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by

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Abstract. We prove a formula for the determinant of Laplacian on an arbitrary compact polyhedral surface of genus one. This formula generalizes the well-known Ray-Singer result for a flat torus. A special case of flat conical metrics given by the modulus of a meromorphic quadratic differential on an elliptic surface is also considered. We study the determinant of Laplacian as a functional on the moduli space $Q_1(1,\dots,1,[-1]^L)$ of meromorphic quadratic differentials with L simple poles and L simple zeros and derive formulas for variations of this functional with respect to natural coordinates on $Q_1(1,\dots,1,[-1]^L)$. We give also a new proof of Troyanov's theorem stating the existence of a conformal flat conical metric on a compact Riemann surface of arbitrary genus with a prescribed divisor of conical points.

1 Introduction

There exist several equivalent ways to look at compact Riemann surfaces: for instance, one can define them via algebraic equations or make use of one of the uniformization theorems, introducing the surface as, say, the quotient of the upper half-plane over the action of a Fuchsian group. Another possibility to get a Riemann surface comes from Riemannian geometry: a two-dimensional Riemannian manifold carries the natural complex structure defined via isothermal local parameters.

Another, simple and elementary, way to represent a Riemann surface is the following: one can consider the boundary of an arbitrary (connected but, generally, not simply connected) polyhedron in the three dimensional Euclidean space. This is a polyhedral surface which carries the structure of a complex manifold (the corresponding system of holomorphic local parameters is obvious for all points except the vertices; near a vertex one should introduce the local parameter $\zeta = z^{2\pi/\alpha}$, where α is the sum of the angles adjacent to the vertex). In this way the Riemann surface comes together with a conformal metric; this metric is flat and has conical singularities at the vertices. Actually, to perform this construction it is not necessary to start from polyhedra embedded in the three dimensional Euclidean space, one can use instead some simplicial complex, thinking of a polyhedral surface as glued from plane triangles.

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Troyanov (see [20]) proved that on any compact Riemann surface there exists a flat conformal conical metric with a prescribed divisor of conical points (see the precise formulation of this theorem below). Moreover, he noticed that any compact Riemann surface with flat conformal conical metric admits a proper triangulation (i. e. each conical point is a vertex of some triangle of the triangulation). This means that the above construction is universal: any compact Riemann surface can be glued from triangles.

The goal of this paper is to study the determinant of the Laplacian (acting in the trivial line bundle over the surface) as a functional on the space of Riemann surfaces with conformal flat conical metrics (polyhedral surfaces). The similar question for *smooth* conformal metrics and arbitrary holomorphic bundles was very popular in the eighties and early nineties being motivated by the string theory. Among the most notable results one can mention the Ray-Singer calculation of the determinant of the Laplacian in arbitrary flat line bundle over flat tori [18], an explicit formula for the determinant of Laplacian in the Arakelov metric found by Dugan and Sonoda [7], the D'Hoker-Phong formula relating the determinant of the Laplacian in the Poincaré metric to Selberg's zeta-function [6], the Zograf-Takhtajan formula for variation of the determinant of Laplacian in the Poincaré metric with respect to moduli of the Riemann surface [22], Fay's formula for variation of the determinant of Laplacian under arbitrary (not necessarily conformal) variation of the metric [9].

The determinants of Laplacians in flat singular metrics are much less studied: among the very few appropriate references we mention [5], where the determinant of the Laplacian in conical metric was defined via some special regularizations of the diverging Liouville integral and the question about the relation of such a definition with the spectrum of the Laplacian remained open, and two papers [10], [1] dealing with flat conical metrics on the Riemann sphere.

In [12] the determinant of the Laplacian was studied as a functional

$$\mathcal{H}_g(k_1, \dots, k_M) \ni (\mathcal{L}, \omega) \mapsto \det \Delta^{|\omega|^2}$$

on the space $\mathcal{H}_g(k_1, \dots, k_M)$ of equivalence classes of pairs $(\mathcal{L}, \omega)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and ω is a holomorphic one-form (an Abelian differential) with M zeros of multiplicities $k_1, \dots, k_M$. Here $\det \Delta^{|\omega|^2}$ stands for the determinant of the Laplacian in the flat metric $|\omega|^2$ having conical singularities at the zeros of ω. The corresponding results for the moduli spaces $Q_g(k_1, \dots, k_M, [-1]^L)$ of quadratic differentials with M zeros of multiplicities $k_1, \dots, k_M$ and L simple poles were stated in [12] without proofs. The flat conical metric $|\omega|^2$ considered in [12] is very special: the divisor of the conical points of this metric is not arbitrary (it should be the canonical one, i. e. coincide with the divisor of a holomorphic one-form) and the conical angles at the conical points are integer multiples of 2π.

In the present paper we study determinants of Laplacians on arbitrary polyhedral surfaces of genus one. Our first main result is formula (3.2) giving an explicit expression for the determinant of the Laplacian on arbitrary polyhedral torus. Then we consider an important special case of flat metrics given as the modulus of a meromorphic quadratic differential on the torus with at most simple poles. In this case we give simple and straightforward proofs of the results announced in [12], in particular, we derive formulas of the Rauch type for variations of basic holomorphic differential and the period of the elliptic surface under variation of the natural holomorphic coordinates on the moduli space of meromorphic quadratic differentials. The second main result of the paper is Theorem 2 below which gives variational formulas for the determinant of the Laplacian as a functional on this moduli space.

Although in this paper we deal with elliptic surfaces only, we start it with a new proof of Troyanov's existence theorem for flat conical metrics on Riemann surfaces of an arbitrary genus; in contrast to previously known proofs of this theorem our proof is constructive.

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2 Flat conical metrics on surfaces

2.1 Troyanov's theorem

Let $\sum_{k=1}^N \beta_k P_k$ be a (generalized, i. e. the coefficients β_k are not necessary integers) divisor on a compact Riemann surface $\mathcal{L}$ of genus g. Let also $\sum_{k=1}^N \beta_k = 2g-2$. Then, according to Troyanov's theorem (see [20]), there exists a (unique up to a homothety) conformal flat metric $\mathbf{m}$ on $\mathcal{L}$ which is smooth in $\mathcal{L}\setminus\{P_1,\dots,P_N\}$ and has simple singularities of order β_k at P_k. The latter means that in a vicinity of P_k the metric $\mathbf{m}$ can be represented in the form

$$\mathbf{m} = e^{u(z,\bar z)}|z|^{2\beta_k}|dz|^2, \tag{2.1}$$

where z is a conformal coordinate and u is a smooth real-valued function. In particular, if $\beta_k > -1$ the point P_k is conical with conical angle $2\pi(\beta_k+1)$. Here we construct the metric $\mathbf{m}$ explicitly, giving an effective proof of Troyanov's theorem.

Fix a canonical basis of cycles on $\mathcal{L}$ (we assume that $g \geq 1$, the case $g=0$ is trivial) and let $E(P,Q)$ be the prime-form (see [8]). Then for any divisor $\mathcal{D} = r_1Q_1 + \dots r_mQ_M - s_1R_1 - \dots - s_NR_N$ of degree zero on $\mathcal{L}$ (here the coefficients r_k, s_k are positive integers) the meromorphic differential

$$\omega_{\mathcal{D}} = d_z \ln \frac{\prod_{k=1}^M E^{r_k}(z,Q_k)}{\prod_{k=1}^N E^{s_k}(z,R_k)}$$

is holomorphic outside $\mathcal{D}$ and has the first order poles at the points of $\mathcal{D}$ with residues r_k at Q_k and $-s_k$ at R_k. Since the prime-form is single-valued along the a-cycles, all the a-periods of the differential $\omega_{\mathcal{D}}$ vanish.

Let $\{v_\alpha\}_{\alpha=1}^g$ be the basis of holomorphic normalized differentials and $\mathbb{B}$ the corresponding matrix of b-periods. Then all the a- and b-periods of the meromorphic differential

$$\Omega_{\mathcal{D}} = \omega_{\mathcal{D}} - 2\pi i \sum_{\alpha,\beta=1}^g ((\Im\mathbb{B})^{-1})_{\alpha\beta} \Im\left(\int_{s_1R_1+\dots s_NR_N}^{r_1Q_1+\dots r_MQ_M} v_\beta\right) v_\alpha$$

are purely imaginary (see [8], p. 4).

Obviously, the differentials $\omega_{\mathcal{D}}$ and $\Omega_{\mathcal{D}}$ have the same structure of poles: their difference is a holomorphic 1-form.

Choose a base-point P_0 on $\mathcal{L}$ and introduce the following quantity

$$\mathcal{F}_{\mathcal{D}}(P) = \exp\int_{P_0}^P \Omega_{\mathcal{D}}.$$

Clearly, $\mathcal{F}_{\mathcal{D}}$ is a meromorphic section of some *unitary* flat line bundle over $\mathcal{L}$, the divisor of this section coincides with $\mathcal{D}$.

Now we are ready to construct the metric $\mathbf{m}$. Choose any holomorphic differential w on $\mathcal{L}$ with, say, only simple zeros $S_1, \dots, S_{2g-2}$. Then one can set $\mathbf{m} = |u|^2$, where

$$u(P) = w(P)\mathcal{F}_{(2g-2)S_0 - S_1 - \dots S_{2g-2}}(P) \prod_{k=1}^{N} [\mathcal{F}_{P_k - S_0}(P)]^{\beta_k} \tag{2.2}$$

and S_0 is an arbitrary point.

Notice that in case $g = 1$ the second factor in (2.2) is absent and the remaining part is nonsingular at the point S_0.

2.2 Distinguished local parameter

In a vicinity of a conical point the flat metric (2.1) takes the form

$$\mathbf{m} = |g(z)|^2 |z|^{2\beta} |dz|^2$$

with some holomorphic function g such that $g(0) \neq 0$. It is easy to show (see, e. g., [20], Proposition 2) that there exists a holomorphic change of variable $z = z(w)$ such that in the local parameter w

$$\mathbf{m} = |w|^{2\beta} |dw|^2 .$$

We shall call the parameter w (unique up to a constant factor c, $|c| = 1$) *distinguished.* In case $\beta > -1$ the existence of the distinguished parameter means that in a vicinity of conical point the surface $\mathcal{L}$ is isometric to the standard cone with conical angle $2\pi(\beta + 1)$.

3 Flat conical metrics on tori and determinants of Laplacians

3.1 Determinants of Laplacians

From now on $\mathcal{L}$ is an elliptic ($g = 1$) Riemann surface and it is assumed that $\mathcal{L}$ is the quotient of the complex plane $\mathbb{C}$ by the lattice generated by 1 and σ, where $\Im\sigma > 0$. The differential dz on $\mathbb{C}$ gives rise to a holomorphic differential v_0 on $\mathcal{L}$ with periods 1 and σ.

Let $\sum_{k=1}^{N} \beta_k P_k$ be a generalized divisor on $\mathcal{L}$ with $\sum_{k=1}^{N} \beta_k = 0$ and assume that $\beta_k > -1$ for all k. Let $\mathbf{m}$ be a flat conical metric corresponding to this divisor via Troyanov's theorem. Clearly, it has a finite area and is defined uniquely when this area is fixed. Fixing numbers $\beta_1, \dots, \beta_N > -1$ such that $\sum_{k=1}^{N} \beta_k = 0$, we define the space $\mathcal{M}(\beta_1, \dots, \beta_N)$ as the moduli space of pairs $(\mathcal{L}, \mathbf{m})$, where $\mathcal{L}$ is an elliptic surface and $\mathbf{m}$ is a flat conformal metric on $\mathcal{L}$ having N conical singularities with conical angles $2\pi(\beta_k + 1)$, $k = 1, \dots, N$. The space $\mathcal{M}(\beta_1, \dots, \beta_N)$ is a connected orbifold of real dimension $2N + 3$.

Let $z = x + iy$ be a conformal coordinate on $\mathcal{L}$ and let $\mathbf{m} = \rho^{-2}(z, \bar{z})\widehat{dz} = \rho^{-2} dx\, dy$. Denote by $\Delta^{\mathbf{m}}$ the Friedrichs extension of the operator

$$C_0^\infty(\mathcal{L} \setminus \{P_1, \dots, P_N\}) \ni f \mapsto 4\rho^2 \partial^2_{z\bar{z}} f.$$

The determinant of $\Delta^{\mathbf{m}}$ for flat metrics with conical singularities was first defined in [10]. Briefly, this definition looks as follows. Cheeger's theorem ([4]) states that the spectrum, $\{\lambda_k\}$, of $\Delta^{\mathbf{m}}$ is discrete (with each eigenvalue having finite multiplicity) and its counting function, $N(\lambda)$, obeys the standard spectral asymptotics $N(\lambda) = O(|\lambda|)$ at the infinity. Moreover, from the results of Brüning and Seeley [2] it follows that the analytic continuation of the corresponding operator zeta-function

$$\zeta_{\Delta^{\mathbf{m}}}(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s}$$

(the latter series converges to a holomorphic function of s in the half-plane $\{\Re s > 1\}$) is meromorphic in the complex plane and has no pole at $s = 0$. Therefore, one can define the determinant of the operator $\Delta^{\mathbf{m}}$ via the standard Ray-Singer regularization:

$$\det\Delta^{\mathbf{m}} = \exp\{-\zeta'_{\Delta^{\mathbf{m}}}(0)\}\,.$$

The main result of the present paper, stated below as Theorem 1, is an explicit formula for the function

$$\mathcal{M}(\beta_1, \dots, \beta_N) \ni (\mathcal{L}, \mathbf{m}) \mapsto \det\Delta^{\mathbf{m}}\,.$$

Write the normalized holomorphic differential v_0 on the elliptic surface $\mathcal{L}$ in the the distinguished local parameter w_k near the conical point P_k $(k = 1, \dots, N)$ as

$$v_0 = f_k(w_k) dw_k$$

and define

$$\mathbf{f}_k := f_k(w_k)|_{w_k=0},\ \ k = 1, \dots, N\,. \tag{3.1}$$

Theorem 1 *The following formula holds true*

$$\det\Delta^{\mathbf{m}} = C|\Im\sigma|\,\text{Area}(\mathcal{L}, \mathbf{m})\,|\eta(\sigma)|^4 \prod_{k=1}^{N} |\mathbf{f}_k|^{-\beta_k/6}, \tag{3.2}$$

where C is a constant depending only on $\beta_1, \dots, \beta_N$, and η is the Dedekind eta-function.

The proof of this theorem will be given in the next section.

Remark 1 An analogous statement for genus 0 polyhedral surfaces was obtained in [1]. When the flat metric $\mathbf{m}$ is everywhere nonsingular formula (3.2) reduces to the well-known Ray-Singer result [18].

3.2 Proof of Theorem 1

The proof uses three basic technical tools: the Burghelea-Friedlander-Kappeler analytic surgery, the Polyakov formula and the Ray-Singer calculation of the determinant of Laplacian corresponding to smooth flat metric on the elliptic surface.

3.2.1 Analytic surgery

Take $\epsilon > 0$ and introduce the disks $D_k(\epsilon) = \{|w_k| \leq \epsilon\}$, centered at the conical points P_k, $k = 1, \dots, N$. Let $\Sigma_\epsilon = \mathcal{L} \setminus \cup_{k=1}^N D_k(\epsilon)$. Let also $g_k : \overline{\mathbb{R}}_+ \to \mathbb{R}$, $k = 1, \dots, N$ be smooth positive functions such that

1. $\int_0^1 g_k^2(r) r dr = \int_0^1 r^{2\beta_k+1} dr = \frac{1}{2\beta_k+2}$,

2. $g_k(r) = r^{\beta_k}$ for $r \geq 1$.

Define the family of *smooth* conformal metrics $\mathbf{m}_\epsilon$ on $\mathcal{L}$ via

$$\mathbf{m}_\epsilon(z) = \begin{cases} \epsilon^{2\beta_k} g_k^2(|w_k|/\epsilon)|dw_k|^2, & z \in D_k(\epsilon), \quad k = 1, \dots, N \\ \mathbf{m}(z), & z \in \Sigma_\epsilon \end{cases}$$

The metrics $\mathbf{m}_\epsilon$ converge to $\mathbf{m}$ in $\mathcal{L} \setminus \{P_1, \dots, P_N\}$ as $\epsilon \to 0$ and

$$\text{Area}(\mathcal{L}, \mathbf{m}_\epsilon) = \text{Area}(\mathcal{L}, \mathbf{m}).$$

Lemma 1 *Let ∂_t be the differentiation with respect to one of the coordinates on $\mathcal{M}(\beta_1, \dots, \beta_N)$ and let $\text{det}\Delta^{\mathbf{m}_\epsilon}$ be the standard ζ-regularized determinant of the Laplacian corresponding to the smooth metric $\mathbf{m}_\epsilon$. Then*

$$\partial_t \ln \text{det}\Delta^{\mathbf{m}} = \partial_t \ln \text{det}\Delta^{\mathbf{m}_\epsilon}. \tag{3.3}$$

Proof. For simplicity suppose first that $N = 1$. Let $(\Delta^{\mathbf{m}_\epsilon}|D)$ and $(\Delta^{\mathbf{m}_\epsilon}|\Sigma)$ be the operators of the Dirichlet boundary problem for $\Delta^{\mathbf{m}_\epsilon}$ in domains $D := D_1(\epsilon)$ and $\Sigma := \Sigma_\epsilon$ respectively. Define the Neumann jump operator (a pseudodifferential operator on ∂D of order 1) $R : C^\infty(\partial D) \to C^\infty(\partial D)$ by

$$R(f) = \partial_\nu(V^- - V^+),$$

where ν is the outward normal to ∂D, the functions V^- and V^+ are the solutions of the boundary value problems $\Delta^{\mathbf{m}_\epsilon} V^- = 0$ in D, $V^-|_{\partial D} = f$ and $\Delta^{\mathbf{m}_\epsilon} V^+ = 0$ in Σ, $V^+|_{\partial D} = f$.

In what follows it is crucial that the Neumann jump operator does not change if we vary the metric within the same conformal class. Due to Theorem B^* from [3], we have

$$\text{det}\Delta^{\mathbf{m}_\epsilon} = \det(\Delta^{\mathbf{m}_\epsilon}|D)\det(\Delta^{\mathbf{m}_\epsilon}|\Sigma)\,\text{det}R\,\{\text{Area}(\mathcal{L}, \mathbf{m}_\epsilon)\}\,\{l(\partial D)\}^{-1}, \tag{3.4}$$

where $l(\partial D)$ is the length of the contour ∂D in the metric $\mathbf{m}_\epsilon$ [1].

Analogous statement holds if the metric defining the Laplacian has a conical singularity inside D (see [12]). One has the surgery formula for the operator $\Delta^{\mathbf{m}}$:

$$\text{det}\Delta^{\mathbf{m}} = \det(\Delta^{\mathbf{m}}|D)\det(\Delta^{\mathbf{m}}|\Sigma)\,\text{det}R\,\{\text{Area}(\mathcal{L}, \mathbf{m})\}\,\{l(\partial D)\}^{-1}. \tag{3.5}$$

Notice that the variations of the logarithms of the first factors in right hand sides of (3.4) and (3.5) vanish (these factors are independent of t) whereas the variations of logarithms of all the remaining factors coincide. This leads to (3.3). To consider the general case ($N > 1$) one should apply an obvious generalization of the surgery formula for several non-overlapping discs; similar result can be found in ([17], remark on page 326). □

3.2.2 Polyakov's formula

We state this result in the form given in ([9], p. 62). Let $\mathbf{m}_0 = \rho_0^{-2}(z, \bar{z})\widehat{dz}$ and $\mathbf{m}_1 = \rho_1^{-2}(z, \bar{z})\widehat{dz}$ be two *smooth* conformal metrics on $\mathcal{L}$ and let $\text{det}\Delta^{\mathbf{m}_0}$ and $\text{det}\Delta^{\mathbf{m}_0}$ be the determinants of the corresponding Laplacians (defined via the standard Ray-Singer regularization). Then

$$\frac{\text{det}\Delta^{\mathbf{m}_1}}{\text{det}\Delta^{\mathbf{m}_0}} = \frac{\text{Area}(\mathcal{L}, \mathbf{m}_1)}{\text{Area}(\mathcal{L}, \mathbf{m}_0)} \exp\left\{\frac{1}{3\pi}\int_{\mathcal{L}} \ln\frac{\rho_1}{\rho_0}\partial^2_{z\bar{z}}\ln(\rho_1\rho_0)\widehat{dz}\right\}. \tag{3.6}$$

[1] We have excluded the zero modes of an operator from the definition of its determinant, so we are using the same notation $\det A$ for the determinants of operators A with and without zero modes. In [3] the determinant of an operator A with zero modes is always equal to zero, and what we call here $\det A$ in [3] is called the modified determinant and denoted by $\det^* A$.

3.2.3 Ray-Singer formula

Let Δ be the Laplacian on $\mathcal{L}$ corresponding to the flat smooth metric $|v_0|^2$, where v_0 is the normalized holomorphic differential. The following formula for $\det\Delta$ was proved in [18]:

$$\det\Delta = C|\Im\sigma|^2|\eta(\sigma)|^4, \tag{3.7}$$

where C is a σ-independent constant.

3.2.4 Proof of Theorem 1

By virtue of Lemma 1 one has the relation

$$\partial_t\left\{\ln\frac{\det\Delta^{\mathbf{m}}}{\text{Area}(\mathcal{L},\mathbf{m})} - \ln\frac{\det\Delta}{\Im\sigma}\right\} = \partial_t\left\{\ln\frac{\det\Delta^{\mathbf{m}_\epsilon}}{\text{Area}(\mathcal{L},\mathbf{m}_\epsilon)} - \ln\frac{\det\Delta}{\Im\sigma}\right\}. \tag{3.8}$$

Applying to the r. h. s. of (3.8) Polyakov's formula, we get

$$\partial_t\left\{\ln\frac{\det\Delta^{\mathbf{m}}}{\text{Area}(\mathcal{L},\mathbf{m})} - \ln\frac{\det\Delta}{\Im\sigma}\right\} = \sum_{k=1}^{N}\frac{1}{3\pi}\partial_t\int_{D_k(\epsilon)}(\ln G_k)_{w_k\bar{w}_k}\ln|f_k|\widehat{dw_k}, \tag{3.9}$$

where $G_k(w_k) = \epsilon^{-\beta_k}g_k^{-1}(|w_k|/\epsilon)$. Notice that the function G_k coincides with $|w_k|^{-\beta_k}$ in a vicinity of the circle $\{|w_k| = \epsilon\}$ and the Green formula implies that

$$\int_{D_k(\epsilon)}(\ln G_k)_{w_k\bar{w}_k}\ln|f_k|\widehat{dw_k} = \frac{i}{2}\left\{\oint_{|w_k|=\epsilon}(\ln|w_k|^{-\beta_k})_{\bar{w}_k}\ln|f_k|d\bar{w}_k + \right.$$

$$\left. + \oint_{|w_k|=\epsilon}\ln|w_k|^{-\beta_k}(\ln|f_k|)_{w_k}dw_k + \int_{D_k(\epsilon)}(\ln|f_k|)_{w_k\bar{w}_k}\ln G_k dw_k\wedge d\bar{w}_k\right\}$$

and, therefore,

$$\partial_t\int_{D_k(\epsilon)}(\ln G_k)_{w_k\bar{w}_k}\ln|f_k|\widehat{dw_k} = -\frac{\beta_k\pi}{2}\partial_t\ln|\mathbf{f}_k| + o(1) \tag{3.10}$$

as $\epsilon\to 0$. Formula (3.2) follows from (3.8), (3.10) and (3.7). $\square$

4 Spaces of meromorphic quadratic differentials on elliptic surfaces

Here we study reductions of formula (3.2) to the case of flat conical metrics $|W|$, where W is a meromorphic quadratic differential on $\mathcal{L}$ having only simple poles. For simplicity we assume that the zeroes of W are also simple, although with a little more effort one can consider the general case of arbitrary multiplicities. Notice that the metric $|W|$ is flat and has conical points with conical angles 3π at the zeroes of W and π at the poles of W and, of course, the divisor of conical points is not arbitrary — it should be linearly equivalent to zero (since the canonical divisor of an elliptic surface coincides with the principle one).

Following [14], [15], introduce the space $\mathcal{Q}_1(1,...,1,[-1]^L)$ of equivalence classes of pairs $(\mathcal{L},W)$, where $\mathcal{L}$ is an elliptic surface and W is a meromorphic quadratic differential on $\mathcal{L}$ with L simple zeroes and L simple poles[2]. The space $\mathcal{Q}_1(1,...,1,[-1]^L)$ is known to be a connected complex orbifold [14]. (It should be noted that the space $\mathcal{Q}_1(1,-1)$ is empty.)

Notice that due to modular properties of Dedekind's eta-function the product $|\Im\sigma||\eta(\sigma)|^4$ depends only on the conformal class of the elliptic surface $\mathcal{L}$ (and not on the choice of the canonical basis of cycles on $\mathcal{L}$). So one can introduce the function

$$\mathcal{T} : \mathcal{Q}_1(1,...,1,[-1]^L) \ni (\mathcal{L}, W) \mapsto \frac{\det\Delta^{|W|^2}}{|\Im\sigma||\eta(\sigma)|^4 \,\mathrm{Area}\,(\mathcal{L},|W|^2)}$$

and by (3.2) we have

$$\mathcal{T}(\mathcal{L}, W) = C\,|\tau|^2,$$

with C being a constant independent of $(\mathcal{L}, W)$ and τ given by

$$\tau = \left(\frac{\prod_{k=1}^L \mathbf{h}_k}{\prod_{k=1}^L \mathbf{f}_k}\right)^{\frac{1}{24}}. \tag{4.1}$$

Here $\mathbf{f}_k$ (respectively $\mathbf{h}_k$) is the value of some chosen (say, normalized differential v_0) holomorphic differential on $\mathcal{L}$ at the k-th zero (respectively k-th pole) of the quadratic differential W calculated in the distinguished local parameter. Now, in contrast to Theorem 1, we split the conical points into two types (with angle π and with angle 3π), that is why we use the new notation for the values of v_0 at the conical points with angle π.

The main goal of the remaining part of this paper is to study τ as a function of moduli (the holomorphic coordinates on $\mathcal{Q}_1(1,...,1,[-1]^L)$).

4.1 Local coordinates on $\mathcal{Q}_1(1,...,1,[-1]^L)$

For any pair $(\mathcal{L}, W)$ from $\mathcal{Q}_1(1,...,1,[-1]^L)$ one can construct the so-called canonical two-fold covering

$$\pi : \tilde{\mathcal{L}} \to \mathcal{L}$$

such that $\pi * W = \omega^2$, where ω is a holomorphic 1-differential on $\tilde{\mathcal{L}}$. This covering is ramified over the poles and zeroes of W.

Let $R_1,...,R_L$ be the zeroes of a quadratic differential W and let $S_1,...,S_L$ be its poles. The only zeroes of the holomorphic differential ω on $\tilde{\mathcal{L}}$ are the double zeroes at $R_1,...,R_L$, therefore, one has the relation $2\tilde{g}-2 = 2L$ for the genus $\tilde{g}$ of the surface $\tilde{\mathcal{L}}$ and $\tilde{g} = L+1$.

Denote by $*$ the holomorphic involution on $\tilde{\mathcal{L}}$ interchanging the sheets of the canonical covering. The differential $\omega(P)$ is anti-invariant with respect to involution $*$:

$$\omega(P^*) = -\omega(P). \tag{4.2}$$

Here $\omega(P)$ and $\omega(P^*)$ stand for values of the differential ω in any local parameter lifted from the base of the canonical covering.

Due to ([8], p. 85), one can choose a canonical basis of cycles

$$\{a_\alpha, b_\alpha, a_{\alpha'}, b_{\alpha'}, a_m, b_m\}, \qquad \alpha, \alpha' = 1; \;\; m = 1,...,L-1$$

on $\tilde{\mathcal{L}}$ such that

[2] Two pairs $(\mathcal{L}_1, W_1)$ and $(\mathcal{L}_2, W_2)$ are called equivalent if there exists a biholomorphic map $f : \mathcal{L}_1 \to \mathcal{L}_2$ such that $f_* W_2 = W_1$

- The pair $(\pi a_\alpha, \pi b_\alpha)$ forms a canonical basis on $\mathcal{L}$.
- The following invariance properties under the involution $*$ hold:

$$a_\alpha^* + a_{\alpha'} = b_\alpha^* + b_{\alpha'} = 0 \tag{4.3}$$

and

$$a_m^* + a_m = b_m^* + b_m = 0. \tag{4.4}$$

Remark 2 The symbols denoting the basic cycles $a_\alpha, b_\alpha, a_{\alpha'}, b_{\alpha'}$ are provided with (extrinsic) indices α, α' in order to make our notation agree with that of [8], where the base of the two-fold covering may have arbitrary genus.

Remark 3 It is convenient to keep in mind the following informal representation of the canonical covering $\tilde{\mathcal{L}}$: take the standard picture of a hyperelliptic covering of the Riemann sphere branched at $2L$ points $R_1, \dots, R_L, S_1, \dots, S_L$ with the usual canonical basis of cycles (see, e. g., [16], p. 76) $\{a_m, b_m\}$, $m = 1, \dots, L-1$. Then make two holes on two different sheets (one under another). Now the sheets are two tori and in order to get a canonical basis on the obtained two-fold covering of the torus one have to add to the cycles $\{a_m, b_m\}$, $m = 1, \dots, L-1$ two pairs of cycles $\{a_\alpha, b_\alpha\}$ and $\{a_{\alpha'}, b_{\alpha'}\}$ lying one under another on different sheets of the covering (each pair forms a canonical basis on the corresponding torical sheet).

For corresponding basis of normalized holomorphic differentials $u_\alpha, u_{\alpha'}, u_m$ on $\tilde{\mathcal{L}}$ we have as a corollary of (4.3, 4.4):

$$u_\alpha(P^*) = -u_{\alpha'}(P), \qquad u_m(P^*) = -u_m(P). \tag{4.5}$$

According to [14], the complex dimension of the space $\mathcal{Q}_1(1^L, [-1]^L)$ is $2L$. As it is explained in ([15], §4.2; see, also, [14], §2) one can choose a system of local coordinates on this space as follows:

$$A_\alpha := \oint_{a_\alpha} \omega, \qquad B_\alpha := \oint_{b_\alpha} \omega, \qquad A_m := \oint_{a_m} \omega, \qquad B_m := \oint_{b_m} \omega \tag{4.6}$$

for $\alpha = 1, m = 1, ..., L-1$. (The above coordinates are called in [14] Kontsevich's cohomological coordinates.)

In what follows we shall refer to the cycles $\{a_m, b_m\}$ and the coordinates A_m, B_m as *Latin* and to the cycles $\{a_\alpha, b_\alpha\}$ and the coordinates A_α, B_α as *Greek*.

4.2 Projective connections and canonical meromorphic bidifferential

Having fixed a canonical basis of cycles on a Riemann surface, one can introduce the prime-form $E(P,Q)$ and the canonical meromorphic bidifferential $B(P,Q) = d_P d_Q \ln E(P,Q)$ (see [8]). Recall that the canonical meromorphic bidifferential $B(P,Q)$ is singular on the diagonal $P = Q$ and has the following local behavior as $P \to Q$:

$$B(x(P), x(Q)) = \left(\frac{1}{(x(P) - x(Q))^2} + \frac{1}{6} S_B(x(P)) + o(1) \right) dx(P) dx(Q) \tag{4.7}$$

Here $x(P)$ is a local parameter of a point $P \in \mathcal{L}$ and the term $S_B(x(P))$ is a projective connection. This projective connection is called the *Bergman projective connection*. Recall, that a projective connection S is a quantity transforming under the coordinate change $z = z(t)$ as follows:

$$S(t) = S(z) \left(\frac{dz}{dt} \right)^2 + \{z, t\},$$

where

$$\{z,t\} = \frac{z'''(t)z'(t) - \frac{3}{2}(z''(t))^2}{(z'(t))^2}$$

is the Schwarzian derivative.

In what follows we denote by S_B (respectively $\tilde{S}_B$) and B (respectively $\tilde{B}$) the Bergman projective connection and the canonical meromorphic differential on the elliptic surface $\mathcal{L}$ (respectively on the canonical covering $\tilde{\mathcal{L}}$ of genus $\tilde{g} = L + 1$). The canonical basis of cycles on $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are chosen as it is explained in the previous section.

With σ denoting the b-period of the normalized holomorphic differential v_0 on $\mathcal{L}$, introduce the function $\tilde{\eta}$ by the equation

$$\tilde{\eta}(\sigma) = \frac{d}{d\sigma} \ln \eta(\sigma),$$

where η is the Dedekind eta-function. Then the canonical meromorphic bidifferential on $\mathcal{L}$ has the following explicit expression:

$$B(x,y) = \Big[\wp(\int_x^y v_0) - 4\pi i \tilde{\eta}(\sigma)\Big] v_0(x) v_0(y), \tag{4.8}$$

where $\wp$ is the Weierstrass $\wp$-function (see [8]).

4.3 Rauch type formulas on the space $\mathcal{Q}_1(1^L, [-1]^L)$

Varying the coordinates of the pair $(\mathcal{L}, W)$ in the space $\mathcal{Q}_1(1^L, [-1]^L)$, we change the conformal class of the elliptic surface $\mathcal{L}$. The following two propositions describe the behavior of the normalized holomorphic differential v_0 on $\mathcal{L}$ under variations of the coordinates. Let, as before, ω be the holomorphic differential on $\tilde{\mathcal{L}}$ such that $\omega^2 = W$. Then one can introduce the following local coordinate on $\tilde{\mathcal{L}}$ (outside the divisor (ω)):

$$z(P) = \int_{R_1}^P \omega.$$

Below in order to simplify the notation we always make the following agreement.

Under the expression $v_0(P)$ with the argument P belonging to the canonical covering one should understand the lift $\pi_* v_0$ of the one-form v_0 on the base $\mathcal{L}$ to the canonical covering $\tilde{\mathcal{L}}$. The same agreement holds for the canonical meromorphic bidifferential $B(P,Q)$ on $\mathcal{L}$: if P (or Q or both P and Q) belongs to the canonical covering one should apply the corresponding lift.

Proposition 1 *If $z(P)$ is kept fixed under the differentiation then the basic differential v_0 on $\mathcal{L}$ depends on the coordinates A_α and B_α as follows*

$$\frac{\partial v_0(P)}{\partial A_\alpha}\Big|_{z(P)} = -\frac{1}{2\pi i} \oint_{b_\alpha} \frac{v_0(Q) B(P,Q)}{\omega(Q)}, \qquad \frac{\partial v_0(P)}{\partial B_\alpha}\Big|_{z(P)} = \frac{1}{2\pi i} \oint_{a_\alpha} \frac{v_0(Q) B(P,Q)}{\omega(Q)}. \tag{4.9}$$

Proof. Let us prove the first formula of (4.9). The differential $\frac{\partial v_0(P)}{\partial A_\alpha}\Big|_{z(P)}$ has a jump on $\tilde{\mathcal{L}}$ only on the cycle b_α and all the a-periods of this differential vanish. Therefore, one can restore this differential in terms of the canonical meromorphic differential $\tilde{B}(P,Q)$ on $\tilde{\mathcal{L}}$:

$$\frac{\partial v_0(P)}{\partial A_\alpha}\Big|_{z(P)} = \frac{1}{2\pi i} \oint_{b_\alpha} \frac{v_0(Q) \tilde{B}(P,Q)}{\omega(Q)}$$

(cf., [23]). Recall that

$$b_m = -b_m^*, \quad \omega(Q^*) = -\omega(Q), \quad \omega_\alpha(Q^*) = -\omega_\alpha(Q) \tag{4.10}$$

and that the canonical meromorphic differential on $\tilde{\mathcal{L}}$ satisfies

$$\tilde{B}(P^*, Q^*) = \tilde{B}(P, Q) \tag{4.11}$$

for any $P, Q \in \mathcal{L}$ and is related to the meromorphic differential $B(P, Q)$ on $\mathcal{L}$ as follows:

$$B(P, Q) = \tilde{B}(P, Q) + \tilde{B}(P, Q^*), \quad P, Q \in \mathcal{L} \tag{4.12}$$

(see [8]). Therefore,

$$\oint_{b_\alpha} \frac{v_0(Q)\tilde{B}(P, Q)}{\omega(Q)} = \frac{1}{2}\left\{\oint_{b_\alpha} \frac{v_0(Q)\tilde{B}(P, Q)}{\omega(Q)} + \oint_{b_\alpha} \frac{v_0(Q)\tilde{B}(P, Q^*)}{\omega(Q)}\right\} = \frac{1}{2}\oint_{b_\alpha} \frac{v_0(Q)B(P, Q)}{\omega(Q)}$$

The second formula of (4.9) can be proved in the same way.

Before writing variational formulas with respect to remaining Latin coordinates we have to introduce some new notation and make an agreement about the choice of Latin cycles.

Let us specify the form of the distinguished local parameters at the points S_i and R_k, $\ i, k = 1, ..., L$ and introduce the local parameters near the same points considered as points of the canonical covering.

The distinguished local parameter (on the base $\mathcal{L}$) near the point R_k will be denoted by λ_k: one has

$$\lambda_k = \left(\int_{R_k}^P \omega\right)^{2/3}.$$

For a neighborhood of R_k on the covering $\tilde{\mathcal{L}}$ we define the local parameter $\tilde{\lambda}_k$ to be $\tilde{\lambda}_k = \left(\int_{R_k}^P \omega\right)^{1/3}$.

The distinguished local parameter near the S_i on $\mathcal{L}$ will be denoted by ϑ_i: one has

$$\vartheta_i = \left(\int_{S_i}^P \omega\right)^2.$$

For a neighborhood of R_k on the covering $\tilde{\mathcal{L}}$ we define the local parameter $\tilde{\vartheta}_i$ to be $\tilde{\vartheta}_i = \int_{S_i}^P \omega$.

Assume for definiteness that the Latin cycles are chosen in the following way: we split the zeros and poles $R_1, \dots, R_L, S_1, \dots, S_L$ into L pairs (R_k, S_k), $k = 1, \dots, L$ and choose the cycle a_k, $k = 1, \dots L-1$ encircling the pair $(R_{k+1}, S_{k+1}$; the cycle b_k intersects the cuts $[R_1, S_1]$ and $[R_{k+1}, S_{k+1}]$ (cf. [16], p. 76). Under this assumption we have the following expressions for $z(P)$ when P belongs to the divisor (ω):

$$\begin{gathered} z(S_1) = \sum_{m=1}^{L-1} \frac{A_m}{2}, \quad z(S_2) = \frac{A_1 - B_1}{2}, \quad z(R_2) = -\frac{B_1}{2}, \\ z(S_k) = -\frac{B_{k-1}}{2} + \sum_{j=1}^{k-2} \frac{A_j}{2}, \quad z(R_k) = -\frac{B_{k-1}}{2} + \sum_{j=1}^{k-1} \frac{A_j}{2}. \end{gathered} \tag{4.13}$$

It will be convenient to use the following agreement: if, say, R_k is the point of the divisor (ω) then $v_0(R_k)$ and $v_0'(R_k)$ are the coefficients in the expansion of v_0 near the point R_k of the *canonical covering*:

$$v_0(P) = (v_0(R_k) + v_0'(R_k)\tilde{\vartheta}_k + \dots)d\tilde{\vartheta}_k.$$

Analogously, for points P outside the divisor (ω): the quantities $v(P)$ and $v'(P)$ are defined via the expansion

$$v(Q) = (v(P) + v'(P)(z(Q) - z(P)) + \dots)dz(Q)$$

near the point P of the canonical covering. The expressions $\omega'(P)$, $\omega''(P)$, $B(P, R_k)$ etc. are understood in the same way. Now we are ready to continue the list of variational formulas.

Proposition 2 *If $z(P)$ is kept fixed under the differentiation and the projection of the point P on the base of canonical covering lies outside the projection of the contour b_m on the base for the first formula and outside the projection of a_m on the base for the second one* [3] *then the basic differential v_0 on $\mathcal{L}$ depends on the coordinates A_m and B_m as follows*

$$\frac{\partial v_0(P)}{\partial A_m}\Big|_{z(P)} = -\frac{1}{4\pi i}\oint_{b_m} \frac{v_0(Q)B(P,Q)}{\omega(Q)}, \qquad \frac{\partial v_0(P)}{\partial B_m}\Big|_{z(P)} = \frac{1}{4\pi i}\oint_{a_m} \frac{v_0(Q)B(P,Q)}{\omega(Q)}. \tag{4.14}$$

If the projection of P on the base lies inside the projection of the contour b_m than the variational formula for v_0 with respect to A_m will look as follows:

$$\frac{\partial v_0(P)}{\partial A_m}\Big|_{z(P)} = -\frac{1}{4\pi i}\oint_{b_m} \frac{v_0(Q)B(P,Q)}{\omega(Q)} + \frac{1}{2}\frac{v_0'(P)\omega(P) - v_0(P)\omega'(P)}{\omega^2(P)}. \tag{4.15}$$

Similarly, if the projection of P lies inside the projection of the contour a_m then

$$\frac{\partial v_0(P)}{\partial B_m}\Big|_{z(P)} = \frac{1}{4\pi i}\oint_{a_m} \frac{v_0(Q)B(P,Q)}{\omega(Q)} + \frac{1}{2}\frac{v_0'(P)\omega(P) - v_0(P)\omega'(P)}{\omega^2(P)}. \tag{4.16}$$

Proof. The proof of formulas (4.14) is similar to the proof of (4.9) in the Proposition 1. Let us prove (4.16). For P in a neighborhood of the point R_k one has the expansion

$$v_0(P) = (\mathbf{f}_k + \mathbf{f}_{k,1}\lambda_k(P) + \dots)d\lambda_k(P), \quad P \to R_k \tag{4.17}$$

Using the relation between the local parameters λ and $\tilde{\lambda}$ we get that $d\lambda_k = 2\tilde{\lambda}_k d\tilde{\lambda}_k$. Taking into account that

$$d\tilde{\lambda}_k = \frac{1}{3}[z(P) - z(R_k)]^{-2/3}dz = \frac{dz}{3\tilde{\lambda}_k^2},$$

we rewrite $v_0(P)$ in the following way:

$$v_0(P) = \frac{2}{3}\left(\frac{\mathbf{f}_k}{\tilde{\lambda}_k} + \mathbf{f}_{k,1}\tilde{\lambda}_k^2 + \dots\right)dz\,.$$

Differentiate this equation with respect to B_m and making use of the relation

$$\frac{\partial \tilde{\lambda}_k(P)}{\partial B_m} = \frac{1}{3}[z(P) - z(R_k)]^{-2/3}\frac{\partial z(R_k)}{\partial B_m}$$

[3]This refers to the the picture explained in Remark 3. To avoid this referring, one has to note that the cycles a_m and $-a_m^*$ (as well as b_m and $-b_m^*$) are freely homotopic and, therefore, by virtue of Theorem 2.5 [19] bound a (uniquely defined) ring domain. The point P should lie outside this domain.

and formulas (4.13), we see that the differential $\frac{\partial v_0(P)}{\partial B_m}$ has the pole of the second order at R_{m+1},

$$\frac{\partial v_0}{\partial B_m}\Big|_{z(P)} = -\frac{1}{3}v_0(R_{m+1})\frac{d\tilde{\lambda}_{m+1}}{\tilde{\lambda}^2_{m=1}} + \dots,$$

and the only other singularity of $\frac{\partial v_0(P)}{\partial B_m}$ on $\tilde{\mathcal{L}}$ is the jump on the cycle a_m. Thus,

$$\frac{\partial v_0}{\partial B_m}\Big|_{z(P)} = \frac{1}{2\pi i}\oint_{a_m}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)} - \frac{1}{3}\mathbf{f}_{m+1}\tilde{B}(P,R_{m+1}). \tag{4.18}$$

Then from (4.10), (4.11) and (4.12) it follows that:

$$\oint_{a_m}\frac{v_0(Q)\tilde{B}(P,Q^*)}{\omega(Q)} = \oint_{a_m^*}\frac{v_0(Q^*)\tilde{B}(P,Q)}{\omega(Q^*)} = -\oint_{a_m^*}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)}.$$

Therefore,

$$\begin{aligned}&\oint_{a_m}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)} - \oint_{a_m}\frac{v_0(Q)\tilde{B}(P,Q^*)}{\omega(Q)} = \oint_{a_m}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)} + \oint_{a_m^*}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)}\\ &= 2\pi i\left[\text{res}\Big|_{Q=P}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)} + \text{res}\Big|_{Q=R_{m+1}}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)}\right]\\ &= 2\pi i\left[\frac{v_0'(P)\omega(P - v_0(P)\omega'(P))}{\omega^2(P)} + \frac{1}{3}v_0'(R_{m+1})\tilde{B}(P,R_{m+1})\right].\end{aligned}$$

Hence,

$$\begin{aligned}\oint_{a_m}\frac{v_0(Q)\tilde{B}(P,Q)}{\omega(Q)} &= \frac{1}{2}\oint_{a_m}\frac{v_0(Q)(\tilde{B}(P,Q^*) + \tilde{B}(P,Q))}{\omega(Q)}\\ &+\pi i\left[\frac{v_0'(P)\omega(P) - v_0(P)\omega'(P)}{\omega^2(P)} + \frac{1}{3}v_0'(R_{m+1})\tilde{B}(P,R_{m+1})\right].\end{aligned} \tag{4.19}$$

Finally, substituting (4.19) into (4.18) we arrive at (4.16). Similarly, one can prove formula (4.15). □

Integrating formulas (4.9) and (4.14–4.16) over the b-cycles of $\mathcal{L}$, we get the following result which presents an analog of the well-known Rauch formulas.

Corollary 1 *The b-periods σ of the Riemann surface $\mathcal{L}$ depend on the coordinates A_α, B_α, A_m, B_m as follows:*

$$\frac{\partial\sigma}{\partial A_\alpha} = -\oint_{b_\alpha}\frac{v_0^2}{\omega}, \qquad \frac{\partial\sigma}{\partial B_\alpha} = \oint_{a_\alpha}\frac{v_0^2}{\omega},$$

$$\frac{\partial\sigma}{\partial A_m} = -\frac{1}{2}\oint_{b_m}\frac{v_0^2}{\omega}, \qquad \frac{\partial\sigma}{\partial B_m} = \frac{1}{2}\oint_{a_m}\frac{v_0^2}{\omega}.$$

Our last technical result is the list of variational formulas for quantities $\mathbf{f}_k$ and $\mathbf{h}_k$.

Lemma 2 *The following variational formulas hold:*

$$\frac{\partial \mathbf{f}_k}{\partial A_\alpha} = -\frac{1}{2\pi i}\oint_{b_\alpha} \frac{v_0(Q)B(R_k,Q)}{\omega(Q)}, \qquad \frac{\partial \mathbf{h}_i}{\partial A_\alpha} = -\frac{1}{2\pi i}\oint_{b_\alpha} \frac{v_0(Q)B(S_i,Q)}{\omega(Q)} \tag{4.20}$$

$$\frac{\partial \mathbf{f}_k}{\partial B_\alpha} = \frac{1}{2\pi i}\oint_{a_\alpha} \frac{v_0(Q)B(R_k,Q)}{\omega(Q)}, \qquad \frac{\partial \mathbf{h}_i}{\partial B_\alpha} = \frac{1}{2\pi i}\oint_{a_\alpha} \frac{v_0(Q)B(S_i,Q)}{\omega(Q)} \tag{4.21}$$

$$\frac{\partial \mathbf{f}_k}{\partial A_m} = -\frac{1}{4\pi i}\oint_{b_m} \frac{v_0(Q)B(R_k,Q)}{\omega(Q)}, \qquad \frac{\partial \mathbf{h}_i}{\partial A_m} = -\frac{1}{4\pi i}\oint_{b_m} \frac{v_0(Q)B(S_i,Q)}{\omega(Q)} \tag{4.22}$$

$$\frac{\partial \mathbf{f}_k}{\partial B_m} = \frac{1}{4\pi i}\oint_{a_m} \frac{v_0(Q)B(R_k,Q)}{\omega(Q)}, \qquad \frac{\partial \mathbf{h}_i}{\partial B_m} = \frac{1}{4\pi i}\oint_{a_m} \frac{v_0(Q)B(S_i,Q)}{\omega(Q)} \tag{4.23}$$

Proof. The proofs of these formulas are similar, let us prove, say, the second formula of (4.23). The proof splits into two cases depending whether the projection of the point P on the base of the canonical covering lies inside or outside of the projection of the basic cycle a_m. For brevity consider only the case when the projection of P lies inside the projection of a_m. In the neighborhood of S_{m+1} one has the expansion

$$v_0(P) = 2\left[\mathbf{h}_{m+1}\left(z(P) + \frac{B_m}{2} - \sum_{j=1}^{m-1}\frac{A_j}{2}\right) + \dots\right] dz.$$

Differentiating this equality with respect to B_m and using the first variational formula of (4.14) for v_0 we get

$$\frac{1}{4\pi i}\oint_{a_m} \frac{v_0(Q)B(S_{m+1},Q)}{\omega(Q)} d\vartheta_{m+1} + \frac{1}{2}\frac{v_0'(S_{m+1})\omega(S_{m+1}) - v_0(S_{m+1})\omega'(S_{m+1})}{\omega^2(S_{m+1})}$$
$$= 2\left[\frac{1}{2}\mathbf{h}_{m+1} + \mathbf{h}'_{m+1\,B_m}\left(z(P) + \frac{B_m}{2} - \sum_{j=1}^{m-1}\frac{A_j}{2}\right) + \dots\right] dz, \quad P \to S_{m+1}$$

Notice that $v_0(S_{m+1}) = 0$ (recall that this is true on the canonical covering and not on the base, where the differential v_0 has neither zero nor poles) and $d\vartheta_{m+1}$ can be rewritten in terms of z-coordinate as $d\vartheta_{m+1} = 2(z(P) - z(S_{m+1}))dz$. Hence,

$$\frac{1}{4\pi i}\oint_{a_m} \frac{v_o(Q)B(S_{m+1},Q)}{\omega(Q)} \cdot 2(z(P) - z(S_{m+1})) + \mathbf{h}_{m+1} = \mathbf{h}_{m+1} + 2\mathbf{h}'_{m+1\,B_m} + \dots, \quad P \to S_{m+1}.$$

Taking the limit $P \to S_{m+1}$, we obtain formula (4.23). □

4.4 Wirtinger tau-function on $\mathcal{Q}_1(1, ..., 1, [-1]^L)$

Let $\xi : \mathbb{C} \longrightarrow \mathbb{C}/\{1, \sigma\} = \mathcal{L}$ be the natural projection and let x be some local parameter on $\mathcal{L}$. Then the Schwarzian derivative $\{\xi^{-1}(x), x\}$, being independent of the choice of the branch of the multivalued map ξ^{-1}, defines a projective connection on $\mathcal{L}$. This projective connection is called (see, e. g., [21]) *the invariant Wirtinger projective connection*: in contrast to the Bergman projective connection it does not depend on the choice of canonical basis of cycles on $\mathcal{L}$. In what follows we denote this projective connection by $S_{\rm Wirt}$. One can also put into correspondence to a quadratic differential W on $\mathcal{L}$ a projective connection S_ω on $\mathcal{L}$ via the equation

$$S_\omega(x(P)) = \left\{ \int^P \omega, x(P) \right\}. \tag{4.24}$$

(The Schwarzian derivative at the r. h. s. is independent of the choice of the branch of $\omega = \sqrt{W}$.)

Notice that the difference between two projective connections $S_{\rm Wirt}$ and S_ω is a meromorphic quadratic differential on L with poles at the zeroes of W. This quadratic differential can be lifted to $\tilde{\mathcal{L}}$, so we may define the the following quantities:

$$H_{A_\alpha} = \frac{1}{12\pi i}\oint_{b_\alpha} \frac{S_{\rm Wirt} - S_\omega}{\omega}, \quad H_{B_\alpha} = -\frac{1}{12\pi i}\oint_{a_\alpha} \frac{S_{\rm Wirt} - S_\omega}{\omega},$$
$$H_{A_m} = \frac{1}{24\pi i}\oint_{b_m} \frac{S_{\rm Wirt} - S_\omega}{\omega}, \quad H_{B_m} = -\frac{1}{24\pi i}\oint_{a_m} \frac{S_{\rm Wirt} - S_\omega}{\omega}$$

Lemma 3 *Introduce the 1-form by*

$$\Omega = H_{A_\alpha} dA_\alpha + H_{B_\alpha} dB_\alpha + \sum_{m=1}^{L-1} (H_{A_m} dA_m + H_{B_m} dB_m).$$

Then

- *the 1-form Ω is independent of the choice of the canonical basis with properties (4.3, 4.4) and therefore is defined on the space $\mathcal{Q}_1(1, ..., 1, [-1]^L)$.*
- *$d\Omega = 0$.*

In the next section we shall prove that

$$\Omega = d \ln \tau, \tag{4.25}$$

where τ is given by (4.1). Since τ is a (multivalued) function on $\mathcal{Q}_1(1, ..., 1, [-1]^L)$ having at most constant multiplicative twists along nontrivial loops in $\mathcal{Q}_1(1, ..., 1, [-1]^L)$ (actually its 24-th power is single-valued on $\mathcal{Q}_1(1, ..., 1, [-1]^L)$), equation (4.25) implies the Lemma.

However, we notice that the direct proof of the Lemma is also possible: the first statement follows from a somewhat cumbersome calculation which uses nothing but linear algebra, whereas the second one can be proved via Rauch type formulas and manipulations with singular double integrals – the proof of a similar statement can be found in [12].

From Lemma 3 it follows that the connection

$$d_{\rm Wirt} = d + \Omega$$

in the trivial line bundle over $\mathcal{Q}_1(1, ..., 1, [-1]^L)$ is flat. This flat connection defines a character of the fundamental group of $\mathcal{Q}_1(1, ..., 1, [-1]^L)$ which in its turn defines a flat line bundle Ξ over $\mathcal{Q}_1(1, ..., 1, [-1]^L)$.

Definition 1 *A horizontal holomorphic section of the bundle Ξ is called Wirtinger tau-function* [4] *on the space $\mathcal{Q}_1(1, ..., 1, [-1]^L)$.*

In the next section the Wirtinger tau-function will be identified with the (multivalued) function τ from (4.1).

[4] It should be noted that its direct analog in case when the space of quadratic differentials on tori is replaced by the moduli space of meromorphic functions on tori has the meaning of the isomonodromic tau-function of Jimbo-Miwa [11].

4.5 Calculation of Wirtinger tau-function.

The following proposition gives an explicit expression for the Wirtinger tau-function on $\mathcal{Q}_1(1,...,1,[-1]^L)$.

Proposition 3 *Let a pair $(\mathcal{L}, W)$ belong to the space $\mathcal{Q}_1(1,...,1,[-1]^L)$. The Wirtinger tau-function on the stratum $\mathcal{Q}_1(1,...,1,[-1]^L)$ of the space of quadratic differentials over the Riemann surface $\mathcal{L}$ is given by the expression*

$$\tau(\mathcal{L}, W) = \left[\frac{\prod_{k=1}^L \mathbf{h}_k}{\prod_{i=1}^L \mathbf{f}_i}\right]^{1/24}. \tag{4.26}$$

In particular, the 24-th power of τ is a single-valued holomorphic function on $\mathcal{Q}_1(1,...,1,[-1]^L)$.

Proof. Let

$$\mathbf{T}(A_\alpha, \{A_m\}) := \ln\left\{\frac{\prod_{k=1}^L \mathbf{h}_k}{\prod_{i=1}^L \mathbf{f}_i}\right\} = (24 \ln \tau).$$

Define the (multivalued) map $R : t \mapsto z$ by $z = \int^P \omega$ and $t = \int^P v_0$. Clearly, the derivative $R'(t)$ is a single-valued function. Then the one-form $(S_{\rm Wirt} - S_\omega)/\omega$ can be rewritten as

$$-\frac{\{R,t\}}{R'}dt,$$

where $\{R,t\}$ is the Schwarzian derivative, and, therefore, the statement of the proposition is equivalent to the following equalities:

$$\frac{\partial T}{\partial A_\alpha} = -\frac{2}{\pi i}\oint_{b_\alpha}\frac{\{R,t\}}{R'}dt, \quad \frac{\partial T}{\partial B_\alpha} = \frac{2}{\pi i}\oint_{a_\alpha}\frac{\{R,t\}}{R'}dt,$$

$$\frac{\partial T}{\partial A_m} = -\frac{1}{\pi i}\oint_{b_m}\frac{\{R,t\}}{R'}dt, \quad \frac{\partial T}{\partial B_m} = \frac{1}{\pi i}\oint_{a_m}\frac{\{R,t\}}{R'}dt.$$

The proof of these four formulas coincide verbatim. For example, let us prove the first one.

Using Lemma 2 and the representation (4.8) of the canonical meromorphic bidifferential on an elliptic surface, we get

$$\frac{\partial T}{\partial A_\alpha} = \sum_{k=1}^L \frac{\mathbf{h}'_k}{\mathbf{h}_k} - \sum_{i=1}^L \frac{\mathbf{f}'_i}{\mathbf{f}_i} = -\frac{1}{2\pi i}\oint_{b_\alpha}\frac{v_0(Q)}{\omega(Q)}\Big\{\sum_{k=1}^L \frac{B(R_k,Q)}{\mathbf{h}_k} + \sum_{i=1}^L \frac{B(S_i,Q)}{\mathbf{f}_i}\Big\} =$$

$$\frac{1}{2\pi i}\oint_{b_\alpha}\Big\{\sum_{k=1}^L \frac{v_0(Q)}{\omega(Q)\mathbf{h}_k d\vartheta_k(P)}\Big[\wp(\int_P^Q v_0) - 4\pi i\tilde\eta(\sigma)\Big]v_0(P)v_0(Q)\Big\}\Big|_{P=R_k}$$

$$-\frac{1}{2\pi i}\oint_{b_\alpha}\Big\{\sum_{i=1}^L \frac{v_0(Q)}{\omega(Q)\mathbf{f}_i d\lambda_i(P)}\Big\{\wp(\int_P^Q v_0) - 4\pi i\tilde\eta(\sigma)\Big\}v_0(P)v_0(Q)\Big\}\Big|_{P=S_i} =$$

$$-\frac{1}{2\pi i}\oint_{b_\alpha}\frac{v_0^2(Q)}{\omega(Q)}\sum_{k=1}^L\Big[\wp(\int_{S_k}^Q v_0) - \wp(\int_{R_k}^Q v_0)\Big].$$

Observe that the sum under the last integral coincides with

$$\frac{d}{dt}\Big(\frac{\mathcal{R}''(t)}{\mathcal{R}'(t)}\Big),$$

where $\mathcal{R}'$ is defined by the relation $W = \mathcal{R}'(t)(dt)^2$.

Since $\mathcal{R}'(t) = [R'(t)]^2$, we get

$$\frac{\partial T}{\partial A_\alpha} = -\frac{1}{\pi i}\oint_{b_\alpha}\left(\frac{R'''}{(R')^2} - \frac{(R'')^2}{(R')^3}\right) dt\,. \tag{4.27}$$

It remains to notice that

$$\oint_{b_\alpha}\frac{R'''}{(R')^2}dt = -\oint_{b_\alpha} R'' d\left(\frac{1}{(R')^2}\right) = 2\oint_{b\alpha}\frac{(R'')^2}{(R')^3}dt\,, \tag{4.28}$$

$$\oint_{b_\alpha}\frac{\{R,t\}}{R'} = \oint_{b_\alpha}\frac{R'''}{(R')^2}dt - \frac{3}{2}\oint_{b_\alpha}\frac{(R'')^2}{(R')^3}dt = 2\oint_{b_\alpha}\frac{(R'')^2}{(R')^3}dt - \frac{3}{2}\oint_{b_\alpha}\frac{(R'')^2}{(R')^3}dt = \frac{1}{2}\oint_{b_\alpha}\frac{(R'')^2}{(R')^3}dt \tag{4.29}$$

and the desired statement follows. □

4.6 Variational formulas for the determinant of the Laplacian

Let a pair $(\mathcal{L}, W)$ belong to $\mathcal{Q}_1(1, ..., 1, [-1]^L)$. Introduce the quantity

$$Q(\mathcal{L}, W) = \frac{\det\Delta^{|W|}}{\{\Im\sigma\}\mathrm{Area}(\mathcal{L}, |W|)}$$

(this is the inverse to the Quillen norm on the determinant line). The following Theorem describes variations of $Q(\mathcal{L}, W)$ with respect to coordinates on the space $\mathcal{Q}_1(1, ..., 1, [-1]^L)$.

Theorem 2 *The variational formulas hold:*

$$\frac{\partial \ln Q}{\partial A_\alpha} = \frac{1}{12\pi i}\oint_{b_\alpha}\frac{S_B - S_\omega}{\omega}, \qquad \frac{\partial \ln Q}{\partial B_\alpha} = -\frac{1}{12\pi i}\oint_{a_\alpha}\frac{S_B - S_\omega}{\omega},$$

$$\frac{\partial \ln Q}{\partial A_m} = \frac{1}{24\pi i}\oint_{b_m}\frac{S_B - S_\omega}{\omega}, \qquad \frac{\partial \ln Q}{\partial B_m} = -\frac{1}{24\pi i}\oint_{a_m}\frac{S_B - S_\omega}{\omega},$$

where $m = 1, \ldots, L-1$ and S_B is the Bergman projective connection.

Proof. Recall that there is the following relation between the invariant Wirtinger and the Bergman projective connections on the elliptic surface $\mathcal{L}$:

$$S_{\mathrm{Wirt}}(x) = S_B(x) + 24\pi i\tilde{\eta}(\sigma)v_0^2(x) \tag{4.30}$$

(see, e. g., [8] p. 35; since Fay uses another normalization of the basic differential, the coefficient near $\tilde{\eta}v_0^2$ in (4.30) differs from that in [8]). By virtue of Proposition 3, relation (4.30) and the Rauch formula from Corollary 1, we have

$$\frac{\partial \ln Q}{\partial A_\alpha} = \frac{\partial \ln(|\eta(\sigma)|^4|\tau(\mathcal{L}, W)|^2)}{\partial A_\alpha} = \frac{\partial \ln(\eta^2(\sigma)\tau(\mathcal{L}, W))}{\partial A_\alpha} = \frac{1}{12\pi i}\oint_{b_\alpha}\frac{S_{\mathrm{Wirt}} - S_\omega}{\omega} + 2\tilde{\eta}(\sigma)\frac{\partial\sigma}{\partial A_\alpha} =$$

$$\frac{1}{12\pi i}\oint_{b_\alpha}\frac{S_{\mathrm{Wirt}} - S_\omega}{\omega} - 2\tilde{\eta}(\sigma)\oint_{b_\alpha}\frac{v_0^2}{\omega} = \frac{1}{12\pi i}\oint_{b_\alpha}\frac{S_B - S_\omega}{\omega},$$

which gives the first variational formula. The remaining variational formulas can be proved in the same way. □

5 Summary and outlook

In this paper we study the determinant of the Laplacian on a polyhedral surface of genus one. The method we use here (see the proof of Theorem 1) can be considered as a generalization of the Polyakov formula, relating the determinants of Laplacians in two smooth conformal metrics, to the case when one of the metrics is flat conical and another is flat and everywhere nonsingular.

Using a further generalization of the Polyakov formula to the case of two flat conical metrics and the results of [12], it is possible to write a closed expression for the determinant of Laplacian on a polyhedral surface of an arbitrary genus. We hope to address this question in the near future.

It is also interesting to look at extremal properties of the determinants of Laplacians in conical metrics; the only known result in this direction is contained in [13], where it was solved the problem of the maximization of the determinant of the Laplacian on the Riemann sphere over the set of flat metrics of area 1 with four conical points of conical angle π.

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