

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

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(revised version: February 2007)

by

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Preprint no.: 68

2006





# A $\Gamma$ -convergence result for thin martensitic films in linearized elasticity

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February 20, 2007

## Abstract

The elastic energy of a thin film  $\Omega_h$  of thickness  $h$  with displacement  $u$  is given by the functional  $E^h(u) = \int_{\Omega_h} W(\nabla u) dx$ . We consider materials whose energy density  $W$  is linearly frame indifferent and vanishes on two linearized wells which are compatible in the plane but incompatible in the thickness direction. We prove compactness of displacement sequences  $u^{(h)}$  satisfying  $E^h(u^{(h)}) \leq Ch^2$  and show that the limiting two-dimensional displacements can only have two different interface directions. Our main result is the derivation of the  $\Gamma$ -limit of the functionals  $\frac{1}{h^2}E^h$  as  $h \rightarrow 0$ . It is given by a weighted sum over the lengths of the interfaces.

## 1 Introduction

The study of solid-solid phase transitions in thin nonlinearly elastic films leads to functionals of the form

$$E^h = \int_{\Omega_h} W(\nabla v(x)) dx, \quad (1)$$

where  $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$  is a cylindrical domain of thickness  $h$ ,  $S \subset \mathbb{R}^2$  is a Lipschitz domain,  $v : \Omega_h \rightarrow \mathbb{R}^3$  is the elastic deformation, and  $W$  is a frame-indifferent free energy density with  $n$  energy minima  $F_i$ , i.e.  $W(F_i) = 0$  for  $i = 1, \dots, n$  and  $W(RF) = W(F)$  for all  $R \in SO(3)$  and all  $F \in \mathbb{R}^{3 \times 3}$ . In [8] Bhattacharya and James observed that for many materials which undergo austenite-martensite phase transitions, the low-energy states of very thin samples of material display a much richer variety of structures than bulk samples made of the same material. The reason is that three dimensional compatibility requires a plane on which two juxtaposed affine deformations coincide, i.e. that their gradients be rank-one connected. In contrast, two dimensional compatibility is already satisfied if there exists one in-plane vector on which the two deformations agree, so a rank-two connection between the gradients suffices. Roughly speaking, this weakened two-dimensional compatibility requirement is inherited by thin films with finite but small thickness  $h > 0$ . This fact leads to the existence of many nontrivial low-energy states, including laminates, tunnels and tents; see [8] for a detailed analysis.

Recently, Chaudhuri and Müller [11] showed that nontrivial (in the sense that both phases are present) thin-film deformations arise as limits of three dimensional thin film deformations whose energy (1) scales like  $h^2$ . Their main result is that in the case of strongly incompatible wells, there exists a positive constant  $c$  such that any sequence  $v^{(h)}$  of thin-film deformations converging to  $y$  in a suitable sense satisfies

$$\liminf_{h \rightarrow 0} \frac{1}{h^2} E^h(v^{(h)}) \geq c \inf\{\text{Per}_S U : U \subset U_1, S \setminus U \subset U_2\},$$

where the sets  $U_1, U_2$  denote the different phases. These two sets might not be disjoint because they are related to the convex hulls of the corresponding energy wells. The scaling  $h^2$  is interesting since it lies just between the membrane scaling and the plate scaling. The derivation of the  $\Gamma$ -limit for this scaling remains open.

In this article we complete the picture for the analogous problem within the framework of linearized elasticity, i.e. when the zero set of the energy density  $W$  consists of two linearized wells. (Two-well materials in the linearly elastic setting have been studied e.g. in [20, 36], and the  $\Gamma$ -limit of linearly elastic thin films and rods for single-well materials was derived in [3] for the membrane and the plate scaling.) We obtain the same scaling law  $E^h \sim h^2$  as in the nonlinear case, and moreover we prove compactness for low energy sequences and derive the full  $\Gamma$ -limit of the functionals  $\frac{1}{h^2} E^h$  as the film thickness  $h$  converges to zero. Our compactness result ensures that the structure of possible limiting displacements becomes rather restricted. Exploiting this fact, we will show that the functionals

$$I^h(u; S) = \begin{cases} \frac{1}{h^2} \int_{S \times (-\frac{h}{2}, \frac{h}{2})} W(\nabla u(x)) \, dx & \text{if } u \in W^{1,2}(S \times (-\frac{h}{2}, \frac{h}{2}); \mathbb{R}^3) \\ +\infty & \text{otherwise} \end{cases}$$

$\Gamma$ -converge to

$$I^0(w; S) = \begin{cases} \int_J k(\nu(x)) d\mathcal{H}^1(x) & \text{if } w \in \mathcal{A}(S), \\ +\infty & \text{otherwise,} \end{cases}$$

where the class  $\mathcal{A}(S)$  of admissible limiting displacements is given in (24) below,  $J$  denotes the jump set of  $\text{sym } \nabla' w$  and  $\nu$  denotes the normal to it, which by the compactness result can only assume two values  $\nu_1$  and  $\nu_2$ . The function  $k$  is a ‘‘surface tension’’ which depends on the normal and which we define in (23) below. To state the precise result, let us write  $v'$  to denote the first two entries of  $v \in \mathbb{R}^3$  and let us call a domain  $S \subset \mathbb{R}^2$  strictly star-shaped if there is  $z \in S$  such that for all  $z' \in \bar{S}$  the open segment  $(z, z')$  is contained in  $S$ .

**Theorem 1.** *Let  $A, B \in \mathbb{R}^{3 \times 3}$  satisfy (i) through (iv) from Section 2, let  $W$  satisfy the conditions (4) through (6) below and let  $S \subset \mathbb{R}^2$  be a bounded strictly star-shaped Lipschitz domain. Then a  $\Gamma$ -type convergence*

$$I^h(\cdot; S) \xrightarrow{\Gamma} I^0(\cdot; S)$$

holds in the following sense:

- (i) Ansatz-free lower bound: Let  $w \in L^2(S; \mathbb{R}^2)$ ,  $h_n \rightarrow 0$  and let  $v_n \in W^{1,2}(S \times I_{h_n}; \mathbb{R}^3)$  be such that

$$\int_{(-\frac{h_n}{2}, \frac{h_n}{2})} v'_n \, dx_3 \rightarrow w \text{ in } L^2(S; \mathbb{R}^2).$$

Then

$$\liminf_{n \rightarrow \infty} I^{h_n}(v_n; S) \geq I^0(w; S).$$

(ii) Existence of recovery sequences: Let  $w \in L^2(S; \mathbb{R}^2)$  and  $h_n \rightarrow 0$ . Then there is a sequence  $v_n \in W^{1,2}(S \times (-\frac{h_n}{2}, \frac{h_n}{2}); \mathbb{R}^3)$  such that

$$\int_{(-\frac{h_n}{2}, \frac{h_n}{2})} v'_n dx_3 \rightarrow w \text{ strongly in } W^{1,2}(S; \mathbb{R}^2)$$

and

$$\lim_{n \rightarrow \infty} I^{h_n}(v_n; S) = I^0(w; S).$$

This theorem is complemented by the compactness result for low energy sequences stated in Theorem 6 below.

The  $\Gamma$ -limit obtained in Theorem 1 has the same structure as that derived in [15]. The reason for this is that the functionals  $I^h$  turn out to be related to singularly perturbed functionals of the form

$$J^{(h)}(u; S) = \int_S \bar{W}(\nabla u(x')) + h^2 |\nabla^2 u(x')|^2 dx'. \quad (2)$$

Starting with the classical work [32], the asymptotic behaviour of singularly perturbed functionals has been extensively studied in the literature, but an important feature of (2) is that the relevant quantity is the gradient of some function. A simplified case has been addressed in [13], where frame indifference of  $\bar{W}$  is dropped altogether. Recently, Conti and Schweizer were able to derive the  $\Gamma$ -limit of the functionals (2) both under the assumption of linearized frame indifference [15] and under nonlinear frame indifference [16] in two dimensions. One major task addressed below will be to make the relationship between the original functionals (1) and the singularly perturbed functionals (2) precise enough to apply some of their results.

It would be interesting to derive a similar  $\Gamma$ -convergence result as the one presented here within the framework of fully nonlinear elasticity. This could lead to a  $\Gamma$ -limit of the form  $\int_{\partial^* E} k(\nu(x)) d\mathcal{H}^1$  or some similar expression, where  $\nu$  denotes the normal to the interface and  $k : S^1 \rightarrow \mathbb{R}$  is some ‘‘surface tension’’. While the analogous problem for rods, studied in [33] can be handled by standard arguments, for nonlinearly elastic thin films the situation seems to be much more delicate.

This paper is organized as follows. In Section 2 we introduce the precise definitions and reduce the problem to a canonical form. Then we prove a two-well analogue of Korn’s inequality, Theorem 3, which applies to incompatible linear wells. Then we apply this result to deduce the main compactness result Theorem 6. In Section 3 we obtain the lower bound, Theorem 12, with the use of the compactness result and abstract scaling arguments. Finally, in Section 4 we derive the upper bound by constructing appropriate recovery sequences. This step relies on several interpolation arguments and on a rigidity result for compatible wells in two dimensions provided by Conti and Schweizer in [15]. The proof of Theorem 1 closes this section.

**Notation.** We use the letter  $C$  to denote constants depending only on the domain and on  $W$ . Within an expression the explicit value of  $C$  may change from line to line. We use

the notation  $h \rightarrow 0$  to denote a sequence  $(h_n) \subset \mathbb{R}^+$  which converges to zero as  $n \rightarrow \infty$ . A bar above a given  $3 \times 3$ -matrix denotes its upper left  $2 \times 2$  submatrix, and in general we use barred letters to denote  $2 \times 2$  matrices. Primes on 3-vectors will denote the 2-vector consisting of the first two entries, so in particular  $x = (x', x_3)$ . For a matrix  $A$  we write  $\text{sym } A = \frac{1}{2}(A + A^T)$ , skew  $A = \frac{1}{2}(A - A^T)$  and  $|A| = |\text{Tr}(A^T A)|$ , where  $\text{Tr}$  denotes the trace.

By a subscript  $\cdot_i$  we will denote the partial derivative with respect to the  $x_i$ -variable. By  $\nabla'$  we denote the in-plane gradient, that is  $\nabla' w = (w_{,1}|w_{,2})$ . For  $h > 0$  we set  $I_h = (-\frac{h}{2}, \frac{h}{2})$ . We use a dashed integral sign  $\int$  to denote the average. Often we will simply write  $\{f = a\}$  instead of  $\{x \in S : f(x) = a\}$ , and for  $U, V$  open and bounded, the notation  $U \subset\subset V$  means that the closure  $\bar{U}$  of  $U$  is contained in  $V$ . For  $a \in \mathbb{R}$  the notation  $[a]$  denotes the largest integer which is not greater than  $a$ .

## 2 Preliminaries and compactness

We consider the functional, defined for any Lipschitz domain  $U \subset \mathbb{R}^2$ ,

$$I^h(u; U) = \begin{cases} \frac{1}{h^2} \int_{U \times I_h} W(\nabla u) & \text{if } u \in W^{1,2}(U \times I_h; \mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases} \quad (3)$$

Here  $W$  denotes the geometrically linear energy density which is assumed to satisfy the following conditions:

$$W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R} \text{ is continuous.} \quad (4)$$

$$\text{Linearized frame indifference: } W(F) = W(\text{sym } F) \text{ for all } F \in \mathbb{R}^{3 \times 3}. \quad (5)$$

$$\text{Quadratic growth and coercivity: } c_0 W_0(F) \leq W(F) \leq C_0 W_0(F). \quad (6)$$

where  $c_0, C_0$  are positive constants. Here we have introduced the standard energy density

$$W_0(F) = \text{dist}^2(\text{sym } F, \{A, B\}),$$

where  $A$  and  $B$  are symmetric  $3 \times 3$ -matrices to be specified below. We define the reduced functional

$$I_{2D}^h(w; U) = \begin{cases} \int_U \frac{1}{h} W_{2D}(\nabla' w) + h \left| \nabla'^2 w \right|^2 & \text{if } w \in W^{2,2}(U; \mathbb{R}^2) \\ +\infty & \text{otherwise} \end{cases},$$

where  $W_{2D}(\bar{F}) = \text{dist}^2(\text{sym } \bar{F}, \{\bar{A}, \bar{B}\})$ . We make the following assumptions on the wells  $A$  and  $B$ :

- (i)  $A$  and  $B$  are symmetric  $3 \times 3$ -matrices.
- (ii) *Incompatibility in bulk.*  $\text{rank}(A - B + T) \geq 2$  for all skew symmetric  $3 \times 3$  matrices  $T$ .
- (iii) *Compatibility in the plane.* There exists a skew symmetric  $2 \times 2$  matrix  $\bar{T}$  such that  $\text{rank}(\bar{A} - \bar{B} + \bar{T}) \leq 1$ .

(iv) *Nondegeneracy.*  $\det(\bar{A} - \bar{B}) \neq 0$ .

Item (iii) is satisfied if and only if there exists  $t \in \mathbb{R}$  such that

$$0 = \det\left(\bar{A} - \bar{B} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}\right) = \det(\bar{A} - \bar{B}) + t^2,$$

whence (iii) is equivalent to  $\det(\bar{A} - \bar{B}) \leq 0$  with equality if and only if  $\bar{A}$  and  $\bar{B}$  are rank-one connected. Thus (iii) and (iv) together are equivalent to  $\det(\bar{A} - \bar{B}) < 0$ . Table 11.1 in [6] shows that conditions (i) through (iv) are generically satisfied by real materials (in a linearized framework).

Let us now reduce the set of all matrices satisfying (i)-(iv) to a canonical form. Let  $\tilde{A}, \tilde{B}$  satisfy conditions (i) through (iv) but be arbitrary otherwise. Then there is an orthogonal matrix  $R \in O(3)$  with  $Re_3 = e_3$  such that

$$R^T(\tilde{B} - \tilde{A})R = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \sum_{i=1}^3 \tilde{\mu}_i \frac{e_i \otimes e_3 + e_3 \otimes e_i}{2},$$

where  $\lambda_i$  are the eigenvalues of the matrix  $\bar{B} - \bar{A}$  and  $\tilde{\mu}_i$  are some real numbers. By possibly choosing  $R$  differently (by interchanging the first two columns), we may assume that  $\lambda_1 \geq \lambda_2$ , so since  $\det(\bar{A} - \bar{B}) < 0$ , we must in fact have  $\lambda_1 > 0 > \lambda_2$ . Let  $Q = \text{diag}(|\lambda_1|^{-\frac{1}{2}}, |\lambda_2|^{-\frac{1}{2}}, 1)$  and set  $\hat{B} = QR^T(\tilde{B} - \tilde{A})RQ$ . This gives  $\hat{B} = e_1 \otimes e_1 - e_2 \otimes e_2 + \sum_{i=1}^3 \hat{\mu}_i \frac{e_i \otimes e_3 + e_3 \otimes e_i}{2}$ , where  $\hat{\mu}_i$  are related to  $\tilde{\mu}_i$  and  $\lambda_i$ . Now we can find a rotation  $\hat{Q} \in SO(3)$  with eigenvector  $e_3$  such that

$$B = \hat{Q}^T \hat{B} \hat{Q} = e_1 \otimes e_2 + e_2 \otimes e_1 + \sum_{i=1}^3 \mu_i \frac{e_i \otimes e_3 + e_3 \otimes e_i}{2} \quad (7)$$

for some  $\mu_i \in \mathbb{R}$ . Since the structural assumptions on the energy density  $W$  and on the shape of the domain (i.e. strict star-shapedness with respect to the origin and a cylindrical form  $S \times I_h$ ) are invariant under the transformations introduced above, we obtain

**Lemma 2.** *If Theorem 1 is shown for the special pairs  $A, B$  given by  $A = 0$  and  $B$  as in (7), then it holds for all possible choices of  $A$  and  $B$  which satisfy conditions (i)-(iv).*

## 2.1 Korn's Inequality for two incompatible strains

In [24] Friesecke, James and Müller derived a nonlinear version of Korn's inequality, which was generalized to a two-well setting by Chaudhuri and Müller in [10]. A simpler proof was later obtained in [19], where it is shown that in the case of two compact incompatible wells the two-well estimate can be reduced to the corresponding one-well estimate. A generalization of Korn's inequality to the case of two incompatible linearized wells (which are not compact) is provided by the following theorem. A related but non-quantitative version of this result can be found in [20], compare also [36]. A related rigidity result for compatible energy wells is proven in [15]. In that work the authors provide an example which shows that no Korn-type rigidity like the one derived here can be expected in the case of two compatible wells.

**Theorem 3.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded connected Lipschitz domain,  $n \geq 2$  and  $K = (A + \text{Skew}) \cup (B + \text{Skew})$ , where  $A$  and  $B$  are incompatible strains, i.e.  $(B - A) + \text{Skew}$  does not contain rank-one matrices. Then there exists a positive constant  $C(\Omega, A, B)$  with the following property: For every  $u \in W^{1,2}(\Omega; \mathbb{R}^n)$ , there exists an associated  $R \in K$  such that*

$$\|\nabla u - R\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq C(\Omega, A, B) \|\text{dist}(\nabla u, K)\|_{L^2(\Omega; \mathbb{R}^{n \times n})}.$$

This theorem will follow from the interior estimate provided by the following lemma.

**Lemma 4.** *With assumptions as in Theorem 3 and  $U \subset\subset \Omega$  Lipschitz and connected, there is a constant  $C(U, \Omega, A, B)$  such that the following holds: For every  $u \in W^{1,2}(\Omega; \mathbb{R}^n)$ , there exists an associated  $R \in K$  such that*

$$\|\nabla u - R\|_{L^2(U; \mathbb{R}^{n \times n})} \leq C(U, \Omega, A, B) \|\text{dist}(\nabla u, K)\|_{L^2(\Omega; \mathbb{R}^{n \times n})}.$$

**Proof.** By setting  $\tilde{B} = B - A$  and applying the lemma to  $\tilde{u}(x) = u(x) - Ax$  we may assume without loss of generality that  $A = 0$ . Define  $d(F) = \text{dist}(F, \{0, B\})$  and set  $\varepsilon^2 = \int_{\Omega} \text{dist}^2(\nabla u, K)$ . Notice that  $\text{dist}^2(F, K) = d^2(\text{sym } F)$  for all  $F \in \mathbb{R}^{n \times n}$ . Denote by  $P : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  the orthogonal projection onto the orthogonal complement of the subspace  $(\text{span}\{B\}) \oplus \text{Skew}$ . By the incompatibility of the matrix  $B$  with the zero matrix, the linear space  $(\text{span}\{B\}) \oplus \text{Skew}$  intersects the cone of all rank-one matrices only at zero. Thus we have  $|P(a \otimes b)|^2 > 0$  for all  $a, b \neq 0$ . Hence, by continuity and by compactness of the sphere,  $P$  satisfies the Legendre-Hadamard ellipticity condition  $\Lambda|a|^2|b|^2 \geq |P(a \otimes b)|^2 \geq \lambda|a|^2|b|^2$  for some  $\lambda, \Lambda > 0$ . Now let  $w \in W^{1,2}(\Omega; \mathbb{R}^n)$  be a weak solution of the linear elliptic system with constant coefficients

$$\begin{aligned} \text{div } P(\nabla w) &= 0 \text{ in } \Omega \\ w &= u \text{ on } \partial\Omega. \end{aligned} \tag{8}$$

Set  $z = u - w$ . Then  $z \in W_0^{1,2}(\Omega; \mathbb{R}^n)$  is a weak solution of  $\text{div } P(\nabla z) = \text{div } P(\nabla u)$ . Testing with  $z$  itself gives

$$\int_{\Omega} P(\nabla z) : \nabla z = \int_{\Omega} P(\nabla u) : \nabla z \leq \left( \int_{\Omega} |P(\nabla u)|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla z|^2 \right)^{\frac{1}{2}}.$$

Since by ellipticity the left-hand side of the above inequality is greater than  $\int_{\Omega} \lambda |\nabla z|^2$  we conclude

$$\int_{\Omega} |\nabla z|^2 \leq C \int_{\Omega} |P(\nabla u)|^2 = C \int_{\Omega} \text{dist}^2(\text{sym } \nabla u, \text{span}\{B\}) \leq C\varepsilon^2.$$

Thus it remains to prove that there exists  $R \in K$  such that  $\int_{\Omega} |\nabla w(x) - R|^2 dx \leq C\varepsilon^2$ , where  $C$  is independent of  $w$ .

Denote by  $e_w = \frac{1}{2}(\nabla w + (\nabla w)^T)$  the linear strain of  $w$  and let  $y \in \Omega$  be such that  $B(y, 2r) \subset \Omega$ . By Korn's inequality there is a  $C = C(n)$  (which by scaling invariance is independent of  $r$ ) and a skew symmetric matrix  $W$  such that

$$\int_{B(y, 2r)} |\nabla w - W|^2 \leq C \int_{B(y, 2r)} |e_w|^2. \tag{9}$$



Since by  $P(M) = P(\text{sym } M)$  we have  $P(W) = 0$ , the mapping  $v(x) = w(x) - Wx$  is a weak solution of

$$\begin{aligned} \operatorname{div} P(\nabla v) &= 0 \text{ in } \Omega \\ v &= u - Wx \text{ on } \partial\Omega. \end{aligned} \quad (10)$$

By standard elliptic regularity for linear systems with constant coefficients (see e.g. [26]), we obtain the inequality

$$\int_{B(y,r)} |\nabla^2 v|^2 \leq \frac{C}{r^2} \int_{B(y,2r)} |\nabla v|^2 = \frac{C}{r^2} \int_{B(y,2r)} |\nabla w - W|^2. \quad (11)$$

We have  $|\nabla e_w|^2 = \frac{1}{4} \sum_{i,j,k} (w_{i,jk} + w_{j,ik})^2 \leq |\nabla^2 w|^2$ . Hence by the choice of the matrix  $W$  and since  $|\nabla^2 w|^2 = |\nabla^2 v|^2$  on  $B(y, 2r)$ , we conclude from (9, 11) that

$$\int_{B(y,r)} |\nabla e_w|^2 \leq \frac{C}{r^2} \int_{B(y,2r)} |e_w|^2. \quad (12)$$

This inequality holds for all  $y \in \Omega$  with  $B(y, 2r) \subset \Omega$ .

Set  $r_0 = \frac{\operatorname{dist}(U, \partial\Omega)}{4}$  and assume for the moment that  $\varepsilon < 1$ . Covering  $\bar{U}$  with finitely many balls of radius  $\frac{1}{3} \operatorname{dist}(U, \partial\Omega)$  and applying (12) shows that  $\int_U |\nabla e_w|^2$  is bounded by a constant independent of  $u$ . Hence by Lemma 5 below with  $K_1 = \{0\}$ ,  $K_2 = \{B\}$  applied to the strain  $e_w$  we obtain

$$\begin{aligned} \min \left\{ \int_U |e_w - B|^2, \int_U |e_w|^2 \right\} &\leq C \left( \int_U d^2(e_w) \int_U |\nabla e_w|^2 \right)^{\frac{n}{2(n-1)}} + \int_U d^2(e_w) \\ &\leq C(\varepsilon^2 + \varepsilon^{\frac{n}{n-1}}). \end{aligned} \quad (13)$$

Now let us assume (the other case can be treated analogously) that  $B$  is the minority phase, i.e. the set  $E = \{x \in U : |e_w(x) - B|^2 \leq \rho^2\}$ , where  $\rho = \frac{|B|}{2}$ , satisfies  $|E| \leq |\{x \in U : |e_w(x)|^2 \leq \rho^2\}|$ . In particular, this implies  $|E| \leq |E \setminus U|$  by the choice of  $\rho$ . We have  $\rho^2 |E| \leq \int_U |e_w|^2$  by the definition of  $\rho$  and also  $\rho^2 |E| \leq \rho^2 |U \setminus E| \leq \int_U |e_w - B|^2$ . Thus by (13), for all  $\varepsilon < 1$  we have

$$|E| \leq C_1 \left( \varepsilon^{\frac{n}{n-1}} + \varepsilon^2 \right) \quad (14)$$

for some constant  $C_1$  independent of  $u$ . Now we fix  $\varepsilon_0 \in (0, 1)$  such that  $C_1 \left( \varepsilon_0^{\frac{n}{n-1}} + \varepsilon_0^2 \right) < \frac{|B_{r_0}|}{2}$  and assume from now on that  $\varepsilon \leq \varepsilon_0$ ; the other case is treated at the end of this proof. From (14) we deduce that  $|E| < \frac{|B_{r_0}|}{2}$ . Our aim is to show that in fact

$$|E| \leq C\varepsilon^2 \quad (15)$$

for a constant  $C$  independent of  $u$ . Using the scaling invariant Lemma 5 together with (12) one can prove that for all  $x \in \Omega$  and for all  $r > 0$  such that  $B_{2r}(x) \subset \Omega$

$$\begin{aligned} \min \left\{ \int_{B_r(x)} |e_w|^2, \int_{B_r(x)} |e_w - B|^2 \right\} \\ \leq C \left[ \left( \mathcal{M}(|e_w|^2)(x) \int_{B_r(x)} d^2(e_w) \right)^{\frac{n}{2(n-1)}} + \int_{B_r(x)} d^2(e_w) \right], \end{aligned} \quad (16)$$

where  $\mathcal{M}$  denotes the Hardy-Littlewood maximal function,  $\mathcal{M}(f)(x) = \sup_{r>0} \int_{B_r(x)} |f|$ . Above and in the sequel we extend  $e_w$  by zero outside  $\Omega$ .

**Claim #1.** For  $D \geq 8|B|$ , the set  $A_\infty = \{x \in \Omega : \mathcal{M}(|e_w|^2)(x) \geq D^2\}$  satisfies  $|A_\infty| \leq C\varepsilon^2$ .

In fact,  $\mathcal{M}(|e_w|^2) \leq \mathcal{M}(|e_w|^2 - \frac{D^2}{2})_+ + \frac{D^2}{2}$ , whence  $x \in A_\infty$  implies  $\mathcal{M}(|e_w|^2 - \frac{D^2}{2})_+(x) \geq \frac{D^2}{2}$ . For  $D$  as assumed one can show that  $d^2(e_w) \geq \frac{1}{4}(|e_w|^2 - \frac{D^2}{2})_+$ , whence  $A_\infty \subset \{\mathcal{M}(d^2(e_w)) \geq \frac{D^2}{8}\}$ . Thus by the Hardy-Littlewood maximal theorem ([27] Chapter 4),  $|A_\infty| \leq |\{\mathcal{M}(d^2(e_w)) \geq \frac{D^2}{8}\}| \leq C \int_\Omega d^2(e_w)$ , which proves Claim #1.

For almost every  $x \in E \setminus A_\infty$  there is an  $r_x \leq r_0$  such that

$$\frac{|E \cap B_{r_x}(x)|}{|B_{r_x}(x)|} = \frac{1}{2}. \quad (17)$$

This follows by continuity from the fact that the left-hand side converges to one as  $r \rightarrow 0$  and that it is not bigger than  $\frac{|E|}{|B_r|}$ , which is less than  $1/2$  for  $r > r_0$  by the choice of  $\varepsilon_0$ . In particular,  $B_{2r_x}(x) \subset \Omega$  for every  $x$  as above, by the definition of  $r_0$ .

By Vitali's covering theorem we can choose countably many such  $x_i \in E \setminus A_\infty$  such that

$$|E \setminus A_\infty| \leq C \sum |B_{r_{x_i}}(x_i)| \quad (18)$$

with pairwise disjoint balls on the right-hand side. By (17), for every  $i$  we have

$$\begin{aligned} \frac{\rho^2}{2} &\leq \frac{1}{|B_{r_{x_i}}|} \min \left\{ \int_{B_{r_{x_i}}(x_i) \cap E} |e_w|^2, \int_{B_{r_{x_i}}(x_i) \cap U \setminus E} |e_w - B|^2 \right\} \\ &\leq C \left[ \left( \int_{B_{r_{x_i}}(x_i)} d^2(e_w) \right)^{\frac{n}{2n-2}} + \int_{B_{r_{x_i}}(x_i)} d^2(e_w) \right], \end{aligned}$$

where we have used (16) and the definition of  $A_\infty$  in the second inequality. A simple calculation shows that this implies  $|B_{r_{x_i}}(x_i)| \leq C \int_{B_{r_{x_i}}(x_i)} d^2(e_w)$ . Summing over  $i$  and using disjointness of  $B_{r_{x_i}}(x_i)$  and (18) we conclude that  $|E \setminus A_\infty| \leq C\varepsilon^2$ , which by Claim #1 implies (15) in the case  $\varepsilon \leq \varepsilon_0$ . But the case  $\varepsilon > \varepsilon_0$  is already covered by (14) by enlarging the constant  $C_1$ , which then depends on  $\varepsilon_0$ . Thus (15) also holds in this case. Using (15) we can finally estimate

$$\int_U |e_w|^2 = \int_{U \setminus E} |e_w|^2 + \int_E |e_w|^2 \leq C \left[ \int_{U \setminus E} d^2(e_w) + |E| + \int_E d^2(e_w) \right] \leq C\varepsilon^2.$$

The desired estimate now follows from Korn's inequality.  $\square$

The proof of Theorem 3 is completed using a cube decomposition of  $\Omega$  and applying a weighted Poincaré inequality exactly as in the proof of Theorem 2 in [10]. We have used the following

**Lemma 5.** *Let  $n \geq 2$ ,  $\Omega \subset \mathbb{R}^n$  be a bounded Lipschitz domain and let  $K_1, K_2$  be compact disjoint subsets of  $\mathbb{R}^{n \times n}$ ,  $K = K_1 \cup K_2$ . Then there is a constant  $C = C(K, \Omega)$ , such that*

for any  $F \in W^{1,2}(\Omega; \mathbb{R}^{n \times n})$

$$\min_{i=1,2} \int_{\Omega} \text{dist}^2(F, K_i) \leq C(K, \Omega) \left( \int_{\Omega} \text{dist}^2(F, K) \int_{\Omega} |\nabla F|^2 \right)^{\frac{n}{2(n-1)}} \quad (19)$$

$$+ C(K, \Omega) \int_{\Omega} \text{dist}^2(F, K). \quad (20)$$

The proof is the same as that of Lemma 2.4 in [10], where one can replace  $\nabla w$  throughout by an arbitrary matrix-valued  $W^{1,2}$ -function  $F$ .

## 2.2 Compactness

The following theorem provides the compactness result which complements the  $\Gamma$ -convergence result of Theorem 1. Its proof uses ideas from [11] and is rather different from the usual Young-measure arguments applied, for example, in the compactness results of [22], [13], [15] or [16].

**Theorem 6.** *Let  $S, A, B$  and  $K$  be as in Theorem 3, let  $h \rightarrow 0$  and suppose a sequence  $u^{(h)} \in W^{1,2}(\Omega_h; \mathbb{R}^3)$  satisfies*

$$\limsup_{h \rightarrow 0} \frac{1}{h^2} \int_{\Omega_h} \text{dist}^2(\nabla u^{(h)}, K) < \infty.$$

Set  $w^{(h)}(x') = \int_{I_h} (u^{(h)}(x', x_3))' dx_3$ . Then there exist a subsequence (not relabelled) and affine mappings  $f^{(h)} : S \rightarrow \mathbb{R}^2$  with skew symmetric gradient such that  $w^{(h)} + f^{(h)}$  converges strongly in  $W^{1,2}(S; \mathbb{R}^2)$ . Moreover, the limit function  $w_0$  satisfies  $\text{sym } \nabla' w_0 \in BV(S; \{\bar{A}, \bar{B}\})$ .

**Proof.** We introduce a piecewise constant approximation of the displacement  $\nabla u^{(h)}$  using Theorem 3. Consider a lattice of squares  $S_{a,h} = a + (-\frac{h}{2}, \frac{h}{2})^2$ ,  $a \in h\mathbb{Z}^2$ , and let  $S'_h = \bigcup_{S_{a,h} \subset S} S_{a,h}$ . Now apply Theorem 3 to  $u^{(h)}$  restricted to each cube  $a + (-\frac{h}{2}, \frac{h}{2})^3$ . This yields a piecewise constant map  $R^{(h)} : S'_h \rightarrow K$  such that

$$\int_{S_{a,h} \times I_h} |\nabla u^{(h)} - R^{(h)}|^2 \leq C \int_{S_{a,h} \times I_h} \text{dist}^2(\nabla u^{(h)}, K). \quad (21)$$

Define the piecewise constant map  $G^{(h)} : S'_h \rightarrow \{A, B\}$  by setting  $G^{(h)}(x) = \text{sym } R^{(h)}(x)$ . Let  $\varepsilon > 0$  be sufficiently small (to be fixed below). We divide the family of squares  $S_{a,h}$  into three different groups:

$$a \in \mathcal{A}_0 \text{ if and only if } \int_{S_{a,h} \times I_h} \text{dist}^2(\nabla u^{(h)}, K) \geq \varepsilon h^3.$$

If  $a \notin \mathcal{A}_0$ , then the matrix  $G^{(h)}(a) \in \{A, B\}$  is such that  $\frac{1}{h^3} \int_{S_{a,h} \times I_h} \left| \text{sym } \nabla u^{(h)} - G^{(h)}(a) \right|^2 \leq C\varepsilon$ . This follows from (21) by the definition of  $G^{(h)}(a)$ . Now define

$$\begin{aligned} a \in \mathcal{A}_1 & \text{ if and only if } a \notin \mathcal{A}_0 \text{ and } G^{(h)}(a) = A \\ a \in \mathcal{A}_2 & \text{ if and only if } a \notin \mathcal{A}_0 \text{ and } G^{(h)}(a) = B. \end{aligned}$$

For  $\varepsilon$  small enough, each square  $S_{a,h}$  belongs to exactly one of these three groups. Thus the sets  $\Omega_i^h = \text{int} \left( \bigcup_{a \in \mathcal{A}_i} \bar{S}_{a,h} \right)$ ,  $i = 0, 1, 2$  are disjoint and cover  $S'_h$  up to a Lebesgue null set.

As in [11] one can prove that, for  $\varepsilon$  small enough, the following implication holds:

$$a \in \mathcal{A}_i, a' \in \mathcal{A}_j \text{ and } S_{a',h} \text{ is a neighbour of } S_{a,h} \implies j \in \{0, i\}, \quad (22)$$

that is, a square of type  $\mathcal{A}_1$  can only have neighbouring squares of type  $\mathcal{A}_1$  or  $\mathcal{A}_0$ , but never of type  $\mathcal{A}_2$  and the analogous statement holds with  $\mathcal{A}_1$  and  $\mathcal{A}_2$  swapped. But from the definition of  $\mathcal{A}_0$  and the scaling of the energy, the cardinality of  $\mathcal{A}_0$  is of the order  $\frac{1}{h}$ . Since the side-length of each square  $S_{a,h}$  is just  $h$ , this leads to the estimate (see [11])  $\mathcal{H}^1(\partial\Omega_1^h \setminus \partial S) \leq C$ . This implies that the characteristic functions  $\chi_{\Omega_1^h}$  are bounded in  $BV(S)$ , whence they have a subsequence converging strongly in  $L^1(S)$ , hence (by interpolation) in all  $L^p(S)$  with  $p < \infty$ . Since the cardinality of  $\mathcal{A}_0$  is of the order  $\frac{1}{h}$ , the area of  $\Omega_0^h$  is of the order  $h$ , whence  $\chi_{\Omega_0^h} \rightarrow 0$  in  $L^1(\mathbb{R}^2)$ . Hence we also have strong convergence of  $\chi_{\Omega_2^h}$ . Note that the respective limit functions  $\chi_{\Omega_1}$  and  $\chi_{\Omega_2}$  both belong to  $BV(S)$ .

On the other hand  $G^{(h)} = A\chi_{\Omega_1^h} + B\chi_{\Omega_2^h} + G^{(h)}\chi_{\Omega_0^h}$ . Let us extend  $G^{(h)}$  by zero to all of  $S$ . By the convergence  $\chi_{S'_h} \rightarrow 1$  in  $L^1(S)$  we obtain that  $G^{(h)} \rightarrow G$  strongly in  $L^2(S; \mathbb{R}^{2 \times 2})$ , where  $G = \chi_{\Omega_1}A + \chi_{\Omega_2}B \in BV(S; \{A, B\})$ . By (21), the hypothesis on  $u^{(h)}$  and Jensen's inequality we have  $\int_{S'_h} |\text{sym } \nabla' w^{(h)} - \bar{G}^{(h)}|^2 \leq Ch$ . Using  $G^{(h)} = 0$  on  $S \setminus S'_h$  and applying Jensen's inequality again, we find  $\int_{S \setminus S'_h} |\text{sym } \nabla' w^{(h)} - \bar{G}^{(h)}|^2 \leq C|S \setminus S'_h| + Ch$ .

We conclude that  $\text{sym } \nabla' w^{(h)} \rightarrow \bar{G}$  strongly in  $L^2(S; \mathbb{R}^{2 \times 2})$ . Since the subspace of symmetrized gradients is strongly closed in  $L^2(S; \mathbb{R}^{2 \times 2})$ , there is a  $w_0 \in W^{1,2}(S; \mathbb{R}^2)$  such that  $\text{sym } \nabla' w_0 = \bar{G} \in BV(S; \{\bar{A}, \bar{B}\})$ . An application of Korn's and of Poincaré's inequalities on  $S$  yield the claim.  $\square$

**Remark.** Let  $y^{(h)}(x', x_3) = u^{(h)}(x', hx_3)$  be the rescaled displacement defined on  $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$  and define the rescaled gradient by  $\nabla_h y = (\nabla' y|_{\frac{1}{h}y, 3})$ . Then the previous proof in fact shows that  $\text{sym } \nabla_h y^{(h)} \rightarrow G$  strongly in  $L^2(\Omega; \mathbb{R}^{3 \times 3})$ .

In [15], Proposition 2.2, the following characterization is provided for functions whose symmetrized gradient has bounded variation and is supported on two incompatible matrices  $\bar{A}, \bar{B}$ .

**Proposition 7.** Let  $S \subset \mathbb{R}^2$  be a bounded Lipschitz domain. Let  $\bar{A}, \bar{B}$  satisfy (iii), (iv) from the beginning of Section 2, let  $\nu_1, \nu_2$  be linearly independent solutions to  $\bar{A} - \bar{B} + t(e_1 \otimes e_2 - e_2 \otimes e_1) = a \otimes \nu_i$ , where  $a \in \mathbb{R}^3$  and  $t \in \mathbb{R}$ , and let  $w \in W^{1,2}(S; \mathbb{R}^2)$  satisfy  $\text{sym } \nabla' w \in BV(S; \{\bar{A}, \bar{B}\})$ . Then the jump set  $J$  of  $\text{sym } \nabla' w$  consists of countably many disjoint segments whose endpoints belong to  $\partial S$  and which have normal directions  $\nu_1$  or  $\nu_2$ . In addition,  $\nabla' w$  is constant on each connected component of  $S \setminus J$ .

Together with this proposition, Theorem 6 gives a good description of the admissible limit functions.

### 3 Lower bound

In this section we prove that the limiting energy of any sequence  $v^{(h)}$  is bounded from below by the functional  $I^0$ . From now on we assume that the matrices  $A$  and  $B$  satisfy (i) through (iv) in Section 2. In view of Lemma 2 we will restrict the following analysis to the canonical form of the matrices  $A$  and  $B$  given in Section 2, i.e.  $A = 0$  and  $B$  as in (7). This choice allows exactly two different normal directions orthogonal to each other: Setting  $T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  we have  $\bar{B} + T_1 = 2e_2 \otimes e_1$ , giving the normal  $\nu_1 = e_1$ , and  $\bar{B} + T_2 = 2e_1 \otimes e_2$ , giving the normal  $\nu_2 = e_2$ . Clearly  $T_1$  and  $T_2$  are the only skew matrices  $T$  such that  $\bar{B} + T$  is singular. For the rest of this article we set  $K = \text{Skew} \cup (B + \text{Skew})$ . Define the matrix-valued piecewise constant function

$$F_i^\pm(x') = \begin{cases} 0 & \text{for } \pm x' \cdot \nu_i < 0 \\ \bar{B} + T_i & \text{otherwise} \end{cases}$$

and set  $w_i^\pm(x') = F_i^\pm(x')x'$ . Note that  $w_i^\pm \in W^{1,\infty}(S; \mathbb{R}^2)$ . For open intervals  $J \subset \mathbb{R}$  and  $\varepsilon > 0$  we define the quantities

$$\begin{aligned} \mathcal{F}_1^\pm(J; \varepsilon) &= (\Gamma - \liminf_{h \rightarrow 0} I^h)(w_1^\pm; J \times (-\varepsilon, \varepsilon)) \\ &= \inf \left\{ \liminf_{n \rightarrow \infty} I^{h_n}(u_n; J \times (-\varepsilon, \varepsilon)) : (h_n, u_n) \text{ is admissible for } \mathcal{F}_1^\pm \right\}. \end{aligned}$$

The quantities  $\mathcal{F}_2^\pm$  are defined analogously, with  $(-\varepsilon, \varepsilon) \times J$  replacing  $J \times (-\varepsilon, \varepsilon)$ . We have used the following

**Definition 8.** *Let  $J$  be an open interval and  $\varepsilon > 0$ . A pair of sequences  $(h_n, u_n)$  is admissible for  $\mathcal{F}_1^\pm(J; \varepsilon)$  if  $h_n \in (0, 1)$ ,  $h_n \rightarrow 0$ ,*

$$u_n \in W^{1,2}(J \times (-\varepsilon, \varepsilon) \times I_{h_n}; \mathbb{R}^3)$$

and

$$\int_{I_{h_n}} u'_n dx_3 \rightarrow w_1^\pm \text{ in } L^2(J \times (-\varepsilon, \varepsilon); \mathbb{R}^2).$$

*Admissibility for  $\mathcal{F}_2^\pm$  is defined analogously with  $w_2^\pm$  replacing  $w_1^\pm$  and with  $(-\varepsilon, \varepsilon) \times J$  replacing  $J \times (-\varepsilon, \varepsilon)$  throughout.*

We define

$$k(\nu_i) = \mathcal{F}_i^+ \left( \left( -\frac{1}{2}, \frac{1}{2} \right); \frac{1}{2} \right). \quad (23)$$

**Lemma 9.** *Let  $J \subset \mathbb{R}$  be an interval, let  $i \in \{1, 2\}$  and let  $\varepsilon > 0$ . Then*

$$\mathcal{F}_i^+(J; \varepsilon) = \mathcal{F}_i^-(J; \varepsilon) = k(\nu_i) |J|.$$

**Proof.** As in [13] Lemma 4.3. (and later in [15] Lemma 3.2) one proves the lemma by showing the following facts:

- (i)  $\mathcal{F}_i^+(J; \varepsilon) = \mathcal{F}_i^-(J; \varepsilon) =: \mathcal{F}(J; \varepsilon)$ .

- (ii) Translation invariance:  $\mathcal{F}_i(x' + J; \varepsilon) = \mathcal{F}_i(J; \varepsilon)$ .
- (iii) Monotonicity: If  $J_1 \subset J_2$  then  $\mathcal{F}_i(J_1; \varepsilon) \leq \mathcal{F}_i(J_2; \varepsilon)$ , and if  $\delta > \varepsilon > 0$ , then  $\mathcal{F}_i(J; \delta) \geq \mathcal{F}_i(J; \varepsilon)$ .
- (iv) Homogeneity: If  $\alpha > 0$  then  $\mathcal{F}_i(\alpha J; \alpha \varepsilon) = \alpha \mathcal{F}_i(J; \varepsilon)$ .
- (v) Concentration:  $\mathcal{F}_i(J; \varepsilon)$  does not depend on  $\varepsilon > 0$ .

□

**Definition 10.** An admissible sequence  $(u_n, h_n)$  for  $\mathcal{F}_i^\pm((-\frac{1}{2}, \frac{1}{2}); \frac{1}{2})$  is called a recovery sequence for  $w_i^\pm$  if

$$\lim_{n \rightarrow \infty} I^{h_n} \left( u_n; \left(-\frac{1}{2}, \frac{1}{2}\right)^2 \right) = k(\nu_i).$$

For given  $(h_n)$  we will also call  $u_n$  a recovery sequence for  $w_i^\pm$  if the condition of Definition 10 is satisfied. In Lemma 13 below we will show that there exist recovery sequences for any given sequence  $h_n \rightarrow 0$ . As an immediate result of Lemma 9 (v) we obtain

**Corollary 11.** Let  $\sigma \in \{+, -\}$ ,  $i \in \{1, 2\}$ , let  $(u_n, h_n)$  be a recovery sequence for  $w_i^\sigma$  on a rectangle  $S = J_1 \times J_2 \subset \mathbb{R}^2$ . Let  $Q \subset S$  be a rectangle covering the interface of  $w_i^\sigma$  with two sides parallel to it. Then the energy concentrates at the interface:  $\lim_{n \rightarrow \infty} I^{h_n}(u_n; S \setminus Q) = 0$ .

Now we define the set of admissible limiting functions as

$$\mathcal{A}(S) = \left\{ w \in W^{1,2}(S; \mathbb{R}^2) : \text{sym } \nabla' w \in BV(S; \{0, \bar{B}\}) \right\} \quad (24)$$

and the limiting functional

$$I^0(w; S) = \begin{cases} \int_J k(\nu(x)) d\mathcal{H}^1(x) & \text{if } w \in \mathcal{A}(S), \\ \infty & \text{otherwise.} \end{cases} \quad (25)$$

Here  $J$  denotes the jump set of  $\text{sym } \nabla' w$ , also called the phase interface, and  $\nu$  denotes the normal (the sign does not matter), which up to a sign can only assume the values  $\nu_1 = e_1$  and  $\nu_2 = e_2$ .

**Theorem 12.** (Lower bound.) Let  $S \subset \mathbb{R}^2$  be a bounded Lipschitz domain and  $w \in L^2(S; \mathbb{R}^2)$ . Then, for all  $h_n \rightarrow 0$  and all  $u_n \in L^2(S \times I_{h_n}; \mathbb{R}^3)$  satisfying  $\int_{I_{h_n}} u'_n dx_3 \rightarrow w$  in  $L^2(S; \mathbb{R}^2)$  one has the lower bound

$$\liminf_{n \rightarrow \infty} I^{h_n}(u_n; S) \geq I^0(w; S). \quad (26)$$

**Proof.** If the limes inferior on the left-hand side of (26) is infinite, then there is nothing to prove. Otherwise, by passing to a subsequence (not relabelled) we may assume that the sequence  $I^{h_n}(u_n; S)$  converges, so in particular also  $\limsup_{n \rightarrow \infty} I^{h_n}(u_n; S) < \infty$ . After passing to a further subsequence, the compactness result of Theorem 6 implies that there is a sequence of skew-affine functions  $f_n$  such that  $w_n + f_n \rightarrow w_0$  in  $W^{1,2}(S; \mathbb{R}^2)$  for some

$w_0 \in \mathcal{A}(S)$ , where we have set  $w_n = \int_{I_{h_n}} u'_n dx_3$ . Since  $w_n \rightarrow w$  in  $L^2(S; \mathbb{R}^2)$ , we deduce that  $f_n$  converges in  $L^2(S; \mathbb{R}^2)$ , whence there is a skew matrix  $\bar{T}$  and a vector  $c \in \mathbb{R}^2$  such that  $f_n(x') \rightarrow c + \bar{T}x'$  pointwise. Hence  $w = w_0 - \bar{T}x' - c$  and in particular we have  $w \in \mathcal{A}(S)$ . By the strong  $W^{1,2}$ -convergence of both  $w_n + f_n$  and  $f_n$  we also deduce  $w_n \rightarrow w$  in  $W^{1,2}(S; \mathbb{R}^2)$ .

The jump set of  $\text{sym } \nabla' w$  consists of a countable union of disjoint segments  $J_k$  with normal  $\nu_1$  or  $\nu_2$ . The rest of the proof is similar to that of Proposition 3.1 in [15]: One covers each  $J_k$  with a box, applies Lemma 9 to each box separately and uses the minimality of  $\mathcal{F}_i^\pm$ .  $\square$

## 4 Upper bound

In this section we will show that for any admissible limit function  $w$  and for any given sequence  $h_n \rightarrow 0$  one can find a recovery sequence  $v^{(h_n)}$  whose vertical average converges to  $w$  and whose thin-film energy converges to the limiting energy  $I^0(w)$ . As in [15] we will first show that given *any* sequence  $h_n \rightarrow 0$  one can find a sequence of test functions  $v_{i,n}^\pm$  such that  $(h_n, v_{i,n}^\pm)$  is a recovery sequence for  $w_i^\pm$  in the sense of Definition 10. The rough idea in the following proof is similar to that of Proposition 5.5 in [15]: One restricts the recovery sequence furnished by Definition 10 to a cuboid with the right aspect ratio on which both phases are used and on which the limiting energy is the same (up to scaling) as in that of the original sequence and then one rescales it. Our argument is simpler than that given in [15] and, moreover, it does not require the modification steps of Lemmas 15 and 16 below.

**Lemma 13.** *Let  $S = (-\frac{1}{2}, \frac{1}{2})^2$  be the unit square,  $\sigma \in \{+, -\}$ ,  $i \in \{1, 2\}$  and let  $H_n \rightarrow 0$ . Then we have*

$$k(\nu_i) = \inf \left\{ \liminf_{n \rightarrow \infty} I^{H_n}(u_n; S) : u_n \in W^{1,2}(S \times I_{H_n}; \mathbb{R}^3) \right. \\ \left. \int_{I_{H_n}} u'_n dx_3 \rightarrow w_i^\sigma \text{ in } L^2(S; \mathbb{R}^2) \right\}.$$

**Proof.** Clearly we must only prove the “ $\geq$ ”-inequality. Further, let us restrict to the case  $\sigma = +$  and  $i = 1$ , so the interface normal is  $\nu_1 = e_1$  and the phase “0” is used on the left,  $\{\nabla' w_1^+ = 0\} = S \cap \{x_1 < 0\}$  and  $\{\nabla' w_1^+ = \bar{B} + T_1\} = S \cap \{x_1 > 0\}$ ; the other cases are similar.

First note that the infimum in the definition of  $k(\nu_i)$  is attained, i.e. there is a sequence  $h_n \rightarrow 0$  and  $v_n \in W^{1,2}(S \times I_{h_n}; \mathbb{R}^3)$  such that  $\int_{I_{h_n}} v'_n dx_3 \rightarrow w_1^+$  in  $L^2(S; \mathbb{R}^2)$  and  $\lim_{n \rightarrow \infty} I^{h_n}(v_n; S) = k(\nu_1)$ . Since after passing to subsequences this equality remains valid, we may assume without loss of generality that  $h_n \ll H_n$ , so  $\alpha_n = \frac{H_n}{h_n} \rightarrow \infty$ . In the sequel we will simply write  $h, H$  omitting their subindex  $n$ .

Set  $y_1^{(n)} = \frac{1}{2\alpha_n} - \frac{1}{2}$  and  $y_{m+1}^{(n)} = y_m^{(n)} + \frac{1}{\alpha_n}$ ,  $m = 1, \dots, [\alpha_n] - 1$ , and let  $S_m^{(n)} = (-\frac{1}{2}, \frac{1}{2}) \times (y_m^{(n)} - \frac{1}{2\alpha_n}, y_m^{(n)} + \frac{1}{2\alpha_n})$ . We will use notation and results from the proof of Theorem 6. Define  $S_{m,1}^{(n)} = S_m^{(n)} \cap \Omega_1^h \cap \{\nabla' w_1^+ = 0\}$  and  $S_{m,2}^{(n)} = S_m^{(n)} \cap \Omega_2^h \cap \{\nabla' w_1^+ = \bar{B} + T_1\}$ . It follows from the proof of Theorem 6 that  $\Omega_1^h \rightarrow \{\nabla' w_1^+ = 0\}$  and  $\Omega_2^h \rightarrow \{\nabla' w_1^+ = \bar{B} + T_1\}$

in the sense that the corresponding characteristic functions converge in  $L^1$ . Now denote by

$$G_n = \left\{ m = 1, \dots, [\alpha_n] : |S_{m,1}^{(n)}| > \frac{1}{4\alpha_n} \text{ and } |S_{m,2}^{(n)}| > \frac{1}{4\alpha_n} \right\}$$

the index set of “good” stripes. We claim that the cardinality of  $G_n$  satisfies  $\#G_n \rightarrow \infty$ . To prove this, first note that by the definitions

$$\left( \Omega_1^h \cap \{\nabla' w_1^+ = 0\} \right) \cup \left( \Omega_2^h \cap \{\nabla' w_1^+ = \bar{B} + T_1\} \right) \subset \left( S \setminus \bigcup_{m=1}^{[\alpha_n]} S_m^{(n)} \right) \cup \bigcup_{m=1}^{[\alpha_n]} (S_{m,1}^{(n)} \cup S_{m,2}^{(n)}).$$

Taking measures on both sides we obtain

$$\begin{aligned} \left| \Omega_1^h \cap \{\nabla' w_1^+ = 0\} \right| + \left| \Omega_2^h \cap \{\nabla' w_1^+ = \bar{B} + T_1\} \right| &\leq \left| S \setminus \bigcup_{m=1}^{[\alpha_n]} S_m^{(n)} \right| + \left| \bigcup_{m=1}^{[\alpha_n]} (S_{m,1}^{(n)} \cup S_{m,2}^{(n)}) \right| \\ &\leq \frac{\alpha_n - [\alpha_n]}{\alpha_n} + \sum_{m=1}^{[\alpha_n]} (|S_{m,1}^{(n)}| + |S_{m,2}^{(n)}|). \end{aligned}$$

Thus

$$\begin{aligned} \left| \Omega_1^h \cap \{\nabla' w_1^+ = 0\} \right| + \left| \Omega_2^h \cap \{\nabla' w_1^+ = \bar{B} + T_1\} \right| &\leq \frac{\alpha_n - [\alpha_n]}{\alpha_n} + \sum_{m \notin G_n} \frac{3}{4\alpha_n} + \sum_{m \in G_n} \frac{1}{\alpha_n} \\ &\leq \frac{\alpha_n - [\alpha_n]}{\alpha_n} + [\alpha_n] \frac{3}{4\alpha_n} + \frac{\#G_n}{\alpha_n}. \end{aligned}$$

As  $n \rightarrow \infty$  the left-hand side converges to one, the first term on the right-hand side to zero and the second term on the right-hand side to  $\frac{3}{4}$ , whence  $\frac{\#G_n}{\alpha_n} \rightarrow \frac{1}{4}$ . In particular  $\#G_n \rightarrow \infty$ .

From this one can now deduce by a simple argument by contradiction that we can always find one among these “good” stripes which, in addition, has low energy. Precisely, we have

$$\limsup_{n \rightarrow \infty} \left( \min_{m \in G_n} I^h(v_n; S_m^{(n)}) - \frac{1}{\alpha_n} I^h(v_n; S) \right) \leq 0. \quad (27)$$

Now choose  $m_n \in \operatorname{argmin}_{m \in G_n} I^h(v; S_m^{(n)})$  and set  $\hat{y}_n = y_{m_n}$ . Let  $\sigma_n = \left(-\frac{l}{2\alpha_n}, \frac{l}{2\alpha_n}\right) \times \left(\hat{y}_n - \frac{1}{2\alpha_n}, \hat{y}_n + \frac{1}{2\alpha_n}\right)$  with  $l > 1 + \frac{k(\nu_1)}{k(\nu_2)}$  fixed. Consider the mapping  $g_n : x_1 \mapsto \int_{(x_1, 0) + \sigma} \chi_{\Omega_1^h}$ . Since  $m_n \in G_n$ , there are  $x_1 < 0$  (resp.  $x_1 > 0$ ) such that  $g_n(x_1) > \frac{|\sigma_n|}{2}$  (resp.  $g_n(x_1) < \frac{|\sigma_n|}{2}$ ). Since  $g_n$  is continuous, we conclude that there is some  $\hat{x}_n$  with  $g_n(\hat{x}_n) = \frac{|\sigma_n|}{2}$ .

Thus the rectangle  $\hat{S}_n = \left(\hat{x}_n - \frac{l}{2\alpha_n}, \hat{x}_n + \frac{l}{2\alpha_n}\right) \times \left(\hat{y}_n - \frac{1}{2\alpha_n}, \hat{y}_n + \frac{1}{2\alpha_n}\right)$  has low energy, and it contains fifty percent of phase “0”. By the definition of  $\mathcal{A}_0$  in the proof of Theorem 6, the amount of high-energy-phase  $\Omega_0^h \cap \hat{S}_n$  is controlled by the energy, whence its area is of order  $o\left(\frac{h}{\alpha_n}\right)$ . This is negligible with respect to the area of  $\hat{S}_n$ , which is of order  $\alpha_n^{-2} \gg \frac{h}{\alpha_n}$ . Hence in the limit there are equal amounts of phases “0” and “ $\bar{B} + T_1$ ” present in  $\hat{S}_n$ , that is

$$\frac{|\Omega_i^h \cap \hat{S}_n|}{|\hat{S}_n|} \rightarrow \frac{1}{2} \text{ for } i = 1, 2. \quad (28)$$



Now set  $V_n(x) = \alpha_n v_n(\frac{x}{\alpha_n} + (\hat{x}_n, \hat{y}_n))$  and  $S' = (-l/2, l/2) \times (-\frac{1}{2}, \frac{1}{2})$ . Then  $V_n \in W^{1,2}(S' \times (-\frac{H_n}{2}, \frac{H_n}{2}); \mathbb{R}^3)$  and  $\limsup_{n \rightarrow \infty} I^{H_n}(V_n; S') \leq k(\nu_1)$ . By Theorem 6, there is an affine mapping  $F_n$  with skew symmetric gradient such that  $W_n + F_n \rightarrow W_0$  strongly in  $W^{1,2}(S'; \mathbb{R}^2)$ , where  $W_n = \int_{I_{H_n}} V_n' dx_3$  and  $W_0$  in  $\mathcal{A}(S')$ . Application of Theorem 12 together with the upper bound on the energy implies that  $I^0(W_0; S') \leq k(\nu_1)$ . By the geometry of  $S'$  and by Proposition 7,  $W_0$  has either only interfaces with normal  $\nu_1 = e_1$  or only interfaces with normal  $\nu_2 = e_2$ . If it had an interface of the latter type, however, we would obtain the contradiction  $I^0(W_0; S') \geq lk(\nu_2) > k(\nu_1)$ , by the choice of  $l$ . Thus  $W_0$  has only interfaces with normal  $\nu_1$ , and by the bound on the limiting energy there can be at most one such interface. On the other hand, by construction, there are two phases present in the limit. We conclude that  $W_0$  has exactly one interface with normal  $\nu_1$ . By possibly reflecting the function  $V_n$  we may also assume that it has the right orientation, i.e. phase “0” on the left. Further, by possibly adding the same affine function with skew symmetric matrix to each  $F_n$ , we may assume that  $W_0 = w_1^+$ . Indeed, notice that by (28) the interface of  $W_0$  must be centered. Finally define  $\tilde{V}_n(x) = V_n(x) + \begin{pmatrix} F_n(x') \\ 0 \end{pmatrix}$ . This sequence satisfies  $\int_{I_{H_n}} \tilde{V}_n' dx_3 \rightarrow w_1^+$  in  $L^2(S; \mathbb{R}^2)$  and recovers the optimal limiting energy  $k(\nu_1)$ .  $\square$

**Lemma 14.** *Let  $l, d > 0$  and  $S = (-l, l) \times (-d, d)$ ,  $i \in \{1, 2\}$ ,  $\sigma \in \{+, -\}$ ,  $h \rightarrow 0$  and let  $v^{(h)} \in W^{1,2}(S \times I_h; \mathbb{R}^3)$  be a recovery sequence for  $w_i^\sigma$ . Subdivide  $S$  into  $N \in \mathbb{N}$  stripes normal to the interface. Then for any stripe  $S_j$  we have*

$$\lim_{h \rightarrow 0} I^h(v^{(h)}; S_j) = \frac{1}{N} \lim_{h \rightarrow 0} I^h(v^{(h)}; S) = \frac{1}{N} I^0(w_i^\sigma; S)$$

**Proof.** The claim follows by noting that otherwise there would be one stripe  $S_{j_0}$  with lower limiting energy, which would contradict the lower bound in Theorem 12, since  $\int_{I_h} (v^{(h)})' dx_3 \rightarrow w_i^\sigma$  in  $W^{1,2}(S_{j_0}; \mathbb{R}^2)$ .  $\square$

In a first modification step we will change the recovery sequence furnished by Definition 10 and Lemma 13 in such a way that its vertical averages become smooth away from the interface.

**Lemma 15.** *Let  $S = (-\frac{1}{2}, \frac{1}{2})^2$ ,  $\sigma \in \{+, -\}$  and  $i \in \{1, 2\}$ , and let  $Q \subset S$  be a rectangle covering the interface of  $w_i^\sigma$  with two sides parallel to this interface. Then there exists a recovery sequence  $u^{(h)} \in W^{1,2}(S \times I_h; \mathbb{R}^3)$  for  $w_i^\sigma$  whose vertical averages  $w^{(h)}(x') = \int_{I_h} (u^{(h)})'(x) dx_3$  and  $\tau^{(h)}(x') = \int_{I_h} u_3^{(h)}(x) dx_3$  are smooth on  $S \setminus Q$  and satisfy*

$$\lim_{h \rightarrow 0} I_{2D}^h(w^{(h)}; S \setminus Q) + h \int_{S \setminus Q} \left| \nabla'^2 \tau^{(h)}(x') \right|^2 dx' = 0. \quad (29)$$

**Proof.** If we only wanted to prove the existence of smooth recovery sequences, then we could simply mollify each  $v^{(h)}$  on a scale fine enough to ensure that the  $h$ -energy of the mollified displacement is very close to that of the original one. To control the second term in (29), however, one must take a mollification scale of order  $h$  (and not smaller). Hence

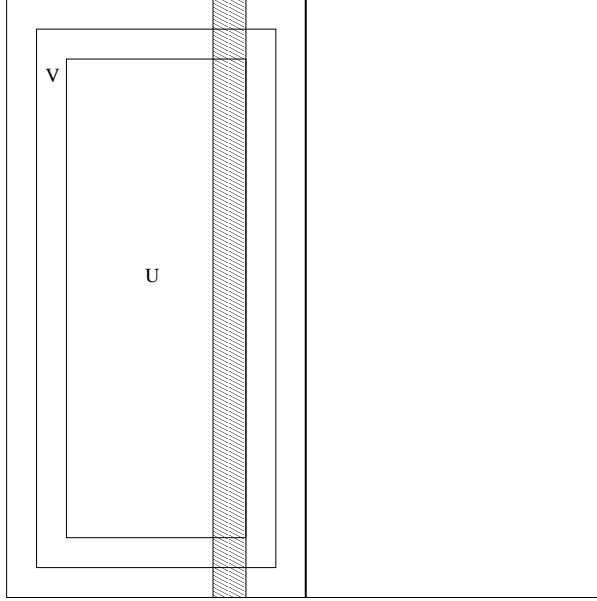


Figure 1: The shaded region represents the interpolation layer.

we cannot approximate the value of  $I^h(v^{(h)})$  separately for each  $h$ , but only have control up to a prefactor. This is why we mollify only away from the interface, and then we must glue the smoothed mapping to the original one.

We will prove the statement for  $i = 1$ ,  $\sigma = +$  only, the other cases being analogous. Recall that  $w_1^+$  has one vertical interface with phase “0” to its left. Let  $v^{(h)}$  be a recovery sequence for  $w_1^+$ , so  $\lim_{h \rightarrow 0} I^h(v^{(h)}; S) = k(\nu_1)$ . Fix a small  $a > 0$  satisfying  $(-3a, 3a) \times (-\frac{1}{2}, \frac{1}{2}) \subset Q$  and set  $V = (-\frac{1}{2} + \frac{a}{2}, -\frac{a}{2}) \times (-\frac{1}{2} + \frac{a}{2}, \frac{1}{2} - \frac{a}{2})$  and let  $U = (-\frac{1}{2} + a, -a) \times (-\frac{1}{2} + a, \frac{1}{2} - a)$ , so we have  $U \subset\subset V \subset\subset \{\nabla' w_1^+ = 0\} \cap S$ . The situation is depicted in Figure 1. Instead of mollifying in all space dimensions, which would lead to smoothed displacements defined on a film of thickness smaller than  $h$  (which could then be rescaled), we prefer to mollify slicewise in the plane and thus directly obtain displacements with smooth vertical averages defined on the full plate thickness: Let  $\psi$  be a standard mollifier supported on  $(-\frac{1}{2}, \frac{1}{2})^2$  and set  $\psi_h(x') = \frac{1}{h^2} \psi(\frac{x'}{h})$ , which is supported on the  $h$ -square  $I_h^2$ . Define the in-plane convolution

$$\tilde{v}^{(h)}(x) = (\psi_h * v^{(h)}(\cdot, x_3))(x') = \int_{I_h^2} \psi(\frac{y'}{h}) v^{(h)}(x' - y', x_3) dy',$$

which for  $h$  small enough is well defined on the set  $U \times I_h$ . We have

$$\nabla \tilde{v}^{(h)}(x) = (\psi_h * \nabla v^{(h)}(\cdot, x_3))(x'). \quad (30)$$

Let  $R^{(h)}$  be the  $K$ -valued piecewise constant approximation of  $\nabla v^{(h)}$  introduced in the proof of Theorem 6. Adopting the notation introduced there, we have  $V \subset S'_h$  for  $h$  small

enough, so  $R^{(h)}$  is defined everywhere on  $V$ . Using (30) we estimate

$$\begin{aligned}
& \int_{U \times I_h} |\nabla \tilde{v}^{(h)}(x) - R^{(h)}(x')|^2 dx \\
& \leq \frac{C}{h^2} \int_{\text{spt } \psi_h} dy' \int_{U \times I_h} \left( |\nabla v^{(h)}(x' - y', x_3) - R^{(h)}(x' - y')|^2 \right. \\
& \quad \left. + |R^{(h)}(x' - y') - R^{(h)}(x')|^2 \right) dx \\
& \leq C \int_{V \times I_h} W(\nabla v^{(h)}). \tag{31}
\end{aligned}$$

In the second step we have applied Jensen's inequality and have added and subtracted  $R^{(h)}(x' - y')$ . In the last step we used that  $\psi_h$  is supported on a  $h$ -square and applied the estimate

$$\int_U |R^{(h)}(x' + \zeta) - R^{(h)}(x')|^2 dx' \leq C \int_{V \times I_h} W(\nabla v^{(h)}), \tag{32}$$

which holds for all  $\zeta \in \mathbb{R}^2$  with  $|\zeta_1|, |\zeta_2| \leq h$ . The estimate (32) can be derived by arguments similar to the first part of the proof of Theorem 4.1 in [24], with our Theorem 3 replacing their Theorem 3.1. By (31) and Corollary 11 we deduce

$$I^h(\tilde{v}^{(h)}; U) \leq C I^h(v^{(h)}; V) = o(1) \tag{33}$$

Let  $w^{(h)} = \int_{I_h} (v^{(h)})' dx_3$ ,  $\tau^{(h)} = \int_{I_h} (v^{(h)})_3 dx_3$  and  $\tilde{w}^{(h)} = \int_{I_h} (\tilde{v}^{(h)})' dx_3$ ,  $\tilde{\tau}^{(h)} = \int_{I_h} (\tilde{v}^{(h)})_3 dx_3$ . From Jensen's inequality and (31) we easily deduce  $I_{2D}^h(\tilde{w}^{(h)}; U) \leq C I^h(v^{(h)}; V)$ . By (30) we have, for  $\alpha \in \{1, 2\}$ ,

$$(\nabla \tilde{v}^{(h)})_{,\alpha}(x) = \int \psi_{,\alpha}(y') \left( \nabla v(x' - y', x_3) - \nabla v(x', x_3) \right) dy',$$

where we have added a term which is zero by  $\int \nabla' \psi_h = 0$ . Taking squares, integrating and arguing as in (31) we obtain that also  $h \int_U \left| \nabla'^2 \tilde{\tau}^{(h)}(x') \right|^2 dx' \leq C I^h(v^{(h)}; V)$ .

For  $\kappa \in (0, a)$  let  $\phi$  be a smooth cutoff function that decreases from one to zero within the transition layer  $(-a - \kappa, -a)$  with  $\|\phi'\|_\infty \leq \frac{2}{\kappa}$ . Consider the linear interpolation  $u_\kappa^{(h)}(x) = v^{(h)}(x) + \phi(x_1)(\tilde{v}^{(h)}(x) - v^{(h)}(x))$ . Since  $\int_{I_h} (\tilde{v}^{(h)})' dx_3 \rightarrow w_1^+$  in  $L^2(U; \mathbb{R}^2)$  and the same for  $v^{(h)}$ , it follows that also  $\int_{I_h} (u_\kappa^{(h)})' dx_3 \rightarrow w_1^+$  in  $L^2(U; \mathbb{R}^2)$ . Moreover, the energy of  $u_\kappa^{(h)}$  on the transition layer  $T_h = (-a - \kappa, -a) \times (-\frac{1}{2} + a, \frac{1}{2} - a) \times I_h$  is bounded by

$$\begin{aligned}
C \int_{T_h} W_0(\nabla u_\kappa^{(h)}(x)) dx & \leq C \int_{T_h} W_0(\nabla v^{(h)}) + \frac{1}{\kappa^2} \left| \tilde{v}^{(h)} - v^{(h)} \right|^2 + \left| \nabla \tilde{v}^{(h)} - \nabla v^{(h)} \right|^2 dx \\
& \leq \frac{C}{\kappa^2} \int_{T_h} W_0(\nabla v^{(h)}) + \left| \nabla \tilde{v}^{(h)} - \nabla v^{(h)} \right|^2 dx \\
& \leq \frac{C}{\kappa^2} \int_{V \times I_h} W(\nabla v^{(h)}) dx. \tag{34}
\end{aligned}$$

Here we have assumed, by possibly adding a constant  $c^{(h)}$  to  $\tilde{v}^{(h)}$ , that  $\int_{T_h} (\tilde{v}^{(h)}(x) - v^{(h)}(x)) dx = 0$ , so we could apply Poincaré's inequality to estimate the term proportional

to  $|\tilde{v}^{(h)} - v^{(h)}|^2$  (as usual, the varying domain causes no problem: rescale the  $x_3$ -variable such that the rescaled mappings are all defined on the same domain  $T_1$  and apply the Poincaré inequality to these mappings). Note that  $c^{(h)} \rightarrow 0$ , since  $w^{(h)}$  and  $\tilde{w}^{(h)}$  converge to the same limit  $w_1^+$  in  $W^{1,2}(S; \mathbb{R}^2)$ . In the last step in (34) we have used the fact that by (31) both  $\nabla v^{(h)}$  and  $\nabla \tilde{v}^{(h)}$  are  $L^2$ -close to  $R^{(h)}$  on  $U$ , whence they are close to each other with the same bound  $\int_{V \times I_h} W(\nabla v^{(h)})$ .

After applying the analogous construction to the right of the interface (adding a different constant  $c^{(h)}$  to the corresponding  $\tilde{v}^{(h)}$ ), we are almost done. However, notice that  $u^{(h)}$  is not yet defined near the boundary of  $S$ . This minor technical objection is treated as follows: Fix  $\lambda \in (\frac{8}{9}, 1)$  and set  $S' = \lambda S$ . For small  $h$  the mapping  $u^{(h)}$  is well defined on  $S' \times I_h$ . By Lemma 14 and Corollary 11 we have that the sequence  $\hat{u}^{(\lambda h)}(x) = \lambda u^{(h)}(\frac{x}{\lambda})$  is again a recovery sequence for  $w_1^+$  satisfying the statement of the lemma, and it is defined on all of  $S \times I_{\lambda h}$ .  $\square$

Finally we will further modify the recovery sequence such that the resulting functions are affine away from the interface. This is achieved via the two-step interpolation depicted in Figure 2. In the first step the recovery sequence is modified in such a way that it uses only one well away from the interface; namely the one which is being used by the limiting mapping on that region. In a second step, it is further modified to become affine with gradient in the corresponding well.

**Lemma 16.** *Let  $S \subset \mathbb{R}^2$  be a rectangle with sides parallel to the coordinate axes and let  $w \in \mathcal{A}(S)$  have exactly one interface. Then, for any sequence  $h \rightarrow 0$  there is a sequence  $v^{(h)} \in W^{1,2}(S \times I_h; \mathbb{R}^3)$  with*

$$\int_{I_h} (v^{(h)})' dx_3 \rightarrow w \text{ strongly in } W^{1,2}(S; \mathbb{R}^2)$$

and

$$\lim_{h \rightarrow 0} I^h(v^{(h)}; S) = I^0(w; S).$$

Moreover, for any rectangle  $Q \subset S$  covering the interface with two sides parallel to it there is  $h_0 > 0$  such that for  $h \leq h_0$ , the mapping  $v^{(h)}$  is affine on each connected component of  $S \setminus Q$  with  $\nabla v^{(h)} \in K$ .

**Proof.** Suppose we had shown the lemma for the special domain  $(-\frac{1}{2}, \frac{1}{2})^2$  and the special limiting displacement  $w_1^+$ . Now let  $S$  be as in the hypothesis and assume that the interface of  $w$  has the same orientation as that of  $w_1^+$ ; the other cases are treated similarly. By translation invariance we may assume without loss of generality that the interface of  $w$  is given by  $\{0\} \times (-\frac{l}{2}, \frac{l}{2})$ . There exists  $\delta \in (0, \frac{l}{2})$  such that  $S' = (-\delta, \delta) \times (-\frac{l}{2}, \frac{l}{2}) \subset S$ . Let  $(v^{(h/l)}, h/l)$  be a recovery sequence for  $w_1^+$  with the additional properties stated in the conclusion of the lemma. Defining  $\hat{v}^{(h)}(x) = lv^{(h/l)}(\frac{x}{l})$ , the sequence  $(\hat{v}^{(h)}, h)$  is a recovery sequence for  $\hat{w}(x) = lw_1^+(\frac{x}{l})$  on the set  $lS$ . By Lemma 9 and Corollary 11 the restriction  $\hat{v}^{(h)}|_{S' \times I_h}$  is a recovery sequence for  $\hat{w}|_{S'} = w|_{S'} + f$ , where  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is affine with  $\text{sym } \nabla f = 0$ . Hence  $\bar{v}^{(h)}(x) = \hat{v}^{(h)}(x) - \begin{pmatrix} f(x') \\ 0 \end{pmatrix}$  is a recovery sequence for  $w|_{S'}$  which, by construction, is affine away from the interface (in the precise sense stated in the conclusion of the lemma). Hence now we can extend  $\bar{v}^{(h)}$  affinely to all of  $S \times I_h$  to obtain a recovery sequence with the claimed properties. Thus it suffices to prove the lemma for  $S = (-\frac{1}{2}, \frac{1}{2})^2$

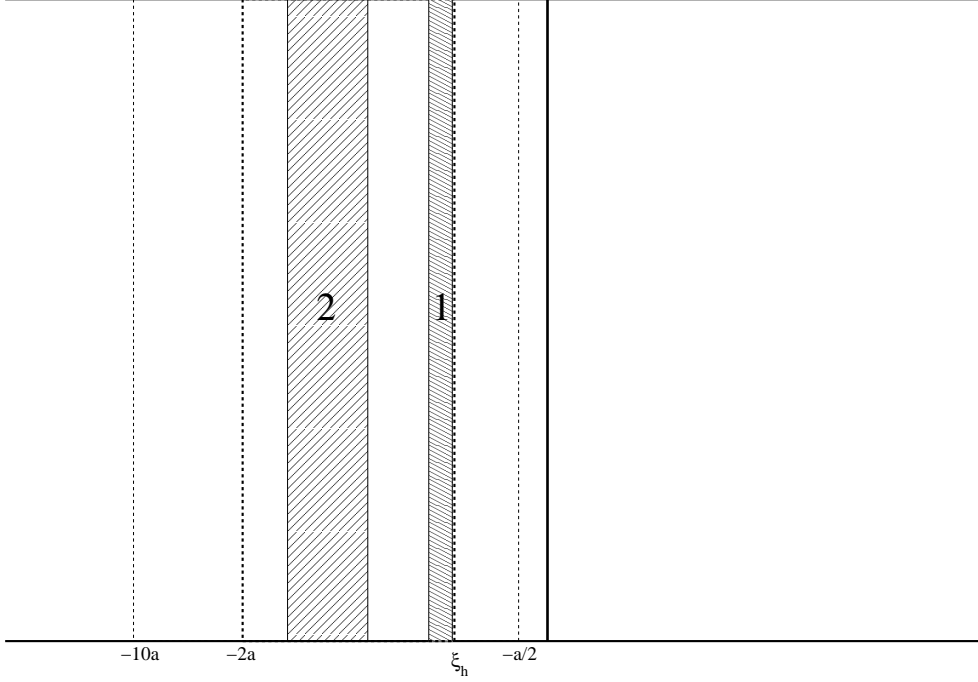


Figure 2: The shaded regions represent the interpolation layers; their numbers correspond to the steps in the proof.

and  $w = w_i^\sigma$ . Again we do the construction only for  $w_1^+$  and only on the left side of the interface; the other cases are similar.

We will first construct a recovery sequence for a fixed rectangle  $Q$ , and only in Step 3 we will take a diagonal sequence which then satisfies the property stated in the conclusion of this lemma. So fix a rectangle  $Q \subset S$  covering the interface with two sides parallel to it and choose some  $a > 0$  such that  $(-10a, 10a) \times (-\frac{1}{2}, \frac{1}{2}) \subset Q$ . Let  $v^{(h)}$  be a recovery sequence for  $w_1^+$  as furnished by Lemma 15, whose vertical averages are smooth at a distance  $a/3$  from the interface. Set  $U = (-5a, -\frac{a}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ . We may assume without loss of generality that  $h < \frac{a}{100}$ , and by Corollary 11 we have  $I^h(v^{(h)}; U) = o(1)$ .

Consider the vertical averages  $w^{(h)}(x') = \int_{I_h} (v^{(h)}(x))' dx_3$  and  $\tau^{(h)} = \int_{I_h} v_3^{(h)}(x) dx_3$ . By the strong convergence  $w^{(h)} \rightarrow w_1^+$  in  $W^{1,2}(S; \mathbb{R}^2)$  and since  $U \subset \{\nabla' w_1^+ = 0\}$  we have  $\int_U |\nabla' w^{(h)}|^2 dx' = o(1)$ . Hence, using Lemma 15, we can write

$$I^h(v^{(h)}; U) + I_{2D}^h(w^{(h)}; U) + h \int_U |\nabla'^2 \tau^{(h)}(x')|^2 dx' + \int_U |\nabla' w^{(h)}(x')|^2 dx' = \eta_h, \quad (35)$$

where  $\eta_h = o(1)$  as  $h \rightarrow 0$ .

**Step 1.** *Interpolation to a displacement with low one-well energy.* Subdivide  $U$  into  $h$ -squares as in the proof of Theorem 6. Let  $G_h = h\mathbb{Z} \cap (-2a, -a)$  and set  $N_h = \#G_h$ . To every  $\xi \in G_h$  define the column  $Z_h(\xi) = (\xi - \frac{h}{2}, \xi + \frac{h}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ . By definition of  $U$  and since  $h < \frac{a}{2}$ , the inclusion  $Z_h(\xi) \subset U$  holds for all  $\xi \in G_h$ . We have

$$\sum_{\xi \in G_h} \int_{Z_h(\xi)} |\nabla' w^{(h)}|^2 dx' \leq \int_U |\nabla' w^{(h)}|^2 dx' \leq \eta_h. \quad (36)$$

Let  $\rho \in (0, 1)$  (to be fixed later). Denote by  $G_h^1$  the set of all  $\xi \in G_h$  with the property that

$$\int_{Z_h(\xi)} |\nabla' w^{(h)}|^2 dx' \leq \frac{\eta_h}{[\rho N_h]}. \quad (37)$$

One can easily check that by (36) the cardinality of  $G_h^1$  is at least  $[(1 - \rho)N_h]$ . Notice that by (37) and since  $N_h \leq \frac{a}{h}$  we have

$$\int_{Z_h(\xi)} |\nabla' w^{(h)}|^2 dx' \leq C\eta_h h. \quad (38)$$

Next, recalling notation and results from the proof of Theorem 6, by (35) the cardinality of the set  $\mathcal{A}_0 \cap U$  is of order  $o(1/h)$ . Hence the set  $G_h^2$  of all  $\xi \in G_h$  with the property that  $Z_h(\xi)$  does not contain any square of type  $\mathcal{A}_0$  satisfies

$$1 \geq \frac{\#(G_h^2)}{N_h} \geq \frac{N_h - \#\mathcal{A}_0}{N_h} \rightarrow 1$$

as  $h \rightarrow 0$ . Note that by (22) such columns must consist either only of squares of type  $\mathcal{A}_1$  or only of squares of type  $\mathcal{A}_2$ . On the other hand, by the convergence  $\chi_{\Omega_1^h} \rightarrow \chi_{\{\nabla' w_1^+ = 0\}}$ , the set  $G_h^3$  of  $\xi \in G_h$  with the property that  $Z_h(\xi)$  contains a square of type  $\mathcal{A}_1$  satisfies  $\frac{\#G_h^3}{N_h} \rightarrow 1$  as well.

Columns  $Z_h(\xi)$  with  $\xi \in G_h^3 \cap G_h^2$  consist only of  $\mathcal{A}_1$ -squares. The reason for picking out such a column  $\xi$  is that on it only the well  $A = 0$  is used. More precisely, using that by definition  $\text{sym } R^{(h)} = 0$  in an  $\mathcal{A}_1$ -type cube, and that  $Z_h(\xi) \times I_h$  is made up of such cubes, using (21) and the definition of  $I^h$  we can estimate, for  $\xi \in G_h^3 \cap G_h^2$ ,

$$\begin{aligned} \int_{Z_h(\xi) \times I_h} |\text{sym } \nabla v^{(h)}(x)|^2 dx &= \int_{Z_h(\xi) \times I_h} |\text{sym } \nabla v^{(h)}(x) - \text{sym } R^{(h)}(\xi, x_2)|^2 dx \\ &\leq Ch^2 I^h(v^{(h)}; U) \leq C\eta_h h^2. \end{aligned} \quad (39)$$

(One could gain one more power of  $h$  in this estimate by choosing a low-energy column in analogy to the definition of  $G_h^1$ , but this estimate will suffice.) Let  $J_2$  be the set of all  $\xi \in (-2a, -a)$  satisfying the property  $(P_h)$  (defined in the statement of Lemma 19 in the Appendix) for  $w^{(h)}$ . Applying Lemma 19 on the domain  $(-2a, -a) \times (-\frac{1}{2}, \frac{1}{2})$ , for  $h$  small enough we have  $\mathcal{H}^1(J_2) \geq \frac{a}{2}$ . For every  $\xi \in J_2$  there is a continuous function  $\tilde{w}^{(h)} \in W^{1,2}(( -2a, \xi) \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{R}^2)$  satisfying  $w^{(h)}(\xi, \cdot) = \tilde{w}^{(h)}(\xi, \cdot)$  and

$$\frac{1}{h} \int_{(-2a, \xi) \times (-\frac{1}{2}, \frac{1}{2})} |\text{sym } \nabla' \tilde{w}^{(h)}|^2 dx' \leq C \left( I_{2D}^h(w^{(h)}; U) + \int_U |\nabla' w^{(h)}|^2 dx' \right) \leq C\eta_h \quad (40)$$

with  $C$  independent of  $\xi$  and  $h$ . (Strictly speaking, Lemma 19 provides mappings  $\tilde{w}^{(h)}$  with the above properties only on a connected component of  $(-2a, \xi) \times (-\frac{1}{4}, \frac{1}{4})$ , i.e. on a stripe of width one-half only. But by Lemmas 13 and 14 we could as well restrict the following construction to the corresponding sub-stripe and then rescale uniformly, compare the arguments at the beginning of this proof and the one at the end of the proof of Lemma 15. Thus, there is no loss of generality if we assume that  $\tilde{w}^{(h)}$  are defined as just claimed,

and it will avoid unnecessary additional notation.)  
Define the thinner columns

$$Z'_h(\xi) = \left(\xi - \frac{h}{4}, \xi\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (41)$$

and consider the neighbourhood  $J_1 = \bigcup_{\xi \in \bigcap_{i=1}^3 G_h^i} \left(\xi - \frac{h}{4}, \xi + \frac{h}{4}\right)$ . of  $\bigcap_{i=1}^3 G_h^i$ . By choosing  $\rho$  above large enough, we have  $\mathcal{H}^1(J_1 \cap (-2a, -a)) > \frac{4a}{9}$ . Since  $J_2 \subset (-2a, -a)$  and  $\mathcal{H}^1(J_2) \geq \frac{a}{2}$  we conclude that, in particular, there is one  $\xi_h \in J_1 \cap J_2$  with  $\xi_h \geq -\frac{3a}{2}$ . Note that, in general,  $\xi_h \notin G_h$ . The reason for taking  $\frac{h}{4}$  instead of  $\frac{h}{2}$  in the definition of  $J_1$  is that this choice ensures that  $Z'_h(\xi_h) \subset Z_h(\xi)$  for some  $\xi \in \bigcap_{i=1}^3 G_h^i$ , by the definition of  $J_1$ . The property of  $\tilde{w}^{(h)}$  using only one well is crucial, since it allows us to apply Korn's inequality in the plane to deduce the existence of a constant skew symmetric matrix  $W_h \in \mathbb{R}^{2 \times 2}$ , given explicitly as

$$W_h = \text{skew} \int_{U_h} \nabla' \tilde{w}^{(h)}(x') dx', \quad (42)$$

with the property that there is an affine mapping  $f^{(h)}$  with gradient  $W_h$  and

$$\int_{U_h} |\tilde{w}^{(h)} - f^{(h)}|^2 + |\nabla' \tilde{w}^{(h)} - W_h|^2 dx' \leq C \int_{U_h} |\text{sym} \nabla' \tilde{w}^{(h)}|^2 dx' \leq Ch\eta_h. \quad (43)$$

Here we have introduced the set

$$U_h = \left(\xi_h - \frac{a}{4}, \xi_h\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right), \quad (44)$$

which satisfies  $U_h \subset [-\frac{7a}{4}, -a) \times (-\frac{1}{2}, \frac{1}{2}) \subset U$  for all  $h$ , since  $\xi_h \in [-\frac{3a}{2}, -a)$ . Notice that in fact  $C$  in (43) is independent of  $h$ , because the constant appearing in Korn's inequality is invariant under translation of the domain. We claim that

$$W_h \rightarrow 0 \text{ in } \mathbb{R}^{2 \times 2}. \quad (45)$$

Indeed, consider any subsequence. By the Trace Inequality and the fact that  $\tilde{w}$  and  $w$  agree on the line  $x_1 = \xi_h$  we have

$$\begin{aligned} \int_{(-\frac{1}{2}, \frac{1}{2})} \left| f^{(h)}(\xi_h, x_2) \right|^2 dx_2 &\leq C \int_{(-\frac{1}{2}, \frac{1}{2})} \left| f^{(h)}(\xi_h, x_2) - \tilde{w}^{(h)}(\xi_h, x_2) \right|^2 dx_2 + C \int_{(-\frac{1}{2}, \frac{1}{2})} \left| w^{(h)}(\xi_h, x_2) \right|^2 dx_2 \\ &\leq C \int_{U_h} \left| f^{(h)} - \tilde{w}^{(h)} \right|^2 + \left| W_h - \nabla' \tilde{w}^{(h)} \right|^2 dx' \\ &\quad + C \int_{U_h} \left| w^{(h)} \right|^2 + \left| \nabla' w^{(h)} \right|^2 dx', \end{aligned}$$

which tends to zero by (43) and the convergence  $w^{(h)} \rightarrow w_1^+$  in  $W^{1,2}(S; \mathbb{R}^2)$ . From this one easily deduces (45) since  $W_h$  is skew symmetric.

Now we extend  $\tilde{w}^{(h)}$  to a three-dimensional displacement  $\tilde{v}^{(h)}$  by defining

$$\tilde{v}^{(h)}(x) = \begin{pmatrix} \tilde{w}^{(h)}(x') \\ \tau^{(h)}(x') \end{pmatrix} + x_3 \begin{pmatrix} -\tau_{,1}^{(h)}(x') \\ -\tau_{,2}^{(h)}(x') \\ 0 \end{pmatrix}. \quad (46)$$

This displacement has good one-well energy on  $U_h \times I_h$ , since by (35) and (40)

$$\begin{aligned} \int_{U_h \times I_h} |\text{sym } \nabla \tilde{v}^{(h)}(x)|^2 dx &\leq h^3 \int_{U_h} |\nabla'^2 \tau^{(h)}(x')|^2 dx' + h \int_{U_h} |\text{sym } \nabla' \tilde{w}^{(h)}(x')|^2 dx' \\ &\leq C\eta_h h^2. \end{aligned} \quad (47)$$

(Later we will repeat this construction on the other side of the interface. Then one must replace the second summand in (46) by  $x_3(\mu_2 - \tau_{,1}^{(h)}(x'), \mu_1 - \tau_{,2}^{(h)}(x'), \mu_3)^T$ .) Now consider the interpolation

$$u^{(h)}(x) = v^{(h)}(x) + \phi^{(h)}(x_1)(\tilde{v}^{(h)}(x) - v^{(h)}(x)), \quad (48)$$

where  $\phi^{(h)}$  denotes a smooth cutoff function that decreases from one to zero within the interval  $(\xi_h - \frac{h}{4}, \xi_h)$ . We take  $h/4$  here to make sure that we stay within the chosen  $h$ -column. We claim that

$$\int_{U_h \times I_h} |\text{sym } \nabla u^{(h)}|^2 dx \leq C\tilde{\eta}_h h^2, \quad (49)$$

where  $\tilde{\eta}_h = \eta_h + |W_h|^2$  converges to zero as  $h \rightarrow 0$ .

To prove (49), recall that by (44) and (41) we have  $Z'_h(\xi_h) \subset U_h$ . Now notice that  $u^{(h)} = \tilde{v}^{(h)}$  on  $(U_h \times I_h) \setminus (Z'_h(\xi_h) \times I_h)$ , whence by (47) we have  $\int_{(U_h \times I_h) \setminus (Z'_h(\xi_h) \times I_h)} |\text{sym } \nabla u^{(h)}|^2 \leq C\eta_h h^2$ . It remains to prove (49) on the interpolation layer  $Z'_h(\xi_h)$ . We make a standard calculation to obtain

$$\begin{aligned} &\int_{Z'_h(\xi_h) \times I_h} |\text{sym } \nabla u^{(h)}|^2 dx \\ &\leq C \int_{Z'_h(\xi_h) \times I_h} |\text{sym } \nabla v^{(h)}|^2 + |\text{sym } \nabla \tilde{v}^{(h)}|^2 + \frac{1}{h^2} |\tilde{v}^{(h)} - v^{(h)}|^2 dx. \end{aligned} \quad (50)$$

The first term on the right-hand side is estimated by (39) - this is where the fact is used that  $Z'_h(\xi_h) \subset Z_h(\xi)$  for some  $\xi \in G_h^2 \cap G_h^3$ , i.e. we are inside a column consisting only of squares of type  $\mathcal{A}_1$ . By (47) the second term in (50) satisfies  $\int_{Z'_h(\xi_h) \times I_h} |\text{sym } \nabla \tilde{v}^{(h)}|^2 dx \leq C\eta_h h^2$ .

Let us estimate the third term in (50). Since

$$\begin{pmatrix} w^{(h)}(x') \\ \tau^{(h)}(x') \end{pmatrix} = \int_{I_h} v^{(h)}(x) dx_3 = \int_{I_h} v^{(h)}(x) + x_3(\nabla' \tau^{(h)}(x'))^T dx_3$$

and since  $w^{(h)} = \tilde{w}^{(h)}$  on a line, we can apply a Poincaré inequality (see e.g. [12] Theorem 6.1-8) to estimate the second term in the last step in (51) below. The first term in that step is estimated by the usual Poincaré inequality in the  $x_3$ -direction:

$$\begin{aligned} &\int_{Z'_h(\xi_h) \times I_h} \frac{1}{h^2} |v^{(h)}(x) - \tilde{v}^{(h)}(x)|^2 dx \\ &\leq \frac{C}{h^2} \int_{Z'_h(\xi_h) \times I_h} \left| v^{(h)}(x) + x_3 \begin{pmatrix} \tau_{,1}^{(h)}(x') \\ \tau_{,2}^{(h)}(x') \\ 0 \end{pmatrix} - \begin{pmatrix} w^{(h)}(x') \\ \tau^{(h)}(x') \end{pmatrix} \right|^2 + |w^{(h)}(x') - \tilde{w}^{(h)}(x')|^2 dx \\ &\leq C \int_{Z'_h(\xi_h) \times I_h} \left| v_{,3}^{(h)}(x) + \begin{pmatrix} \tau_{,1}^{(h)}(x') \\ \tau_{,2}^{(h)}(x') \\ 0 \end{pmatrix} \right|^2 + |\nabla' w^{(h)}(x') - \nabla' \tilde{w}^{(h)}(x')|^2 dx \end{aligned} \quad (51)$$



To estimate the first term in (51), we observe that  $(\tau_{,1}^{(h)}(x'), \tau_{,2}^{(h)}(x'), 0) = \nabla \int_{I_h} v_3^{(h)}(x) dx_3$ , so we have

$$\begin{aligned} & \int_{Z'_h(\xi_h) \times I_h} \left| v_{,3}^{(h)}(x) + \begin{pmatrix} \tau_{,1}^{(h)}(x') \\ \tau_{,2}^{(h)}(x') \\ 0 \end{pmatrix} \right|^2 dx \\ & \leq C \int_{Z'_h(\xi_h) \times I_h} \left| v_{,3}^{(h)}(x) - \int_{I_h} v_{,3}^{(h)}(x', z) dz \right|^2 + \left| \int_{I_h} v_{,3}^{(h)}(x', z) dz + (\nabla \int_{I_h} v_3^{(h)}(x', z) dz)^T \right|^2 dx \\ & \leq C \eta_h h^2. \end{aligned}$$

In the last step we have applied (39) again and the fact that, by the  $x_3$ -independence of  $R^{(h)}$ , by Jensen's inequality and by (35),

$$\begin{aligned} & \int_{Z'_h(\xi_h) \times I_h} \left| v_{,3}^{(h)}(x) - \int_{I_h} v_{,3}^{(h)}(x', z) dz \right|^2 dx \\ & \leq C \int_{Z'_h(\xi_h) \times I_h} |v_{,3}^{(h)}(x) - R_3^{(h)}(x')|^2 + \left| \int_{I_h} R_3^{(h)}(x') - v_{,3}^{(h)}(x', z) dz \right|^2 dx \\ & \leq C \eta_h h^2. \end{aligned}$$

The second term in (51) is bounded by

$$C \int_{Z'_h(\xi_h) \times I_h} |\nabla' w^{(h)}(x')|^2 + |\nabla' \tilde{w}^{(h)}(x') - W_h|^2 + |W_h|^2 dx \leq C \eta_h h^2 + |Z'_h(\xi_h) \times I_h| |W_h|^2$$

We have applied (45) and (38) multiplied by  $h$ , since here we are integrating over the thickness on the left-hand side. This proves (49) and finishes the first interpolation step.

**Step 2. Interpolation to an affine displacement.** We apply Lemma 17 to the mapping  $u^{(h)}$  defined in (48) with  $J_h = (\xi_h - \frac{a}{4}, \xi_h)$  instead of  $J$ , so  $J_h \times (-\frac{1}{2}, \frac{1}{2}) = U_h$ . Instead of the interpolation layer  $(t, t+a)$  for which Lemma 17 is stated, we consider the interpolation layer  $(\xi_h - \frac{3a}{16}, \xi_h - \frac{a}{8})$ . Notice that on the set  $J_h \times (-\frac{1}{2}, \frac{1}{2})$  in fact  $u^{(h)} = \tilde{v}^{(h)}$ , since we are to the left of the first interpolation layer, compare Figure 2. Lemma 17 furnishes a mapping  $\tilde{u}^{(h)}$  which agrees with  $u^{(h)}$  on the set  $\{x \in S : x_1 > \xi_h - \frac{a}{8}\}$  and equals an affine function  $f^{(h)}$  with skew symmetric gradient  $T^{(h)}$  on the set  $\{x \in S : x_1 < \xi_h - \frac{3a}{16}\}$  (the mapping  $\tilde{u}^{(h)}$  is at first not defined on  $\{x_1 < \xi_h - \frac{a}{4}\}$ , but since it is affine on  $(\xi_h - \frac{3a}{16}, \xi_h - \frac{a}{4})$  we can extend it affinely). Moreover,  $\tilde{u}^{(h)}$  satisfies

$$\int_{U_h \times I_h} |\text{sym } \nabla \tilde{u}^{(h)}(x)|^2 dx \leq \frac{C}{a^4} \int_{U_h \times I_h} |\text{sym } \nabla u^{(h)}(x)|^2 dx.$$

Combining this with (49) we conclude that

$$I^h(\tilde{u}^{(h)}; S) \rightarrow k(\nu_1). \quad (52)$$

**Step 3. Convergence.** Now we apply Steps 1 and 2 with obvious modifications also on the other side of the interface. Then we have shown the following: For any given rectangle  $Q \subset S$  as in the statement of this lemma, there exists a sequence  $\tilde{u}_Q^{(h)}$  satisfying  $I^h(\tilde{u}_Q^{(h)}; S) \rightarrow$

$k(\nu_1)$ , and each  $\tilde{u}_Q^{(h)}$  is affine with  $\text{sym } \nabla \tilde{u}_Q^{(h)} = 0$  on the left and  $\text{sym } \nabla \tilde{u}_Q^{(h)} = B$  on the right connected component of  $S \setminus Q$ . Now set  $Q_j = (-\frac{1}{j}, \frac{1}{j}) \times (-\frac{1}{2}, \frac{1}{2})$ . By Proposition 20 there exists a sequence  $j_n \rightarrow \infty$  such that

$$\limsup_{n \rightarrow \infty} I^{h_n}(\tilde{u}_{Q_{j_n}}^{(h_n)}; S) = \limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} I^{h_n}(\tilde{u}_{Q_j}^{(h_n)}; S) = k(\nu_1). \quad (53)$$

Theorem 6 implies that there exist skew affine mappings  $f_n$  and  $w \in \mathcal{A}(S)$  such that

$$\bar{w}_{Q_{j_n}}^{(h_n)} + f_n \rightarrow w \text{ strongly in } W^{1,2}(S; \mathbb{R}^2), \quad (54)$$

and by Theorem 12 and (53) the limiting function  $w$  satisfies  $I^0(w; S) \leq k(\nu_1)$ . But by (54) necessarily  $\text{sym } \nabla' w = 0$  to the left and  $\text{sym } \nabla' w = B$  to the right of the  $x_2$ -axis. Hence by possibly adding the same skew affine mapping to each  $f_n$  we may assume that the convergence (54) holds with  $w = w_1^+$ , whence  $\tilde{u}_{Q_{j_n}}^{(h_n)} + \begin{pmatrix} f_n \\ 0 \end{pmatrix}$  is the sought-for recovery sequence.  $\square$

**Lemma 17.** *Let  $I$  and  $J$  be open bounded intervals and set  $U = J \times I$ . There is a constant  $C$  (invariant under translations of  $U$ ) such that the following holds: For every  $h \in (0, 1)$ , for every  $\tilde{v}^{(h)} \in W^{1,2}(U \times I_h; \mathbb{R}^3)$ , for every  $t \in J$  and for every  $b \in (0, 1)$  satisfying  $t + b \in J$ , there exists a skew symmetric matrix  $T^{(h)} \in \mathbb{R}^{3 \times 3}$ ,  $c^{(h)} \in \mathbb{R}^3$  and  $\tilde{u}^{(h)} \in W^{1,2}(U \times I_h; \mathbb{R}^3)$  such that  $\tilde{u}^{(h)} = \tilde{v}^{(h)}$  on  $\{x_1 < t\}$  and  $\tilde{u}^{(h)} = T^{(h)}x + c^{(h)}$  on  $\{x_1 > t + b\}$  and such that*

$$\int_{U \times I_h} |\text{sym } \nabla \tilde{u}^{(h)}|^2 dx \leq \frac{C}{b^4} \int_{U \times I_h} |\text{sym } \nabla \tilde{v}^{(h)}|^2 dx.$$

One can take

$$T^{(h)} = \text{skew} \int_{U \times I_h} \nabla \tilde{v}^{(h)}(x) dx \text{ and } c^{(h)} = \int_{U \times I_h} \tilde{v}^{(h)}(x) dx. \quad (55)$$

The analogous statements hold reversing the order of  $t$  and  $b$  or considering the well  $B + \text{Skew}$  instead of  $\text{Skew}$ .

**Proof.** By Proposition 18 (ii) and Poincaré's inequality the mappings  $f^{(h)}(x) = T^{(h)}x + c^{(h)}$  with  $T^{(h)}$  and  $c^{(h)}$  as in (55) satisfy

$$\int_{U \times I_h} |\tilde{v}^{(h)}(x) - f^{(h)}|^2 + |\nabla \tilde{v}^{(h)}(x) - T^{(h)}|^2 dx \leq \frac{C}{h^2} \int_{U \times I_h} |\text{sym } \nabla \tilde{v}^{(h)}(x)|^2 dx. \quad (56)$$

By Proposition 18 (iii) also

$$\int_{U \times I_h} |(\tilde{v}^{(h)})'(x) - (f^{(h)})'(x)|^2 dx \leq C \int_{U \times I_h} |\text{sym } \nabla \tilde{v}^{(h)}(x)|^2 dx. \quad (57)$$

Fix a smooth cutoff function  $\phi(x_1)$  which decreases from one to zero within the interval  $(t, t + b)$ . Define the interpolation

$$\tilde{u}^{(h)}(x) = f^{(h)}(x) + \phi(x_1)(\tilde{v}^{(h)}(x) - f^{(h)}(x)) - x_3 \phi'(x_1)(\tilde{v}_3^{(h)}(x) - f_3^{(h)}(x))e_1.$$

Then

$$\begin{aligned}\nabla \tilde{u}^{(h)}(x) &= T^{(h)} + \phi(x_1)(\nabla \tilde{v}^{(h)}(x) - T^{(h)}) + \begin{pmatrix} (\tilde{v}^{(h)} - f^{(h)})'(x) \\ 0 \end{pmatrix} \otimes e_1 \phi'(x_1) \\ &\quad + (\tilde{v}^{(h)} - f^{(h)})_3(x) e_3 \otimes e_1 \phi'(x_1) - (\tilde{v}^{(h)} - f^{(h)})_3(x) e_1 \otimes e_3 \phi'(x_1) \\ &\quad - x_3 \left( \phi''(x_1) (\tilde{v}^{(h)} - f^{(h)})_3(x) e_1 \otimes e_1 + \phi'(x_1) e_1 \otimes (\nabla \tilde{v}_3^{(h)}(x) - (T^{(h)})^T e_3) \right).\end{aligned}$$

Upon taking the symmetric part of the above expression, the second line cancels and we obtain

$$\begin{aligned}\int_{U \times I_h} |\text{sym } \nabla \tilde{u}^{(h)}|^2 dx &\leq C \int_{U \times I_h} |\text{sym } \nabla \tilde{v}^{(h)}|^2 + \frac{1}{b^2} |(\tilde{v}^{(h)} - f^{(h)})'|^2 dx \\ &\quad + h^2 \int_{U \times I_h} \left( \frac{1}{b^4} |\tilde{v}^{(h)} - f^{(h)}|^2 + \frac{1}{b^2} |\nabla \tilde{v}^{(h)} - T^{(h)}|^2 \right) dx,\end{aligned}$$

since  $\phi' \sim \frac{1}{b}$  and  $\phi'' \sim \frac{1}{b^2}$ . The last term is controlled by (56) and the  $(\tilde{v}^{(h)} - f^{(h)})'$ -term is controlled by (57).  $\square$

**Proof of Theorem 1.** By Lemma 2 we must prove the theorem only for the special case  $A = 0$  and  $B$  as in (7). Statement (i) just rephrases the content of Theorem 12. The proof of (ii) is similar to that of Proposition 5.1 in [15]. We sketch it here for the convenience of the reader. First one recalls that by Proposition 7,  $w$  is piecewise affine with straight interfaces  $J_i$  separating affine regions and with interface normal either equal to  $e_1$  or to  $e_2$ . By translation we may assume without loss of generality that  $S$  is strictly star-shaped with respect to the origin. Thus strict star-shapedness implies that the scaled domain  $\eta S$  satisfies  $\bar{S} \subset \eta S$  for any  $\eta > 1$ . Hence the restriction of  $w_\eta(x) = \eta w(\frac{x}{\eta})$  to  $S$  is well defined. The limiting function  $w_\eta|_S$  only has finitely many intervals  $J_i$ ,  $i = 1, \dots, n$ , which have a positive distance from each other. Moreover, each  $J_i$  can be covered by a rectangle  $R_i$  with two sides parallel to  $J_i$  and with the other two sides lying outside  $\bar{S}$ . Moreover, the  $R_i$  can be chosen to be pairwise disjoint. Now Lemma 16 applied to  $R_i$  furnishes a recovery sequence  $w_i^{(h)}$  with each  $w_i^{(h)}$  affine with gradient in  $K$  near those sides  $\partial R_i$  which are parallel to the interface covered by  $R_i$ . Hence the each restriction  $w_i^{(h)}|_{R_i}$  can be affinely extended into the region  $S \setminus R_i$ . There are only finitely many interfaces. Starting at one local recovery sequence, one can glue it to the local recovery sequences of the neighbouring interfaces by adding affine functions with skew symmetric gradient to them, so they agree identically with the first one away from their interfaces. Then one proceeds inductively. The composite mapping  $v^{(h)}$  obtained by this procedure is a recovery sequence for  $w_\eta|_S$ . But as explained in [2] p. 3 this implies the limes superior-part of Theorem 1, since  $w_\eta|_S \rightarrow w$  in  $W^{1,2}(S)$  and  $I^0(w_\eta|_S; S) \rightarrow I^0(w; S)$  as  $\eta \downarrow 1$  (i.e. mappings with finitely many interfaces are energy dense).  $\square$

## 5 Appendix

For a film of thickness  $h$ , the constant in the two-well Korn inequality derived in Theorem 3 deteriorates as  $h^{-2}$  as  $h$  tends to zero. The same is true for the classical Korn inequality,

as is also shown in [3]. A combination of this latter result with a Poincaré estimate in the thickness direction leads to (iii) in the following proposition, which was needed in the proof of Lemma 17. Statement (ii) is Korn's inequality for thin films as presented in [3].

**Proposition 18.** *Let  $S \subset \mathbb{R}^2$  be a bounded Lipschitz domain and let  $A, B \in \mathbb{R}^{3 \times 3}$  be such that  $\text{rank}(A - B + F) \geq 2$  for all skew-symmetric  $F \in \mathbb{R}^{3 \times 3}$ . Then there is a constant  $C(S)$  such that for all  $h \in (0, 1)$  and for all  $v^{(h)} \in W^{1,2}(S \times I_h; \mathbb{R}^3)$  the following hold:*

(i) *There exists a matrix  $T^{(h)} \in K$  such that*

$$\int_{S \times I_h} |\nabla v^{(h)}(x) - T^{(h)}|^2 dx \leq \frac{C(S)}{h^2} \int_{S \times I_h} \text{dist}^2(\text{sym } \nabla v^{(h)}(x), \{A, B\}) dx.$$

(ii) *The estimate*

$$\int_{S \times I_h} |\nabla v^{(h)}(x) - T^{(h)}|^2 dx \leq \frac{C(S)}{h^2} \int_{S \times I_h} |\text{sym } \nabla v^{(h)}(x)|^2 dx$$

*holds for  $T^{(h)} = \text{skew } \int_{S \times I_h} \nabla v^{(h)} dx$ .*

(iii) *There exists  $c^{(h)} \in \mathbb{R}^2$  such that*

$$\int_{S \times I_h} |(v^{(h)})'(x) - (T^{(h)}x)' - c^{(h)}|^2 dx \leq C(S) \int_{S \times I_h} |\text{sym } \nabla v^{(h)}(x)|^2 dx$$

*for the same  $T^{(h)}$  as in (ii).*

**Proof.** The proof of (i) is analogous to that of Theorem 10 in [25], with Theorem 3 replacing their geometric rigidity theorem. Statement (ii) can be proven in the same way, with Korn's inequality for one well replacing their geometric rigidity theorem. Another proof is given in [3] and [30]. Notice that if (ii) holds for some skew matrix, then it will also hold for the special choice  $T^{(h)} = \text{skew } \int_{S \times I_h} \nabla v^{(h)} dx$ .

To prove statement (iii) set  $w^{(h)}(x') = \int_{I_h} (v^{(h)})'(x', x_3) dx_3$ . From Korn's inequality in the plane and from Jensen's inequality we obtain

$$\int_S \left| \nabla' w^{(h)}(x') - \bar{T}^{(h)} \right|^2 dx' \leq C \int_S \left| \text{sym } \nabla' w^{(h)}(x') \right|^2 dx' \leq \frac{C}{h} \int_{S \times I_h} |\text{sym } \nabla v^{(h)}|^2 dx. \quad (58)$$

With  $c^{(h)} = \int_S w^{(h)}(x') - \bar{T}^{(h)} x' dx'$  we obtain

$$\begin{aligned} & \int_{S \times I_h} \left| (v^{(h)})'(x) - (T^{(h)}x)' - c^{(h)} \right|^2 dx \\ & \leq C \int_{S \times I_h} \left| (v^{(h)})'(x) - (T_3^{(h)})' x_3 - w^{(h)}(x') \right|^2 + \left| w^{(h)}(x') - \bar{T}^{(h)} x' - c^{(h)} \right|^2 dx, \end{aligned}$$

where  $T_3^{(h)}$  denotes the third column of  $T^{(h)}$ . The second term is estimated by applying Poincaré's inequality on  $S$  and then (58). To estimate the first term, notice that since the integration domain is symmetric, we have  $w^{(h)}(x') = \int_{I_h} (v^{(h)}(x) - (T_3^{(h)})' x_3) dx_3$ . Applying

Poincaré's inequality in the  $x_3$ -direction for almost every  $x'$  and subsequently using (ii) shows that the first term is controlled by  $\int_{S \times I_h} |\text{sym } \nabla v^{(h)}|^2$ .  $\square$

The following lemma is a corollary of Proposition 4.1 in [15]. Notice that their  $\varepsilon$  corresponds to our  $h$ .

**Lemma 19.** *Let  $l, d > 0$ , let  $U = (-l, l) \times (-d, d)$ , let  $\bar{A} = 0$  and  $\bar{B} = e_1 \otimes e_2 + e_2 \otimes e_1$ , and let  $\bar{F} \in \{\bar{A}, \bar{B}\}$ . Given  $\rho \in (0, 1)$ , there are constants  $\eta_0, C_0 > 0$  such that for every  $h \in (0, 1)$  and  $w \in W^{2,2}(U; \mathbb{R}^2)$  with*

$$I_{2D}^h(w; U) \leq \eta_0 \text{ and } \int_U |\text{sym } \nabla' w - \bar{F}|^2 dx' \leq \eta_0$$

the set of  $\xi \in (-l, l)$  satisfying property  $(P_h)$  for  $w$  has measure not smaller than  $l$ .

We say  $\xi \in (-l, l)$  satisfies property  $(P_h)$  for  $w \in W^{2,2}(U; \mathbb{R}^2)$  if, denoting by  $U_1, U_2$  the connected components of  $(-l, l) \times (-\frac{d}{2}, \frac{d}{2}) \setminus \{x_1 = \xi\}$ , for each  $i = 1, 2$  there exist  $\tilde{w}_i \in W^{1,2}(U_i; \mathbb{R}^2)$  with  $\tilde{w}(\xi, \cdot) = w(\xi, \cdot)$  on  $(-d/2, d/2)$  and

$$\frac{1}{h} \int_{U_i} |\text{sym } \nabla' \tilde{w}_i(x') - \bar{F}|^2 dx' \leq C_0 \left( I_{2D}^h(w; U) + \int_U |\text{sym } \nabla' w(x') - \bar{F}|^2 dx' \right)$$

An analogous result holds for lines of the form  $\{x_2 = \xi\}$ .

**Proof.** Set  $\eta = I_{2D}^h(w; U) + \int_U |\text{sym } \nabla' w - \bar{F}|^2 dx'$ . By Proposition 4.1 in [15] there exists a Borel set  $\Sigma \subset (-l, l)$  with  $|\Sigma| \geq l$  and such that for all  $\xi \in \Sigma$  there exists an affine mapping  $w_\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $\text{sym } \nabla' w_\xi = \bar{F}$  and

$$\|w(\xi, \cdot) - w_\xi(\xi, \cdot)\|_{H^{1/2}((-\frac{d}{2}, \frac{d}{2}); \mathbb{R}^2)}^2 \leq Ch\eta.$$

By the properties of the  $H^{1/2}$ -norm (see e.g. the appendix of [29] for a review), for  $i = 1, 2$  there exist  $v_i \in W^{1,2}(U_i; \mathbb{R}^2)$  such that

$$\int_{U_i} |\nabla' v_i|^2 dx' \leq \|w(\xi, \cdot) - w_\xi(\xi, \cdot)\|_{H^{1/2}((-\frac{d}{2}, \frac{d}{2}); \mathbb{R}^2)}^2$$

and  $v_i(\xi, \cdot) = w(\xi, \cdot) - w_\xi(\xi, \cdot)$  on  $(-\frac{d}{2}, \frac{d}{2})$  in the trace sense. Setting  $\tilde{w}_i = v_i + w_\xi$  we find

$$\frac{1}{h} \int_{U_i} |\text{sym } \nabla' \tilde{w}_i - \bar{F}|^2 dx' \leq \frac{C}{h} \int_{U_i} |\nabla' v_i|^2 + |\text{sym } \nabla' w_\xi - \bar{F}|^2 dx' \leq C\eta$$

and  $\tilde{w}_i(\xi, \cdot) = w(\xi, \cdot)$  on  $(-\frac{d}{2}, \frac{d}{2})$  in the trace sense.  $\square$

The following proposition is a standard diagonalization lemma, compare [4] Corollary 1.16. or [9] Lemma 7.2.

**Proposition 20.** *Let  $a_{k,j}$  be a doubly indexed sequence of real numbers,  $k, j \rightarrow \infty$ . Then there exists a subsequence  $k_j \rightarrow \infty$  such that*

$$\limsup_{j \rightarrow \infty} a_{k_j, j} = \limsup_{k \rightarrow \infty} \limsup_{j \rightarrow \infty} a_{k, j}.$$

**Acknowledgements.** Support by the EU research and training network MULTIMAT MRTN-CT-2004-505226 is gratefully acknowledged. This work is part of my PhD-thesis [28] supervised by Prof. S. Müller at the Max-Planck-Institute for Mathematics in the Sciences, Leipzig, whom I thank for his steady advice. I also wish to thank Prof. S. Conti for interesting discussions on the topic and for a lot of helpful comments on a preliminary version of this work.

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