_M - 1) \psi \, dx = 0 \quad \text{if } m(\psi) = 0.$$

Hence  $-\Delta_N c = w - P_0 \phi'(c) \in L^2_{(0)}(\Omega)$ , where  $\Delta_N$  is the weak Neumann-Laplace operator as above. Thus  $c \in H^2(\Omega)$  and  $c$  is the unique strong solution of (2.2)-(2.3) with  $c = u$  and  $f = w - P_0 \phi'(c)$  and  $\|c\|_{H^2} \leq C \|\phi'\|_2$ . Using this, (4.6), and  $\|c\|_{H^1}^2 \leq C \|c\|_2 \|c\|_{H^2}$ , we obtain (4.2). Moreover, the previous observations imply that  $\partial F(c) = -\Delta c + P_0 \phi'(c)$  is single-valued and the characterization of the domain. This finishes the proof.  $\blacksquare$

**Corollary 4.4** *Let  $F$  be defined as above and extend  $F$  to a functional  $\tilde{F}: H_{(0)}^{-1}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  by setting  $\tilde{F}(c) = F(c)$  if  $c \in \text{dom } F$  and  $\tilde{F}(c) = +\infty$  else. Then  $\tilde{F}$  is a proper, convex, and lower semi-continuous functional,  $\partial \tilde{F}$  is a maximal monotone operator with  $\partial \tilde{F}(c) = -\Delta_N \partial F(c)$  and*

$$\mathcal{D}(\partial \tilde{F}) = \{c \in \mathcal{D}(\partial F) : \partial F(c) = \mu_0 = -\Delta c + P_0 \phi'(c) \in H_{(0)}^1(\Omega)\}. \quad (4.8)$$

**Proof:** The lower semi-continuity is proved in the same way as in Lemma 4.1. Then the fact that  $\partial \tilde{F}$  is a maximal monotone operator follows from Corollary 1.2 and Lemma 1.3 in [19, Chapter IV].

First let  $c \in \mathcal{D}(\partial \tilde{F})$  and let  $w \in \partial \tilde{F}(c)$ , i.e.,

$$(w, c' - c)_{H_{(0)}^{-1}} \leq \tilde{F}(c') - \tilde{F}(c) \quad \text{for all } c' \in H_{(0)}^{-1}(\Omega). \quad (4.9)$$

Now let  $\mu_0 = -\Delta_N^{-1} w$  and choose  $c' \in L^2(\Omega)$ . Then

$$\begin{aligned} (\mu_0, c' - c)_{L^2} &= -(\nabla \mu_0, \nabla \Delta_N^{-1}(c' - c))_{L^2} = (\nabla \Delta_N^{-1} w, \nabla \Delta_N^{-1}(c' - c))_{L^2} \\ &= (w, c' - c)_{H_{(0)}^{-1}} \leq \tilde{F}(c') - \tilde{F}(c) = F(c') - F(c) \end{aligned}$$

for all  $c' \in L^2(\Omega)$ . Hence  $\mu_0 = -\Delta c + P_0 \phi'(c) \in \mathcal{D}(\partial F)$ . On the other hand,  $\mu_0 = -\Delta_N^{-1} w \in H_{(0)}^1(\Omega)$ . This implies that  $\partial \tilde{F}(c) = -\Delta_N \partial F(c)$  and

$$\mathcal{D}(\partial \tilde{F}) \subseteq \{c \in \mathcal{D}(\partial F) : \mu_0 = -\Delta c + P_0 \phi'(c) \in H_{(0)}^1(\Omega)\}.$$

Conversely, let  $c \in \mathcal{D}(\partial F)$  such that  $\mu_0 = -\Delta c + P_0 \phi'(c) \in H_{(0)}^1(\Omega)$ . Then one easily verifies that  $w = -\Delta_N \mu_0$  satisfies (4.9) by the same calculations as above. Hence  $c \in \mathcal{D}(\partial \tilde{F})$  and (4.8) is proved.  $\blacksquare$



## 5 Existence of Unique Solutions

First of all, we can assume w.l.o.g. that

$$m(c_0) = \frac{1}{|\Omega|} \int_{\Omega} c_0 dx = 0. \quad (5.1)$$

As in the previous section we can always reduce to this case by a simple shift. Since (5.1) implies that any solution of (1.4)-(1.5) as in Theorem 1.2 satisfies

$$\frac{d}{dt} \int_{\Omega} c(x, t) dx = \int_{\Omega} \Delta \mu dx = 0,$$

we conclude  $m(c(t)) = 0$  for almost all  $t > 0$ .

We will consider (1.4)-(1.7) as an evolution equation on  $H_{(0)}^{-1}(\Omega)$  in the following way:

$$\partial_t c + \mathcal{A}(c) + \mathcal{B}c = 0, \quad t > 0, \quad (5.2)$$

$$c|_{t=0} = c_0, \quad (5.3)$$

where

$$\begin{aligned} \langle \mathcal{A}(c), \varphi \rangle_{H_{(0)}^{-1}, H_{(0)}^1} &= (\nabla \mu, \nabla \varphi)_{L^2} \quad \text{with } \mu = -\Delta c + \phi'(c) \\ \langle \mathcal{B}c, \varphi \rangle_{H_{(0)}^{-1}, H_{(0)}^1} &= d(\nabla c, \nabla \varphi)_{L^2}, \quad \varphi \in H_{(0)}^1(\Omega), \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \{c \in H^2(\Omega) : c(x) \in [a, b] \text{ for all } x \in \Omega, \phi'(c) \in L^2(\Omega), \\ &\quad \phi''(c)|\nabla c|^2 \in L^1(\Omega), \mu = -\Delta c + \phi'(c) \in H^1(\Omega), \partial_\nu c|_{\partial\Omega} = 0\} \\ \mathcal{D}(\mathcal{B}) &= H_{(0)}^1(\Omega) \subset H_{(0)}^{-1}(\Omega). \end{aligned}$$

In other words

$$\mathcal{A}(c) = \Delta_N(\Delta c - P_0 \phi'(c)), \quad \mathcal{B}c = d\Delta_N c,$$

where  $\Delta_N: H_{(0)}^1(\Omega) \subset H_{(0)}^{-1}(\Omega) \rightarrow H_{(0)}^{-1}(\Omega)$  is the Laplace operator with Neumann boundary conditions as above, which is considered as an unbounded operator on  $H_{(0)}^{-1}(\Omega)$ .

In order to apply Theorem 3.1 we use that by Corollary 4.4  $\mathcal{A} = \partial \tilde{F}$  is a maximal monotone operator with  $\tilde{F} = \varphi_1 + \varphi_2$ ,

$$\begin{aligned} \varphi_1(c) &= \frac{1}{2} \int_{\Omega} |\nabla c(x)|^2 dx, \quad \text{dom } \varphi_1 = H_{(0)}^1(\Omega), \\ \varphi_2(c) &= \int_{\Omega} \phi(c(x)) dx, \\ \text{dom } \varphi_2 &= \text{dom } \varphi = \{c \in H_{(0)}^1(\Omega) : c(x) \in [a, b] \text{ a.e. in } \Omega\} \end{aligned}$$

Obviously,  $\varphi_1|_{H_{(0)}^1(\Omega)}$  is a bounded, coercive quadratic form on  $H_{(0)}^1(\Omega)$ .

**Proof of Theorem 1.2:** We apply Theorem 3.1 to the choice  $H_1 = H_{(0)}^1(\Omega)$ ,  $H_0 = H_{(0)}^{-1}(\Omega)$ ,  $f = 0$ , and  $\varphi_1, \varphi_2$  as above, where we assume w.l.o.g. that  $\phi(c) \geq 0$ . This gives the existence of a unique solution  $c: [0, \infty) \rightarrow H_0$  of (5.2)-(5.3) such that  $c \in W_2^1(0, T, H_0) \cap L^\infty(0, T; H_1)$ ,  $\varphi(c) \in L^\infty(0, T)$  for every  $T > 0$  and  $c(t) \in \mathcal{D}(\mathcal{A})$  for almost all  $t > 0$ .

In order to prove (1.9), we use that

$$E(c(t)) = \tilde{F}(c(t)) - \frac{d}{2}\|c(t)\|_{L^2}^2.$$

Because of Lemma 4.3 in [19, Chapter IV], we have

$$\frac{d}{dt}\tilde{F}(c(t)) = (\partial\tilde{F}(c(t)), \partial_t c(t))_{H_{(0)}^{-1}} = -\|\partial_t c(t)\|_{H_{(0)}^{-1}}^2 - (\mathcal{B}c(t), \partial_t c(t))_{H_{(0)}^{-1}}.$$

Moreover,

$$\begin{aligned} (\mathcal{B}c(t), c(t))_{H_{(0)}^{-1}} &= -d(\Delta_N c(t), \partial_t c(t))_{H_{(0)}^{-1}} = d(\nabla c(t), \nabla \Delta_N^{-1} \partial_t c(t))_{L^2} \\ &= -d\langle \partial_t c(t), c(t) \rangle_{H_{(0)}^{-1}, H_{(0)}^1} = -\frac{d}{2} \frac{d}{dt} \|c(t)\|_{L^2}^2 \end{aligned}$$

due to [22, Proposition 23.23] and  $\|\partial_t c(t)\|_{H_{(0)}^{-1}} = \|\Delta_N \mu(t)\|_{H_{(0)}^{-1}} = \|\mu(t)\|_{H_{(0)}^1}$ . Hence integration on  $[0, T]$  yields (1.9). In particular, (1.9) implies  $\partial_t c = \Delta_N \mu \in L^2(0, \infty; H_{(0)}^{-1}(\Omega))$  and  $c \in L^\infty(0, \infty; H_{(0)}^1(\Omega))$ .

In order to derive the higher regularity, we apply  $\partial_t^h$  to (5.2) and take the inner product with  $\partial_t^h c$  in  $H_{(0)}^{-1}(\Omega)$ , where  $\partial_t^h f(t) = \frac{1}{h}(f(t+h) - f(t))$ ,  $t, h > 0$ . This gives

$$\begin{aligned} &\frac{1}{2}\|\partial_t^h c(t)\|_{H_{(0)}^{-1}}^2 + \int_s^t (\nabla \partial_t^h c(\tau), \nabla \partial_t^h c(\tau))_{L^2} d\tau \\ &\leq d \int_s^t \|\partial_t^h c(\tau)\|_{L^2}^2 d\tau + \frac{1}{2}\|\partial_t^h c(s)\|_{H_{(0)}^{-1}}^2 \\ &\leq C \int_s^t \|\partial_t^h c(\tau)\|_{H_{(0)}^1} \|\partial_t^h c(\tau)\|_{H_{(0)}^{-1}} d\tau + \frac{1}{2}\|\partial_t^h c(s)\|_{H_{(0)}^{-1}}^2 \end{aligned}$$

because of

$$(\partial_t^h \mathcal{A}(c(\tau)), \partial_t^h c(\tau))_{H_{(0)}^{-1}} \geq (\partial_t^h c(\tau), \partial_t^h c(\tau))_{H^1}$$

and (2.1). Furthermore, since  $\partial_t c \in L^2(0, \infty; H_{(0)}^{-1}(\Omega))$ , we have

$$\|\partial_t^h c(s)\|_{H_{(0)}^{-1}} \leq \frac{1}{h} \int_s^{s+h} \|\partial_t c(\tau)\|_{H_{(0)}^{-1}} d\tau \xrightarrow{h \rightarrow 0} \|\partial_t c(s)\|_{H_{(0)}^{-1}}$$

for almost every  $s > 0$  and  $\|\partial_t^h c\|_{L^2(0, \infty; H_{(0)}^{-1})} \leq \|\partial_t c\|_{L^2(0, \infty; H_{(0)}^{-1})}$ . Hence  $\|\partial_t^h c\|_{L^2(s, t; H_{(0)}^1)}$  and  $\|\partial_t^h c(t)\|_{H_{(0)}^{-1}}$  are uniformly bounded in  $h > 0, t > 0$  for all  $s > 0$ . On the other

hand  $\partial_t^h c \rightarrow_{h \rightarrow 0} \partial_t c$  in  $L^2(0, \infty; H_{(0)}^{-1}(\Omega))$ . Thus the uniform bounds on  $\partial_t^h c$  yield that  $\partial_t c \in L^2(s, \infty; H_{(0)}^1(\Omega)) \cap L^\infty(s, \infty; H_{(0)}^{-1}(\Omega))$  for every  $s > 0$ .

In order to derive the estimate near  $t = 0$ , we apply again  $\partial_t^h$  to (5.2) and take the inner product with  $t\partial_t^h c$ , which gives

$$\frac{1}{2} \|\partial_t^h c(t)\|_{H_{(0)}^{-1}}^2 + \int_0^t \tau \|\partial_t^h c(\tau)\|_{H_{(0)}^1} d\tau \leq C \int_0^t \tau \|\partial_t^h c(\tau)\|_{L^2}^2 d\tau$$

proceeding as above, yields  $t^{\frac{1}{2}} \partial_t c \in L^2(0, 1; H_{(0)}^1(\Omega)) \cap L^\infty(0, 1; H_{(0)}^{-1}(\Omega))$

This implies  $\kappa\mu = \kappa\Delta_N^{-1} \partial_t c \in L^\infty(0, \infty; H_{(0)}^1(\Omega))$ . Thus (4.2) yields  $\kappa\phi'(c), \kappa\nabla^2 c \in L^\infty(0, \infty; H^2(\Omega))$  since  $\kappa\partial F(c) = \kappa\mu + \kappa dc \in L^\infty(0, \infty; H^1(\Omega))$ .

It remains to prove the continuity  $c_0 \mapsto c(t)$ . To this end let  $c_0^j \in Z_0$ ,  $j = 1, 2$ , and let  $c_j(t)$  be the unique solutions of (5.2) with initial values  $c_j|_{t=0} = c_0^j$ . Then

$$\partial_t(c_1 - c_2) + \mathcal{A}(c_1) - \mathcal{A}(c_2) + \mathcal{B}(c_1 - c_2) = 0.$$

Now multiplying the latter identity by  $c_1 - c_2$  in  $H_{(0)}^{-1}(\Omega)$  we conclude

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|c_1(t) - c_2(t)\|_{H_{(0)}^{-1}}^2 + (\mathcal{A}(c_1(t)) - \mathcal{A}(c_2(t)), c_1(t) - c_2(t))_{H_{(0)}^{-1}} \\ \leq -(\mathcal{B}(c_1 - c_2), c_1 - c_2)_{H_{(0)}^{-1}} \leq C \|c_1(t) - c_2(t)\|_{L^2}^2. \end{aligned}$$

Using  $\|w\|_{L^2}^2 \leq \|w\|_{H_{(0)}^1} \|w\|_{H_{(0)}^{-1}} \leq C_\varepsilon \|w\|_{H_{(0)}^{-1}}^2 + \varepsilon \|w\|_{H^1}^2$  and

$$\frac{1}{C} \|c_1(t) - c_2(t)\|_{H_{(0)}^1}^2 \leq (\mathcal{A}(c_1(t)) - \mathcal{A}(c_2(t)), c_1(t) - c_2(t))_{H_{(0)}^{-1}},$$

we derive

$$\frac{1}{2} \frac{d}{dt} \|c_1(t) - c_2(t)\|_{H_{(0)}^{-1}}^2 + \frac{1}{C} \|c_1(t) - c_2(t)\|_{H^1}^2 \leq C \|c_1(t) - c_2(t)\|_{H_{(0)}^{-1}}^2.$$

Hence the Lemma of Gronwall implies

$$\|c_1(t) - c_2(t)\|_{H_{(0)}^{-1}}^2 \leq e^{2Ct} \|c_0^1 - c_0^2\|_{H_{(0)}^{-1}}^2,$$

which implies the strong continuity of  $c_0 \mapsto c(t)$  w.r.t. to the  $H_{(0)}^{-1}$ -norm. Finally, since  $Z_0 \ni c_0 \mapsto c(t) \in H^2(\Omega)$  is a bounded mapping, interpolation yields the continuity  $c_0 \mapsto c(t)$  w.r.t. to the  $H^1$ -norm. This finishes the proof.  $\blacksquare$

## 6 Convergence to Equilibrium

Again we assume w.l.o.g. (5.1).

From Theorem 1.1 it follows that  $c \in L_\infty(J_\delta; H^2(\Omega))$  and  $\partial_t c \in L_2(J_\delta; H^1(\Omega))$ , where  $J_\delta := [\delta, T]$ ,  $T < \infty$  and  $\delta \in (0, T)$  may be arbitrarily small. Hence by Sobolev embedding we obtain  $c \in C^{\frac{1}{2}}(J_\delta; H^1(\Omega))$  and interpolation yields  $c \in C(J_\delta; H^{2r}(\Omega))$  for all  $r \in [0, 1)$ . Observe that  $\delta > 0$  does not depend on the initial value  $c_0$ . We set

$$Z \equiv Z_0 = \{z \in H_{(0)}^1(\Omega) : E(z) < \infty\},$$

and define a family of operators  $\mathcal{S} := \{S(t)\}_{t>0}$  by

$$S(t): Z \rightarrow Z, \quad S(t)c_0 = c(t; c_0), \quad t \in \mathbb{R}_+.$$

Here  $c(t; c_0)$  denotes the solution due to Theorem 1.1 with initial value  $c_0 \in Z$ . In the sequel we will use the abbreviation  $c(t)$  for the solution  $c(t; c_0)$ . The fact that the range of  $S(t)$  is a subset of  $Z$  follows from the energy equation (1.9). By the strong continuity of the mapping  $c_0 \mapsto c(t)$  w.r.t. to the  $H^1$ -norm, cf. Theorem 1.2 it holds that  $S(t) \in C(Z; Z)$ . Moreover,  $c(t) \xrightarrow{t \rightarrow 0} c_0$  in  $H^1(\Omega)$  since  $\lim_{t \rightarrow 0} c(t) = c_0$  in  $H_{(0)}^{-1}(\Omega)$  and  $c(t)$  is uniformly bounded in  $H^1(\Omega)$ . Together with  $\lim_{t \rightarrow 0} \|u(t)\|_{H^1} = \|u_0\|_{H^1}$  due to (1.9) this implies  $\lim_{t \rightarrow 0} c(t) = c_0$  in  $H^1(\Omega)$ . Combining this with  $c \in C^{\frac{1}{2}}(J_\delta; H^1(\Omega))$ , we see that  $S(\cdot)z \in C([0, \infty); Z)$ . Hence  $\mathcal{S}$  is a dynamical system on  $Z$  in the sense of [4, Definition 9.1.1].

Furthermore, we define the  $\omega$ -limit set  $\omega(c_0)$  by

$$\omega(c_0) = \{z \in H^{2r}(\Omega) \cap Z : \exists (t_n) \nearrow \infty \text{ s.t. } c(t_n) \rightarrow z \text{ in } H^{2r}(\Omega)\}, \quad (6.1)$$

where  $r \in [1/2, 1)$ . The estimate  $\kappa c \in L_\infty(0, \infty; H^2(\Omega))$  from Theorem 1.2 yields that the orbit  $\{c(t)\}_{t \geq \delta}$  is relatively compact in  $H^{2r}(\Omega)$ ,  $r \in [1/2, 1)$ , therefore  $\omega(c_0) \neq \emptyset$  and  $\omega(c_0)$  is connected (see for instance [4, Theorem 9.1.8]). Since the definition (6.1) is equivalent to

$$\omega(c_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \{c(t)\}} = \bigcap_{s \geq \delta} \overline{\bigcup_{t \geq s} \{c(t)\}},$$

it follows immediately that  $\omega(c_0)$  is compact in  $H^{2r}(\Omega)$  for each  $r \in [1/2, 1)$ , since  $\omega(c_0)$  is the intersection of a decreasing sequence of compact sets. Moreover, it follows from relative compactness of the orbits and (6.1) that

$$\lim_{t \rightarrow \infty} \text{dist}(S(t)c_0, \omega(c_0)) = 0 \quad \text{in } H^{2r}(\Omega).$$

We call  $z \in Z$  a *stationary point* of  $\mathcal{S}$ , if  $S(t)z \equiv z$  for all  $t \in \mathbb{R}_+$  and the set of all stationary points will be denoted by  $\mathcal{E}$ . A function  $E: Z \rightarrow \mathbb{R}$  is called a *Lyapunov function* for  $\mathcal{S}$ , if  $E(S(t)z) \leq E(z)$  for a.e.  $t \geq 0$  and all  $z \in Z$ . In addition  $E$  is a *strict Lyapunov function*, if the identity  $E(S(t)z) = E(z)$  for all  $t \in \mathbb{R}_+$  implies that  $z$  is a stationary point.

Let us recall the energy functional from the introduction

$$E(c) = \frac{1}{2} \int_{\Omega} |\nabla c|^2 dx + \int_{\Omega} f(c) dx,$$

where we take the space  $Z$  from above as the underlying domain of definition. Due to the formula (1.9) it holds that  $E: Z \rightarrow \mathbb{R}$  is a strict Lyapunov function for  $\mathcal{S}$ . Following the lines of the proof of [4, Theorem 9.2.7] it holds that each  $z \in \omega(c_0)$  is a stationary point and therefore  $\omega(c_0) \subset \mathcal{E}$  and

$$\lim_{t \rightarrow \infty} \text{dist}(S(t)z, \mathcal{E}) = 0 \quad \text{in } H^{2r}(\Omega).$$

The first higher order estimate in Theorem 1.1 implies that  $c(t) \in H^2(\Omega)$  for a.e.  $t > 0$ . Hence if  $z \in \mathcal{E}$ , i.e.  $S(t)z = z$  for all  $t > 0$ , we have necessarily  $z \in H^2(\Omega)$  and  $z$  is a solution of the stationary system. On the other hand, if  $z \in H^2(\Omega) \cap Z$  is a solution of the stationary system, then by uniqueness of solutions,  $S(t)z = z$  for all  $t \geq 0$ , hence  $z \in \mathcal{E}$ . Therefore the set  $\mathcal{E}$  is characterized by

$$\mathcal{E} = \{z \in H^2(\Omega) \cap Z : z \text{ solves (1.10)-(1.12)}\}.$$

Note that by Sobolev embedding we have  $H^{2s}(\Omega) \hookrightarrow C(\bar{\Omega})$ , whenever  $s > n/4$ .

**Proposition 6.1** *For each  $u \in \mathcal{E}$  there are constants  $M_j$ ,  $j = 1, 2$ , such that*

$$a < M_1 \leq u(x) \leq M_2 < b,$$

for all  $x \in \bar{\Omega}$ . Moreover, for all  $u \in \omega(c_0)$ , there are uniform constants  $\tilde{M}_j$ ,  $j = 1, 2$ , such that

$$a < \tilde{M}_1 \leq \|u\|_\infty \leq \tilde{M}_2 < b,$$

provided that  $r \in (n/4, 1)$  in (6.1).

**Proof:** W.l.o.g. we may assume that  $a = -1$  and  $b = 1$  and we will only consider the case that there is a point  $x_0 \in \bar{\Omega}$ , s.t.  $u(x_0) = 1$ .

First we assume that there is such a point  $x_0 \in \Omega$  with  $u(x_0) = 1$ . Since  $u \in C(\bar{\Omega})$  and therefore  $\lim_{x \rightarrow x_0} f'(u(x)) = \infty$ , we may use the stationary problem to find a ball  $B_R(x_0) \subset \subset \Omega$ , such that  $\Delta u(x) \geq 0$  for a.e.  $x \in B_R(x_0)$ . Since  $u(x) \in (-1, 1)$  a.e. in  $\Omega$  it holds that

$$1 = \sup_{x \in B_R(x_0)} u(x) = \sup_{x \in \Omega} u(x) \geq 0.$$

Now we are in a situation to apply the strong maximum principle, cf. Gilbarg and Trudinger [14, Theorem 8.19], to conclude  $u(x) \equiv 1$  for all  $x \in \Omega$ , which contradicts the fact that  $u(x) \in (-1, 1)$  for a.e.  $x \in \Omega$ .

The next step is to take care of the points at the boundary  $\partial\Omega \in C^3$ . Assume that there is a point  $x_0 \in \partial\Omega$ , s.t.  $u(x_0) = 1$ . We choose a ball  $B_R(x_0)$  with radius  $R$  and a  $C^2$ -diffeomorphism

$$\Phi: B_R(x_0) \rightarrow V(z_0) \subset \mathbb{R}^n,$$

which maps  $\Omega \cap B_R(x_0)$  onto  $\mathbb{R}_+^n \cap V(z_0)$  for some neighborhood  $B_R(x_0)$  of  $x_0$  and which leaves the normal direction unchanged (see e.g. Giga [13, p. 318]), i.e.

$$\partial_{z_n} \Theta(z)|_{z_n=0} = \nu(\Theta(z))|_{z_n=0}, \quad z \in V(z_0),$$

where  $\Theta := \Phi^{-1}$ . We denote by  $\Theta^*$  the operator

$$(\Theta^*u)(z) = u(\Theta(z)), \quad z \in V(z_0) \cap \mathbb{R}_+^n$$

and define furthermore,  $\Theta_* = (\Theta^{-1})^*$  as well as

$$v(z) = (\Theta^*u)(z), \quad z \in V(z_0) \cap \mathbb{R}_+^n.$$

Since ellipticity of an operator is invariant under diffeomorphisms, it follows that the transformed operator

$$Lv := \Theta^* \Delta(\Theta_*v)$$

is again elliptic. We extend the function  $v(z) = v(z', z_n)$ ,  $z_n \geq 0$  to a function  $\tilde{v}(z)$  by reflection, i.e.

$$\tilde{v}(z) = \tilde{v}(z', z_n) = \begin{cases} v(z', z_n), & z_n \geq 0, \\ v(z', -z_n), & z_n \leq 0, \end{cases}$$

where  $z' := (z_1, \dots, z_{n-1})$ . Since  $\Delta u(x) = f'(u(x)) + \text{const.}$ ,  $x \in \Omega$ , it holds that  $\Delta u(x) \geq M(R)$  for a.a.  $x \in B_R(x_0) \cap \Omega$ , with  $M(R) \rightarrow \infty$ , as  $R \rightarrow 0$ , due to the continuity of  $u$  in  $\bar{\Omega}$  and the fact that  $f'(s) \rightarrow \infty$ , if  $s \rightarrow 1$ . Obviously for the transformed  $v$  we obtain  $Lv(z', z_n) \geq M(R)$  for a.a.  $z \in V(z_0) \cap \mathbb{R}_+^n$ . Choosing  $R > 0$  sufficiently small this yields  $L\tilde{v}(z) \geq 0$  for a.a.  $z \in V(z_0)$ . Furthermore  $|\tilde{v}(z)| \leq 1$  for all  $z \in V(z_0)$  since  $|u(x)| \leq 1$  for all  $x \in B_R(x_0) \cap \Omega$ . By the construction of the diffeomorphism  $\Phi$  and by  $\partial_\nu u|_{\partial\Omega} = \partial_n v|_{x_n=0} = 0$ , the function  $\tilde{v}$  is an element of  $H^2(V(z_0))$  and  $L\tilde{v}(z) \geq 0$  for a.a.  $z \in V(z_0)$ , with  $v(z_0) = 1$ . Therefore  $\tilde{v}$  attains its maximum at the interior point  $z_0 \in V(z_0)$ . Hence again by the strong maximum principle it follows that  $\tilde{v}(z) \equiv 1$  for all  $z \in V(z_0)$ , which is a contradiction.

The existence of the uniform constants  $\tilde{M}_j$  in the second statement of the proposition is an immediate consequence of the compactness of  $\omega(c_0)$  in  $L^\infty(\Omega)$  if  $r \in (n/4, 1)$ . ■

Since the  $\omega$ -limit set  $\omega(c_0)$  is compact in  $H^{2r}(\Omega)$ ,  $r \in (n/4, 1)$ , there exists an open set  $U \supset \omega(c_0)$ , such that for all  $u \in U$  we have

$$a < \tilde{M}_1 - \varepsilon < \|u\|_\infty < \tilde{M}_2 + \varepsilon < b, \quad \varepsilon > 0,$$

where  $\tilde{M}_j$ ,  $j = 1, 2$ , are the uniform constants from Proposition 6.1. Furthermore, it follows from  $\text{dist}(c(t), \omega(c_0)) \rightarrow 0$  in  $H^{2r}(\Omega)$ , as  $t \rightarrow \infty$ , that there is a  $t^* \geq 0$ , with  $c(t) \in U$  for all  $t \geq t^*$ , provided that  $r \in (n/4, 1)$ . Due to this fact, the singularities of  $f$  and its derivatives play no longer any role in our investigations as we are only interested in the behavior of the solution  $u(t)$ , as  $t \rightarrow \infty$ . Therefore we may alter

the function  $f$  outside the interval  $J_\varepsilon := [\tilde{M}_1 - \varepsilon, \tilde{M}_2 + \varepsilon]$  in such a way that for the extension  $\tilde{f}$  of  $f$  it holds that  $\tilde{f} \in C^3(\mathbb{R})$  and additionally  $|\tilde{f}^{(j)}(s)|$ ,  $j = 1, 2, 3$ , are uniformly bounded on  $\mathbb{R}$ . Observe that  $\tilde{f}|_{J_\varepsilon} = f$  and  $f$  is analytic in  $(\tilde{M}_1 - \varepsilon, \tilde{M}_2 + \varepsilon)$ , hence for each  $\varphi \in \omega(c_0)$  and by Proposition 6.1 there exists a neighborhood  $W$  of  $\varphi(x)$ , such that  $f$  is analytic in  $W$ . Now consider the functional  $\tilde{E} : V \rightarrow \mathbb{R}$ , defined by

$$\tilde{E}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \tilde{f}(u) dx,$$

where

$$V = H_{(0)}^1(\Omega) \hookrightarrow H = L_{(0)}^2(\Omega).$$

Observe that  $\tilde{E}|_{U \cap V} = E$ , where  $E$  is the energy functional from Theorem 1.1. The first two Frechét derivatives of  $\tilde{E}$  read as follows.

$$\langle \tilde{E}'(u), h \rangle_{V^*, V} = \int_{\Omega} \nabla u \nabla h dx + \int_{\Omega} \tilde{f}'(u) h dx \quad (6.2)$$

and

$$\langle \tilde{E}''(u) h_1, h_2 \rangle_{V^*, V} = \int_{\Omega} \nabla h_1 \nabla h_2 dx + \int_{\Omega} \tilde{f}''(u) h_1 h_2 dx, \quad (6.3)$$

for all  $u, h, h_j \in V$ ,  $j = 1, 2$ . We omit the proof of these formulas since the derivation is a direct result of the properties of  $\tilde{f}$ . Firstly, we make use of (6.2) to show that each  $\varphi \in \omega(c_0)$  is a critical point of  $\tilde{E}$ .

$$\begin{aligned} \langle \tilde{E}'(\varphi), h \rangle_{V^*, V} &= \int_{\Omega} \nabla \varphi \nabla h dx + \int_{\Omega} \tilde{f}'(\varphi) h dx \\ &= \int_{\Omega} \nabla \varphi \nabla h dx + \int_{\Omega} f'(\varphi) h dx \\ &= \int_{\Omega} (-\Delta \varphi + f'(\varphi)) h dx \\ &= \int_{\Omega} (-\Delta \varphi + P_0 f'(\varphi)) h dx = 0 \end{aligned}$$

for all  $h \in V$ . Hence the claim follows. Secondly, let  $A = -\Delta_N : V \rightarrow V^*$  and let  $A_2$  the part of  $A$  in  $H$ , i.e.,

$$\mathcal{D}(A_2) = \{u \in V : Au \in H\}$$

and  $A_2 u = Au$  for all  $u \in \mathcal{D}(A_2)$ . Then as summarized in Section 2.1

$$\mathcal{D}(A_2) = \{u \in H^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}.$$

Fix  $u \in V$  and let  $h_1 \in \text{Ker } \tilde{E}''(u)$ . It follows from (6.3) that

$$Ah_1 = -\tilde{f}''(u) h_1$$

for all  $h_1 \in \text{Ker } \tilde{E}''(u)$  and thus  $Ah_1 \in H$ . Hence  $\text{Ker } \tilde{E}''(u) \subset \mathcal{D}(A_2)$ .

Note that the embeddings  $V \hookrightarrow H$  and  $\mathcal{D}(A_2) \hookrightarrow H$  are compact, thus  $A$  and  $A_2$  have compact resolvents in  $V^*$  and  $H$ , respectively. Furthermore, by (6.3), the operator  $\tilde{E}''(u)$  and  $\tilde{E}''(u)|_{\mathcal{D}(A_2)}$  are bounded perturbations of  $A$  and  $A_2$ , respectively. Now it follows from [6, Proof of Proposition 6.6] that  $\text{Ker } \tilde{E}''(u)$  is finite dimensional,  $\text{Rg } \tilde{E}''(u)$  and  $\text{Rg } \tilde{E}''(u)|_{\mathcal{D}(A_2)}$  are closed in  $V^*$  and  $H$ , resp.,  $V^*$  is the direct orthogonal sum of  $\text{ker } \tilde{E}''(u)$  and  $\text{Rg } \tilde{E}''(u)$ , and  $H$  is the direct orthogonal sum of  $\text{ker } \tilde{E}''(u)$  and  $\text{Rg } \tilde{E}''(u)|_{\mathcal{D}(A_2)}$ . To prove the validity of the *Lojasiewicz-Simon inequality* for the functional  $\tilde{E}$ , we want to apply Chill [5, Corollary 3.11]. To this end we define  $X = \mathcal{D}(A_2)$ ,  $Y = H$ ,  $W = V^*$  and denote by  $P: V \rightarrow V$  the continuous orthogonal projection with  $\text{Rg } P = \text{ker } \tilde{E}''(u)$ . Since  $f$  is real analytic in  $J_\varepsilon$  it holds that for each  $\varphi \in \omega(c_0)$  the function  $\tilde{E}'$  is real analytic in a neighborhood of  $\varphi$  in  $X$  (see e.g. [5, Proof of Corollary 4.6]). By the above considerations all the assumptions of [5, Corollary 3.11] can be readily checked. This results in the following

**Proposition 6.2 (Lojasiewicz-Simon inequality)** *Let  $c$  be the global solution of (1.4)-(1.7) in the sense of Theorem 1.2 and suppose that  $\varphi \in \omega(c_0)$ . Then there exist constants  $\theta \in (0, \frac{1}{2}]$ ,  $C, \sigma > 0$  such that*

$$|\tilde{E}(u) - \tilde{E}(\varphi)|^{1-\theta} \leq C \|\tilde{E}'(u)\|_{V^*},$$

whenever  $\|u - \varphi\|_V \leq \sigma$ .

Now we are in the position to prove Theorem 1.3:

**Proof of Theorem 1.3:** First of all we note that the functional  $\tilde{E}|_{\omega(c_0)} = E|_{\omega(c_0)}$  is constant. This follows from (1.9) and the embedding  $H^{2r}(\Omega) \hookrightarrow L^\infty(\Omega)$ . We denote this constant by  $\tilde{E}_\infty$ . By Proposition 6.2 the Lojasiewicz-Simon inequality holds for each  $\varphi \in \omega(c_0)$ . Hence, by compactness of the  $\omega$ -limit set, we may choose finitely many balls  $B_{\sigma_i}(\varphi_i)$ , such that

$$\tilde{U} := \bigcup_{i=1}^N B_{\sigma_i}(\varphi_i) \supset \omega(c_0)$$

and in every ball the Lojasiewicz-Simon inequality is valid. Therefore we are allowed to pick uniform constants  $C > 0, \theta \in (0, 1/2]$ , such that

$$|\tilde{E}(u) - \tilde{E}_\infty|^{1-\theta} \leq C \|\tilde{E}'(u)\|_{V^*},$$

for all  $u \in \tilde{U}$ . Obviously, again by a compactness argument, there exists  $\tilde{t} \geq 0$ , so that  $u(t) \in \tilde{U}$ , whenever  $t \geq \tilde{t}$ . Recall that there is an open set  $U \supset \omega(c_0)$  and a time  $t^* \geq 0$ , such that

$$c(t, \Omega) \subseteq J_\varepsilon,$$

for all  $t \geq t^*$ . Set  $\bar{t} = \max\{t^*, \tilde{t}\}$ . After these preliminaries we define a function  $H: [\bar{t}, \infty) \rightarrow \mathbb{R}_+$  by

$$H(t) = (\tilde{E}(c(t)) - \tilde{E}_\infty)^\theta.$$



Since  $\tilde{E}|_{U \cap V} = E$  it holds by (1.9) that  $H$  is a non-increasing and nonnegative function. We compute

$$\begin{aligned} -\frac{d}{dt} H(t) &= -\theta \frac{d}{dt} \tilde{E}(c(t)) |\tilde{E}(c(t)) - \tilde{E}_\infty|^{\theta-1} \\ &= -\theta \frac{d}{dt} E(c(t)) |\tilde{E}(c(t)) - \tilde{E}_\infty|^{\theta-1} \\ &\geq C \frac{\|\nabla \mu(t)\|_2^2}{\|\tilde{E}'(c(t))\|_{V^*}}. \end{aligned} \tag{6.4}$$

The term  $\|\tilde{E}'(c(t))\|_{V^*}$  may be estimated as follows.

$$\begin{aligned} \|\tilde{E}'(c(t))\|_{V^*} &= \sup_{\|h\|_V \leq 1} |\langle \tilde{E}'(c(t)), h \rangle_{V^*, V}| \\ &= \sup_{\|h\|_V \leq 1} \left| \int_{\Omega} (-\Delta c(t) + f'(c(t))) h \, dx \right| \\ &= \sup_{\|h\|_V \leq 1} \left| \int_{\Omega} \mu(t) h \, dx \right| = \sup_{\|h\|_V \leq 1} \left| \int_{\Omega} P_0 \mu(t) h \, dx \right| \\ &\leq \sup_{\|h\|_V \leq 1} \|P_0 \mu(t)\|_2 \|h\|_2 \leq C \|\nabla \mu(t)\|_2 \sup_{\|h\|_V \leq 1} \|h\|_V = C \|\nabla \mu(t)\|_2. \end{aligned}$$

Thus we obtain from (6.4) the estimate

$$-\frac{d}{dt} H(t) \geq c \|\nabla \mu(t)\|_2.$$

Integrating the latter inequality from  $\bar{t}$  to  $\infty$  yields  $\nabla \mu \in L_1(\mathbb{R}_+; L_2(\Omega))$  and by equation (1.4) we obtain  $\partial_t c \in L_1(\mathbb{R}_+; H_{(0)}^{-1}(\Omega))$ . Therefore  $\lim_{t \rightarrow \infty} c(t) =: \varphi$  exists in  $H_{(0)}^{-1}(\Omega)$  and by relative compactness the limit even exists in  $H^{2r}(\Omega)$ . The fact that  $\varphi$  is a solution of the stationary system has already been proven. ■

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