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Scaling laws of domain walls in magnetic
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Abstract

This paper investigates magnetic 180 degree domain walls in thin wires. It establishes a crossover between two scaling regimes for the energy as a function of the radius R . For small radii the optimal scaling can be realized by a transverse wall for which the magnetization is constant on each cross section. For large radii a vortex wall yields the optimal scaling. Moreover we show that for $R \rightarrow 0$ the energy minimization problem Γ -converges to a local, one dimensional problem where the energy is given by $\pi \|\partial_x m\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})}^2$.

1 Introduction

1.1 Motivation from physics

In the last years several groups have succeeded in the production and investigation of magnetic wires with less than 100 nm diameter. Arrays of such magnetic nanowires are in consideration as future high density storage devices [AXF⁺05]. The time necessary to change the magnetization of a nanowire is directly related to the writing and reading speed of such a device. Therefore it is important to understand the reversal process of magnetic nanowires. It is known that the reversal of the magnetization starts at one end of the wire and then a domain wall separating the already reversed part from the not yet reversed part is propagating through the wire. However, there are few experimental results about the speed of the wall [AAX⁺03, BNK⁺05, HG95, NTM03] and there are no experimental results about the form of the wall.

In numerical simulations of the magnetic reversal process of nanowires, several groups, e.g. [FSS⁺02, HK04, WNU04], have observed two different reversal modes. These modes depend on the wire thickness and correspond to very different switching speeds. For thin wires the transverse mode is observed: the magnetization is constant on each cross section, rotating and moving along the wire (Figure 1). For thick wires the vortex mode is observed: the magnetization is approximately tangential to the boundary and

forms a vortex which moves along the wire. (Figure 2). In some simulations, when looking more closely, one can see additional effects like the periodic creation and annihilation of singularities [HK04]. The vortex mode is much faster than the transverse mode.

For nickel the transition from the transverse mode to the vortex mode occurs at a radius of about 25 nm.

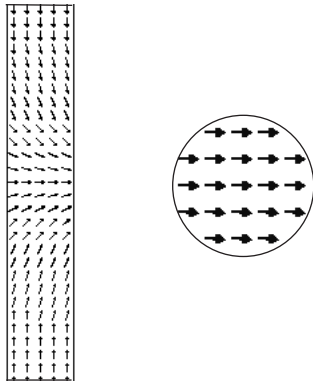


Figure 1: Transverse Mode: longitudinal section and cross section

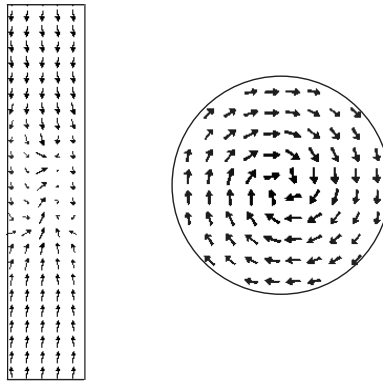


Figure 2: Vortex Mode: longitudinal section and cross section

Forster et al. [FSS⁺02] suggest that the reversal modes correspond to minimizers of the static energy functional. In this work we investigate the static energy functional and show a crossover in the scaling of the energy as a function of the radius. This crossover corresponds to the change from the scaling of the energy of transversal walls to the scaling the energy of the vortex walls.

1.2 The model

We work in the framework of micromagnetism. This is a mesoscopic continuum theory that assigns a nonlocal nonconvex energy to each magnetization m from the domain Σ to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. If necessary, m is extended by zero outside Σ . Experimentally observed ground states of the magnetization correspond to minimizers of the micromagnetic energy functional

$$E(m) := \underbrace{\int_{\Sigma} A |\nabla m|^2}_{\text{exchange energy}} + \underbrace{\int_{\mathbb{R}^3} K_d |\nabla u|^2}_{\text{stray field energy}} + \underbrace{E_{\text{an}}(m)}_{\text{anisotropy energy}} - \underbrace{\int_{\Sigma} h \cdot m}_{\text{external field energy}}$$

Here h is the external field and u is the weak solution of $\Delta u = \text{div } m$ in \mathbb{R}^3 , i.e., $\nabla u = H(m)$ is the projection of m on gradient fields. We refer to [DKMO05, HS00] for a general discussion of the micromagnetic model.

In this paper we study static domain walls, so the external field h is zero. We additionally assume that the material is magnetically soft, i.e., without anisotropy. The wire is represented by

$$\Sigma := \Sigma(R) := \mathbb{R} \times D_R := \mathbb{R} \times \{y \in \mathbb{R}^2 : |y| < R\}.$$

The magnetization $m: \Sigma \rightarrow \mathbb{S}^2$ has the components m_x, m_{y_1}, m_{y_2} , where m_x is the component in direction of the wire. To simplify the calculations, we measure distances in multiples of the characteristic length $\sqrt{\frac{A}{K_d}}$, also called the exchange length or the Bloch line width, and energies as multiples of $\sqrt{\frac{A^3}{K_d}}$. In these units the energy E is

$$E(m) := E(m, R) := E_{\text{ex}}(m) + E_H(m) := \int_{\Sigma} |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla u|^2,$$

where u is again the weak solution of $\Delta u = \text{div } m$. We define the admissible set

$$\mathcal{M} := \mathcal{M}(R) := \{m: \Sigma(R) \rightarrow \mathbb{S}^2 \mid E(m) < \infty\}.$$

We are interested in magnetizations with a 180 degree domain wall, so we would like to consider a subset \mathcal{M}_l of \mathcal{M} with $\lim_{x \rightarrow -\infty} m(x, \cdot) = -\vec{e}_x$ and $\lim_{x \rightarrow \infty} m(x, \cdot) = \vec{e}_x$. Initially it is not clear in which sense the limits should be understood. However, in Section 4 the set \mathcal{M} will be characterized in the following way.

Theorem 1. *Set*

$$\chi: \mathbb{R} \rightarrow [-1, 1], \quad x \mapsto \begin{cases} x & \text{if } |x| < 1 \\ \text{sign}(x) & \text{otherwise.} \end{cases}$$

A function $m: \Sigma \rightarrow \mathbb{S}^2$ is in \mathcal{M} if and only if one of the four maps $m \pm \vec{e}_x$, $m \pm \chi \vec{e}_x$ is in $H^1(\Sigma)$.

This motivates the definition

$$\mathcal{M}_l := \mathcal{M}_l(R) := \{m \in \mathcal{M}(R) \mid m - \chi \vec{e}_x \in H^1(\Sigma)\}.$$

To study transverse walls and vortex walls, we consider the following restricted classes of admissible maps

$$\begin{aligned} \mathcal{T} &:= \mathcal{T}(R) := \{m \in \mathcal{M}(R) \mid m \text{ is constant on each cross section}\}, \\ \mathcal{V} &:= \mathcal{V}(R) := \left\{ m \in \mathcal{M}(R) \left| \begin{array}{l} m_y(x, y_1, y_2) \text{ is parallel to } (-y_2, y_1) \\ \text{and } |m_y| \text{ depends only on } x, |y| \end{array} \right. \right\}, \\ \mathcal{T}_l &:= \mathcal{T}_l(R) := \mathcal{T}(R) \cap \mathcal{M}_l(R), \quad \mathcal{V}_l := \mathcal{V}_l(R) := \mathcal{V}(R) \cap \mathcal{M}_l(R), \end{aligned}$$

and the infima of the energies

$$E_{\mathcal{M}_l}(R) := \inf_{m \in \mathcal{M}_l(R)} E(m), \quad E_{\mathcal{T}_l}(R) := \inf_{m \in \mathcal{T}_l(R)} E(m), \quad E_{\mathcal{V}_l}(R) := \inf_{m \in \mathcal{V}_l(R)} E(m).$$

To get an idea why transverse walls are energetically favorable in thin wires and vortex walls are energetically favorable in thick wires we rescale and set $m_k(x, y) := m(kx, ky)$. Then

$$E(m_k) = \int_{\Sigma(kR)} |\nabla m_k|^2 + \int_{\Sigma(kR)} m_k \cdot H(m_k) = kE_{\text{ex}}(m) + k^3 E_H(m).$$

This calculation suggests that for small radii the biggest contribution to the energy is the exchange energy E_{ex} whereas for big radii the magnetostatic energy becomes important. In order to reduce the magnetostatic energy, it is favorable to avoid surface charges like in the vortex wall. In order to reduce exchange energy, it is favorable to have constant magnetization on a cross section like in the transverse wall. In this work we will investigate this idea in more detail.

1.3 The main results

We discuss the question of existence of optimal wall profiles, the scaling of the energy and the shape of the optimal wall profile.

Theorem 2 (Existence). *For each radius $R > 0$ there exist minimizers of the energy E in $\mathcal{M}_l(R)$, $\mathcal{T}_l(R)$ and $\mathcal{V}_l(R)$.*

The energy of the optimal wall profile scales like $E_{\mathcal{T}_l}$ when the radius goes to zero and scales like $E_{\mathcal{V}_l}$ for radius to infinity.

Theorem 3 (Energy scaling). *There exist constants c, C such that*

$$\begin{aligned} \text{for } R < 2 : \quad & cR^2 \leq E_{\mathcal{M}_l}(R) \leq E_{\mathcal{T}_l}(R) \leq CR^2, \\ \text{for } R > 2 : \quad & cR^2 \sqrt{\ln(R)} \leq E_{\mathcal{M}_l}(R) \leq E_{\mathcal{V}_l}(R) \leq CR^2 \sqrt{\ln(R)}. \end{aligned}$$

Neither $E_{\mathcal{T}_l}$ nor $E_{\mathcal{V}_l}$ has the optimal scaling in the opposite regime: There exists a constant \tilde{c} such that for all $R \in \mathbb{R}^+$ we have

$$E_{\mathcal{T}_l}(R) \geq \tilde{c}R^{\frac{8}{3}} \quad \text{and} \quad E_{\mathcal{V}_l}(R) \geq \tilde{c}R.$$

This shows that the transverse wall is energetically favorable for small radii and the vortex wall is energetically favorable for big radii. However, the constants are not sharp enough to get good estimates for the critical radius where the crossover occurs.

Now we come to the optimal wall profile. To capture the essence of the energy minimizing problem for small radii, we use the notion of Γ -convergence as described in [DM93]. We rescale the energy E by a factor of $\frac{1}{R^2}$ and rescale the maps $m: \Sigma \rightarrow \mathbb{S}^2$ to $\hat{m}: \mathbb{R} \times D_1 \rightarrow \mathbb{S}^2$, $(x, y) \mapsto m(x, \frac{y}{R})$.

In the limit we get a reduced problem where the admissible functions are maps from \mathbb{R} to \mathbb{S}^2 and where the energy simplifies to

$$E_{\text{red}}(m) = \pi \|\partial_x m\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})}^2.$$

The minimizer m^{\min} of the reduced problem exists and is unique up to translation and rotation. Its energy is $\sqrt{8}\pi$, its profile is that of a Bloch wall, i.e.,

$$m^{\min} = \left(\tanh\left(\frac{x}{\sqrt{2}}\right), \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)}, 0 \right).$$

Since Γ -convergence implies the convergence of minimizers as well as convergence of the minimal energies, we can conclude that for small radii minimizers of E are almost constant on the cross section and have a profile that resembles a Bloch wall. Their energy can be approximated by $\sqrt{8}\pi R^2$.

For $R \gg 1$ we do not know the shape of minimizer. However, we have example functions in \mathcal{V}_l whose energies have the optimal scaling. They have a square root type singularity and the width of their transition regions scales like $R^2 \sqrt{\ln(R)}$. The latter is in contrast to the regime $R \ll 1$ where the thickness of the transition region of the optimal walls is of order 1.

1.4 Outline of the paper

In this paper we often exploit the close connection between the full problem, the reduced one-dimensional problem and the full problem restricted to \mathcal{T} , the functions that are constant on each cross section. Therefore we do not prove the results in the same order as they are stated above.

Since the main difficulty in analyzing the functional E consists in finding good estimates for the stray field energy E_H , we start by collecting some general results about the stray field in Section 2.

In Section 3, we study the restricted class of transverse walls. We establish a lower bound for $E_{\mathcal{T}_l}$ and show, that the transverse component of the magnetization is bounded by the energy. As a corollary we obtain a characterization of the set \mathcal{T} similar to Theorem 1. In this section we use the representation of the stray field energy via a Fourier multiplier. The calculations regarding the Fourier multiplier can be found in Section 8.

In Section 4, we use the characterization theorem for transverse walls to show Theorem 1 and the existence of minimizers of the energy in \mathcal{M}_l .

In Section 5, we investigate the case of small radii. We establish the energy scaling of $E_{\mathcal{M}_l} \sim E_{\mathcal{T}_l} \sim R^2$ and find the Γ -limit for $R \rightarrow 0$.

In Section 6, we find the lower bound $E_{\mathcal{M}_l} \geq cR^2 \sqrt{\ln(R)}$ for all R that are large enough.

In Section 7, we calculate upper and lower bounds for $E_{\mathcal{V}_l}$ with elementary methods. In particular we get the estimates $E_{\mathcal{V}_l} \leq CR^2 \sqrt{\ln(R)}$ for all $R \geq 2$ and $E_{\mathcal{V}_l} \geq CR$ for all $R \in \mathbb{R}^+$. Combining the first estimate with the result of Section 6, we see that $E_{\mathcal{M}_l}(R)$ scales like $E_{\mathcal{V}_l}(R) \sim R^2 \sqrt{\ln(R)}$ for $R \rightarrow \infty$.

1.5 Definitions and notation

We will use the following conventions. The letter p denotes a point in \mathbb{R}^3 and has the components $p = (x, y_1, y_2) = (x, y)$. A map f with values in \mathbb{R}^3 has the components $f = (f_x, f_{y_1}, f_{y_2})$. We write f_y for $(0, f_{y_1}, f_{y_2})$, i.e., we view f_y as a map to $\{0\} \times \mathbb{R}^2$. For finite and infinite cylinders we use the symbols $\Sigma_l := [-l, l] \times D_R$ and $\Sigma := \mathbb{R} \times D_R$, for the characteristic function of a set Ω we use $\mathbb{1}_\Omega$. Moreover, generalizing the definition in Theorem 1, we define the functions

$$\chi_{c^-}^{c^+} : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto c_{\text{sign}(x)} \min(1, |x|) \vec{e}_x, \quad \chi := \chi_{-1}^1.$$

Now let m be a function $\Sigma \rightarrow \mathbb{R}^3$. The divergence of m consists of two parts: the *body charges* ρ in the interior of the cylinder and the *surface charges* σ , the divergence from the normal component of the magnetization on the surface,

$$\rho(p) = \begin{cases} -\text{div } m(p) & \text{if } p \in \Sigma \\ 0 & \text{otherwise} \end{cases}, \quad \sigma(p) = m \cdot \vec{e}_\nu \text{ for all } p \in \partial\Sigma.$$

The map u is by definition a weak solution of

$$\Delta u = \text{div } m, \quad \text{in } \mathbb{R}^3, \quad (1)$$

if and only if $\nabla u \in L^2(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \nabla u \nabla \eta = \int_{\Sigma} \rho \eta + \int_{\partial\Sigma} \sigma \eta, \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^3). \quad (2)$$

This defines u only up to a constant. We can remove this ambiguity by requiring $u \in L^6(\mathbb{R}^3)$. Note that there is a constant C such that for all functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ the inequality $\|f - c\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}$ holds for some $c \in \mathbb{R}$ if the right hand side exists.

We can decompose u and define u_ρ, u_σ as those maps in $L^6(\mathbb{R}^3)$ that satisfy

$$\int_{\mathbb{R}^3} \nabla u_\rho \nabla \eta = \int_\Sigma \rho \eta \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^3), \quad (3)$$

$$\int_{\mathbb{R}^3} \nabla u_\sigma \nabla \eta = \int_{\partial\Sigma} \sigma \eta \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^3). \quad (4)$$

Finally we set

$$E_{\rho\rho}(m) := \int_{\mathbb{R}^3} |\nabla u_\rho|^2, \quad E_{\sigma\sigma}(m) := \int_{\mathbb{R}^3} |\nabla u_\sigma|^2, \quad E_{\rho\sigma}(m) := \int_{\mathbb{R}^3} \nabla u_\rho \cdot \nabla u_\sigma.$$

Then we have $E_H(m) = E_{\rho\rho}(m) + E_{\sigma\sigma}(m) + 2E_{\rho\sigma}(m)$.

A special case are functions $m: \Sigma \rightarrow \mathbb{R}^3$, that are constant on each cross section. To simplify notation we will often describe such functions by maps $\tilde{m}: \mathbb{R} \rightarrow \mathbb{R}^3$. For a map $f: \mathbb{R} \rightarrow \mathbb{R}^3$ we therefore define $f_\Sigma: \Sigma \rightarrow \mathbb{R}^3$, $(x, y) \mapsto f(x)$ and

$$\begin{aligned} E(f) &:= E(f_\Sigma), & E_{\rho\rho}(f) &:= E_{\rho\rho}(f_\Sigma) \\ E_{\sigma\sigma}(f) &:= E_{\sigma\sigma}(f_\Sigma), & E_{\rho\sigma}(f) &:= E_{\rho\sigma}(f_\Sigma). \end{aligned}$$

2 The stray field energy

Since we are not working on a finite domain, it is initially not clear under which conditions the solution u of the equation $\Delta u = \operatorname{div} m$ exists and ∇u has a finite L^2 norm. An particularly simple case is the case when $m \in L^2(\Sigma)$ since then $\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq \|m\|_{L^2(\Sigma)}$. We will reduce the general case to this situation. We define the set

$$\mathcal{A} := \left\{ m: \Sigma \rightarrow \mathbb{R}^3 \mid \exists c^-, c^+ \in \mathbb{R} \text{ such that } m - \chi_{c^-}^{c^+} \in H^1(\Sigma) \right\}.$$

Lemma 4 below states that, for all $m \in \mathcal{A}$, Equation (1) has a weak solution and that for such m the stray field energy $E_H(m)$ is finite. Later we will show that \mathcal{M} is a subset of \mathcal{A} .

We define the maps G and $K_i: \mathbb{R}^3 \rightarrow \mathbb{R}$, $i \in \{1, 2, 3\}$, by setting

$$G(p) := \frac{1}{4\pi|p|}, \quad K_i(p) := \partial_i G = -\frac{1}{4\pi} \frac{p_i}{|p|^3}.$$

Lemma 4. For $m \in \mathcal{A}$ define the maps $u, u_\rho, u_\sigma: \mathbb{R}^3 \rightarrow \mathbb{R}$ by setting

$$u_\rho(p) := \int_\Sigma G(p-p') \rho(p') dp', \quad u_\sigma(p) := \int_{\partial\Sigma} G(p-p') \sigma(p') dp', \quad u = u_\rho + u_\sigma. \quad (5)$$

Then the following statements hold.

(i) The map u is a weak solution of (1), ∇u is in $L^2(\mathbb{R}^3)$ and we have

$$\nabla u_\rho(p) = \sum_{i=1}^3 \int_{\Sigma} K_i(p-p') \rho(p') \vec{e}_i dp', \quad \nabla u_\sigma(p) = \sum_{i=1}^3 \int_{\partial\Sigma} K_i(p-p') \sigma(p') \vec{e}_i dp'.$$

(ii) The map u_ρ is continuous, contained in $H_{\text{loc}}^2(\mathbb{R}^3)$ and a strong solution of

$$\Delta u_\rho = \rho. \quad (6)$$

The map u_σ is also continuous and its restriction to $\mathbb{R}^3 \setminus \partial\Sigma$ is arbitrarily often differentiable. We have

$$\Delta u_\sigma = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial\Sigma, \quad (7)$$

$$\lim_{\substack{y \rightarrow y_0 \\ y \in \Sigma}} \partial_r u_\sigma - \lim_{\substack{y \rightarrow y_0 \\ y \notin \Sigma}} \partial_r u_\sigma = -\sigma(y_0) \quad \text{for } y_0 \in \partial\Sigma. \quad (8)$$

(iii) The map u is in $L^2(\Sigma)$ and in $L^2(\partial\Sigma)$. We have

$$\int_{\mathbb{R}^3} |\nabla u(p)|^2 dp = \int_{\Sigma} u(p) \rho(p) dp + \int_{\partial\Sigma} u(p) \sigma(p). \quad (9)$$

Proof. The proof uses standard techniques, for details see [Küh]. \square

For some function $m \in \mathcal{A}$ the following Lemma gives a bound on $E_{\rho\rho}(m)$. Note that for all $m \in \mathcal{T}$ and all $m \in \mathcal{V}$ the condition “ $\text{div } m_y = 0$ in Σ ” is fulfilled.

Lemma 5. *If $m \in \mathcal{A}$ with $\partial_x m_x \geq 0$ and $\text{div } m_y = 0$ in Σ then*

$$E_{\rho\rho}(m) \leq 2\pi^2(c^+ - c^-)^2 R^3.$$

Proof. In this case $\rho = \partial_x m_x$ and we calculate

$$\begin{aligned} E_{\rho\rho}(m) &= \int_{D_R} \int_{\mathbb{R}} u_\rho(x, y) \partial_x m_x(x, y) dx dy \\ &= \int_{D_R} \int_{D_R} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial_x m_x(x, y) \partial_x m_x(x', y')}{\sqrt{(x-x')^2 + |y-y'|^2}} dx' dx dy' dy \\ &\leq \int_{D_R} \int_{D_R} \frac{(c^+ - c^-)^2}{|y-y'|} dy' dy \leq \pi R^2 (c^+ - c^-)^2 \int_0^R 2\pi dr. \end{aligned}$$

\square

The following lemma concerns convergence in $L_{\text{loc}}^2(\overline{\Sigma})$, which, by definition, coincides with convergence in all $L^2(\Sigma_l)$, $l \in \mathbb{N}$.

Lemma 6. For $f, g_n \in \mathcal{A} \cap L^\infty(\Sigma)$ ($n \in \mathbb{N}$) let u_f, u_{g_n} be the weak solutions of (1) for $m = f, m = g_n$, respectively. We assume that u_f exists, $\nabla u_f \in L^2(\mathbb{R}^3)$ and $E(g_n), \|(g_n)_y\|_{L^2(\Sigma)}^2$ as well as $\|g_n\|_{L^\infty(\Sigma)}$ are uniformly bounded by some constant M .

(i) If $(g_n)_{n \in \mathbb{N}}$ converges to zero in $L^2_{\text{loc}}(\bar{\Sigma})$ then $\int_{\mathbb{R}^3} \nabla u_f \nabla u_{g_n}$ converges to zero, too.

(ii) If $(g_n)_{n \in \mathbb{N}}$ converges in $L^2_{\text{loc}}(\bar{\Sigma})$ to $g_0 \in \mathcal{A} \cap L^\infty(\Sigma)$ then $E_H(g_0) \leq \liminf_{n \rightarrow \infty} E_H(g_n)$.

Proof. To prove the first part we have to estimate the interaction between the stray field of different parts of the magnetization and show that this interaction decays fast enough. For details see [Küh]. The second part is an immediate consequence of the first one. \square

3 Transverse walls

In this section we investigate functions that are constant on the cross section. To simplify notation, we describe such functions by maps from \mathbb{R} to \mathbb{R}^3 . In particular we will view the functionals $E, E_{\rho\rho}, E_{\sigma\sigma}$ and $E_{\rho\sigma}$ also as functionals on $\{f: \mathbb{R} \rightarrow \mathbb{R}^3\}$ as described in subsection 1.5. The following lemma simplifies the calculation of E_H .

Lemma 7. If $m: \Sigma \rightarrow \mathbb{R}^3$ is constant on each cross section and $E(m) < 0$ then the following equalities hold:

- (i) $E_{\rho\sigma}(m) = 0$,
- (ii) $E_{\sigma\sigma}(m) = E_{\sigma\sigma}(m_{y_1} \vec{e}_{y_1}) + E_{\sigma\sigma}(m_{y_2} \vec{e}_{y_2})$.

Proof. Since ρ is independent of y , the map u_ρ is rotationally symmetric and since $\sigma(x, y) = -\sigma(x, -y)$ we have $u_\sigma(x, y) = -u_\sigma(x, -y)$. Thus $\int_{\mathbb{R}^3} \nabla u_\rho \nabla u_\sigma = 0$. The same type of symmetry argument works for (ii). \square

So the energy of a map $m: \mathbb{R} \rightarrow \mathbb{R}^3$ is given by

$$E(m) = \pi R^2 \|\partial_x m\|_{L^2(\mathbb{R})}^2 + E_{\sigma\sigma}(m) + E_{\rho\rho}(m).$$

In this section we establish for $E_{\mathcal{T}_l}$ an upper bound and two different lower bounds. First we show that there exists a constant $C(R)$ such that for all $m \in \mathcal{T}_l$ the energy $E(m)$ is bounded from below by $C(R) \|m_y\|_{L^2(\Sigma)}^2$. This implies the characterization theorem for \mathcal{T} . Second we combine this first lower bound with an estimate for $E_{\rho\rho}$ to get a lower bound for $E_{\mathcal{T}_l}$.

To get the estimates, we use the representation of the stray field energy via a Fourier multiplier. The derivation of the Fourier multiplier can be found in Section 8.

All Fourier transforms in this paper refer only to the first argument, we will still denote them by $\hat{f} := \mathcal{F}(f)$. Moreover, we choose the constants in a way that $\|f(\cdot, a)\|_{L^2} = \|\hat{f}(\cdot, a)\|_{L^2}$.

When we apply the Fourier transform to the defining partial differential equations for u_ρ and u_σ , for every $\xi \in \mathbb{R}$ we get ordinary differential equations that can be solved explicitly. Of course, this only works when m is constant on the cross section. Using the explicit representation of the Fourier transforms of \hat{u}_ρ and \hat{u}_σ , we get the Fourier multipliers. The following lemma summarizes their properties. The Fourier multipliers involve the modified Bessel functions I_1 and K_1 . For a definition and for properties of these functions see Section 8.

Theorem 8 (Estimates via Fourier multipliers).

(i) For $m_y \in L^2(\mathbb{R}, \{0\} \times \mathbb{R}^2)$ we have

$$E_{\sigma\sigma}(m_y) = R^2 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 g(\xi R) d\xi := R^2 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 \pi K_1(|\xi R|) I_1(|\xi R|) d\xi$$

In particular, g is a smooth function, monotone decreasing in $|t|$ with $g(0) = \frac{\pi}{2}$. Moreover, we have the inequalities

$$1 \leq g(t) \leq \frac{\pi}{2} \text{ for } |t| \leq 1 \quad \text{and} \quad \frac{1}{t} \leq g(t) \leq \frac{\pi}{2t} \text{ for } |t| \geq 1.$$

(ii) Let $m_x: \mathbb{R} \rightarrow \mathbb{R}$ be a map such that $\rho := \partial_x m_x$ is in $L^2(\Sigma)$. We have

$$\begin{aligned} E_{\rho\rho}(m_x \vec{e}_x) &= R^4 \int_{\mathbb{R}} |\hat{\rho}(\xi)|^2 h(\xi R) d\xi \\ &:= R^4 \int_{\mathbb{R}} |\hat{\rho}(\xi)|^2 \frac{\pi}{|\xi R|^2} (1 - 2I_1(|\xi R|)K_1(|\xi R|)) d\xi. \end{aligned}$$

In particular, h is a smooth function with

$$\frac{\pi}{2} \leq \frac{\pi}{2} |\ln(t)| \leq h(t) \leq \pi |\ln(t)| \quad \text{for } t \leq \frac{1}{2}$$

and

$$\frac{0.4}{t^2} \leq h(t) \leq \frac{\pi}{t^2} \quad \text{for } t > \frac{1}{2}.$$

As a Corollary of (i) we directly get an upper bound on $E_{\mathcal{I}_l}$.

Corollary 9. We have $E_{\mathcal{I}_l} \leq \sqrt{8}\pi R^2 + 2\pi^2 R^3$.

Proof. Let m^{red} be the minimizer of $E_{\text{red}} := \pi \left(\|\partial_x m\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|m_y\|_{L^2(\mathbb{R})}^2 \right)$ in \mathcal{I}_l . This minimizer can be calculated explicitly (cf. Lemma 21 below). It is monotone increasing and we have $E_{\text{red}}(m^{\text{red}}) = \sqrt{8}\pi$. So the combination of Theorem 8 (i) and Lemma 5 yields the estimate. \square

We use Theorem 8 now to bound the L^2 -norm of m_y from below.

Lemma 10. *Let $m_y: \mathbb{R} \rightarrow \{0\} \times \mathbb{R}^2$ be a map with $|m_y| \leq 1$ for which $E_{\sigma\sigma}(m) + \|\partial_x m_y\|_{L^2(\Sigma)}^2$ is finite. Then $\|m_y\|_{L^2(\mathbb{R})}$ is finite and*

$$E_{\sigma\sigma}(m) + \|\partial_x m_y\|_{L^2(\Sigma)}^2 \geq \|m_y\|_{L^2(\mathbb{R})}^2 \cdot \min\{R^2, 2R^{\frac{4}{3}}\}. \quad (10)$$

Proof. First, we assume that $\|m_y\|_{L^2(\Sigma)}$ is finite. In this case we can apply the estimate from Theorem 8 (i) and the equality $\|\xi \hat{m}_y\|_{L^2(\Sigma)} = \|\partial_x m_y\|_{L^2(\Sigma)}$. Since $\min_{\xi \in \mathbb{R}} \left(\frac{R}{|\xi|} + \pi R^2 \xi^2 \right) \geq 2R^{\frac{4}{3}}$ we have

$$\begin{aligned} & E_{\sigma\sigma}(m) + \|\partial_x m_y\|_{L^2(\Sigma)}^2 \\ & \geq \int_{-\frac{1}{R}}^{1/R} R^2 |\hat{m}_y(\xi)|^2 d\xi + \int_{\mathbb{R} \setminus [-\frac{1}{R}, \frac{1}{R}]} \left(\frac{R}{|\xi|} + \pi R^2 \xi^2 \right) |\hat{m}_y(\xi)|^2 d\xi \\ & \geq \int_{-\frac{1}{R}}^{1/R} R^2 |\hat{m}_y(\xi)|^2 d\xi + \int_{\mathbb{R} \setminus [-\frac{1}{R}, \frac{1}{R}]} 2R^{\frac{4}{3}} |\hat{m}_y(\xi)|^2 d\xi \\ & \geq \|m_y\|_{L^2(\mathbb{R})}^2 \cdot \min\{R^2, 2R^{\frac{4}{3}}\}. \end{aligned}$$

In order to treat the general case, we fix some number $k > 0$ and decompose \mathbb{R} in three subsets

$$I_1 := [-k, k], \quad I_2 := ([-k-1, -k] \cup [k, k+1]), \quad I_3 := (]-\infty, k-1] \cup [k+1, \infty[).$$

Set $m_y^i := m_y \mathbb{1}_{I_i}$, and define u_i as the corresponding solution of (4). Then $E_{\sigma\sigma}(m) = \|\nabla u_1 + \nabla u_2 + \nabla u_3\|_{L^2(\mathbb{R}^3)}^2$. A direct calculation shows that there is a constant $C(R)$ such that $E_{\sigma\sigma}(m) \geq \frac{1}{2} \|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 - C(R)$. For details see [Küh]. The support of m_y^1 is bounded, so we can calculate

$$\begin{aligned} E_{\sigma\sigma}(m) + \|\partial_x m_y\|_{L^2(\Sigma)}^2 & \geq \frac{1}{2} \left(\|\partial_x m_y\|_{L^2(I_1)}^2 + \|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 \right) - C(R) \\ & \geq \frac{1}{2} \|m_y\|_{L^2([-k, k])}^2 \min\{R^2, 2R^{\frac{4}{3}}\} - C(R). \end{aligned}$$

Since k was arbitrary, $E_{\sigma\sigma}(m) + \|\partial_x m_y\|_{L^2(\Sigma)}$ can only be finite if $\|m_y\|_{L^2(\mathbb{R})}$ is finite. \square

Corollary 11. *A map $m: \Sigma \rightarrow \mathbb{S}^2$ is in \mathcal{T} if and only if m is constant on each cross section and one of the four functions $m \pm \vec{e}_x$, $m \pm \chi \vec{e}_x$ is in $H^1(\Sigma)$.*

Proof. If one of the four functions $m \pm \vec{e}_x$, $m \pm \chi \vec{e}_x$ is in $H^1(\Sigma)$, then m is in \mathcal{A} as defined in Section 2. So, according to Lemma 4 (i), $E_H(m)$ is finite, and thus $E(m)$ is finite.

To show the other implication we assume that $m \in \mathcal{T}$. Then $E_{\text{ex}}(m)$ is finite, therefore ∇m is in $L^2(\Sigma)$. Moreover we have

$$\|1 - |m_x|\|_{L^2(\mathbb{R})}^2 \leq \|1 - m_x^2\|_{L^1(\mathbb{R})} = \|m_y^2\|_{L^1(\mathbb{R})} = \|m_y\|_{L^2(\mathbb{R})}^2.$$

So either one of the four functions $m \pm \vec{e}_x$, $m \pm \chi \vec{e}_x$ is in $H^1(\Sigma)$ or m_x oscillates infinitely often between $+1$ and -1 . In the latter case we have an infinite sequence of disjoint intervals $(I_n)_{n \in \mathbb{N}}$ such that $m_x(I_n) = [-\frac{1}{2}, \frac{1}{2}]$ for all $n \in \mathbb{N}$. But by assumption both

$$E(m) \geq E_{\text{ex}}(m) \geq \sum_{n=1}^{\infty} \frac{1}{|I_n|} \quad \text{and}$$

$$E(m) \geq E_{\sigma\sigma}(m) + \|\partial_x m_y\|_{L^2(\Sigma)}^2 \geq \|m_y\|_{L^2(\mathbb{R})}^2 \min \left\{ R^2, 2R^{\frac{4}{3}} \right\} \sum_{n=1}^{\infty} \frac{1}{4} |I_n|$$

have to be finite. This is impossible. \square

The following lemma concerns monotone increasing rearrangement as defined in [Alb00]. One result in that article is the decrease of a certain energy functional under monotone increasing rearrangement. We apply this result to the functional $E_{\rho\rho}$.

Lemma 12. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a map such that $f - \chi \in H^1(\mathbb{R})$ and let f_{mon} be the monotone increasing rearrangement of f as defined in [Alb00]. Then $E_{\rho\rho}(f\vec{e}_x) \geq E_{\rho\rho}(f_{\text{mon}}\vec{e}_x)$.*

Proof. Let u be the weak solution of $\Delta u = (\partial_x f)\mathbb{1}_{D_R}$ and set $G: \mathbb{R}^3 \rightarrow \mathbb{R}$, $p \mapsto \frac{1}{4\pi|p|}$, as before. Then

$$E_{\rho\rho}(f\vec{e}_x) = \int_{\Sigma} f \partial_x u = \int_{\Sigma} f(x) \partial_x \left(\int_{\Sigma} G(p - p') \partial_{x'} f(x') dp' \right) dx.$$

First, assume $f - \chi \in C_c^\infty(\mathbb{R})$ and integrate by parts carefully. For details of the calculation see [Küh]. Then

$$E_{\rho\rho}(f\vec{e}_x) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x) - f(x'))^2 h(x - x') dx' dx,$$

where

$$h(x) := \int_{D_R} \int_{D_R} \partial_{xx} G(x, y - y') dy' dy$$

is a positive, integrable function. Thus we are in a situation where we can apply [Alb00, Theorem 2.11] and get $E_{\rho\rho}(f\vec{e}_x) \geq E_{\rho\rho}(f_{\text{mon}}\vec{e}_x)$.

When $f - \chi \in H^1(\mathbb{R})$ we use an approximation argument. In [Cor84] it is shown that symmetric rearrangement is continuous in $H^1(\mathbb{R})$. This result can be easily generalized to the case of monotone increasing rearrangement. \square

We use the preceding lemma to estimate $E_{\rho\rho}$ from below.

Lemma 13. *Let $m_x : \mathbb{R} \rightarrow [-1, 1]$ be a function such that one of the four functions $m_x \pm 1$, $m_x \pm \chi$ is in $H^1(\mathbb{R})$. Then*

$$E_{\rho\rho}(m_x \vec{e}_x) \geq \min \left\{ \frac{4}{3} \frac{R^4}{\|1 - m_x^2\|_{L^1(\mathbb{R})}}, \frac{1}{3} R^3 \right\}.$$

In particular, for $m \in \mathcal{T}$ we have

$$E_{\rho\rho}(m) \geq \min \left\{ \frac{4}{3} \frac{R^4}{\|m_y\|_{L^2(\mathbb{R})}^2}, \frac{1}{3} R^3 \right\}.$$

Proof. We assume that m_x is monotone increasing, since monotone increasing rearrangement of m_x decreases $E_{\rho\rho}(m_x \vec{e}_x)$. The estimates for Fourier multiplier in Theorem 8 (ii) yield

$$E_{\rho\rho}(m_x \vec{e}_x) \geq \frac{\pi}{2} R^4 \int_{-\frac{1}{2R}}^{\frac{1}{2R}} \hat{\rho}^2(\xi) d\xi.$$

We set $c := \min \left\{ \frac{\hat{\rho}(0)}{\|\partial_\xi \hat{\rho}\|_{L^\infty(\mathbb{R})}}, \frac{1}{2R} \right\}$. Then

$$\begin{aligned} E_{\rho\rho}(m_x \vec{e}_x) &\geq \frac{\pi}{2} R^4 \int_{-c}^c (\hat{\rho}(0) - \|\partial_\xi \hat{\rho}\|_{L^\infty(\mathbb{R})} |\xi|)^2 d\xi \\ &\geq \pi R^4 \int_0^c \left(\hat{\rho}(0) - \frac{\hat{\rho}(0)}{c} \xi \right)^2 d\xi \\ &= \frac{\pi}{3} R^4 \hat{\rho}(0)^2 c = \min \left\{ \frac{\pi}{3} R^4 \frac{\hat{\rho}(0)^3}{\|\partial_\xi \hat{\rho}\|_{L^\infty(\mathbb{R})}}, \frac{\pi}{6} R^3 \hat{\rho}(0)^2 \right\}. \end{aligned}$$

We calculate

$$\hat{\rho}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \rho = \frac{2}{\sqrt{2\pi}}.$$

For all $t > 0$ we have

$$\begin{aligned} \int_{-t}^t |\rho(x)x| dx &= t m_x(t) - t m_x(-t) - \int_{-t}^t |m_x(x)| dx \\ &\leq \int_{-t}^t 1 - |m_x(x)| dx \leq \int_{-t}^t 1 - m_x(x)^2 dx. \end{aligned}$$

In the limit $t \rightarrow \infty$ we get

$$\|\partial_\xi \hat{\rho}\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|x\rho\|_{L^1(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|1 - m_x^2\|_{L^1(\mathbb{R})},$$

thus

$$E_{\rho\rho}(m_x \vec{e}_x) \geq \min \left\{ \frac{4}{3} R^4 \frac{1}{\|1 - |m_x|^2\|_{L^1(\mathbb{R})}}, \frac{1}{3} R^3 \right\}$$

□

Combining Lemma 10 and Lemma 13 we get a lower bound for $E_{\mathcal{T}_l}$.

Theorem 14. *We have $E_{\mathcal{T}_l} \geq \min \left\{ \frac{1}{3} R^3, 3 R^{\frac{8}{3}} \right\}$.*

Proof. For all $m \in \mathcal{T}_l$ we have

$$\begin{aligned} E(m) &= \pi \|\partial_x m\|_{L^2(\mathbb{R})}^2 + E_{\sigma\sigma}(m) + E_{\rho\rho}(m) \\ &\geq \|m_y\|_{L^2(\mathbb{R})}^2 \min \left\{ R^2, 2R^{\frac{4}{3}} \right\} + \min \left\{ \frac{4}{3} \frac{R^4}{\|m_y\|_{L^2(\mathbb{R})}}, \frac{1}{3} R^3 \right\} \\ &\geq \min \left\{ 2\sqrt{\frac{4}{3}} R^3, 2\sqrt{\frac{8}{3}} R^{\frac{8}{3}}, \frac{1}{3} R^3 \right\} \geq \min \left\{ 3R^{\frac{8}{3}}, \frac{1}{3} R^3 \right\}. \end{aligned}$$

□

4 The characterization theorem for \mathcal{M}_l and the existence of minimizers

We first show an estimate for $\|m_y\|_{L^2(\Sigma)}$. We set $\bar{m}(x) := \frac{1}{|D_R|} \int_{D_R} m(x, y) dy$ and $\tilde{m}(x, y) := m(x, y) - \bar{m}(x)$. We use the estimates from Section 3 to bound $\|\bar{m}_y\|_{L^2(\Sigma)}$ and use the exchange energy to bound $\|\tilde{m}_y\|_{L^2(\Sigma)}$.

Lemma 15. *There exist constants C_1, C_2 that depend only on R such that*

$$\|m_y\|_{H^1(\Sigma)}^2 \leq C_1 E(m), \quad \|\sigma\|_{L^2(\partial\Sigma)}^2 \leq C_2 E(m)$$

Proof. Since $\int_{D_R} \tilde{m}(\cdot, y) dy \equiv 0$ we have

$$\begin{aligned} E_{\text{ex}}(m) &= E_{\text{ex}}(\bar{m}) + E_{\text{ex}}(\tilde{m}) + 2 \int_{\mathbb{R}} (\partial_x \bar{m}(x)) \partial_x \left(\int_{D_R} \tilde{m}(x, y) dy \right) dx \\ &= E_{\text{ex}}(\bar{m}) + E_{\text{ex}}(\tilde{m}). \end{aligned}$$

For almost all $x \in \mathbb{R}$ the map $\nabla_y m(x, \cdot)$ is in $L^2(D_R)$. Using the Poincaré inequality we get $\|\tilde{m}_y(x, \cdot)\|_{L^2(D_R)}^2 \leq 16R^2 \|\nabla_y m_y(x, \cdot)\|_{L^2(D_R)}^2$ almost everywhere. Integration over x yields

$$E_H(\tilde{m}) \leq \|\tilde{m}\|_{L^2(\Sigma)}^2 \leq 16R^2 \|\nabla_y m\|_{L^2(\Sigma)}^2 \leq 16R^2 E(m), \quad (11)$$

thus

$$E(\overline{m}) \leq E_{\text{ex}}(m) + 2E_H(\tilde{m}) + 2E_H(m) \leq (32R^2 + 2)E(m) < \infty.$$

Using (11) and Lemma 10 we get the estimate

$$\begin{aligned} \|m_y\|_{L^2(\Sigma)}^2 &\leq 2 \left(\|\tilde{m}_y\|_{L^2(\Sigma)}^2 + \|\overline{m}_y\|_{L^2(\Sigma)}^2 \right) \\ &\leq 32R^2 E(m) + \frac{2E(\overline{m})}{c_1} \leq \left(32R^2 + \frac{64R^2 + 4}{c_1} \right) E(m) \end{aligned}$$

where $c_1 = \min\{R^2, 2R^{\frac{4}{3}}\}$ which implies the first statement. The second statement is a consequence of the trace estimate for Sobolev spaces. \square

Like in Corollary 11, this estimate implies directly the characterization of maps $m: \Sigma \rightarrow \mathbb{S}^2$ with finite energy.

Theorem 16. *A map $m: \Sigma \rightarrow \mathbb{S}^2$ is in \mathcal{M} if and only if one of the four functions $m \pm \vec{e}_x$, $m \pm \chi \vec{e}_x$ is in $H^1(\Sigma)$.*

We use this result to show the existence of minimizers.

Theorem 17. *For every $R > 0$ there exist minimizers of E in \mathcal{M}_l , \mathcal{T}_l and \mathcal{V}_l .*

Proof. We use the direct method to find a minimizer in \mathcal{M}_l . Let $(m^n)_{n \in \mathbb{N}}$ be a minimizing sequence in \mathcal{M}_l . Since the problem is invariant under translations we can choose the functions m^n in a way that $\overline{m}_x^n(0) = 0$ and $\overline{m}_x^n(x) \leq 0$ for $x \leq 0$. The energy $E(m^n)$ is bounded, therefore $\|\nabla m^n\|_{L^2(\Sigma)}$ is bounded. So there is a map $m^{\text{lim}}: \Sigma \rightarrow \mathbb{S}^2$ and a subsequence, denoted with $(m^n)_{n \in \mathbb{N}}$ as well, such that ∇m^n converges weakly to ∇m^{lim} in $L^2(\Sigma)$ and m^n converges strongly to m^{lim} in $L^2_{\text{loc}}(\overline{\Sigma})$. Then in particular $\overline{m}_x^{\text{lim}}(0) = 0$. The functional E_{ex} is lower semicontinuous with respect to weak L^2 convergence of $(\nabla m^n)_{n \in \mathbb{N}}$, and the functional E_H is lower semicontinuous with respect to convergence in $L^2_{\text{loc}}(\overline{\Sigma})$ (Lemma 6). Thus $E(m^{\text{lim}}) \leq \liminf_{n \rightarrow \infty} E(m^n)$ and we only have to show $m^{\text{lim}} \in \mathcal{M}_l$. Since $E(m^{\text{lim}})$ is finite and $\overline{m}_x^{\text{lim}} \leq 0$ for $x \leq 0$ we have that either $m^{\text{lim}} \in \mathcal{M}_l$ or $m^{\text{lim}} + \vec{e}_x \in H^1(\Sigma)$.

We now assume $m^{\text{lim}} + \vec{e}_x \in H^1(\Sigma)$ in order to show by contradiction $m^{\text{lim}} \in \mathcal{M}_l$. The proof will be in the spirit of concentration compactness: If the sequence $(m^n)_{n \in \mathbb{N}}$ converges to a map $m^{\text{lim}} \notin \mathcal{M}_l$ the maps m^n “split” into two parts. We show that the sum of the energies of the parts is strictly greater than the energy that can be obtained when the splitting does not occur.

Clearly $E(m^{\text{lim}}) > 0$. Indeed, if $E(m^{\text{lim}})$ is zero, m^{lim} has to be constant on Σ with $m_y^{\text{lim}} \equiv 0$ (Lemma 15). Thus $m^{\text{lim}} \equiv \vec{e}_x$ or $m^{\text{lim}} \equiv -\vec{e}_x$. This is in contradiction to $\overline{m}_x^{\text{lim}}(0)$ being zero.

In the case $m + \vec{e}_x \in H^1(\Sigma)$ we can construct sequences of maps $(g^n)_{n \in \mathbb{N}}$ and $(h^n)_{n \in \mathbb{N}}$ with the following properties:

1. $g^n \in \mathcal{M}_l$, $(g^n)_{n \in \mathbb{N}}$ converges to $-\vec{e}_x$ in $L^2_{\text{loc}}(\overline{\Sigma})$, $E(g^n)$ is uniformly bounded.
2. $(h^n)_{n \in \mathbb{N}}$ converges to m^{lim} in $L^2(\Sigma)$.
3. $\lim_{n \rightarrow \infty} \int_{\Sigma} \nabla g_n \cdot \nabla h_n = 0$.
4. $m^n = g^n + h^n + \vec{e}_x$.

We give an explicit construction of $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ below, as the last part of the proof.

Let u_{g^n} , u_{h^n} be weak solutions of (1) for $m = g^n$, $m = h^n$ respectively. Then $\nabla u_{g^n} = \nabla u_{g^n + \vec{e}_x}$ and, using Lemma 6, we get

$$\left| \int_{\mathbb{R}^3} \nabla u_{g^n} \nabla u_{h^n} \right| \leq \left| \int_{\mathbb{R}^3} m^{\text{lim}} \cdot (\nabla u_{g^n}) \right| + \|h^n - m^{\text{lim}}\|_{L^2(\Sigma)} \sqrt{E_H(g^n)} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore

$$\lim_{n \rightarrow \infty} E(m^n) = \lim_{n \rightarrow \infty} (E(h^n) + E(g^n)) \geq E(m^{\text{lim}}) + E_{\mathcal{M}_l} > E_{\mathcal{M}_l}.$$

This is a contradiction to $(m^n)_{n \in \mathbb{N}}$ being a minimizing sequence in \mathcal{M}_l .

The limit of a sequence whose elements are all in \mathcal{T} , \mathcal{V} , respectively, is in that class, too. Therefore we can find minimizers in \mathcal{T}_l and \mathcal{V}_l in exactly the same way as we have found minimizers in \mathcal{M}_l .

Construction of $(g^n)_{n \in \mathbb{N}}$ and $(h^n)_{n \in \mathbb{N}}$. Since $\|\nabla m^n\|_{L^2(\Sigma)}$ is uniformly bounded and $(m^n + \vec{e}_x)_{n \in \mathbb{N}}$ converges in $L^2_{\text{loc}}(\overline{\Sigma})$ to a map $m^{\text{lim}} + \vec{e}_x \in H^1(\Sigma)$ there exists a sequence $l_n \rightarrow \infty$, such that

$$\|m^n(-l_n, \cdot) + \vec{e}_x\|_{H^1(D_R)} + \|m^n(l_n, \cdot) + \vec{e}_x\|_{H^1(D_R)} + \|m^n - m^{\text{lim}}\|_{L^2(\Sigma_{l_n})} \xrightarrow{n \rightarrow \infty} 0.$$

Then, in view of the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, the sequence

$$\left(\|m^n(-l_n, \cdot) + \vec{e}_x\|_{L^\infty(D_R)} + \|m^n(l_n, \cdot) + \vec{e}_x\|_{L^\infty(D_R)} \right)_{n \in \mathbb{N}}$$

converges to zero as well. We set

$$g^n(x, y) := \begin{cases} m^n(x, y) & \text{if } x \in \mathbb{R} \setminus [-l_n, l_n], \\ -\vec{e}_x & \text{if } x \in [-l_n + 1, l_n - 1], \\ \frac{\alpha^n(x)}{|\alpha^n(x)|} & \text{if } x \in [-l_n, -l_n + 1], \\ \frac{\beta^n(x)}{|\beta^n(x)|} & \text{if } x \in [l_n - 1, l_n], \end{cases} \quad h^n := m^n - g^n - \vec{e}_x,$$

where

$$\begin{aligned}\alpha^n(x, y) &:= (l_n + x)(-\vec{e}_x) + (-l_n - x + 1)m^{\text{lim}}(-l_n, y), \\ \beta^n(x, y) &:= (1 + x - l_n)(-\vec{e}_x) + (-x + l_n)m^{\text{lim}}(l_n, y).\end{aligned}$$

Then 1. and 4. are surely fulfilled and $\lim_{n \rightarrow \infty} \|g_n + \vec{e}_x\|_{H^1(\Sigma_{l_n} \setminus \Sigma_{l_{n-1}})} = 0$. On $(\Sigma \setminus \Sigma_{l_n}) \cup \Sigma_{l_{n-1}}$ either $\nabla g_n \equiv 0$ or $\nabla h_n \equiv 0$. Since $\|\nabla m_n\|_{L^2(\Sigma)}$ is uniformly bounded we have

$$\lim_{n \rightarrow \infty} \int_{\Sigma} \nabla g_n \cdot \nabla h_n = \lim_{n \rightarrow \infty} \int_{\Sigma_{l_n} \setminus \Sigma_{l_{n-1}}} \nabla g_n \cdot (\nabla m_n - \nabla g_n) = 0.$$

Moreover,

$$\begin{aligned}& \lim_{n \rightarrow \infty} \left\| h^n - m^{\text{lim}} \right\|_{L^2(\Sigma)}^2 \\ & \leq \lim_{n \rightarrow \infty} \left(\left\| m^n - m^{\text{lim}} \right\|_{L^2(\Sigma_{l_{n-1}})}^2 + \left\| m^{\text{lim}} + \vec{e}_x \right\|_{L^2(\Sigma \setminus \Sigma_{l_n})}^2 \right. \\ & \quad \left. + \left\| m^n - g^n - \vec{e}_x - m^{\text{lim}} \right\|_{L^2(\Sigma_{l_n} \setminus \Sigma_{l_{n-1}})}^2 \right) \\ & \leq \lim_{n \rightarrow \infty} \left(\left\| m^n - m^{\text{lim}} \right\|_{L^2(\Sigma_{l_n})}^2 + \left\| m^{\text{lim}} + \vec{e}_x \right\|_{L^2(\Sigma \setminus \Sigma_{l_n})}^2 \right. \\ & \quad \left. + \left\| g^n + \vec{e}_x \right\|_{L^2(\Sigma_{l_n} \setminus \Sigma_{l_{n-1}})}^2 \right) \\ & = 0.\end{aligned}$$

Thus the maps g^n and h^n maps have the required properties 1. to 4.. \square

5 Energy scaling and Γ -convergence for $R \rightarrow 0$

In this section we look at sequences of radii that converge to zero. We prove that $\frac{1}{R^2}E(m)$ Γ -converges to a reduced, one dimensional problem whose minimizer can be calculated explicitly. Γ -convergence implies in particular convergence of the minimal energies. Therefore we do not only get the estimate $cR^2 \leq E_{\mathcal{M}_l} \leq CR^2$ for all $R \leq R_0$ and some fixed $c, C, R_0 > 0$ but we moreover know that for $R_0 \rightarrow 0$ the constants c, C both converge to the minimal energy of the reduced problem. In our case this energy is $\sqrt{8}\pi$.

In this section make implicit dependences on the radius R explicit. Instead of Σ we write $\Sigma(R)$, instead of $E(m)$ we write $E(m, R)$, etc. In this section we show how the variational problem we considered so far converges to a reduced variational problem where the magnetization depends only on the x -coordinate and where the nonlocal part of the energy E_H reduces to a local term.

Definition 18. (i) The admissible set for the *full variational problem* for $R \in \mathbb{R}^+$ is

$$\mathcal{M}(R) = \{m: \Sigma(R) \rightarrow \mathbb{S}^2 \mid E(m, R) < \infty\}.$$

For each admissible function $m \in \mathcal{M}(R)$ we set

$$\acute{m}: \Sigma(1) \rightarrow \mathbb{S}^2, \quad \acute{m}\left(x, \frac{y}{R}\right) := m(x, y).$$

After rescaling, the energy functional of the full variational problem is

$$\frac{1}{R^2}E(m, R) = \int_{\Sigma(1)} \left(|\partial_x \acute{m}(p)|^2 + \frac{1}{R^2} |\nabla_y \acute{m}(p)|^2 \right) dp + \frac{1}{R^2}E_H(m, R).$$

(ii) The energy functional for the *reduced variational problem* is

$$E_{\text{red}}(m) := \pi \|\partial_x m\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})}^2.$$

The admissible set is

$$\mathcal{M}(0) = \{m: \mathbb{R} \rightarrow \mathbb{S}^2 \mid E_{\text{red}}(m) < \infty\}.$$

(iii) We use the following notion of convergence: Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers that converges to zero, let $m^n \in \mathcal{M}(R_n)$ and let $m^0 \in \mathcal{M}(0)$. We say the sequence $(m^n)_{n \in \mathbb{N}}$ converges to m^0 if

- $\nabla_y \acute{m}^n$ converges to 0 strongly in $L^2(\Sigma(1))$ and
- $\partial_x \acute{m}^n$ converges to $\partial_x m^0$ weakly in $L^2(\Sigma(1))$ and
- \acute{m}^n converges to m^0 strongly in $L^2_{\text{loc}}(\bar{\Sigma})$.

Theorem 19. *The reduced variational problem (Definition 18 (ii)) is the Γ -limit of the full variational problem (Definition 18 (i)) under the convergence stated in Definition 18 (iii). This means*

- **Compactness:** *Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to zero, let $m^n \in \mathcal{M}(R_n)$ and let $(E(m^n, R_n))_{n \in \mathbb{N}}$ be bounded. Then there exists a subsequence of $(m^n)_{n \in \mathbb{N}}$ which converges in the sense of Definition 18 (iii) to some $m^0 \in \mathcal{M}(0)$.*
- **Lower semicontinuity:** *For every convergent sequence $(m^n)_{n \in \mathbb{N}}$ ($m^n \in \mathcal{M}(R_n)$) with limit $m^0 \in \mathcal{M}(0)$ we have*

$$E_{\text{red}}(m^0) \leq \liminf_{n \rightarrow \infty} \frac{1}{R_n^2} E(m^n, R_n).$$

- **Construction:** For each $m^0 \in \mathcal{M}(0)$ and each sequence $(R_n)_{n \in \mathbb{N}}$ of positive numbers converging to zero there is a sequence $(m^n)_{n \in \mathbb{N}}$ with $m^n \in \mathcal{M}(R_n)$ such that

$$E_{\text{red}}(m^0) = \lim_{n \rightarrow \infty} \frac{1}{R_n^2} E(m^n, R_n).$$

To show Theorem 19 we need the following lemma.

Lemma 20. For all $m \in \mathcal{T}(R)$ we have

$$\begin{aligned} \lim_{R \rightarrow 0} \frac{1}{R^2} E_{\sigma\sigma}(m, R) &= \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})}^2 \\ \lim_{R \rightarrow 0} \frac{1}{R^2} E_{\rho\rho}(m, R) &= 0. \end{aligned}$$

Proof. Let g and h be as in Theorem 8. Then the estimates for g imply $\frac{1}{R^2} E_{\sigma\sigma}(m, R) \leq \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})}^2$ and for all $t > 0$ we have

$$\lim_{R \rightarrow 0} \frac{1}{R^2} E_{\sigma\sigma}(m, R) \geq \lim_{R \rightarrow 0} \int_{-t}^t |\hat{m}_y(\xi)|^2 g(|\xi R|) d\xi = \frac{\pi}{2} \|\hat{m}_y\|_{L^2([-t, t])}^2,$$

thus $\frac{1}{R^2} E_{\sigma\sigma}(m, R) = \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})}^2$.

To show the second statement we note that, since $E_{\rho\rho}$ is finite, $\int_{-1}^1 |\hat{\rho}(\xi)|^2 |\ln(\xi)| d\xi$ is finite as well (Theorem 8). Thus we can estimate $E_{\rho\rho}$ as follows.

$$\begin{aligned} &\lim_{R \rightarrow 0} \frac{1}{R^2} E_{\rho\rho}(m, R) \\ &\leq \lim_{R \rightarrow 0} \left(\pi R^4 \int_{-\frac{1}{2R}}^{\frac{1}{2R}} -|\hat{\rho}(\xi)|^2 \ln(|\xi R|) d\xi + \underbrace{\pi R^4 (2R)^2 \int_{\mathbb{R} \setminus [-\frac{1}{2R}, \frac{1}{2R}]} |\hat{\rho}(\xi)|^2 d\xi}_{\rightarrow 0} \right) \\ &= \lim_{R \rightarrow 0} \pi R^4 \left(-\ln(R) \int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{\rho}(\xi)|^2 d\xi + \int_{-1}^1 -|\hat{\rho}(\xi)|^2 \ln(|\xi|) d\xi \right. \\ &\quad \left. + \int_{-\frac{1}{2R}}^{-1} \underbrace{-|\hat{\rho}(\xi)|^2 \ln(|\xi|)}_{< 0} d\xi + \int_1^{\frac{1}{2R}} \underbrace{-|\hat{\rho}(\xi)|^2 \ln(|\xi|)}_{< 0} d\xi \right) \\ &\leq 0. \end{aligned}$$

□

Proof of Theorem 19. We have to check the three properties of Γ -convergence: compactness, lower semicontinuity and construction.

Showing compactness and constructing a suitable sequence is easy. If $\frac{1}{R_n^2} E(m^n, R_n)$ is bounded, then $\nabla_y \hat{m}^n$ converges to zero in $L^2(\Sigma(1))$ and $\|\partial_x \hat{m}^n\|_{L^2(\Sigma(1))}$ is

bounded. Thus we can find a subsequence, denoted with $(m^n)_{n \in \mathbb{N}}$ as well, which converges in the sense of Definition 18 (iii) to a map $m^0: \mathbb{R} \rightarrow \mathbb{S}^2$ with $E_0(m) < \infty$. Thus we have shown compactness. Moreover Lemma 20 ensures that for a sequence $(R_n)_{n \in \mathbb{N}}$ converging to zero and $m^0 \in \mathcal{M}(0)$ we can construct a recovery sequence by setting $m^n: \Sigma(R_n) \rightarrow \mathbb{S}^2$, $(x, y) \mapsto m^0(x)$. To show lower semicontinuity let $(m^n)_{n \in \mathbb{N}}$, $m^n \in \mathcal{M}(R_n)$ be a sequence that converges in the sense of Definition 18 to some $m^{\text{lim}}: \mathbb{R} \rightarrow \mathbb{S}^2$. Without loss of generality we can assume $R_n < \frac{1}{4}$ and that $\frac{1}{R_n^2} E(m^n)$ is bounded by some number $M \geq 1$. We split m^n as in Section 4 into $m = \bar{m} + \tilde{m}$. The Poincaré inequality yields

$$E_H(\tilde{m}) \leq \|\tilde{m}^n\|_{L^2(\Sigma(R_n))}^2 \leq 16R_n^2 \|\nabla_y \tilde{m}\|_{L^2(\Sigma(R_n))}^2 \leq 16R_n^4 M.$$

Using this estimate we can calculate

$$\begin{aligned} \frac{1}{R_n^2} E_H(m^n, R_n) &\geq \frac{1}{R_n^2} \left(E_H(\bar{m}^n, R_n) - 2\sqrt{E_H(\bar{m}^n, R_n) E_H(\tilde{m}^n, R_n)} \right) \\ &\geq \frac{1}{R_n^2} \left(E_{\sigma\sigma}(\bar{m}^n, R_n) - 2\sqrt{2E_H(m) + 2E_H(\tilde{m})} \sqrt{E_H(\tilde{m})} \right) \\ &\geq \frac{1}{R_n^2} E_{\sigma\sigma}(\bar{m}^n, R_n) - 3 \left(R_n \sqrt{M} + 4R_n^2 M \right) 4M \\ &\geq \frac{1}{R_n^2} E_{\sigma\sigma}(\bar{m}^n, R_n) - 24MR_n. \end{aligned}$$

To bound $E_{\sigma\sigma}(\bar{m}_n, R_n)$ from below we use the Fourier multiplier of Theorem 8 and fix some arbitrary $t > 0$.

$$\begin{aligned} E_{\sigma\sigma}(\bar{m}_n, R_n) &= \int_{\mathbb{R}} g(R_n \xi) \hat{\bar{m}}_y^n(\xi)^2 d\xi \\ &= \int_{\mathbb{R}} g(R_n \xi) \left(\hat{m}_y^{\text{lim}}(\xi)^2 + (\hat{\bar{m}}_y^n - m_y^{\text{lim}})^2 + 2\hat{m}_y^{\text{lim}}(\xi) \left(\hat{\bar{m}}_y^n(\xi) - \hat{m}_y^{\text{lim}}(\xi) \right) \right) d\xi \\ &\geq \int_{-t}^t g(R_n t) \hat{m}_y^{\text{lim}}(\xi)^2 d\xi - 2g(0) \left| \int_{\mathbb{R}} \hat{m}_y^{\text{lim}}(\xi) \left(\hat{m}_y^{\text{lim}}(\xi) - \hat{\bar{m}}_y^n(\xi) \right) d\xi \right| \\ &= \int_{-t}^t g(R_n t) \hat{m}_y^{\text{lim}}(\xi)^2 d\xi - \pi \left| \int_{\mathbb{R}} m_y^{\text{lim}}(x) \left(m_y^{\text{lim}}(x) - \bar{m}_y^n(x) \right) dx \right| \end{aligned}$$

Since

$$\frac{1}{R^2} \left(E_{\sigma\sigma}(\bar{m}_y) + \|\partial_x \bar{m}_y\|_{L^2(\Sigma(R_n))}^2 \right) \leq \frac{1}{R_n^2} E(\bar{m}^n, R_n) + 24MR_n$$

is bounded, $\|\bar{m}_y^n\|_{L^2(\mathbb{R})}$ is bounded (Lemma 10). Thus \bar{m}_y^n converges weakly to m_y^{lim} and we have the estimate $\lim_{n \rightarrow \infty} E_{\sigma\sigma}(\bar{m}_n, R_n) \geq g(0) \|m^{\text{lim}}\|_{L^2([-t, t])}^2$ for all $t > 0$. Letting t tend to infinity we can conclude

$$\lim_{n \rightarrow \infty} \frac{1}{R_n^2} E_H(m^n, R_n) \geq \lim_{n \rightarrow \infty} E_{\sigma\sigma}(\bar{m}_n, R_n) \geq \frac{\pi}{2} \|m^{\text{lim}}\|_{L^2(\mathbb{R})}^2.$$

Moreover, we surely have

$$\lim_{n \rightarrow \infty} \frac{1}{R^2} E_{ex}(m_n) \geq \lim_{n \rightarrow \infty} \pi \|\partial_x \bar{m}_n\|_{L^2(\mathbb{R})}^2 \geq \pi \|\partial_x m^{\text{lim}}\|_{L^2(\mathbb{R})}^2,$$

so $\lim_{n \rightarrow \infty} \frac{1}{R_n^2} (E_H(m_n, R_n) + E_{ex}(m_n, R_n)) \geq E^{\text{red}}(m^{\text{lim}})$, as claimed. \square

We now determine the minimizer of the reduced problem.

Lemma 21. *The minimizer of E_{red} is unique up to translation and rotation. It is given by*

$$m^{\text{min}}: \mathbb{R} \rightarrow \mathbb{S}^2, \quad x \mapsto (\tanh(x), \cosh(x)^{-1}, 0)$$

and its energy is $\sqrt{8}\pi$.

Proof. To find minimizers of E_{red} we parameterize m by the angle $\theta: \Sigma \rightarrow [0, 1]$ and set $m_x = -\cos(\pi\theta)$. Using the Modica Mortola trick, we get

$$E_{\text{red}}(m) \geq \pi \int_{\mathbb{R}} (\pi \partial_x \theta)^2 + \frac{1}{2} \sin^2(\pi\theta) \geq \pi \int_{\mathbb{R}} \sqrt{2}\pi |\sin(\pi\theta) \partial_x \theta| \geq \sqrt{8}\pi.$$

Assume that $|\nabla m| = \pi|\nabla\theta|$, i.e., that the direction m_y does not change. Then the first inequality is an equality. Assume moreover that θ is a monotone increasing solution of

$$\partial_x \theta = \frac{1}{\sqrt{2}\pi} \sin(\pi\theta), \tag{12}$$

then the second inequality is an equality. Such a map is unique up to translation and rotation and we have

$$\begin{aligned} \theta^{\text{min}}(x) &= \frac{2}{\pi} \arctan\left(e^{\frac{x}{\sqrt{2}}}\right) = \frac{2}{\pi} \arccos\left(\frac{1}{\sqrt{1+e^{\sqrt{2}x}}}\right), \\ m^{\text{min}}(x) &= \left(-\cos(\pi\theta^{\text{min}}), \sin(\pi\theta^{\text{min}}), 0\right) = \left(\tanh\left(\frac{x}{\sqrt{2}}\right), \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)}, 0\right). \end{aligned}$$

\square

Theorem 22. *Let m^{min} be as in Lemma 21. For each positive sequence $(R_n)_{n \in \mathbb{N}}$ converging to zero and each sequence of minimizers $m^n \in \mathcal{M}_l(R_n)$ the rescaled energy $\frac{1}{R_n^2} E(m^n, R_n)$ converges to $\sqrt{8}\pi$. Moreover, there is a sequence of translations T^n such that a subsequence of $(T^n(m_n))_{n \in \mathbb{N}}$ converges, up to a rotation, to m^{min} in the sense of Definition 18 (iii).*

Proof. For Γ -limits, the following statement is true [DM93, Corollary 7.17, p.78]: Let $(r_n)_{n \in \mathbb{N}}$ be a sequence converging to zero and let m^n be the minimizer of the full problem for r_n . Then every accumulation point of $(m^n)_{n \in \mathbb{N}}$ is a minimizer of the reduced problem. Moreover the full energy of m_n converges to the reduced energy of m . This statement implies the theorem directly. \square

6 A lower bound for the energy scaling for $R \rightarrow \infty$

In this section we look at the scaling of $E_{\mathcal{M}_l}$ for big radii. We find a lower bound, which will be complemented by an upper bound on $E_{\mathcal{V}_l}$ in the next section. To simplify the calculations, instead of the functional E we consider the functional

$$I: \mathcal{M}(1) \times \mathbb{R}^+ \rightarrow \mathbb{R}, \quad I(m, R) := E_{\text{ex}}(m) + R^2 E_H(m).$$

Then we have for all $m \in \mathcal{M}(1)$ the relation $E(m^R, R) = R I(m, R)$, where $m^R := m(Rx, Ry)$.

Theorem 23. *There are constants $C, R_0 \in \mathbb{R}^+$ such that for all $R \geq R_0$*

$$\inf_{m \in \mathcal{M}_l(1)} I(m, R) \geq CR\sqrt{\ln(R)}, \quad \text{i.e.,} \quad E_{\mathcal{M}_l} \geq CR^2\sqrt{\ln(R)}.$$

Proof. Let $R \geq R_0 := 2e$ and let m be a minimizer of $I(\cdot, R)$. We define

$$\bar{m}: \mathbb{R} \rightarrow \mathbb{R}^3, \quad x \mapsto \frac{1}{|D_1|} \int_{D_1} m(x, y) \, dy, \quad \bar{\bar{m}}: \mathbb{R} \rightarrow \mathbb{R}^3, \quad x \mapsto \frac{1}{R} \int_{-\frac{R}{2}}^{\frac{R}{2}} \bar{m}(x+t) \, dt,$$

$$a := \sup \left\{ x \in \mathbb{R} : \bar{\bar{m}}_x(x) \leq -\frac{1}{\sqrt{2}} \right\}, \quad b := \inf \left\{ x > a : \bar{\bar{m}}_x(x) \geq \frac{1}{\sqrt{2}} \right\},$$

and set $\tilde{m} := m - \bar{m}$, $d := b - a$. We have $d \geq \frac{R}{2}$ since otherwise

$$\begin{aligned} \bar{\bar{m}}_x(b) - \bar{\bar{m}}_x(a) &= \left(\frac{1}{R} \int_b^{b+\frac{R}{2}} \bar{m}_x \right) - \left(\frac{1}{R} \int_{a-\frac{R}{2}}^a \bar{m}_x \right) \\ &\leq \frac{1}{2} + \frac{1}{2} < \frac{2}{\sqrt{2}} = \bar{\bar{m}}_x(b) - \bar{\bar{m}}_x(a). \end{aligned}$$

We distinguish three different cases.

Case 1: $E_{\text{ex}} \geq \frac{d}{200}$. We use a test function to show $E_H \sim \frac{\ln(d)}{d}$ and complement the estimate with the lower bound on the exchange energy. We

define

$$\begin{aligned}\phi'_1 : \mathbb{R} &\rightarrow \mathbb{R}, & x &\mapsto \frac{1}{R} \mathbb{1}_{[a-\frac{R}{2}, a+\frac{R}{2}]}(x) - \frac{1}{R} \mathbb{1}_{[b-\frac{R}{2}, b+\frac{R}{2}]}(x), \\ \phi_1 : \mathbb{R} &\rightarrow \mathbb{R}, & x &\mapsto \int_{-\infty}^x \phi'_2(t) dt, \\ \psi_1 : \mathbb{R}^2 &\rightarrow \mathbb{R}, & y &\mapsto \mathbb{1}_{D_1}(y) + \mathbb{1}_{D_d \setminus D_1}(y) \frac{\ln(d/|y|)}{\ln(d)}.\end{aligned}$$

Then

$$\begin{aligned}\|\phi_1\|_{L^2(\mathbb{R})}^2 &\leq d, & \|\phi'_1\|_{L^2(\mathbb{R})}^2 &\leq \frac{2}{R} \\ \|\nabla_y \psi_1\|_{L^2(\mathbb{R}^2)}^2 &= \int_1^d \frac{2\pi r}{\ln(d)^2 r^2} dr = \frac{2\pi}{\ln(d)}, \\ \|\psi_1\|_{L^2(\mathbb{R}^2)}^2 &= \int_1^d 2\pi r \left(\frac{\ln(\frac{1}{r}d)}{\ln(d)} \right)^2 dr = \frac{2\pi d^2}{\ln(d)^2} \int_{\frac{1}{d}}^1 r \ln(r)^2 dr \leq \frac{\pi d^2}{2 \ln(d)^2}.\end{aligned}$$

Let u be the weak solution of $\Delta u = \operatorname{div} m$. Then we have for all differentiable functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ the equality $\int_{\Sigma} m \cdot \nabla f = \int_{\mathbb{R}^3} \nabla u \cdot \nabla f$. So we can calculate

$$\begin{aligned}\sqrt{2\pi} &= \pi (\overline{m}_x(a) - \overline{m}_x(b)) = \int_{\Sigma} m_x \phi'_1 \psi_1 \\ &= \int_{\Sigma} m_x \phi'_1 \psi_1 + \phi_1 (\nabla_y \psi_1) \cdot m_y = \int_{\Sigma} \nabla(\phi_1 \psi_1) \cdot \nabla u \\ &\leq \sqrt{\|\phi'_1\|_{L^2(\mathbb{R})}^2 \|\psi_1\|_{L^2(\mathbb{R}^2)}^2 + \|\phi_1\|_{L^2(\mathbb{R})}^2 \|\nabla_y \psi_1\|_{L^2(\mathbb{R}^2)}^2} \|\nabla u\|_{L^2(\mathbb{R}^3)} \\ &\leq \sqrt{\frac{\pi}{R} \frac{d^2}{\ln(d)^2} + \frac{2\pi d}{\ln(d)}} \|\nabla u\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Since $d \geq \frac{1}{2}R \geq e$ and $E_{ex} \geq \frac{d}{200}$ we have

$$\begin{aligned}I(m, R) &\geq \inf_{d \geq \frac{R}{2}} \left(\frac{d}{200} + R^2 \left(\frac{1}{2\pi R} \frac{d^2}{\ln(d)^2} + \frac{d}{\pi \ln(d)} \right)^{-1} \right) \\ &\geq \frac{1}{2} \min \left\{ \inf_{d \geq \frac{R}{2}} \left(\frac{d}{200} + 2\pi R^3 \frac{\ln(d)^2}{d^2} \right), \inf_{d \geq \frac{R}{2}} \left(\frac{d}{200} + \pi R^2 \frac{\ln(d)}{d} \right) \right\} \\ &\geq \frac{1}{2} \min \left\{ \inf_{d \geq \frac{R}{2}} \left(\frac{d}{200} + 2\pi R^3 \frac{\ln(\frac{R}{2})^{\frac{3}{2}}}{d^2} \right), \inf_{d \geq \frac{R}{2}} \left(\frac{d}{200} + \pi R^2 \frac{\ln(\frac{R}{2})}{d} \right) \right\} \\ &= \frac{1}{2} \min \left\{ \frac{3}{20} \left(\frac{\pi}{10} \right)^{\frac{1}{3}} R \sqrt{\ln\left(\frac{R}{2}\right)}, 2\sqrt{\frac{\pi}{200}} R \sqrt{\ln\left(\frac{R}{2}\right)} \right\} \\ &\geq \frac{1}{40} R \sqrt{\ln(R)}.\end{aligned}$$

Case 2: $E_{\text{ex}} < \frac{d}{200}$ and $\|\widehat{m}_y\|_{L^2(\Sigma)}^2 \geq \frac{1}{5}d$. We prove a bound on $\|\widehat{m}_y\|_{L^2([-1,1])}$ and use Theorem 8 (i). For the Fourier transform \widehat{m}_y and the Fourier transform of the derivative $\xi \widehat{m}_y$ we have the inequality

$$\|\xi \widehat{m}_y\|_{L^2(\Sigma)}^2 \leq \frac{1}{200}d \leq \frac{1}{40}\|\widehat{m}_y\|_{L^2(\Sigma)}^2,$$

so $\|\widehat{m}_y(\xi)\|_{L^2[-1,1]}^2 \geq (1 - \frac{1}{40}) \|\widehat{m}_y\|_{L^2(\Sigma)}^2$. Using 8 (i) we get

$$E_{\sigma\sigma}(\overline{m}_y) \geq \|\widehat{m}_y(\xi)\|_{L^2[-1,1]}^2 \geq 0.195d.$$

The Poincaré inequality yields $\|\tilde{m}\|_{L^2(\Sigma)}^2 \leq \frac{16}{200}d$, thus

$$\begin{aligned} I(m, R) &\geq R^2 E_H(m) \geq R^2 \left(\sqrt{E_H(\overline{m})} - \|\tilde{m}\|_{L^2(\Sigma)} \right)^2 \\ &\geq R^2 d (\sqrt{0.195} - \sqrt{0.08})^2 \geq \frac{1}{40} R^2 d \geq \frac{1}{80} R^3. \end{aligned}$$

Case 3: $E_{\text{ex}} < \frac{d}{200}$ and $\|\overline{m}_y\|_{L^2(\Sigma)}^2 \leq \frac{1}{5}d$. First we show that $\int_{a-\frac{R}{2}}^{b+\frac{R}{2}} \overline{m}_x^2$ is large. For all $x \in \mathbb{R}$ we have

$$1 = \frac{1}{\pi} \int_{D_1} |\overline{m}(x) + \tilde{m}(x, y)|^2 dy = |\overline{m}(x)|^2 + \frac{1}{\pi} \|\tilde{m}(x, \cdot)\|_{L^2(D_1)}^2.$$

Since $\|\tilde{m}\|_{L^2(\Sigma)}^2 \leq \frac{16}{200}d$, in particular

$$\int_{a-\frac{R}{2}}^{b+\frac{R}{2}} \overline{m}_x^2 \geq R+d - \frac{1}{\pi} \|\overline{m}_y\|_{L^2(\Sigma)}^2 - \frac{1}{\pi} \|\tilde{m}\|_{L^2(\Sigma)}^2 \geq R+d - \frac{d}{5\pi} - \frac{16d}{200\pi} \geq R+0.9d.$$

Thus there is at least one $x_0 \in [a, b]$ such that $\int_{x_0-\frac{R}{2}}^{x_0+\frac{R}{2}} \overline{m}_x^2 \geq 0.9R$. We proceed similar to Case 1. We set

$$\begin{aligned} \phi'_2 : \mathbb{R} &\rightarrow \mathbb{R}, & x &\mapsto \frac{1}{R} \mathbb{1}_{[x_0-\frac{R}{2}, x_0+\frac{R}{2}]}(x) (\overline{m}_x(x) - \overline{\overline{m}}_x(x_0)), \\ \phi_2 : \mathbb{R} &\rightarrow \mathbb{R}, & x &\mapsto \int_{-\infty}^x \phi'_2(t) dt, \\ \psi_2 : \mathbb{R}^2 &\rightarrow \mathbb{R}, & y &\mapsto \mathbb{1}_{D_1}(y) + \mathbb{1}_{D_R \setminus D_1}(y) \frac{\ln(R/|y|)}{\ln(R)}. \end{aligned}$$

Then

$$|\phi'_2| \leq \frac{2}{R}, \quad \phi_2 \equiv 0 \quad \text{on } \mathbb{R} \setminus \left[x_0 - \frac{R}{2}, x_0 + \frac{R}{2} \right], \quad |\phi_2| \leq 1.$$

$$\|\nabla_y \psi_1\|_{L^2(\mathbb{R}^2)}^2 = \frac{2\pi}{\ln(R)}, \quad \|\psi_1\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{\pi R^2}{2\ln(R)^2}.$$

We have

$$\begin{aligned} \pi(0.9 - 0.5) &\leq \frac{\pi}{R} \int_{x_0 - \frac{R}{2}}^{x_0 + \frac{R}{2}} \overline{m}_x(x)^2 - \overline{m}_x(x_0)^2 dx \\ &= \frac{\pi}{R} \int_{x_0 - \frac{R}{2}}^{x_0 + \frac{R}{2}} \overline{m}_x(x) (\overline{m}_x(x) - \overline{m}_x(x_0)) dx \\ &= \int_{\Sigma} \overline{m}_x \phi_2' \psi_2 = \int_{\Sigma} m \cdot \nabla(\phi_2 \psi_2) = \int_{\mathbb{R}^3} \nabla u \cdot \nabla(\phi_2 \psi_2) \\ &\leq \sqrt{\|\phi_2'\|_{L^2(\mathbb{R})}^2 \|\psi\|_{L^2(\mathbb{R}^2)}^2 + \|\phi_2\|_{L^2(\mathbb{R})}^2 \|\nabla_y \psi\|_{L^2(\mathbb{R}^2)}^2} \|\nabla u\|_{L^2(\mathbb{R}^3)} \\ &\leq \sqrt{\frac{2\pi R}{\ln(R)^2} + \frac{2\pi R}{\ln(R)}} \|\nabla u\|_{L^2(\mathbb{R}^3)} \leq \sqrt{4\pi \frac{R}{\ln(R)}} \|\nabla u\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Thus

$$I(m, R) \geq R^2 \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \geq \frac{\pi 0.4^2}{4} R \ln(R) \geq \frac{1}{10} R \ln(R).$$

In all three cases we have the inequality $I(m, R) \geq \frac{1}{40} R \sqrt{\ln(R)}$ for $R \geq 2e$. \square

7 The vortex wall - an example of a set of functions with low energy for big radii

In this section we show upper and lower bounds for the energy $E_{\mathcal{V}_l}$. We see that for large radii the upper bound has the same scaling as the lower bound for $E_{\mathcal{M}_l}$. This shows that this scaling is indeed optimal and that for large radii the energy of the minimizers in \mathcal{V}_l is at most a constant factor larger than the energy of the minimizers in \mathcal{M}_l . For small radii $E_{\mathcal{V}_l}$ scales like R and is thus much larger than $E_{\mathcal{M}_l} \sim E_{\mathcal{T}_l} \sim R^2$.

Because of the symmetry of the functions $m \in \mathcal{V}$, we use spherical coordinates x, r, ϕ in the domain and polar coordinates $\theta : \Sigma \rightarrow [0, 1]$, $\gamma : [0, \pi]$ in the image,

$$p = \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ r \cos \phi \\ r \sin \phi \end{pmatrix}, \quad m = \begin{pmatrix} m_x \\ m_{y_1} \\ m_{y_2} \end{pmatrix} = \begin{pmatrix} -\cos(\pi\theta) \\ \sin(\pi\theta) \cos \gamma \\ \sin(\pi\theta) \sin \gamma \end{pmatrix}.$$

For $m \in \mathcal{V}$ we have $\gamma = \phi + \frac{\pi}{2}$, so m is uniquely determined by the angle θ . The exchange energy is

$$\|\nabla m\|_{L^2(\Sigma)}^2 = \int_{\Sigma} \frac{1}{|y|^2} \sin^2(\pi\theta(p)) + (\pi \partial_x \theta(p))^2 + (\pi \nabla_y \theta(p))^2 dp.$$

7.1 Bounds on $E_{\mathcal{V}_l}$ for small radii

The following Theorem gives upper and lower bounds for $E_{\mathcal{V}_l}$ that are valid for all $R > 0$. However, they are reasonably sharp only for small R .

Theorem 24. *We have*

$$8\pi R \leq E_{\mathcal{V}_l} \leq 12\pi R + 2\pi R^3.$$

Proof. Set

$$E_1(\theta) := \int_{\Sigma} \pi^2 (\partial_x \theta)^2 + \frac{1}{r^2} \sin^2(\pi\theta),$$

Using the Modica Mortola trick, we find for all functions $\theta : \Sigma \rightarrow [0, 1]$ with $\lim_{x \rightarrow -\infty} \theta(x, y) = 0$ and $\lim_{x \rightarrow \infty} \theta(x, y) = 1$

$$E_1(\theta) \geq \int_{\Sigma} \frac{2\pi}{r} |\sin(\pi\theta) \partial_x \theta| \geq \int_0^R \int_{-\infty}^{\infty} 4\pi \partial_x (\cos \pi\theta) dx dr = 8\pi R. \quad (13)$$

In particular we have $E(m) \geq 8\pi R$ for all $m \in \mathcal{V}_l$.

A function θ fulfills equation (13) with equality if and only if it is a monotone increasing solution of

$$\partial_x \theta = \frac{1}{\pi r} \sin(\pi\theta).$$

This solution is unique up to translation. It is given by

$$\theta_1(x, r) := \frac{2}{\pi} \arctan(e^{\frac{x}{r}}).$$

Let $m^1 \in \mathcal{V}_l$ be the magnetisation corresponding to θ_1 . We calculate $E(m_1)$. Since

$$|\partial_r \theta_1(x, r)| = \frac{2x e^{-\frac{x}{r}}}{\pi r^2 (1 + e^{-\frac{2x}{r}})} \leq \frac{2x}{\pi r^2} e^{\frac{-|x|}{r}},$$

we have

$$\int_{\Sigma} \pi^2 (\partial_r \theta_1)^2 \leq \int_0^R \int_{-\infty}^{\infty} 2\pi^3 r \left(\frac{2x e^{\frac{-|x|}{r}}}{\pi r^2} \right)^2 = 4\pi R.$$

Finally, using Lemma 5, we get

$$E_{\mathcal{V}_l} \leq E(m_1) \leq E_1(\theta_1) + \int_{\Sigma} \pi^2 (\partial_r \theta_1)^2 + E_{\rho\rho}(m) \leq 12\pi R + 8\pi^2 R^3.$$

□

7.2 An upper bound on $E_{\mathcal{V}_l}$ for big radii

In this subsection we use a family of maps $m_\alpha^R \in \mathcal{V}_l(R)$ and show that for an appropriate choice of α and large R we have the estimate $E(m_\alpha^R) \leq C R^2 \sqrt{\ln(R)}$.

First, we estimate an integral over a cylindrical surface.

Lemma 25. For $r, l \in \mathbb{R}^+$, with $l \geq r$, and $p := (x, y)$ with $|y| \geq r$ and

$$Z_{r,l} := \{(x', y') \in [-l, l] \times \partial D_r\}$$

we have

$$\int_{Z_{r,l}} \frac{1}{|p - p'|} dp' \leq 2\pi^2 r \left(1 + \ln \left(\frac{1}{r} l \right) \right).$$

Proof. The Lemma is shown by direct calculation, for details see [Küh]. \square

We now set

$$\theta_\alpha : \mathbb{R} \times D_1 \rightarrow \mathbb{R} \quad \theta_\alpha(x, r) := \begin{cases} 0 & \text{if } x < -\alpha\sqrt{r} \\ 0.5 + \frac{x}{2\alpha\sqrt{r}} & \text{if } -\alpha\sqrt{r} \leq x \leq \alpha\sqrt{r} \\ 1 & \text{if } x > \alpha\sqrt{r}, \end{cases}$$

let $m_\alpha \in \mathcal{V}_l(1)$ be the rotationally symmetric magnetization corresponding to θ_α and define $m_\alpha^R : \Sigma(R) \rightarrow \mathbb{S}^2$, $(x, y) \mapsto m_\alpha \left(\frac{x}{R}, \frac{y}{R} \right)$. See Figure 7.2 for a plot of θ_1 . Note that m_α^R has a square root type singularity and that the size of the transition region is $R\alpha$.

Theorem 26. For $R \geq e$ and $\alpha_0 := R\sqrt{\ln(R)}$ we have

$$E_{\mathcal{V}_l} \leq E(m_{\alpha_0}^R) \leq 38R^2 \sqrt{\ln(R)}.$$

Proof. Let I be as in Section 6,

$$I : \mathcal{M}(1) \times \mathbb{R}^+ \rightarrow \mathbb{R}, \quad I(m, R) := E_{\text{ex}}(m) + R^2 E_H(m).$$

Then $E(m_\alpha^R, R) = RI(m_\alpha, R)$. First, we estimate $I(m_\alpha)$ for general $\alpha \geq 1$ and then choose a suitable α_0 . According to Lemma 4 we have

$$\begin{aligned} E_{\rho\rho}(m_\alpha) &= \int_{\Sigma} u_\alpha(p) \rho_\alpha(p) dp = \int_{\Sigma} \int_{\Sigma} \frac{\rho_\alpha(p) \rho_\alpha(p')}{4\pi|p - p'|} dp' dp \\ &= \frac{1}{2\pi} \int_{\Sigma} \int_0^{|y|} \int_{\partial D_t} \int_{\mathbb{R}} \frac{\rho_\alpha(x', y') \rho_\alpha(p)}{|(x', y') - p|} dx' dy' dt dp. \end{aligned}$$

Since $\rho_\alpha = \text{div } m_\alpha = \partial_x(m_\alpha)_x$, we have

$$\rho_\alpha(x', y') = \begin{cases} \frac{\pi}{2\alpha\sqrt{|y'|}} \sin \left(\frac{\pi x'}{2\alpha\sqrt{|y'|}} \right) \leq \frac{\pi}{2\alpha\sqrt{|y'|}} & \text{if } |x'| < \alpha\sqrt{|y'|} \\ 0 & \text{otherwise.} \end{cases}$$

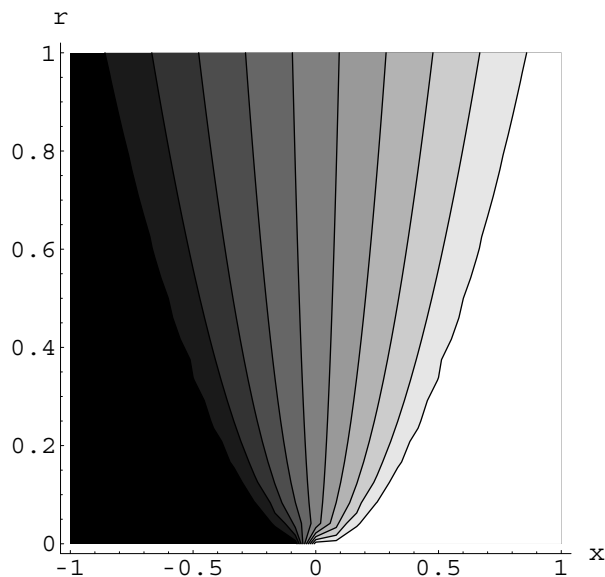


Figure 3: Contour plot of the function θ_1

Using this estimate and applying Lemma 25 with $l = \alpha\sqrt{r'}$ yields

$$\begin{aligned}
E_{\rho\rho}(m_\alpha) &\leq \frac{1}{2\pi} \int_{\Sigma} \rho_\alpha(p) \int_0^{|y|} \int_{\partial D_{r'}} \int_{-\alpha\sqrt{r'}}^{\alpha\sqrt{r'}} \frac{\pi}{2\alpha\sqrt{r'}} \frac{1}{|(x', y') - p|} dx' dy' dr' dp \\
&\leq \frac{1}{2\pi} \int_{\Sigma} \rho_\alpha(p) \int_0^{|y|} \frac{\pi}{2\alpha\sqrt{r'}} 2\pi^2 r' \left(1 + \ln\left(\frac{\alpha\sqrt{r'}}{r'}\right)\right) dr' dp \\
&= \frac{\pi^2}{2\alpha} \int_{\Sigma} \rho_\alpha(p) \int_0^{|y|} \sqrt{r'} \left(1 + \ln(\alpha) - \ln(\sqrt{r'})\right) dr' dp \\
&\leq \frac{\pi^2}{2\alpha} \int_{\Sigma} \rho_\alpha(p) (2 + \ln(\alpha)) dp \leq \frac{\pi^3}{\alpha} (2 + \ln(\alpha)).
\end{aligned}$$

The exchange energy is

$$\begin{aligned}
E_{\text{ex}}(m_\alpha) &= \int_{\Sigma} \pi^2 (\partial_x \theta_\alpha)^2 + \pi^2 (\nabla_y \theta_\alpha)^2 + \frac{1}{r^2} \sin^2(\pi \theta_\alpha) dp \\
&= \int_0^1 2\pi r \int_{-\alpha\sqrt{r}}^{\alpha\sqrt{r}} \frac{\pi^2}{4\alpha^2 r} + \pi^2 \left(\frac{x}{4\alpha} r^{-\frac{3}{2}}\right)^2 + \frac{1}{r^2} \sin^2\left(\frac{\pi x}{2\alpha\sqrt{r}}\right) dx dr \\
&= \int_0^1 \frac{\pi^3}{\alpha} \sqrt{r} + \frac{\pi^3}{8\alpha^2} r^{-2} \frac{2\alpha^3 \sqrt{r}^3}{3} + \frac{2\pi\alpha\sqrt{r}}{r} dr \\
&= \frac{2\pi^3}{3\alpha} + \frac{\pi^3\alpha}{6} + 4\pi\alpha.
\end{aligned}$$

For $R > e$ we choose $\alpha := R\sqrt{\ln R}$. Then $\ln(\alpha) \leq 2\ln(R)$ and

$$E_{\rho\rho}(m_\alpha) \leq \frac{\pi^3}{R\sqrt{\ln(R)}}(2 + 2\ln(R)) \leq \frac{4\pi^3}{e^2}R\sqrt{\ln(R)} \leq 17R\sqrt{\ln(R)},$$

$$E_{\text{ex}}(m_\alpha) \leq \frac{2\pi^3}{3R\sqrt{\ln(R)}} + \left(\frac{\pi^3}{6} + 4\pi\right)R\sqrt{\ln(R)} \leq 21R\sqrt{\ln(R)}$$

Thus

$$I(m_\alpha, R) \leq 38R\sqrt{\ln(R)}, \quad E(m_\alpha^R) \leq 38R^2\sqrt{\ln(R)}.$$

□

8 The Fourier multiplier

For functions that are constant on the cross section, we use a partial Fourier transform to find estimates for $E_{\sigma\sigma}$ and $E_{\rho\rho}$. As in section 3, we view E , $E_{\rho\rho}$ and $E_{\sigma\sigma}$ not only as functionals on \mathcal{M} but also on $\{f: \mathbb{R} \rightarrow \mathbb{R}^3\}$.

As before we apply the Fourier transformation only to the first argument of a function. For all functions $f: \mathbb{R} \times A \rightarrow \mathbb{R}^n$ for which $f(\cdot, a)$ lies in $L^1(\mathbb{R})$ for all $a \in A$, we define

$$\hat{f}(\xi, a) := \mathcal{F}(f)(\xi, a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, a)e^{-i\xi x} dx.$$

We generalize this formula in the usual way to functions f with $f(\cdot, a) \in L^2(\mathbb{R})$. With the above normalization we have $\|f(\cdot, a)\|_{L^2} = \|\hat{f}(\cdot, a)\|_{L^2}$. We denote the inverse Fourier transform of a map $g: \mathbb{R} \times A \rightarrow \mathbb{R}^n$ by $\mathcal{F}^{-1}(g) = \check{g}$.

We now formally Fourier transform the defining equations for u . In cylindrical coordinates the formal Fourier transform of equation (7) and (8) is

$$-\xi^2 v_\sigma + \partial_{rr} v_\sigma + \frac{1}{r} \partial_r v_\sigma + \frac{1}{r^2} \partial_{\phi\phi} v_\sigma = 0 \quad \text{if } r \neq R \quad (14)$$

$$\lim_{r \searrow R} \partial_r v_\sigma(\xi, r, \phi) - \lim_{r \nearrow R} \partial_r v_\sigma(\xi, r, \phi) = -\hat{\sigma}(\xi, R, \phi), \quad (15)$$

the formal Fourier transform of equation (6) in cylindrical coordinates is

$$-\xi^2 v + \partial_{rr} v_\rho + \frac{1}{r} \partial_r v_\rho + \frac{1}{r^2} \partial_{\phi\phi} v_\rho = \hat{\rho}. \quad (16)$$

The next Lemma clarifies in which sense v_σ and v_ρ are the Fourier transforms of u_σ and u_ρ , and allows us to calculate $E_{\rho\rho}$ and $E_{\sigma\sigma}$ in Fourier space.

Lemma 27. *Let $f: \{(\xi, y) \in \mathbb{R} \times \mathbb{R}^2\} \rightarrow \mathbb{R}$ be a function such that for all $y \in \mathbb{R}^2$ the map $f(\cdot, y)$ is in $L^2(\mathbb{R})$, and let $g: \{(x, y) \in \mathbb{R} \times \mathbb{R}^2\} \rightarrow \mathbb{R}$ be such that for all $y \in \mathbb{R}^2$ we have $g(\cdot, y) = \mathcal{F}^{-1}(f(\cdot, y))$.*

If $\xi f \in L^2(\mathbb{R}^3)$ and $\nabla_y f \in L^2(\mathbb{R}^3)$ then, in the sense of equality in $L^2(\mathbb{R}^3)$, $\partial_x g = \mathcal{F}^{-1}(i\xi f)$ and $\nabla_y g = \mathcal{F}^{-1}(\nabla_y f)$. In particular we have

$$\|\nabla g\|_{L^2(\mathbb{R}^3)}^2 = \|\xi f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla_y f\|_{L^2(\mathbb{R}^3)}^2.$$

If $\xi f \notin L^2(\mathbb{R}^3)$ or $\nabla_y f \notin L^2(\mathbb{R}^3)$ then $\nabla g \notin L^2(\mathbb{R}^3)$.

Proof. The methods of the proof are standard, details can be found in [Küh]. \square

In the next lemma and the subsequent corollary we will determine the Fourier transform of u_σ . We use Bessel functions to solve the differential equations (14) and (15). Let I_k and K_k denote the modified k^{th} Bessel functions

$$I_k(x) = e^{-\frac{\pi k i}{2}} J_k(ix), \quad K_k(x) = \frac{\pi i}{2} e^{\frac{\pi i}{2}} H_k^{(1)}(ix),$$

where J_k is the k^{th} Bessel function of first kind and $H_k^{(1)}$ is the k^{th} Hankel function of first kind. In particular both, I_k and K_k , are solutions of the differential equation

$$r^2 \partial_{rr} u + r \partial_r u - (r^2 + k^2)u = 0.$$

The function I_k is continuous in zero and K_k vanishes at infinity.

Lemma 28. Let $m_y: \mathbb{R} \rightarrow \{0\} \times \mathbb{R}^2$ be a map, set $m_\kappa := m_y \cdot \vec{e}_\kappa$ where κ is the angle between the unit vector \vec{e}_κ and \vec{e}_{y_1} , and set

$$v_\kappa: \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi[\rightarrow \mathbb{R},$$

$$(\xi, r, \phi) \mapsto \begin{cases} \hat{m}_\kappa(\xi) R K_1(|\xi|R) I_1(|\xi|r) \cos(\phi - \kappa) & \text{if } r \leq R \\ \hat{m}_\kappa(\xi) R I_1(|\xi|R) K_1(|\xi|r) \cos(\phi - \kappa) & \text{if } r > R \end{cases} \quad (17)$$

If m_y is square integrable with $m_y = m_\kappa \vec{e}_\kappa$, then for all $\xi \in \mathbb{R}$ the map $v_\kappa(\xi, \cdot)$ is continuous in \mathbb{R}^2 , differentiable in $\mathbb{R}^2 \setminus \partial D_R$ and fulfills the equations (14) and (15).

Proof. Simple calculation shows that v_κ is fulfills the equation (14) for $r \neq R$ and is continuous at $r = R$. Using the differentiation rules for Bessel functions and the identity

$$K_1(t)(I_0(t) + I_2(t)) + I_1(t)(K_0(t) + K_2(t)) = \frac{2}{t} \quad (18)$$

we see that (15) is fulfilled. \square

Corollary 29. *Let the notation be as in Lemma 28 and set $v := v_0 + v_\pi$. For all $m_y \in L^2(\mathbb{R}, \{0\} \times \mathbb{R}^2)$ the map $v(\xi, \cdot)$ is continuous in \mathbb{R}^2 and differentiable in $\mathbb{R}^2 \setminus \partial D_R$. The map v fulfills the equations (14) and (15). Moreover $v(\cdot, y)$ is in $L^2(\mathbb{R})$ for all $y \in \mathbb{R}^2$ and both ξv and $\nabla_y v$ are in $L^2(\mathbb{R}^3)$. The map $u := \check{v}$ fulfills (4) and we have*

$$E_{\sigma\sigma}(m_y) = \int_{\mathbb{R}^3} |\nabla_y v_0|^2 + \xi^2 |v_0|^2 + \int_{\mathbb{R}^3} |\nabla_y v_\pi|^2 + \xi^2 |v_\pi|^2. \quad (19)$$

Proof. Note that $E_{\sigma\sigma}(v) = E_{\sigma\sigma}(v_0) + E_{\sigma\sigma}(v_\pi)$ (Lemma 7 (ii)). The statements now follow from Lemma 28 and direct calculation. \square

We now use the explicit representation of \hat{u}_σ to find upper and lower bounds on the Fourier multiplier of $E_{\sigma\sigma}$.

Proof of Theorem 8 (i). Using the notation of Lemma 28 we define

$$\begin{aligned} G_{\text{in}}(\kappa, \xi) &:= \int_0^R \int_0^{2\pi} r |\xi v_\kappa|^2 + r |\partial_r v_\kappa|^2 + \frac{1}{r} |\partial_\phi v_\kappa|^2 d\phi dr \\ &= |\hat{m}_\kappa(\xi)|^2 R^2 \xi^2 K_1(|\xi|R)^2 \int_0^R \int_0^{2\pi} \cos^2(\phi - \kappa) r \\ &\quad \left(\frac{1}{4} (I_0(|\xi|r) + I_2(|\xi|r))^2 + I_1(|\xi|r)^2 \right) + \sin^2(\phi - \kappa) \frac{1}{r \xi^2} I_1(|\xi|r)^2 d\phi dr \\ &= |\hat{m}_\kappa(\xi)|^2 R^2 \xi^2 K_1(|\xi|R)^2 \int_0^R \pi r \left(\frac{1}{2} I_0(|\xi|r)^2 + I_1(|\xi|r)^2 + \frac{1}{2} I_2(|\xi|r)^2 \right) dr \\ &= R^2 |\hat{m}_\kappa(\xi)|^2 \frac{\pi}{2} |\xi|R K_1(|\xi|R)^2 (I_0(|\xi|R)I_1(|\xi|R) + I_1(|\xi|R)I_2(|\xi|R)) \end{aligned}$$

Here we have used the recurrence relations for Bessel functions and the equalities

$$\partial_t(tI_0(t)I_1(t)) = tI_0(t)^2 + tI_1(t)^2, \quad \partial_t(tI_1(t)I_2(t)) = tI_1(t)^2 + tI_2(t)^2$$

Similarly we can prove

$$\begin{aligned} G_{\text{out}}(\kappa, \xi) &:= \int_R^\infty \int_0^{2\pi} r |\xi v_\kappa|^2 + r |\partial_r v_\kappa|^2 + \frac{1}{r} |\partial_\phi v_\kappa|^2 d\phi dr \\ &= R^2 |\hat{m}_\kappa(\xi)|^2 \frac{\pi}{2} |\xi|R I_1(|\xi|R)^2 (K_0(|\xi|R)K_1(|\xi|R) + K_1(|\xi|R)K_2(|\xi|R)). \end{aligned}$$

Thus we have $E_{\sigma\sigma}(m_y) = R^2 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 g(\xi R) d\xi$ with

$$g(t) := \frac{\pi}{2} |t| K_1(|t|) I_1(|t|) (K_1(|t|) (I_0(|t|) + I_2(|t|)) + I_1(|t|) (K_0(|t|) + K_2(|t|))).$$

Now (18) yields $g(t) = \pi K_1(|t|) I_1(|t|)$. \square

We now consider $E_{\rho\rho}$. Again we first give an explicit representation of \hat{u}_ρ and then find estimates for the Fourier multiplier.

Lemma 30. *Let $m_x: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\rho := \partial_x m_x$ is in $L^2(\Sigma)$ and set*

$$v: \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi[\rightarrow \mathbb{R},$$

$$(\xi, r, \phi) \mapsto \begin{cases} \frac{\hat{\rho}}{\xi^2} (|\xi R| K_1(|\xi| R) I_0(|\xi| r) - 1) & \text{if } r \leq R \\ \frac{\hat{\rho}}{\xi^2} |\xi R| I_1(|\xi| R) K_0(|\xi| r) & \text{if } r > R \end{cases} \quad (20)$$

Then $v(\xi, \cdot)$ is continuously differentiable and v is a solution of (16). The map $v(\cdot, y)$ is in $L^2(\mathbb{R})$ for all $y \in \mathbb{R}^2$. If both ξv and $\nabla_y v$ are in $L^2(\mathbb{R}^3)$, the map $u := \check{v}$ is a solution of (6) and we have $E_{\rho\rho}(m_x \vec{e}_x) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \xi^2 |v|^2 + |\nabla_y v|^2 dy d\xi$. Otherwise $E_{\rho\rho}(m_x)$ is infinite.

Proof. Simple calculation shows that v is a solution of (16) for $r \neq R$ and continuous at $r = R$. To see that $\partial_r v$ is continuous at $r = R$ we use the differentiation rules for Bessel functions and the identity $I_0(t)K_1(t) + K_1(t)I_0(t) = \frac{1}{t}$. \square

Proof of Theorem 8 (ii). Using the notation of Lemma 28 we define

$$\begin{aligned} H_{\text{in}}(\xi) &:= \int_0^R 2\pi r (|\partial_r v(\xi, r)|^2 + |\xi v(\xi, r)|^2) dr \\ &= R^2 \frac{\hat{\rho}(\xi)^2}{\xi^2} 2\pi \int_0^R r \left(\xi^2 K_1(|\xi| R)^2 I_1(|\xi| r)^2 + \left(\xi K_1(\xi R) I_0(|\xi| r) - \frac{1}{R} \right)^2 \right) dr \\ &= R^4 \hat{\rho}(\xi)^2 \frac{2\pi}{(\xi R)^2} \left(|\xi| R K_1(|\xi| R)^2 I_1(|\xi| R) I_0(|\xi| R) - 2K_1(\xi R) I_1(|\xi| R) + \frac{1}{2} \right) \\ H_{\text{out}}(\xi) &:= \int_R^\infty 2\pi r (|\partial_r v(\xi, r)|^2 + |\xi v(\xi, r)|^2) dr \\ &= R^2 \hat{\rho}^2 2\pi I_1(|\xi| R)^2 \int_R^\infty (r K_1(|\xi| r)^2 + r K_0(|\xi| r)^2) dr \\ &= R^4 \hat{\rho}^2 \frac{2\pi}{|\xi| R} I_1(|\xi| R)^2 K_0(|\xi| R) K_1(|\xi| R) \end{aligned}$$

Thus we have $E_{\rho\rho}(m_x \vec{e}_x) = R^4 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 h(\xi R) d\xi$ with

$$\begin{aligned} h(t) &:= 2\pi \left(\frac{1}{2t^2} + K_1(|t|) I_1(|t|) \left(\frac{1}{|t|} I_0(|t|) K_1(|t|) + \frac{1}{|t|} I_1(|t|) K_0(|t|) - \frac{2}{t^2} \right) \right) \\ &= \frac{\pi}{t^2} (1 - 2K_1(|t|) I_1(|t|)). \end{aligned}$$

\square

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