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Abstract

The nonlinear elastic energy of a thin film of thickness h is given by a functional E^h. Friesecke, James and Müller derived the Γ-limits, as $h \to 0$, of the functionals $h^{-\alpha}E^h$ for $\alpha \geq 3$. In this article we study the invertibility properties of almost minimizers of these functionals, and more generally of sequences with equiintegrable energy density. We show that they are invertible almost everywhere away from a thin boundary layer near the film surface. Moreover, we obtain an upper bound for the width of this layer and a uniform upper bound on the diameter of preimages. We construct examples showing that these bounds are sharp. In particular, for all $\alpha \geq 3$ there exist Lipschitz continuous low energy deformations which are not locally invertible.

1 Introduction

This article is motivated by the derivation of asymptotic thin-film theories from three dimensional elasticity by Friesecke, James and Müller [7, 8]. To simplify the discussion, in this introduction we will focus on plate theory, which was treated in [7]. One of their results is that the asymptotic deformations - which describe deformations of the mid-plane of the thin film - belong to the space of $W^{2,2}$-isometric immersions. The starting point of our analysis, Theorem 2 below, shows that such mappings are Bilipschitz on small subdomains. This raises the question to what extent three dimensional (3D) thin-film deformations with low bending energy share this invertibility property. More generally, we study this question for 3D deformations whose bending energy density is equiintegrable, see Definition 1 below. This condition is satisfied for recovery (or realizing) sequences in the language of Γ-convergence, i.e. 3D deformations $v^{(h)}$ that converge, as the thickness h of the plate converges to zero, to a limiting 2D deformation v such that their energies converge as well,

$$\frac{1}{h^3}E^h(v^{(h)}) \to \left(\Gamma - \lim_{h\to 0}\frac{1}{h^3}E^{(h)}\right)(v).$$

A particular kind of recovery sequences are almost minimizing sequences as defined e.g. in [8].

Apart from the (local) invertibility of the limiting deformations, there is further evidence suggesting that deformations with equiintegrable bending energy density

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might be invertible: The particular recovery sequences constructed in [7] are continuously differentiable and have deformation gradients uniformly close to $SO(3)$, so they are locally invertible. Thus there always exist deformations with equiintegrable bending energy density which are locally invertible. We also recall Pantz' derivation [15] of Kirchhoff's plate theory, which uses John's estimates [11]. These estimates require local invertibility, whereas the derivation given in [7] does not need this a priori assumption on the admissible deformations. One could hope to derive local invertibility of recovery sequences a posteriori.

In view of Ball's examples [2], it is not very surprising that one cannot expect pointwise invertibility: We construct deformations with equiintegrable energy density which are almost everywhere invertible, but not pointwise invertible. The other example we give is more interesting: We construct Lipschitz continuous orientation preserving deformations $v^{(h)}$ with equiintegrable energy density which are not locally invertible on a nice simply connected set of positive volume near the film surface. More precisely, for any given sequence of positive numbers ε_h with $\lim_{h\downarrow 0}\varepsilon_h = 0$ there exist $v^{(h)}$ and a set with volume of the order $\varepsilon_h^3 h^3$ consisting of pairs of points $\bar{z}_h, \tilde{z}_h$ with the same image $v^{(h)}(\bar{z}_h) = v^{(h)}(\tilde{z}_h)$. A fraction of these points satisfies $|\bar{z}_h - \tilde{z}_h| \geq \varepsilon_h h$. The deformed configuration contains a set of positive volume consisting of points y whose preimage $(v^{(h)})^{-1}(y)$ has diameter at least $\varepsilon_h h$. Our construction only works near the boundary of the film, and the preimage diameters cannot be increased from $o(h)$ to $O(h)$.
It turns out that this is necessarily so: Our main positive result, Theorem 3 below, states that deformations with equiintegrable bending energy density are invertible almost everywhere except for a boundary layer of width $o(h)$ located near the surface of the thin film. It also provides a uniform bound on the diameter of preimages, which again is $o(h)$. This can be interpreted as some "macroscopic invertibility". The examples mentioned earlier show that this theorem is optimal. While the almost everywhere invertibility away from the boundary follows from topological arguments and therefore requires $\det \nabla v^{(h)} > 0$, the bound on the diameter of preimages does not need this extra hypothesis. In contrast to the assumptions made e.g. in [2] and in [4], our Theorem 3 does not assume any a priori integral condition or prescribed boundary behaviour of the admissible deformations. Their invertibility properties are derived from smallness of the elastic energy alone. This is why the pathology exhibited by the second example given in Section 5 cannot be excluded.
While in this introduction we have focussed on the bending energy case $\alpha = 3$, in the main text we derive similar results for all scalings $\alpha \geq 3$, and also here positive result and counterexamples complement each other quantitatively. Moreover, all results presented here for thin films also apply to other thin elastic objects, like shells or rods.

Notation. We write $I := (-\frac{1}{2}, \frac{1}{2})$ and $I_h := (-\frac{h}{2}, \frac{h}{2})$. We also use the letter I to denote the identity matrix. We denote the unit basis vectors in $\mathbb{R}^k$ by e_i. For $z \in \mathbb{R}^k$ we define $z^i := e_i \cdot z$. We denote by $z' = (z^1, z^2)$ the in-plane components of $z = (z^1, z^2, z^3) \in \mathbb{R}^3$. The symbol ∇' denotes the in-plane gradient, and $\nabla_h y := (\partial_1 y, \partial_2 y, \frac{1}{h}\partial_3 y)$. When making statements about the measure of a set $A \subset \mathbb{R}^n$, we will always mean its n-dimensional Lebesgue measure, and we will denote it by $|A|$. We do not relabel subsequences.

2 Preliminaries

Throughout this article $S \subset \mathbb{R}^2$ denotes a bounded Lipschitz domain. Define the reference configuration of the thin film of thickness $h > 0$ as $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$. Set $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. A three dimensional deformation $v^{(h)} \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ of this film has the elastic energy

$$E^h(v^{(h)}) = \int_{\Omega_h} W(\nabla v^{(h)}(x))\, dx, \tag{1}$$

with a stored energy function W. As in [8] we assume that W is Borel measurable with values in $[0, \infty]$ and that it satisfies

$$W \in C^2 \text{ in a neighbourhood of } \mathrm{SO}(3), \tag{2}$$

$$W(F) = W(RF) \text{ for all } F \in \mathbb{R}^{3\times 3} \text{ and all } R \in \mathrm{SO}(3), \tag{3}$$

$$W(F) \geq C \operatorname{dist}^2(F, SO(3)) \text{ and } W(I) = 0. \tag{4}$$

In addition, we will sometimes assume the following:

$$W(F) \geq \frac{1}{C}|F|^p - C \text{ for some } C > 0,\ p > 3 \text{ and for all } F \in \mathbb{R}^{3\times 3}. \tag{5}$$

$$W(F) = \infty \text{ if } \det F \leq 0. \tag{6}$$

Definition 1. *Let $h_n \to 0$ and let $v_n \in W^{1,2}(\Omega_{h_n}; \mathbb{R}^3)$ be given. Then v_n is said to have α-equiintegrable energy density (with respect to h_n) if the sequence*

$$x' \mapsto \int_{-\frac{h_n}{2}}^{\frac{h_n}{2}} \frac{W\big(\nabla v_n(x', x_3)\big)}{h_n^\alpha}\, dx_3$$

is equiintegrable in $L^1(S)$.

To motivate this definition, consider e.g. the case $\alpha = 3$. It was shown in [7] that the Γ-limit of $h^{-3}E^h$ is given by Kirchhoff's functional, which we denote by I^0. The set of admissible limiting deformations, i.e. of those $v : S \to \mathbb{R}^3$ for which $I^0(v) < \infty$, agrees with

$$W^{2,2}_{\mathrm{iso}}(S; \mathbb{R}^3) := \left\{ u \in W^{2,2}(S; \mathbb{R}^3) : \nabla u \in O(2,3) \text{ a.e.} \right\},$$

where $O(2,3) = \{F \in \mathbb{R}^{3\times 2} : F^T F = I\}$. A recovery sequence for some given $v \in W^{2,2}_{\mathrm{iso}}(S; \mathbb{R}^3)$ is a sequence $v^{(h)} \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ converging to v in $W^{1,2}(S; \mathbb{R}^3)$ in the sense of vertical averages and such that, in addition, $\lim_{h\to 0} \frac{1}{h^3} E^h(v^{(h)}) = I^0(v)$. By similar arguments as in [8] Section 7.2, one can prove that recovery sequences have 3-equiintegrable energy density. For the case $\alpha > 3$, recovery sequences play the same role of providing the upper bound-part of the Γ-convergence result, i.e. they satisfy [8] Theorem 3 (ii). For such sequences it is shown in [8] Section 7.2 that they have α-equiintegrable energy densities. An important kind of recovery sequences are the α-minimizing sequences as defined in [8].

3 Local injectivity of $W^{2,2}$ isometric immersions

The proof of the following theorem relies on arguments similar to those used in the proof of Lemma 4.2.8 in [14].

Theorem 2. *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain. There exist constants $C_0, c_0 > 0$ such that for each $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ the following holds: There is a constant $\delta_0 > 0$ such that whenever $\bar{z}, \tilde{z} \in S$ satisfy $|\bar{z} - \tilde{z}| \leq \delta_0$, then we have*

$$c_0 \; |\tilde{z} - \bar{z}| \leq |u(\tilde{z}) - u(\bar{z})| \leq C_0 |\tilde{z} - \bar{z}|. \tag{7}$$

Proof. It is easy to see that there is $L > 0$ and a curve $\gamma \in W^{2,\infty}([0,L];S)$, parametrized by arc length, such that $\gamma(0) = \bar{z}$ and $\gamma(L) = \tilde{z}$, such that

$$L \leq C_0 |\bar{z} - \tilde{z}|, \tag{8}$$

where C_0 only depends on S and such that

$$|\gamma' \cdot e_1| \geq c > 0 \tag{9}$$

for some constant c depending only on S and an appropriate choice of coordinates. In fact, if the segment $\bar{z}\tilde{z}$ is contained in S then we can choose $L := |\bar{z} - \tilde{z}|$, choose coordinates such that $\bar{z} = 0$ and $\tilde{z} = (L, 0)$ and define $\gamma(t) := te_1$. If the segment $\bar{z}\tilde{z}$ is not contained in S then the existence of coordinates and of γ as described above follows from the Lipschitz property of S, provided that we choose δ_0 small enough. We leave the details to the reader.
For simplicity we invoke a result due to Kirchheim [12] (cf. also [13]) ensuring that $\nabla u \in C^0(S;\mathbb{R}^{3\times 2})$. So $\nabla u \in O(2,3)$ is defined pointwise. The upper bound in (7) follows from (8) by Jensen's inequality and the fact that $\nabla u(x)$ preserves distances for all $x \in S$. To prove the lower bound, define $\Phi(t,z) := \gamma(t) + (0,z)$. For simplicity we extend u such that $u \in W^{2,2}(\mathbb{R}^2;\mathbb{R}^3)$, see e.g. [16]. (Note, however, that u is an isometry only on S, since this property is not necessarily inherited by the extension. In fact, in general there is no extension preserving the isometry condition, see e.g. [9].) Set $M := \frac{1}{L}\int_0^L \nabla u(\gamma(t))\; dt$, define the function

$$h(t,z) := \nabla u(\Phi(t,z)) - \frac{1}{L}\int_0^L \nabla u(\Phi(s,z))\; ds$$

and set $\varepsilon := \frac{1}{L}\int_0^L |h(t,0)|^2\; dt$. By scaling invariant versions of the Trace Inequality and of Poincaré's inequality we have (notice that $\int_{[0,L]^2} h = 0$)

$$\begin{aligned}\varepsilon &\leq C\frac{1}{L^2}\int_{[0,L]^2} |h(t,z)|^2\; dt\; dz + C\int_{[0,L]^2} |\nabla h(t,z)|^2\; dt\; dz \\ &\leq C\int_{[0,L]^2} |\nabla h(t,z)|^2\; dt\; dz \leq \; C\int_{\Phi([0,L]^2)} |\nabla^2 u(x)|^2\; dx,\end{aligned}$$

since $|\det \nabla\Phi(t,z)| = |\gamma'(t)\cdot e_1|$. And by (9) this is uniformly bounded from below. Since $|\nabla^2 u|^2 \in L^1(\mathbb{R}^2)$ the value of ε can be made arbitrarily small by choosing

δ_0 small enough because then L is small by (8). Now $\mathrm{dist}^2(M, O(2,3)) \leq |M - \nabla u(\gamma(t))|^2$. Integrating over $(0, L)$ and dividing by L this gives $\mathrm{dist}^2(M, O(2,3)) \leq \varepsilon$. Hence, for all ε small enough, the singular values of M are greater than some positive constant. Therefore we can estimate

$$\begin{aligned} c|\bar{z} - \tilde{z}|^2 &\leq \Big| \int_0^L M\gamma'(t)\ dt \Big|^2 \\ &\leq C \Big| \int_0^L (\nabla u(\gamma(t)) - M)\gamma'(t)\ dt \Big|^2 + C \Big| \int_0^L \nabla u(\gamma(t))\gamma'(t)\ dt \Big|^2 \\ &\leq C\Big(L^2\varepsilon + |u(\bar{z}) - u(\tilde{z})|^2 \Big). \end{aligned}$$

By (8), for small ε we can absorb the first term on the right-hand side into the left-hand side to obtain the lower bound in (7). □

4 Invertibility in thin elastic films

The main result of this section is the following theorem.

Theorem 3. *Let $\alpha \in [3, \infty)$, let W satisfy (2, 3, 4, 5). Let $h_n \to 0$ and let $v_n \in W^{1,2}(\Omega_{h_n}; \mathbb{R}^3)$ have α-equiintegrable energy density. Then there exists $\delta > 0$ such that the following holds: If $\tilde{S} \subset S$ is a Lipschiz subdomain with $\mathrm{diam}\, \tilde{S} < \delta$ and if we set $\tilde{\Omega}_{h_n} := \tilde{S} \times I_{h_n}$ and*

$$\tilde{\Omega}^{\rho}_{h_n} := \{x \in \tilde{\Omega}_{h_n} : \mathrm{dist}(x, \partial\tilde{\Omega}_{h_n}) > \rho {h_n}^{\alpha/3}\},$$

then we have:

(i) There exists a sequence $\varepsilon_n \to 0$ such that, for all $n \in \mathbf{N}$,

$$\sup_{y \in v_n(\tilde{\Omega}_{h_n})} \mathrm{diam}(v_n|_{\tilde{\Omega}_{h_n}})^{-1}(y) \leq \varepsilon_n {h_n}^{\alpha/3} \tag{10}$$

(ii) Assume, in addition, that (6) holds. Then there exists a sequence $\varepsilon_n \to 0$ such that, for all $n \in \mathbf{N}$,

$$\#\{z \in \tilde{\Omega}_{h_n} : v_n(z) = y\} = 1 \text{ for almost every } y \in v_n(\tilde{\Omega}^{\varepsilon_n}_{h_n}). \tag{11}$$

Moreover, for all $y \in v_n(\tilde{\Omega}^{\varepsilon_n}_{h_n})$ the preimage $(v_n|_{\tilde{\Omega}_{h_n}})^{-1}(y)$ is connected and has zero volume.

Remarks.

(i) If $\alpha > 3$ then one can take $\delta = \infty$, $\tilde{S} = S$ and $\tilde{\Omega}_{h_n} = \Omega_{h_n}$. For $\alpha = 3$, easy examples show that the limiting deformations are in general not globally invertible. So the same must be true for the corresponding 3D deformations. But by Theorem 2 the limiting deformations are invertible on small subdomains. This is why the constant δ appears in the statement of Theorem 3 and why it must in general be finite if $\alpha = 3$.

(ii) The counterexamples in Section 5 show that the upper bounds on the diameter of preimages (10) and on the width of the boundary layer in part (ii) of Theorem 3 are sharp. They also show that the word "almost" cannot be omitted from (11).

(iii) As e.g. in [2], [4] the coercivity assumption (5) on W restricts admissible deformations to $W^{1,p}$, so cavitation is excluded. In [2] it was shown that orientation preserving $W^{1,p}$-deformations whose boundary values agree with those of a homeomorphism are one-to-one almost everywhere. In [4] the same conclusion was obtained replacing the hypothesis on the boundary values by an integral constraint on the Jacobian of the admissible deformations which in our notation would read $\int_{\Omega_{h_n}} \det \nabla v_n \leq |v_n(\Omega_{h_n})|$. In contrast, Theorem 3 requires no such integral constraint and makes no assumptions on the boundary values. It derives invertibility only from the hypothesis that the deformations have α-equiintegrable energy density. Notice that Theorem 3 (i) does not even require (6), i.e. the admissible deformations need not have positive Jacobian. Accordingly, the conclusion is weaker than that in [2, 4]. In their setting, a pathological behaviour as that of the second example given in Section 5 is excluded.

(iv) Our hypotheses on the energy density do not imply existence of exact minimizers with nontrivial prescribed boundary data or under nonzero loads. However, if such minimizers do exist, and if v_n is a minimizer, then it is natural to ask whether v_n enjoys better invertibility properties than those obtained in Theorem 3. Results for two dimensional deformations in [10] (see also [3]) suggest that such minimizers might indeed be one-to-one, at least if one imposes additional conditions on the deformed configuration (and on the energy density). Notice, however, that the hypotheses and the conclusions of Theorem 3 are "asymptotic" in the sense that they apply to a sequence of deformations and do not make sense for one single deformation. In contrast, the question about exact minimizers is in fact a question about deformations of a fixed (bulk) reference configuration.

The following two results are needed in the proof of Theorem 3 (i). While Lemma 4 only uses convergence of the deformations, leading to very little information about invertibility properties, Proposition 5 exploits the important fact that energy concentration is excluded by equiintegrability.

Lemma 4. *Let $\alpha \in [3, \infty)$ and let W satisfy (2, 3, 4, 5). Let $h_n \to 0$ and let $v_n \in W^{1,2}(\Omega_{h_n}; \mathbb{R}^3)$ be such that*

$$\limsup_{n\to\infty} \frac{1}{h_n^3} E^{h_n}(v_n) < \infty. \tag{12}$$

Then there is a $\delta > 0$ such that, for any sequences $\bar{z}_n$, $\tilde{z}_n \in \Omega_{h_n}$ satisfying $|\bar{z}_n' - \tilde{z}_n'| \leq \delta$ and $v_n(\bar{z}_n) = v_n(\tilde{z}_n)$, the convergence $|\bar{z}_n' - \tilde{z}_n'| \to 0$ holds. If $\lim \frac{1}{h_n^3} E^{h_n}(v_n) = 0$ then one can take $\delta = \infty$.

Proof. Define $y_n(x', x_3) := v_n(x', h_n x_3)$ and $\bar{x}_n := (\bar{z}_n^1, \bar{z}_n^2, \frac{1}{h_n}\bar{z}_n^3)$ and $\tilde{x}_n := (\tilde{z}_n^1, \tilde{z}_n^2, \frac{1}{h_n}\tilde{z}_n^3)$. By the compactness of $\bar{\Omega}$ and the compactness result [7] Theorem 4.1

there is a subsequence (not relabelled) $h_n \to 0$ and $\bar{x}_0, \tilde{x}_0 \in \bar{\Omega}$ and $\bar{y} \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ such that $\nabla y_n \to \nabla \bar{y}$ strongly in $L^2(\Omega;\mathbb{R}^3)$ and $\bar{x}_n \to \bar{x}_0$, $\tilde{x}_n \to \tilde{x}_0$. (Here we identify $\bar{y}$ with the mapping $(x', x_3) \mapsto \bar{y}(x')$.) By (5) and (12), we have $\|\nabla y_n\|_{L^p(\Omega)} \leq C$. So by interpolation and by Poincaré's inequality, after possibly adding a constant to each y_n we deduce that $y_n \to \bar{y}$ in $W^{1,q}(\Omega;\mathbb{R}^3)$ for some $q > 3$. Hence $y_n \to \bar{y}$ uniformly on $\bar{\Omega}$. This implies $0 = y_n(\bar{x}_n) - y_n(\tilde{x}_n) \to \bar{y}(\bar{x}_0') - \bar{y}(\tilde{x}_0')$. But since $|\bar{x}_0' - \tilde{x}_0'| \leq \delta$, for δ small enough Theorem 2 implies that in fact $\bar{x}_0' = \tilde{x}_0'$. Thus $|\bar{x}_n' - \tilde{x}_n'| \to 0$. Since this argument can be applied to any subsequence of the original sequence $\tilde{x}_n$, $\bar{x}_n$, it follows that the full sequence $|\bar{x}_n' - \tilde{x}_n'|$ converges to zero. If $\lim \frac{1}{h_n^3} E^{h_n}(v_n) = 0$ then this argument is true for all δ since then $\bar{y}$ is an affine isometry by Theorem 6.1 in [7]. □

Proposition 5. *Let $\alpha \in [3, \infty)$ and let W satisfy (2, 3, 4, 5). Let $h_n \to 0$ and let $v_n \in W^{1,2}(\Omega_{h_n};\mathbb{R}^3)$ have α-equiintegrable energy density. Let δ be as in the conclusion of Lemma 4. Then the following holds: For every pair of sequences $\bar{z}_n$, $\tilde{z}_n \in \Omega_{h_n}$ satisfying $|\tilde{z}_n - \bar{z}_n| \leq \delta$ and $v_n(\tilde{z}_n) = v_n(\bar{z}_n)$ for all n, we have*

$$\lim_{n\to\infty} \frac{|\bar{z}_n - \tilde{z}_n|}{h_n^{\alpha/3}} = 0. \tag{13}$$

Proposition 5 is a consequence of the following particular case:

Lemma 6. *Proposition 5 is true for $\alpha = 3$.*

Proof. We write h instead of h_n and $v^{(h)}$ instead of v_n. Choose δ such that it satisfies the conclusion of Lemma 4, so $|\tilde{z}_n - \bar{z}_n| \to 0$. If for all n large enough $\tilde{z}_n = \bar{z}_n$ then the result is trivial, so let us assume that this is not the case. Passing to subsequences (which we denote by an index j), we may assume that $|\tilde{z}_j - \bar{z}_j| > 0$ for all j and that $\tilde{z}_j'$ and $\bar{z}_j'$ converge to the same point $z' \in \bar{S}$. We choose coordinates such that z' agrees with the origin and such that

$$S \cap (-\varepsilon_1, \varepsilon_1)^2 = \{(x_1, x_2) \in (-\varepsilon_1, \varepsilon_1)^2 : x_2 < \tau(x_1)\} \tag{14}$$

for some $\varepsilon_1 > 0$ and some Lipschitz function $\tau : \mathbb{R} \to \mathbb{R}$. If $z' \in \partial S$ then such ε_1 and τ exist because S is a Lipschitz domain. If $0 = z' \in S$ then we choose $\varepsilon_1 > 0$ such that $(-\varepsilon_1, \varepsilon_1)^2 \subset S$ and set $\tau(t) := \varepsilon_1$ for all t. We may assume without loss of generality that $|\bar{z}_j'| + |\tilde{z}_j'| < \frac{\varepsilon_1}{4}$ for all j.

For any sequence of positive numbers $\delta_j \downarrow 0$ define the mappings

$$\Phi^{(j)}(x', x_3) := (\tilde{z}_j' + \delta_j x', h_j x_3) \tag{15}$$

and the rescaled deformations

$$w^{(j)}(x) = \frac{1}{\delta_j}\left(v^{(h_j)}(\Phi^{(j)}(x)) - v^{(h_j)}(\tilde{z}_j', 0)\right). \tag{16}$$

Set $\tilde{q}_j := (\Phi^{(j)})^{-1}(\tilde{z}_j)$ and $\bar{q}_j := (\Phi^{(j)})^{-1}(\bar{z}_j)$. Assume that there is $C_2 \in \mathbb{R}$ such that for all j

$$|\bar{q}_j'| \leq \frac{C_2}{2}, \text{ i.e. } |\bar{z}_j' - \tilde{z}_j'| \leq \frac{C_2 \delta_j}{2}. \tag{17}$$

Then (after passing to subsequences) we have $\tilde{q}_j \to \tilde{q}$ and $\bar{q}_j \to \bar{q}$ for some $\bar{q} \in B_{C_2}(0) \times \bar{I}$ and some $\tilde{q} \in \{0\} \times \bar{I}$. In fact, by definition of $\Phi^{(j)}$ we have $\tilde{q}_j' = \tilde{q}' = 0$ for all j. For $t \in \mathbb{R}$ define

$$\tau_j(t) := \min\left\{\frac{1}{\delta_j}\Big(\tau(\tilde{z}_j^1 + \delta_j t) - \tilde{z}_j^2\Big), 2C_2\right\}. \tag{18}$$

Clearly $\operatorname{Lip}\tau_j \le \operatorname{Lip}\tau$ for all j, so after extracting a subsequence the τ_j converge uniformly on $(-C_2, C_2)$ to a Lipschitz function τ_∞. (Notice that the τ_j are uniformly bounded from below because $\tau_j(0) > 0$ by (14).) Define

$$S_j := \{(x_1, x_2) \in (-C_2, C_2)^2 : x_2 < \tau_j(x_1)\}. \tag{19}$$

Then

$$\Phi^{(j)}(S_j \times I) \subset (S \cap (-\varepsilon_1, \varepsilon_1)^2) \times I_{h_j} \text{ for } j \text{ large enough.} \tag{20}$$

In fact, one readily checks that if $x_2 < C_2$ then we have

$$x_2 < \tau_j(x_1) \iff e_2 \cdot \Phi^{(j)}(x) < \tau\Big(e_1 \cdot \Phi^{(j)}(x)\Big). \tag{21}$$

And clearly $\Phi^{(j)}((-C_2, C_2)^2 \times I) = [\tilde{z}_j' + (-\delta_j C_2, \delta_j C_2)^2] \times I_{h_j}$ is contained in $(-\varepsilon_1, \varepsilon_1)^2 \times I_{h_j}$ for large j. So (20) follows from (14) and (21).
Set $\tilde{\tau}_k(t) := \inf_{j \ge k} \tau_j(t)$ for all $t \in \mathbb{R}$. It is easy to check that $\operatorname{Lip}\tilde{\tau}_k \le \operatorname{Lip}\tau$. Moreover, $\tilde{\tau}_k$ converges uniformly to τ_∞ on $(-C_2, C_2)$ as $k \to \infty$. Define

$$Q_k := \{(x_1, x_2) \in (-C_2, C_2)^2 : x_2 < \tilde{\tau}_k(x_1)\}. \tag{22}$$

Obviously $Q_k \subset \bigcap_{j \ge k} S_j$. Since $\bar{z}_j' \in S \cap (-\varepsilon_1, \varepsilon_1)^2$ formula (14) implies $\bar{z}_j^2 < \tau(\bar{z}_j^1)$. So $\bar{q}_j^2 < \tau_j(\bar{q}_j^1)$ by (21) and by (17). Hence $\bar{q}^2 \le \tau_\infty(\bar{q}^1)$. Setting $p_k^2 := \min\{\bar{q}^2, \tilde{\tau}_k(\bar{q}^1) - \frac{1}{k})\}$, we therefore have $p_k^2 \to \bar{q}^2$ because $\tilde{\tau}_k(\bar{q}^1) \to \tau_\infty(\bar{q}^1)$. Define $p_k := (\bar{q}^1, p_k^2, \bar{q}^3)$. Since $p_k \in (-C_2, C_2)^2 \times \bar{I}$ by (17), we conclude

$$p_k \in Q_k \times \bar{I} \text{ for large } k \text{ and } p_k \to \bar{q} \text{ as } k \to \infty. \tag{23}$$

(Notice that in general $\bar{q}' \notin Q_k$ for any k.) Now set

$$H_j := \frac{h_j}{\delta_j} \tag{24}$$

and use $\nabla_{H_j} w^{(j)}(x) = \nabla v^{(h_j)}(\Phi^{(j)}(x))$ and that $Q_k \subset S_j$ for all $j \ge k$ to find

$$\frac{1}{H_j^2}\int_{Q_k \times I} W(\nabla_{H_j} w^{(j)}(x))\, dx \le \frac{C}{h_j^3}\int_{\Phi^{(j)}(S_j \times I)} W(\nabla v^{(h_j)}(x))\, dx \tag{25}$$

for all $j \ge k$. Notice that $\Phi^{(j)}(S_j \times I) = A_j \times I_{h_j}$, where $A_j \subset \mathbb{R}^2$ has area of the order δ_j^2. Thus 3-equiintegrability implies that the right hand side of (25) converges to zero as $j \to \infty$.

Claim #1. There is a subsequence such that $\sup_{j \in \mathbf{N}} \frac{|\bar{z}_j - \tilde{z}_j|}{h_j} < \infty$.

Indeed, suppose the claim were false. Then every subsequence would satisfy $h_j^{-1}|\bar{z}_j - \tilde{z}_j| \to \infty$. Hence also

$$\frac{|\bar{z}_j' - \tilde{z}_j'|}{h_j} \to \infty. \tag{26}$$

We will show that this leads to a contradiction.
Set $C_2 := 2$ and $\delta_j := |\tilde{z}_j' - \bar{z}_j'|$. Then $\delta_j \geq h_j > 0$ for large j by (26). Making the definitions (15, 16, 18, 19, 22, 24), we have that $\bar{q}_j' \in S^1$ for all j (so (17) holds). Moreover, $H_j \to 0$ by (26) and (25) converges to zero. Combining boundedness of (25) with the compactness result of [7] Theorem 4.1, we conclude that there is some $w_k \in W^{2,2}_{\text{iso}}(Q_k; \mathbb{R}^3)$ such that (after possibly adding a constant to each $w^{(j)}$ and passing to a subsequence)

$$w^{(j)} \to w_k \text{ strongly in } W^{1,2}(Q_k \times I; \mathbb{R}^3), \tag{27}$$

where again we have identified $w_k(x') = w_k(x', x_3)$ for all $x_3 \in I$. By the lower bound provided in [7] Theorem 6.1 (i), since (25) even converges to zero we conclude that w_k is affine with gradient $L_k \in O(2,3)$. Hence we can write

$$w_k(x', x_3) = L_k x' + c_k \tag{28}$$

for some $c_k \in \mathbb{R}^3$. By (5) and boundedness of the right-hand side in (25) we have

$$\int_{S_j \times I} |\nabla w^{(j)}|^p \leq C \text{ for all } j. \tag{29}$$

Hence by interpolation (27) implies that $w^{(j)} \to w_k$ strongly in $W^{1,q}(Q_k \times I; \mathbb{R}^3)$ for some $q > 3$. Thus $w^{(j)} \to w_k$ uniformly on $Q_k \times I$.
Let p_k be as in (23). We estimate

$$\begin{aligned} |w_k(\tilde{q}) - w_k(p_k)| \leq\ & |w_k(\tilde{q}) - w^{(j)}(\tilde{q}_j)| + |w^{(j)}(\tilde{q}_j) - w^{(j)}(\bar{q}_j)| \\ & + |w^{(j)}(\bar{q}_j) - w^{(j)}(p_k)| + |w^{(j)}(p_k) - w_k(p_k)|. \end{aligned} \tag{30}$$

First observe that the second term vanishes by assumption. Since $\tilde{q}, \tilde{q}_j, p_k \in Q_k \times I$, from uniform convergence we deduce that the first and fourth terms in (30) vanish as $j \to \infty$.
Since the Lipschitz constants of ∂S_j are bounded by a constant that is independent of j (recall that $\operatorname{Lip} \tau_j \leq \operatorname{Lip} \tau$ for all j), Lemma 5.17 in [1] together with (29) implies that there exists $\lambda > 0$ (independent of j and k) such that

$$|w^{(j)}(\bar{q}_j) - w^{(j)}(p_k)| \leq \frac{1}{\lambda} |\bar{q}_j - p_k|^\lambda. \tag{31}$$

After passing to the limit $j \to \infty$ in (30) we therefore conclude that

$$|\tilde{q}' - p_k'| = |w_k(\tilde{q}) - w_k(p_k)| \leq \frac{1}{\lambda} |\bar{q} - p_k|^\lambda \tag{32}$$

for all k. The first equality holds because by (28) w_k preserves distances after projecting onto the plane. Letting $k \to \infty$ in (32) and using (23), we deduce $\tilde{q}' = \bar{q}'$. But $\tilde{q}' = (0,0)$ and $\bar{q}' \in S^1$. This contradiction proves Claim #1.

Claim #2. No subsequence satisfies $\inf_{j\in\mathbf{N}} \frac{|\bar{z}_j-\tilde{z}_j|}{h_j} > 0$.

In fact, suppose the claim were false. Then, after passing to a further subsequence, by Claim #1 there is a constant C_1 such that

$$c_1 h_j \le |\bar{z}_j - \tilde{z}_j| \le C_1 h_j. \tag{33}$$

Set $C_2 := 2C_1$ and $\delta_j := h_j$. Making the definitions (15, 16, 18, 19, 22, 24), we see that $|\bar{q}_j'| \le C_1$, so (17) is satisfied. Moreover, $H_j = 1$ and (25) converges to zero. Thus, for all k, we have that $\int_{Q_k\times I} \operatorname{dist}^2(\nabla w^{(j)}, SO(3))$ converges to zero as $j \to \infty$. Theorem 3.1 in [7] yields a sequence of constant rotations $R^{(j)} \in SO(3)$ satisfying $\int_{Q_k\times I} |\nabla w^{(j)} - R^{(j)}|^2 \to 0$ as $j \to \infty$. Since $\nabla w^{(j)} - R^{(j)}$ is uniformly bounded in $L^p(Q_k \times I)$ by (5), by interpolation we conclude that there is $q > 3$ such that $\nabla w^{(j)} - R^{(j)}$ converges to zero in $L^q(Q_k \times I)$. Thus by the continuous embedding $W^{1,q} \subset C^0$ and by the Poincaré inequality, there exist constants $c_j \in \mathbb{R}^3$ such that $w^{(j)} + c_j$ converge uniformly on $Q_k \times I$ to a linear mapping w_k with $\nabla w_k := \lim_{j\to\infty} R^{(j)} \in SO(3)$. (This limit exists after possibly passing to a subsequence. We even have $c_j \to 0$ because $w^{(j)}(0) = 0 = w_k(0)$, so in fact $w^{(j)} \to w_k$ uniformly on $Q_k \times I$.) Since $|\tilde{q} - p_k| = |w_k(\tilde{q}) - w_k(p_k)|$ we conclude

$$|\tilde{q} - p_k| \le |w_k(\tilde{q}) - w^{(j)}(\tilde{q}_j)| + |w^{(j)}(\bar{q}_j) - w^{(j)}(p_k)| + |w^{(j)}(p_k) - w_k(p_k)|.$$

The right-hand side converges to zero by uniform convergence since by (23) both p_k and $\bar{q}_j$ converge to $\bar{q}$. But the left hand side converges to $|\tilde{q} - \bar{q}|$, which is nonzero because $|\bar{q}_j - \tilde{q}_j| \ge c_1$ for all j. This contradiction proves Claim #2.

Finally consider the full sequences $\tilde{z}_n, \bar{z}_n$ and suppose they violated (13) (with $\alpha = 3$). Then there would exist subsequences and a positive constant c such that $|\tilde{z}_j - \bar{z}_j| \ge ch_j$, contradicting Claim #2. □

Proof of Proposition 5. Assume that there were a subsequence satisfying

$$h_j^{-\alpha/3}|\bar{z}_j - \tilde{z}_j| \ge c_1 > 0 \text{ for all } j \in \mathbf{N} \tag{34}$$

for some constant $c_1 > 0$. Since $\alpha > 3$ we have $h_h^{-3}E^{h_j}(v_j) \to 0$. So Lemma 6 applies with $\delta = \infty$. It implies that $h_j^{-1}|\bar{z}_j - \tilde{z}_j| \to 0$. Hence $\hat{h}_j := 2|\bar{z}_j - \tilde{z}_j|$ satisfies

$$\hat{h}_j < \frac{h_j}{10} \text{ for large } j. \tag{35}$$

Since $|\tilde{z}_j^3 - \bar{z}_j^3| \le |\tilde{z}_j - \bar{z}_j| = \frac{\hat{h}_j}{2}$ and by (35), there exist $a_j \in \mathbb{R}$ such that $\tilde{z}_j^3, \bar{z}_j^3 \in (a_j - \frac{\hat{h}_j}{2}, a_j + \frac{\hat{h}_j}{2}) \subset I_{h_j}$ for all j. Define $\hat{v}_j : \Omega_{\hat{h}_j} \to \mathbb{R}^3$ by setting

$$\hat{v}_j(x', x_3) := v_j(x', x_3 + a_j) \text{ for all } x' \in S, x_3 \in I_{\hat{h}_j}. \tag{36}$$

Set $\tilde{x}_j := (\tilde{z}_j', \tilde{z}_j^3 - a_j)$ and $\bar{x}_j := (\bar{z}_j', \bar{z}_j^3 - a_j)$. Then $\tilde{x}_j, \bar{x}_j \in \Omega_{\hat{h}_j}$ and $\hat{v}_j(\tilde{x}_j) = \hat{v}_j(\bar{x}_j)$ for all j. Moreover, the sequence $\hat{v}_j$ is 3-equiintegrable with respect to $\hat{h}_j$ because $\frac{1}{\hat{h}_j^3} \le \frac{C}{h_j^\alpha}$ by (34), so

$$\frac{1}{\hat{h}_j^3}\int_{I_{\hat{h}_j}} W(\nabla\hat{v}_j(x', x_3))\, dx_3 \le \frac{C}{h_j^\alpha}\int_{I_{h_j}} W(\nabla v_j(x', x_3))\, dx_3$$

for almost every $x' \in S$ because $a_j + I_{\hat{h}_j} \subset I_{h_j}$ by definition of the a_j. Applying Lemma 6 to $\hat{v}_j$ therefore implies that $\hat{h}_j^{-1}|\bar{z}_j - \tilde{z}_j| = \hat{h}_j^{-1}|\bar{x}_j - \tilde{x}_j| \to 0$. (Notice that now possibly $\delta < \infty$, but we already know that $|\bar{x}_j - \tilde{x}_j| < h_j < \delta$ for large j.) This contradicts the definition of $\hat{h}_j$. □

The following lemma will be used in the proof of Theorem 3 (ii).

Lemma 7. *Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $q > n$, let $K \subset U$ be compact and connected and let $F : \bar{U} \to \mathbb{R}^n$ be continuous on $\bar{U}$ and one-to-one on U. Then there exists an $\varepsilon > 0$ such that for all $w \in W^{1,q}(U;\mathbb{R}^n)$ with $\det \nabla w > 0$ almost everywhere and $\sup_{x\in U} |w(x) - F(x)| < \varepsilon$ one has*

$$\#\{x \in U : w(x) = y\} = 1 \text{ for almost every } y \in w(K).$$

Proof. By [5] Sect. 6.2 Theorem 1, a function in $W^{1,q}(U)$ is differentiable almost everywhere. Since F is a homeomorphism, it is an open mapping by the Invariance of Domain Theorem (cf. Theorem 3.30 in [6]). Thus $F(U)$ is open and $F(\partial U) = \partial F(U)$. Now let $K \subset U$ be compact and connected. Then $F(K)$ is compact and connected as well, and since $F(K) \subset F(U)$ we have $\delta := \operatorname{dist}(F(K), \partial F(U)) > 0$. Denote by G_ε the ε-neighbourhood of $F(K)$ and let w be as in the assumption. Then clearly $w(K) \subset G_\varepsilon$. Thus, for all $\varepsilon < \delta/4$ we have $\operatorname{dist}(w(K), \partial F(U)) \geq 2\varepsilon$. By [6] Theorem 2.3, together with $\|w - F\|_\infty < \varepsilon$ this implies that for all $y \in w(K)$ we have $y \notin w(\partial U)$ and since $\det \nabla w > 0$ that their Brouwer degree satisfies $d(w, U, y) = 1$. On the other hand, combining Theorem 5.27 (iii) and Theorem 5.30 in [6] and recalling the positivity of the Jacobian, for every $f \in L^\infty(\mathbb{R}^n)$ one has

$$\int_{\mathbb{R}^n} f(y)N(y)\ dy = \int_{\mathbb{R}^n} f(y)d(w, U, y)\ dy,$$

where $N(y) := \#\{x \in U : w(x) = y\}$. Inserting $f = \chi_{w(K)}$ we deduce

$$\int_{w(K)} N(y)\ dy = |w(K)|. \tag{37}$$

Since clearly $N(y) \geq 1$ for all $y \in w(K)$, equation (37) implies the claim. □

Proof of Theorem 3. We omit the index n and write h instead of h_n and $v^{(h)}$ instead of v_n. Let δ be as in the conclusion of Lemma 4. To simplify the notation we write S instead of $\tilde{S}$ and Ω_h^ρ instead of $\tilde{\Omega}_h^\rho$.
Suppose that (i) did not hold. Then there would be an $\varepsilon > 0$, a subsequence $h \to 0$ and a sequence $y_h \in v^{(h)}(\Omega_h)$ with $\operatorname{diam}(v^{(h)})^{-1}(y_h) > \varepsilon h^{\alpha/3}$ for all h. Hence for all h there would be $\bar{z}_h, \tilde{z}_h \in (v^{(h)})^{-1}(y_h)$ with $|\bar{z}_h - \tilde{z}_h| \geq \varepsilon h^{\alpha/3}$, contradicting Proposition 5. This proves (i).
Now set

$$D_h = \Big\{ y \in v^{(h)}(\Omega_h) : \#\{x \in \Omega_h : v^{(h)}(x) = y\} \geq 2 \Big\}.$$

Assume for contradiction that there were $\varepsilon > 0$ and a subsequence $h \to 0$ such that along it

$$|v^{(h)}(\Omega_h^\varepsilon) \cap D_h| > 0 \text{ for all } h. \tag{38}$$

Being Lipschitz, S satisfies the interior cone property for the cone $C = \{(x_1, x_2) : x_2 < (-\operatorname{Lip}\partial S)|x_1| \text{ and } x_2 > -1\}$. Set $E = C \times I$, let $E^\varepsilon = \{x \in E : \operatorname{dist}(x, \partial E) > \varepsilon\}$.

It is easy to see that for all $x \in \Omega_h^\varepsilon$ there exist $Q_h(x) \in SO(3)$ (with $Q_h(x)e_3 = e_3$) and $a_h(x) \in \mathbb{R}^3$ such that $x \in a_h(x) + h^{\alpha/3}Q_h(x)(E^\varepsilon)$. Since Ω_h^ε is a countable union of compact sets this implies that there exist countable sets $\Sigma_h \subset \Omega_h^\varepsilon$ such that

$$\Omega_h^\varepsilon = \bigcup_{x \in \Sigma_h} \Big(a_h(x) + h^{\alpha/3}Q_h(x)(E^\varepsilon) \Big). \tag{39}$$

Applying $v^{(h)}$ to both sides of (39) and using that Σ_h is countable, from (38) we conclude that for all h there are constants $a_h \in \mathbb{R}^3$ and $Q_h \in SO(3)$ such that, setting $\Phi^{(h)}(z) := a_h + h^{\alpha/3}Q_h z$, we have

$$|v^{(h)}\Big(\Phi^{(h)}(E^\varepsilon)\Big) \cap D_h| > 0 \text{ for all } h. \tag{40}$$

Now we claim that for all h small enough

$$\#\Big\{x \in \Phi^{(h)}(E) : v^{(h)}(x) = y\Big\} = 1 \text{ for almost every } y \in v^{(h)}\Big(\Phi^{(h)}(E^\varepsilon)\Big). \tag{41}$$

In fact, define $w^{(h)}(x) = h^{-\alpha/3}v^{(h)}(\Phi^{(h)}(x))$. Since $\nabla w^{(h)}(x) = \nabla v^{(h)}(\Phi^{(h)}(x))Q_h$, by (3) we have

$$\int_E W(\nabla w^{(h)}) = \frac{1}{h^\alpha} \int_{\Phi^{(h)}(E)} W(\nabla v^{(h)}). \tag{42}$$

By the equiintegrability hypothesis, the right-hand side converges to zero as $h \to 0$. As in the proof of Proposition 5, by Theorem 3.1 in [7], using (5), interpolating and using embedding into continuous functions, we deduce (after passing to subsequences and adding a constant to each $w^{(h)}$) that there is a rigid motion $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $w^{(h)} \to F$ uniformly on E. Since F is an affine isometry, Lemma 7 implies that, for all h small enough, $\#\{x \in E : w^{(h)}(x) = y\} = 1$ for almost every $y \in w^{(h)}(E^\varepsilon)$. Since $\Phi^{(h)}$ is bijective, the definition of $w^{(h)}$ implies (41).

Now by Theorem 3 (i), for small h and for all $y \in v^{(h)}\Big(\Phi^{(h)}(E^\varepsilon)\Big)$ we have $(v^{(h)})^{-1}(y) \subset \Phi^{(h)}(E)$. Thus by the definition of D_h we deduce that $\#\{x \in \Phi^{(h)}(E) : v^{(h)}(x) = y\} \geq 2$ for all $y \in D_h \cap v^{(h)}\Big(\Phi^{(h)}(E^\varepsilon)\Big)$. (In fact, y has at least two preimages and all preimages must lie in $\Phi^{(h)}(E)$.) Hence by (40) we obtain a contradiction to (41). Thus there is no $\varepsilon > 0$ such that (38) holds. This finishes the proof of (11).

We will now show that for all $y \in v^{(h)}(\Omega_h^\varepsilon)$, the set $(v^{(h)})^{-1}(y)$ is connected. The following argument is taken from [2]. Let $y \in v^{(h)}(\Omega_h^\varepsilon)$ and suppose that $(v^{(h)})^{-1}(y) = M_1 \cup M_2$ for two nonempty sets $M_i \subset \Omega_h$ with $\operatorname{dist}(M_1, M_2) > 0$. Then there exist two disjoint open neighbourhoods $V_i \subset \Omega_h$ of M_i, $i = 1, 2$. Since $y \notin \bigcup_{i=1,2} v^{(h)}(\partial V_i)$, since $\det \nabla v^{(h)} > 0$ almost everywhere and since $y \in v^{(h)}(\Omega_h^\varepsilon)$, the Brouwer degree satisfies $d(v^{(h)}, V_i, y) > 0$. So $d(v^{(h)}, V_i, p) > 0$ for all p in the (open) connected component U of $\mathbb{R}^3 \setminus \bigcup_{i=1,2} v^{(h)}(\partial V_i)$ that contains y. Thus $U \subset v^{(h)}(V_1) \cap v^{(h)}(V_2)$, so each point in U has at least two preimages. Hence by (11) we must have $|U \cap v^{(h)}(\Omega_h^\varepsilon)| = 0$. Since $\det v^{(h)} > 0$ almost everywhere, the preimage of a null set is a null set, cf. Theorem 5.32 in [6]. Thus $|(v^{(h)})^{-1}(U \cap v^{(h)}(\Omega_h^\varepsilon))| = 0$,

i.e. $|(v^{(h)})^{-1}(U) \cap (v^{(h)})^{-1}(v^{(h)}(\Omega_h^\varepsilon))| = 0$. However, the latter set contains the open set $(v^{(h)})^{-1}(U) \cap \Omega_h^\varepsilon$. And this set is nonempty, since $(v^{(h)})^{-1}(y) \subset (v^{(h)})^{-1}(U)$ and by the choice of y also $(v^{(h)})^{-1}(y) \cap \Omega_h^\varepsilon \neq \emptyset$. This contradiction proves connectedness of $(v^{(h)})^{-1}(y)$. Finally, $|(v^{(h)})^{-1}(y)| = 0$ again follows from Theorem 5.32 in [6]. □

5 Counterexamples to invertibility

For the following examples we assume without loss of generality that S contains the origin.

First example. This is a modification of an example given in [2]. It shows that the hypotheses of Lemma 7 do not imply pointwise invertibility. Application to thin films will then show that (11) need not hold for every $y \in v^{(h)}(\Omega_h^\varepsilon)$.
Let $n \geq 2$, write $x' = (x_1, ..., x_{n-1})$ and define the mapping $u : \mathbb{R}^n \to \mathbb{R}^n$ by setting, for all x with $|x'| \leq 1$,

$$u_i(x) = x_i \text{ for } i = 1, 2, ..., n-1$$
$$u_n(x) = \begin{cases} |x'|x_n & \text{if } |x_n| \leq 1 \\ x_n + (|x'| - 1)\operatorname{sgn} x_n & \text{if } |x_n| \geq 1 \end{cases}$$

and by setting $u(x) = x$ for all x with $|x'| \geq 1$. The mapping u is Lipschitz and satisfies $\det \nabla u > 0$ almost everwhere. Now set $u^{(\varepsilon)}(x) = \varepsilon u(\frac{x}{\varepsilon})$. Then we have, for $|x'| \leq \varepsilon$,

$$|\nabla u^{(\varepsilon)}(x) - I|^2 = \begin{cases} x_n^2 + (1 - |x'|)^2 & \text{if } |x_n| \leq \varepsilon \\ 1 & \text{if } |x_n| \geq \varepsilon, \end{cases} \tag{43}$$

and for $\varepsilon \leq 1$ this is uniformly bounded by two. Since for $|x'| \geq \varepsilon$ the left-hand side is zero, setting $Z = \{x \in \mathbb{R}^n : |x'| < 1, |x_n| < 1\}$ we conclude that $\int_Z |\nabla u^{(\varepsilon)}(x) - I|^q \, dx \leq C_q \varepsilon^{n-1}$ for all $q \in [1, \infty)$. Thus $\|u^{(\varepsilon)} - Id\|_{W^{1,q}(Z)} \leq C_q \varepsilon^{\frac{n-1}{q}}$. Hence the hypotheses of Lemma 7 are satisfied, yet $(u^{(\varepsilon)}|_Z)^{-1}(0) = \{x \in Z : x' = 0, |x_n| \leq \varepsilon\}$.
Now let $\alpha \geq 3$, $n = 3$, $h > 0$ and $\Omega_h = S \times I_h$. Choose a sequence ε_h of positive numbers with $\lim_{h \downarrow 0} h^{\frac{1-\alpha}{2}} \varepsilon_h = 0$ and define the thin-film deformations $v^{(h)} := u^{(\varepsilon_h)}|_{\Omega_h}$. Then we have

$$\int_{\Omega_h} |\nabla v^{(h)}(x) - I|^2 \, dx = \int_{B_{\varepsilon_h}(0)} dx' \int_{-\frac{h}{2}}^{\frac{h}{2}} |\nabla v^{(h)} - I|^2 dx_3 \leq Ch\varepsilon_h^2.$$

Thus $v^{(h)}$ is an α-recovery sequence for the identity, each $v^{(h)}$ is Lipschitz with $\det \nabla v^{(h)} > 0$ almost everywhere, but $(v^{(h)}|_{\Omega_h})^{-1}(0) = \{x \in \mathbb{R}^3 : x' = 0, |x_n| \leq \varepsilon_h\}$.

Second example. This example shows that in general one cannot exclude that y has more than one preimage in Ω_h for all y in a subset of $v^{(h)}(\Omega_h)$ with positive volume. By Theorem 3 (ii) this subset must be located near the boundary of Ω_h. More precisely, we will see that, given any $\alpha \geq 3$ and any sequence $\varepsilon_h \to 0$, there exists an α-recovery sequence $v^{(h)}$ for the identity such that $v^{(h)} \in W^{1,\infty}(\Omega_h; \mathbb{R}^3)$,

$\det \nabla v^{(h)} > 0$ almost everywhere and $v^{(h)}$ agrees with the identity outside a half-ball of radius $\varepsilon_h h^{\alpha/3}$. Moreover there is a simply connected set $D_h \subset \Omega_h$ (a half ball of radius $\frac{1}{2}\varepsilon_h h^{\alpha/3}$, in fact) such that $|D_h| \sim \varepsilon_h^3 h^\alpha$ and $\#(v^{(h)})^{-1}(y) = 2$ for all $y \in v^{(h)}(D_h)$. In addition, $\operatorname{diam}(v^{(h)})^{-1}(y) \geq \frac{\varepsilon_h}{4} h^{\alpha/3}$ for all y in a subset of $v^{(h)}(D_h)$ with volume of the order $\varepsilon_h^3 h^\alpha$. Thus the scalings given in Theorem 3 (i) and (ii) are sharp.
To construct such $v^{(h)}$, let $g : \mathbb{R} \to \mathbb{R}$ denote the hat function,

$$g(t) = \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and define $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi(x) = (x_1, x_2, x_3 + g(|x'|))$. Define the mapping $f(x_1, x_2)$ in polar coordinates $r \geq 0$, $\varphi \in [0, 2\pi)$, by

$$f(r\cos\varphi, r\sin\varphi) = \begin{cases} (r\cos 3\varphi, r\sin 3\varphi) & \text{if } \varphi \in [0, \pi] \\ (r\cos\varphi, r\sin\varphi) & \text{if } \varphi \in (\pi, 2\pi). \end{cases}$$

We define $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\tilde{f}(x) = (x_1, f(x_2, x_3))$, which is Lipschitz because so is f. Thus the mapping $w = \Phi^{-1} \circ \tilde{f} \circ \Phi$ from the lower half-space $\{x \in \mathbb{R}^3 : x^3 < 0\}$ into $\mathbb{R}^3$ is Lipschitz as well. Let $\varepsilon_h \to 0$ as $h \to 0$. Then $\varepsilon_h h \leq \frac{h}{2}$ for small h, so the deformations

$$v^{(h)}(z) := \varepsilon_h h^{\alpha/3} w\left(\frac{z_1}{\varepsilon_h h^{\alpha/3}}, \frac{z_2}{\varepsilon_h h^{\alpha/3}}, \frac{z_3 - \frac{h}{2}}{\varepsilon_h h^{\alpha/3}}\right),$$

are well defined on Ω_h, they are Lipschitz, their Lipschitz constant is independent of h and $\det \nabla v^{(h)} > 0$ almost everywhere. It is easy to see that they have the properties mentioned above. Moreover, they agree with the identity outside the set on which they are two-to-one.
Finally let us show that the $v^{(h)}$ satisfy $\lim_{h\to 0} \frac{1}{h^\alpha} E^h(v^{(h)}) = 0$ for any stored energy function W satisfying the conditions (2) through (4). In fact, due to the behaviour of W near $SO(3)$, to every bounded set in $\mathcal{M} \subset \mathbb{R}^{3\times 3}$ there corresponds a constant C such that $W(F) \leq C \operatorname{dist}^2(F, SO(3))$ for all $F \in \mathcal{M}$. Since the sequence $\|\nabla v^{(h)}\|_{L^\infty(\Omega_h)}$ is uniformly bounded, we can use this to estimate

$$\frac{1}{h^\alpha} \int_{\Omega_h} W(\nabla v^{(h)})\, dx \leq C \frac{1}{h^\alpha} \int_{\Omega_h} |\nabla v^{(h)} - I|^2\, dx \leq C\varepsilon_h^3. \tag{44}$$

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