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Cohomologies of unipotent harmonic bundles over noncompact curves

by

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1 Introduction

Let $\overline{S}$ be a compact Riemann surface (holomorphic curve) of genus g. Let $p_1, p_2, \cdots, p_s$ be $s > 0$ points on it; these points define a divisor, and we denote the open Riemann surface $\overline{S} \setminus \{p_1, \ldots, p_s\}$ by S. When $3g - 3 + s > 0$, it carries a complete hyperbolic metric of finite volume, the so-called Poincaré metric; the points $p_1, p_2, \cdots, p_s$ then become cusps at infinity. Even in the remaining cases, that is, for a once or twice punctured sphere, we can equip S with a metric that is hyperbolic in the vicinity of the cusp(s), and for our purposes, the behavior of the metric there is all what counts, and we call such a metric Poincaré-like. In any case, our metric on S is denoted by ω. Denote the inclusion map of S in $\overline{S}$ by j. Let $\rho : \pi_1(S) \to Gl(n, \mathbb{C})$ be a semisimple linear representation of $\pi_1(S)$ which is unipotent near the cusps (for the precise definition, cf. §2.4). Corresponding to such a representation ρ, one has a local system L_ρ over S and a ρ-equivariant harmonic map $h : S \to Gl(n, \mathbb{C})/U(n)$ with a certain special growth condition near the divisor, which is especially of finite energy (for details, see §2.4). For the present case of complex dimension 1, this is actually elementary; it also follows from the general result of [11]. We also remark that if not imposing any growth condition, ρ-equivariant harmonic maps are not unique in general, even of infinite energy (cf. [13]). In order to see explicitly the behavior at the cusps of the harmonic map h used in this note, we explicitly construct an initial map, which has the required asymptotic behavior, and show why it has finite energy. Then the existence of the harmonic map h and its behavior are obtained in a standard manner presently. It should be pointed out that the construction here is essentially the same as that in [11]; but the target manifolds of [11] are very general, so the construction there was getting very complicated. The idea of the explicit construction here will also be used in the case of higher dimension. The harmonic map h obtained above

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can be considered as a Hermitian metric on L_ρ—harmonic metric—so that we have a so-called harmonic bundle (L_ρ, h) [21]. Such a bundle carries interesting structures, e.g. a Higgs bundle structure (E, θ), where $\theta = \partial h$, and it has a log-singularity at the divisor.

The purpose of this note is to investigate various cohomologies of $\overline{S}$ with degenerating coefficients L_ρ (considered as a local system — a flat vector bundle, a Higgs bundle, or a $\mathcal{D}$-module, depending on the context): *the Čech cohomology of* $j_* L_\rho$ (note that in the higher dimensional case, one needs to consider the corresponding intersection cohomology [4]), *the* L^2*-cohomology, the* L^2*-Dolbeault cohomology, and the* L^2*-Higgs cohomology, and the relationships between them.* Here, L^2 is defined by using the Poincaré(-like) metric ω and the harmonic metric h. We want to generalize the results [24] valid for the case of variations of Hodge structures (VHS) to the case of harmonic bundles, as was suggested by Simpson [21]; in principle, in view of our assumption on the representations in question being unipotent and the growth property of the harmonic metric, the situation should be similar to the case of VHS.

Remark. We remark that as mentioned before, the harmonic metric used here satisfies a special growth condition (cf. §2.4); naturally one should ask what the situation is when one uses some other harmonic metrics, e.g. general *tame* harmonic metrics defined by Simpson [21].

This paper is meant to be a part of the general program of studying cohomologies with degenerating coefficients on quasi-projective varieties and their Kählerian generalizations. The general aim here is not restricted to the case of curves nor to the one of representations that are unipotent near the divisor. One of the purposes of this note therefore is to illuminate at this particular case where many of the analytic and geometric difficulties of the general case are not present what differences will appear when we consider unipotent harmonic bundles instead of VHSs. In the future, we wish to consider the case of higher dimensional quasi-projective varieties, but still assuming that the representations of π_1 are unipotent. Geometrically, the unipotent condition is very natural. It is well-known that a rational VHS is unipotent up to a lifting; more generally, if the representation ρ of π_1 is into $Sl(n, \mathbb{Z})$, then a variant of Borel's lemma (cf. e.g. [19], Lemma 4.5) implies that ρ is unipotent near the divisors after a lifting, assuming the existence of an ρ-equivariant pluriharmonic map. For the case of VHSs, the various cohomologies have been considered by various authors [1, 16, 17, 23, 14] and we should say that the question have been well understood by now.

After submitting a revised version of the paper, we were informed by the referee about related work by Sabbah[1]. In his paper, Sabbah is generalizing Saito's mixed Hodge modules to the case of twistor $\mathcal{D}$-module structures, and he wants to prove a decomposition theorem for the higher direct image. Thus, he has to treat some kind of Zucker-like result on curves in the case of polarizable regular twistor $\mathcal{D}$-modules instead of VHSs. As he remarks in Remark 6.2.6 of his paper, he does not follow the proof given by Zucker, namely, he does not prove the corresponding $\overline{\partial}$-Poincaré/Poincaré lemma near punctures. In contrast, in the present paper, we are generalizing Zucker's approach; technically, this means proving the corresponding $\overline{\partial}$-Poincaré/Poincaré lemma near punctures in the case of harmonic bundles, which is itself of interest in its own. So our argument here is a very direct one. We expect that the idea here is also useful for a direct treatment of the case of higher dimension.

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2 The geometry associated with representations of fundamental groups

2.1 The decomposition of a flat connection

In order to make this note an introduction into our general program, we describe some background material in this §§ and the next two §§. The knowledgable reader may skip these §§. Since the discussion below is local, we do not specify whether the domain manifolds X are compact or noncompact, unless stated otherwise. But when the question involves the existence of harmonic metrics (maps), such an assumption will be essential, as will be seen in the following.

Let X be a Riemannian manifold, V a $Gl(n,\mathbb{C})$ bundle on X with a flat connection D or, equivalently, a representation

$$\rho : \pi_1(X) \to Gl(n,\mathbb{C}).$$

A metric $\langle\cdot,\cdot\rangle$ on V is equivalent to a ρ-equivariant map

$$h : \tilde{X} \to Gl(n,\mathbb{C})/U(n) =: \tilde{Y},$$

[1]Claude Sabbah, *Polarizable twistor $\mathcal{D}$-modules*, arXiv: math.AG 0503038.

where h is defined by setting $\langle v, v\rangle = h_{ij}v^i v^j$, and $\tilde{X}$ is the universal covering of X. Afterwards, we use h to mean either a metric or a map, as will be clear from the context. When the map h is harmonic, we call the flat bundle V together with the corresponding metric a *harmonic bundle*, and the metric a *harmonic metric*. The simplest case is, of course, $n = 1$. Then V is a line bundle, and

$$Gl(1, \mathbb{C})/U(1) = \mathbb{C}^*/S^1 \simeq \mathbb{R}^+.$$

Using the isomorphism

$$\log : \mathbb{R}^+ \to \mathbb{R},$$

a metric can then be written locally as

$$h = e^\lambda, \text{ for } \lambda : \tilde{X} \to \mathbb{R},$$

and so $h^{-1}dh = d\lambda$.[2]

Given any metric h on V, the flat connection D will not preserve it in general, and so, we split D by means of the Cartan decomposition as

$$D = D_h + \Theta,$$

where D_h preserves h, i.e.,

$$d\langle v, v\rangle = \langle D_h v, v\rangle + \langle v, D_h v\rangle.$$

We then have $\Theta = 0$ iff D is unitary iff h is constant. The derivative dh measures the deviation of D from being unitary, i.e., $dh = \Theta$. Thus, the energy of the map h is defined by $\int \|\Theta\|^2$, and h is harmonic iff

$$D_h^*\Theta = 0. \tag{1}$$

For $n = 1$, reverting to our $h^{-1}dh$ notation, we have

$$D_h = d - h^{-1}dh, \text{ i.e. } \Theta = h^{-1}dh = d\lambda.$$

The harmonic map equation (1) becomes

$$0 = (d^* - h^{-1}dh)d\lambda = d^*d\lambda - d\lambda \wedge d\lambda = d^*d\lambda,$$

and so λ is a harmonic function on $\tilde{X}$. It should be pointed out that λ is not well defined as a function on X; $d\lambda$, however, is well defined from the above argument, and a harmonic 1-form when the metric h is harmonic.

[2] $h^{-1}dh$ is of course the derivative of the map $h : \tilde{X} \to \tilde{Y}$; usually, we should write dh, but the problem is that d has two meanings, namely on one hand, the exterior derivative d, leading to the notation $h^{-1}dh$, and on the other hand, the differential d of a map between Riemannian manifolds, suggesting to write dh. In the sequel, we use the notation $h^{-1}dh$ only for the case $n = 1$ and write dh else, except for the energy estimate of the initial map in §2.4; also, see the footnote there.

2.2 Variations of Hodge structures

Let us just consider the following geometric setting. Let Z and X be Kähler manifolds, with $\dim Z > \dim X$, $f : Z \to X$ a proper holomorphic map with smooth fibers. Canonically, one has a flat vector bundle H^k over X with the flat connection D and the fibre over $x \in X$ being the k-th cohomology $H^k(Z_x; \mathbb{C})$ of Z_x (cf. e.g. [5]). For each fibre $H^k(Z_x; \mathbb{C})$ over $x \in X$, we have the Hodge decomposition

$$H^k(Z_x; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Z_x). \tag{2}$$

This induces a filtration of $H^k(Z_x; \mathbb{C})$ (resp. H^k)

$$\begin{aligned} &0 \subset F_x^k \subset \cdots \subset F_x^{p+1} \subset F_x^p \subset \cdots \subset F_x^0 = H^k(Z_x), \\ &(\text{resp. } 0 \subset F^k \subset \cdots \subset F^{p+1} \subset F^p \subset \cdots \subset F^0 = H^k) \end{aligned}$$

with $F_x^p = \bigoplus_{i \geq p} H^{i,k-i}(Z_x)$. This filtration defines an element of a subdomain of a Grassmannian of flags. From this, one obtains the Griffiths period map

$$X \to \mathcal{D}/\Gamma \tag{3}$$

into the quotient $\mathcal{D}/\Gamma$ of the Griffiths period domain $\mathcal{D}$ by a certain discrete group Γ (ref. e.g. [5]). In particular, it should be pointed out that the period map can be equivalently explained as a metric on the bundle H^k; this idea will be used in a more general setting, i.e. that of harmonic bundles, as will be seen in the following.

The Griffiths period domain $\mathcal{D}$ is a homogeneous complex manifold, but generally not a Hermitian symmetric space. If $\mathcal{D}$ is not Hermitian symmetric, it is a Riemannian submersion over a (non-Hermitian) symmetric space of noncompact type under the canonical invariant metrics and the corresponding horizontal distribution is holomorphic but not integrable; more importantly, the image of the period map is tangential to this distribution. Furthermore, $\mathcal{D}$ has nonpositive curvature in the horizontal directions (for details, ref. [7], also [9]).

Decompose the flat connection D into the $(1,0)$-part d' and $(0,1)$-part d''. For the above filtration $\{F^p\}$ of H^k, Griffiths showed (cf. e.g. [5]) that F^p is a holomorphic subbundle of H^k under the holomorphic structure d'' and the following transversal property

$$d' : F^p \to \Omega^{1,0}(F^{p-1}).$$

Taking the quotient F^p/F^{p-1}, denoted by $E^{p,q}$, $(p+q=k)$, which is holomorphic under the induced holomorphic structure, d' then induces a map from

$E^{p,q}$ into $\Omega^{1,0}(E^{p-1,q+1})$, denoted by θ. Consequently, one gets a formal decomposition $D = \partial + \bar{\partial} + \theta + \bar{\theta}$ such that

$$\begin{aligned} \partial &: E^{p,q} \to \Omega^{1,0}(E^{p,q}), \\ \overline{\partial} &: E^{p,q} \to \Omega^{0,1}(E^{p,q}), \\ \theta &: E^{p,q} \to \Omega^{1,0}(E^{p-1,q+1}), \\ \overline{\theta} &: E^{p,q} \to \Omega^{0,1}(E^{p+1,q-1}). \end{aligned}$$

Since $D^2 = 0$, in particular, one has $(\overline{\partial} + \theta)^2 = 0$. So $E = \bigoplus_{p+q=k} E^{p,q}$ together with $\overline{\partial} + \theta$ is a special Higgs bundle (for its definition, see the next §§) satisfying

$$\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1(X).$$

Using the idea of the above geometric setting, one can also give the definition of an abstract (complex) variation of Hodge structures, and from such a variation one can formally get a Higgs bundle; since this is not directly related to our present purpose, we omit these; for details, one can refer to e.g. [22].

2.3 Harmonic bundles and Higgs bundles

A *Higgs bundle* (E, θ) over a complex manifold X consists of a holomorphic vector bundle E and a holomorphic morphism $\theta : E \to E \otimes \Omega^1(X)$ satisfying $\theta \wedge \theta = 0$. It was then Simpson's fundamental idea ([20]) to revert the construction in the previous §§by using the Hermitian-Yang-Mills equation, namely, under certain suitable geometric conditions, to construct a harmonic bundle from a Higgs bundle. The understanding of the relationship between harmonic bundles and Higgs bundles initially comes from Hitchin's important paper [8], which considered the above problem in the case of smooth curves.

From now on, we always assume X is Kähler. When X is compact, by means of a theorem by Narasimhan-Seshadri, Hitchin for curves and by Donaldson, Uhlenbeck-Yau, Simpson for higher dimensional varieties, for a stable Higgs bundle (E, θ), one can construct a Hermitian Yang-Mills connection (resp. metric) D_0 (resp. h), which furthermore satisfies that $D := D_0 + \theta + \overline{\theta}$ is flat if all $c_i(E) = 0$, where $\overline{\theta}$ is defined by setting

$$< \overline{\theta} u, v >_h = < u, \theta v >_h . \tag{4}$$

Such a metric h is then a harmonic metric on E when E is considered as a flat bundle with the flat connection D, i.e., a harmonic map from $\tilde{X}$ into the symmetric space $Gl(n, \mathbb{C})/U(n)$, $n = \text{Rank}E$.

Conversely, letting $\rho : \pi_1(X) \to Gl(n, \mathbb{C})$ be a semisimple representation, L_ρ the corresponding flat bundle, Corlette [2] then obtained a ρ-equivariant harmonic map

$$h : \tilde{X} \to Gl(n, \mathbb{C})/U(n);$$

as shown before, this is equivalently a metric on L_ρ so that L_ρ is a harmonic bundle. Furthermore, the harmonic metric h induces a Higgs bundle structure (E,θ) on L_ρ with $\theta = \partial h$. Note that $\theta\wedge\theta = 0$ follows from the pluriharmonicity of h as originally discovered in a somewhat different context by Jost-Yau [10]; on the other hand, since we will assume that $\dim X = 1$ in the present note, $\theta \wedge \theta = 0$ is automatic because θ is a $(1,0)$-form. It should however be pointed out that when X is noncompact Kähler and $\dim X > 1$, the pluriharmonicity of h is a subtle matter.

For sake of convenience later on, let us now make some formal derivations, which are again local, from a harmonic bundle to the corresponding Higgs bundle, provided that h be pluriharmonic (such an assumption is automatic if $\dim X = 1$). As before, decompose our flat connection $D = d' + d''$ into operators of type $(1,0)$ and $(0,1)$ respectively. Let δ' and δ'' be the unique operators of type $(1,0)$ and $(0,1)$ such that the connections $\delta' + d''$ and $\delta'' + d'$ preserve the metric h. Set

$$\begin{aligned} \partial &= (d' + \delta')/2, \quad \overline{\partial} = (d'' + \delta'')/2, \\ \theta &= (d' - \delta')/2, \quad \overline{\theta} = (d'' - \delta'')/2. \end{aligned}$$

It is clear [21] that the connection $\mu = \partial + \overline{\partial}$ also preserves the metric, $\theta = \partial h$ (more precisely, $\theta = h^{-1}\partial h$, see the footnotes 1 and 2), and $\overline{\theta}$ is the conjugate adjoint of θ w.r.t. h (cf. (4)). Also, one has $(\overline{\partial} + \theta)^2 = 0$. Thus we obtain a structure of Higgs bundle on the bundle L_ρ with $\overline{\partial}$ being the holomorphic structure and θ the Higgs field; later on we denote it by $(E, \overline{\partial} + \theta)$. From the above argument, we also have $D = \partial + \overline{\partial} + \theta + \overline{\theta}$, as have seen in the case of VHS.

Correspondingly, one also has the *Kähler identities*, which was observed by Deligne in the case of VHS. Set

$$D''_h = \overline{\partial} + \theta, \quad D'_h = \partial + \overline{\theta}, \quad D^c_h = D''_h - D'_h.$$

Note that $D = D' + D''$ and $D''_h = (D + D^c_h)/2$. Let Λ be the adjoint of the operation of wedging with the Kähler form ω. Then one has the first order Kähler identities

$$\begin{aligned} (D'_h)^* &= \sqrt{-1}[\Lambda, D''_h], \quad (D''_h)^* = -\sqrt{-1}[\Lambda, D'_h] \\ (D^c_h)^* &= -\sqrt{-1}[\Lambda, D], \quad (D)^* = \sqrt{-1}[\Lambda, D^c_h], \end{aligned}$$

where * represents the adjoint of the respective operator. Set $\Delta = DD^* + D^*D$ and $\Delta'' = D''_h(D''_h)^* + (D''_h)^* D''_h$. Using the above first order identities, one then has

$$\Delta = 2\Delta''.$$

This shows that spaces of Δ-harmonic forms valued in the local system L_ρ can be identified with that of Δ''-harmonic forms valued in the Higgs bundle E.

2.4 Harmonic metrics in the case of noncompact curves

When X is no longer compact, the local geometry works as before, but there arise difficulties with the existence of the harmonic map (metric) h and its pluriharmonicity (in the higher dimensional case). On the other hand, the geometry of the related bundles near a compactifying divisor (provided that the harmonic metric exist) leads to very interesting structures which are, in fact, our main interest. As in the Introduction, we from now on assume that X is a noncompact curve, i.e. a compact Riemannian surface deleting finitely many points, and change the symbol X into S, the compactification of which is denoted by $\overline{S}$; $S = \overline{S} \setminus \{p_1, \cdots, p_s\}$ and $j : S \to \overline{S}$ is the inclusion map. Under such a case together with the following assumption about the representation ρ of $\pi_1(S)$, one can easily get a ρ-equivariant harmonic map of finite enegy and its behavior near the punctures $\{p_1, \cdots, p_s\}$; the purpose of this §§ will then be to treat these problems, which will be essential for the later development. It is worth pointing out that in [11], the authors considered these problems without the restriction of dimension. Since there the target manifolds are very general, the construction of the initial metrics are getting very complicated. In our present case, since the representation is linear, we will give an explicit construction though the basic idea is the same as [11]; more importantly, we can see explicitly why the initial map is of finite energy, and hence the analysis of [11] is applicable. The idea of the construction for the initial map will be used in the most general case, where representations need not be unipotent and also the energy of maps need not be finite; here, we should point out that the construction in [12] is also a very special one (cf. [13]).

Throughout this note, we take the Poincaré-like metric on the base manifolds, namely, on punctured disks Δ^* near the divisor (puncture) the metric is isometric to

$$\frac{dt \wedge d\bar{t}}{|t|^2(\log|t|)^2}, \tag{5}$$

Where t is the Euclidean complex coordinate of Δ^*. Such a metric is complete, of finite volume and bounded geometry.

Let $\rho : \pi_1(S) \to GL(n, \mathbb{C})$ be a semisimple linear representation, and restrict ρ to a neighborhood of p_i, say a small punctured disk Δ^* around p_i, which we call the *boundary representation* of ρ, denoted by ρ_i. Throughout this note, we assume that *all the boundary representations of ρ are unipotent.* A result of Borel tells us that this is the case for VHS up to a suitable lifting (cf. e.g. [19], Lemma 4.5). Such an assumption means that if denoting the image under ρ_i of the generator of $\pi_1(\Delta^*)$ by the matrix γ, then γ, under a suitable basis, can be represented by a upper-triangle matrix with all the diagonal entries being 1. As usual, γ is called the *monodromy transformation* of L_ρ at p_i under the standard flat connection D. Take the logarithm of γ, denoted by N, which is also upper-triangle and all the diagonal entries of which are 0.

We now proceed to construct an initial metric on L_ρ, equivalently, an initial ρ-equivariant map from $\tilde{S}$ to $Gl(n,\mathbb{C})/U(n)$. To this end, let us first give some preliminary. Let $\mathcal{P}_n$ be the set of all positive definite hermitian symmetric matrices of order n. $Gl(n,\mathbb{C})$ acts transitively on $\mathcal{P}_n$ by

$$g \circ H =: gH^t\bar{g}, \;\; H \in \mathcal{P}_n, g \in GL(n,\mathbb{C}).$$

Obviously, the action has the isotropic subgroup $U(n)$ at the identity I_n. Thus $\mathcal{P}_n$ can be identified with the coset space $Gl(n,\mathbb{C})/U(n)$, and can be uniquely endowed an invariant metric[3] up to some constants. In particular, under such a metric, the geodesics through the identity I_n are of the form $\exp(tA)$, $t \in \mathbb{R}$, A being a hermitian matrix.

Let the Jordan normal form of N have p Jordan blocks, denoted by $N_j, 1 \le j \le p$. By the Jacobson-Morosov theorem, one can expand each block N_j into an $\mathfrak{sl}_2$-triple $\{Y_j, N_j, N_j^-\}$, i.e., $[Y_j, N_j] = 2N_j, [Y_j, N_j^-] = -2N_j^-$ and $[N_j, N_j^-] = Y_j$; a theorem of Kostant tells us that such a triple, up to conjugations, is unique (cf. e.g. [15]). Take the Euclidean coordinate t on Δ^* with $t(p_i) = 0$ and $t = re^{\sqrt{-1}x}$; also take the universal covering

$$H_\alpha = \{z = x + \sqrt{-1}y \mid x \in \mathbb{R}, \; y > -\log \alpha\}$$

of Δ^* with $y = -\log r$, for a positive number $\alpha < 1$. Corresponding to a (once and for all) fixed flat sections basis of L_ρ, we construct the required Hermitian metric of L_ρ on Δ^* as

$$h_i = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}, \tag{6}$$

where $M_j = \exp(xN_j) \circ \exp((\frac{1}{2}\log|\log r|)Y_j)$. In the following, h_i is considered as both a metric and the above matrix under the fixed basis. Clearly, h_i, as a map from H_α to $\mathcal{P}_n$, is ρ_i-equivariant when changing $\log r$ into y. Geometrically, this is a geodesic ρ_i-equivariant embedding of Δ^* into $\mathcal{P}_n$ which maps the puncture to the infinity of $\mathcal{P}_n$, as can be explicitly seen when one embeds geodesically the upper-half plane into the real hyperbolic 3-space $\mathbf{H}^3$ by using the matrix model of $\mathbf{H}^3$ (for more details, cf. [13]). We also remark that each h_i is actually harmonic. Now one can easily extend the metrics $\{h_i\}$ to

[3] In terms of matrices, such an invariant metric can be defined as follows. At the identity I_n, the tangent elements just are hermitian matrices; let A, B be such matrices, then the Riemannian inner product $< A, B >_{\mathcal{P}_n}$ is defined by $tr(AB)$. In general, let $H \in \mathcal{P}_n$, A, B two tangent elements at H, then the Riemannian inner product $< A, B >_{\mathcal{P}_n}$ is defined by $tr(H^{-1}AH^{-1}B)$.

a global metric on L_ρ, denoted by h_0. Naturally, the corresponding map is ρ-equivariant.

We now want to show that each h_i, and hence h_0, is of finite energy. For simplicity, we may assume here that the Jordan normal form of N has only one Jordan block. Let $\{Y, N, N^-\}$ be the corresponding $\mathfrak{sl}_2$-triple. The semisimple element Y can actually be described as follows. Canonically, $\mathbb{C}^n$ has a filtration

$$0 \subset W_{-(n-1)} \subset W_{-(n-3)} \subset \cdots \subset W_{n-3} \subset W_{n-1} = \mathbb{C}^n, \tag{7}$$

satisfying that $N(W_i) \subset W_{i-2}$, Y preserves each W_i, and all the quotients W_i/W_{i-2} are 1-dimensional. Then the (induced) action of Y on W_i/W_{i-2} is multiplying by i. Actually, one can also choose a basis $\{e_{-(n-1)}, e_{-(n-3)}, \cdots, e_{n-3}, e_{n-1}\}$ of $\mathbb{C}^n$, which is compatible with the above filtration (i.e., $Ne_j = e_{j-2}$ and $\{e_j\}_{j\le i}$ generates W_i) and satisfies $Ye_i = ie_i$. $\exp((\frac{1}{2}\log|\log r|)Y)$, under the above basis, can then be written as

$$\begin{pmatrix} |\log r|^{\frac{-(n-1)}{2}} & 0 & \cdots & 0 & 0 \\ 0 & |\log r|^{\frac{-(n-3)}{2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |\log r|^{\frac{n-3}{2}} & 0 \\ 0 & 0 & \cdots & 0 & |\log r|^{\frac{n-1}{2}} \end{pmatrix}. \tag{8}$$

Then, using the invariant metric of $\mathcal{P}_n$, a simple computation shows that the energy of h_i satisfies

$$E(h_i) = \int_{\Delta^*} |h_i^{-1} dh_i| * 1 \le C \int_0^\alpha |\log r|^{-2} r^{-1} dr < \infty.$$

The above argument gives an important by-product, namely, the norm estimates near the punctures of flat sections of L_ρ under the metric h_0. We continue to restrict ourself to a small punctured disk Δ^* near p_i. As before, the fiber $(L_\rho)_x$ over $x \in \Delta^*$ canonically has a weight filtration $\{W_l\}_{l=-k}^k$ (k is the weight of N) arising from N and satisfying $N(W_l) \subset W_{l-2}$; this filtration is moreover invariant w.r.t. the flat connection D of L_ρ and hence determines a filtration of L_ρ by some local subsystems, denoted by $\mathbf{W}_l, -k \le l \le k$. We remark that $\{W_l\}_{l=-k}^k$ can be decomposed into the direct sum of some sub-filtrations, each of which corresponds to a unique Jordan block of N and is of the form as in (7), if the number of the Jordan blocks of N is > 1. By the construction of h_i and (8), one now has that a flat section v of $\mathbf{W}_l$, if not lying in $\mathbf{W}_{l-1}$, has the following norm estimate[4]

$$\|v\|_{h_0}^2 \sim |\log r|^l, \tag{9}$$

[4]Here and afterwards, we use the notation $\sim$ to mean "is within a bounded multiple of"

on any ray from the puncture of Δ^*.

As mentioned before, since the initial map h_0 is of finite energy, the analysis of [11] works; especially, one has the following

Proposition 1 *Let $\rho : \pi_1(S) \to Gl(n, \mathbb{C})$ be a semisimple representation all the boundary representations of which are unipotent, h_0 be the $\rho-$equivariant map (initial metric) constructed above. Then there exists a ρ-equivariant harmonic map (harmonic metric) of finite energy*

$$h : \tilde{S} \to Gl(n, \mathbb{C})/U(n),$$

which has the same asymptotic behavior as h_0 near the punctures, where $\tilde{S}$ is the universal covering of S; moreover the norm of the derivative dh of h satisfies, when going down to S and measured near the divisor with respect to the Poincaré-like metric (5) and the standard Riemannian symmetric metric on $Gl(n, \mathbb{C})/U(n)$,

$$|dh|^2 \leq C|\log r|^2$$

for some constant $C > 0$, where r is the radial Euclidean coordinate of Δ^*.

A harmonic bundle over S is said to be *tame* (cf. [21]) if, on any ray from a puncture of S, the norm under the harmonic metric of any flat section grows at most polynomially in r. Let (L_ρ, h) be the harmonic bundle with the harmonic metric h in the Proposition 1 and D the flat connection. Since the harmonic metric h has the same behavior as h_0 near the punctures, especially, the harmonic bundle is tame, our discussions in the remaining part of this subsection therefore lie in the framework of Simpson [21](except for the choice of the metric of the base manifold S) and are even simpler, since, as seen from the above construction, we have no the factors like r^b ($b \in \mathbb{R}\setminus\{0\}$) in the norm estimates at the punctures; indeed, it is such a factor that produces infinite energy of maps. On the other hand, the behavior of h also shows that the flat sections of L_ρ have the same estimates as in (9) under the harmonic metric h. For convenience, we here write down again that for a flat section v of $\mathbf{W}_l$, if not lying in $\mathbf{W}_{l-1}$, one has

$$\|v\|_h^2 \sim |\log r|^l, \tag{10}$$

on any ray from the puncture of Δ^*. Furthermore, from the above construction of h_0, we can choose a flat basis of L_ρ, $\{e_1, e_2, \cdots, e_n\}$, which is compatible with the Jordan blocks of N and the corresponding filtrations (in particular $Ne_j \in \{e_1, e_2, \cdots, e_n\}$), and satisfies that if $e_j \in \mathbf{W}_l - \mathbf{W}_{l-1}$, $\|e_j|_{h_0}^2 = |\log r|^l$ and that $< e_j, e_k >_{h_0} = 0, j \neq k$. Thus, under the harmonic metric h, we have that if $e_j \in \mathbf{W}_l - \mathbf{W}_{l-1}$, $\|e_j\|_h^2 \sim |\log r|^l$, and that as $r << 1$,

$$< e_j, e_k >_h \leq \epsilon(\|e_j\|_h \cdot \|e_k\|_h), \; j \neq k, \tag{11}$$

for some sufficiently small ϵ.

As explained in §2.3, from the harmonic bundle (L_ρ, h), one can derive a structure of Higgs bundle $(E, \overline{\partial}+\theta)$ on L_ρ with $\theta = \partial h$. Again since $(E, \overline{\partial}+\theta)$ comes from a tame harmonic bundle, Theorem 1 of [21] works; in particular, the curvature R_μ of the connection $\mu = \partial + \overline{\partial}$ is bounded under the Poincaré-like metric (5) and the harmonic metric h, and the Higgs field θ is of the log-singularity. If using the Euclidean metric of Δ^* as in [21], the estimate of R_μ is read as

$$|R_\mu| \leq \frac{C}{r^2(\log r)^2}. \tag{12}$$

Therefore, by the theory of Griffiths-Cornalba [3] (also cf. [20], §10), one can algebraically extend E across the punctures, denoted by j_*E as usual; in particular, j_*E is coherent. Using the log-singularity of θ, one also has, at p_i,

$$\theta : j_*E \to j_*E \otimes \Omega^1_{\overline{S}}(\log p_i), \tag{13}$$

where $\Omega^1_{\overline{S}}(\log p_i)$ is the sheaf of *logarithmic differentials* at p_i, i.e. that generated by $\frac{dt}{t}$. We call $(j_*E, \overline{\partial}+\theta)$ a *logarithmic Higgs bundle*.

Now, we want to give the norm estimates near the punctures of meromorphic sections of the logarithmic Higgs bundle $(j_*E, \overline{\partial}+\theta)$ under the harmonic metric h, by using the theory of harmonic bundles of Simpson over noncompact curves [21] together with the above norm estimates (10) for flat sections of (L_ρ, h). To this end, we need some preliminary (also cf. §3 of [21]). A *filtered vector bundle* over S is defined as a locally free sheaf V together with filtrations $\{V_\alpha\}_{\alpha\in\mathbb{R}}$ of $(j_*V)_{p_i}$ at the punctures, satisfying that each V_α is coherent, $E_\alpha \subset E_\beta$ whenever $\alpha \geq \beta$, $E_{\alpha-\epsilon} = E_\alpha$ for sufficiently $\epsilon > 0$ (left continuity), and $E_{\alpha+1} = tE_\alpha$. For a holomorphic vector bundle $(E, \overline{\partial})$ with a Hermitian metric K over S, one can construct a filtration $\{E_\alpha\}$ of $(j_*E)_{p_i}$ for each puncture p_i as follows. The germs of sections of E_α at p_i are the holomorphic sections e of E in a small punctured disk around p_i, satisfying $\|e\|_K \leq Cr^{\alpha-\epsilon}$ for every $\epsilon > 0$. Then, a argument of Simpson (cf. [20], §10; also [21], Proposition 3.1), by using the theory of Griffiths-Cornalba [3], tells us that if the curvature of the Hermitian connection $\partial + \overline{\partial}$ of E w.r.t. K satisfies the same estimate as in (12), the above construction forms a filtered vector bundle; in particular, each E_α is coherent.

Due to the curvature estimate of the connection $\mu = \partial + \overline{\partial}$ in (12), one can apply the above construction to our Higgs bundle $(E, \overline{\partial}+\theta)$ together with the harmonic metric h. Thus, we obtain the corresponding filtered vector bundle $(E, \{E^i_\alpha\}_{\alpha\in\mathbb{R}}, i = 1, 2, \cdots, s)$, satisfying

$$\theta : E^i_\alpha \to E^i_\alpha \otimes \Omega^1_{\overline{S}}(\log p_i), \tag{14}$$

where by $\{E^i_\alpha\}_{\alpha\in\mathbb{R}}$ we mean the filtration of $(j_*E)_{p_i}$. Furthermore, we can see that these filtrations actually satisfy

$$E^i_\alpha = E^i_1, \quad \text{for } 0 < \alpha < 1, \tag{15}$$

and hence the filtrations are $\{E^i_q\}_{q\in\mathbb{Z}}$. This is in fact a direct consequence of the table in p.720 of [21] together with the assumption of the representation ρ being unipotent (which implies that the eigenvalues of monodromy transformations of L_ρ are always 1).

Take the quotients $\mathrm{Gr}^i_q(E) = E^i_q/E^i_{q+1}$, $q \in \mathbb{Z}$. We then define the *residue* $\mathrm{Res}_{p_i}E$ of j_*E at the puncture p_i as the direct sum $\oplus_{q\in\mathbb{Z}}\mathrm{Gr}^i_q(E)$. On the other hand, by means of (14), from θ one has a induced homomorphism

$$\mathrm{Gr}^i_q(\theta) : \mathrm{Gr}^i_q(E) \to \mathrm{Gr}^i_q(E) \bigotimes \Omega^1_{\bar{S}}(\log p_i). \tag{16}$$

Since $E^i_{q+1} = tE^i_q$, one can in fact identify all $\mathrm{Gr}^i_q(E)$ (resp. $\mathrm{Gr}^i_q(\theta)$), $q \neq 0$, with $\mathrm{Gr}^i_0(E)$ (resp. $\mathrm{Gr}^i_0(\theta)$). Take the residue of $\mathrm{Gr}^i_0(\theta)$ at puncture p_i on $\mathrm{Gr}^i_0(E)$, denoted by N'. Again, from the table of [21], we have that N' is nilpotent, i.e. all the eigenvalues of N' are 0. Moreover, an argument of Simpson (cf. [21], pp719-721; also, §5,6,7) tells us that N' can be identified with N; equivalently, the corresponding weight filtrations on the corresponding residues can be identified with (for the definition of the residue of L_ρ at the punctures, cf. [21], p719). We just write N' as N afterwards.

Using this identification, we can now give the following precise description of the norm estimates near the punctures of meromorphic sections of j_*E under the harmonic metric h. From the above discussion, we see that one needs only to consider those sections which lie in E^i_0 and the projections of which on $\mathrm{Gr}^i_0(E)$ do not vanish. Let us first assume that the residue N of $\mathrm{Gr}^i_0(\theta)$ on $\mathrm{Gr}^i_0(E)$ has only one Jordan block. One can then choose some holomorphic sections $\{e_{-(n-1)}, e_{-(n-3)}, \cdots, e_{n-3}, e_{n-1}\}$ of E^i_0, the projections of which on $\mathrm{Gr}^i_0(E)$, still denoted by $\{e_{-(n-1)}, e_{-(n-3)}, \cdots, e_{n-3}, e_{n-1}\}$, form a basis of $\mathrm{Gr}^i_0(E)$; moreover, as a basis of $\mathrm{Gr}^i_0(E)$, it is compatible with the weight filtration $W_{-(n-1)} \subset W_{-(n-3)} \subset \cdots \subset W_{n-3} \subset W_{n-1}$ of N on $\mathrm{Gr}^i_0(E)$ (i.e., $Ne_j = e_{j-2}$ and $\{e_j\}_{j\leq l}$ generates W_l). Then, one has, for e_l,

$$\|e_l\|^2_h \sim |\log r|^l. \tag{17}$$

Furthermore, the cross-terms can be controlled very well, i.e., as $r << 1$,

$$< e_j, e_k >_h \leq \epsilon(\|e_j\|_h \cdot \|e_k\|_h), \quad j \neq k, \tag{18}$$

for some sufficiently small ϵ, as can be obtained from the control (11) of the corresponding cross-terms for L_ρ by means of the above identification and the same asymptotic behavior of h as h_0. When N has at least two Jordan blocks, we can do the same discussion as before (11).

In general, let $\{W_l\}_{l=-k}^{k}$ be the weight filtration of N on $\mathrm{Gr}_0^i(E)$, e a holomorphic section of E_0^i, the projections of which on $\mathrm{Gr}_0^i(E)$ lies in W_l but not in W_{l-1}. Then one has

$$\|e\|_h^2 \leq C|\log r|^l. \tag{19}$$

Here, we use $\leq$ instead of $\sim$, since a holomorphic section can have zero-points in general.

3 The L^2-cohomology and the L^2-Higgs cohomology

In this section, we continue to assume that the representation ρ is semisimple and unipotent at infinity, i.e. monodromy transformations of L_ρ around the punctures are unipotent. Therefore we have the corresponding harmonic bundle (L_ρ, h) and the Higgs bundle (E, θ) with $\theta = \partial h$ of the log-singularity at the punctures (Proposition 1); we also have various norm estimates: (10), (17) (19) at the punctures w.r.t. the metric h for flat sections of L_ρ or holomorphic sections of E; the most important thing is that one can identify the logarithmic monodromy transformations N with the residues of θ at the punctures [21]. The purpose of this section is then consider the L^2-cohomologies with coefficients in L_ρ or E and their relationship. The problem in the case of VHS was considered by S. Zucker [24].

3.1 The L^2-cohomology: The L^2-Poincaré Lemma

As before, denoting the inclusion map of S in $\overline{S}$ by j, one has the direct image sheaf j_*L_ρ of the local system L_ρ on $\overline{S}$ and then the Čech cohomology $H^*(\overline{S}, j_*L_\rho)$. On the other hand, using the Poincaré-like metric ω (5) on S and the harmonic metric h on L_ρ, one can define a complex $\{\mathcal{A}_{(2)}^{\cdot}(L_\rho), D\}$ of sheaves over $\overline{S}$ as follows. Let U be an open subset of $\overline{S}$. Then $\mathcal{A}_{(2)}^i(L_\rho)(U)$ is defined as the set of

L_ρ-valued i-forms η on $U \cap S$, with measurable coefficients and measurable exterior derivative $D\eta$, such that η and $D\eta$ have finite L^2 norm on $K \cap S$, for any compact subset K of U, where D is the canonical flat connection of L_ρ.

Since the sheaves $\mathcal{A}_{(2)}^i(L_\rho)$ can regarded as a module over the sheaf of germs of local complex valued smooth functions on $\overline{S}$, using the partition of unit on $\overline{S}$, it is easy to show that these sheaves are fine. So the cohomology of the complex of global sections computes the hypercohomology of $\{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}$ (cf. e.g. [6]), namely

$$\mathbb{H}^*(\overline{S}, \{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}) \cong H^*(\{\Gamma(\mathcal{A}^{\cdot}_{(2)}(L_\rho)), D\}).$$

We call this cohomology the L^2-*cohomology* of $\overline{S}$ with values in L_ρ, denoted by $H^*_{(2)}(\overline{S}, L_\rho)$. The purpose of this subsection is then to establish the following identification

Theorem 1 *There exists a natural identification*

$$H^*(\overline{S}, j_*L_\rho) \cong H^*_{(2)}(\overline{S}, L_\rho).$$

Remarks. If L_ρ comes from a variation of Hodge structure (VHS), the identification was proved by S. Zucker [24]. In the higher dimensional case, instead of the Čech cohomology, one needs to consider the intersection cohomology [4]; the identification in the case of VHS was proved by Cattani-Kaplan-Schmid [1] and Kashiwara-Kawai [16] independently. In general, one has the following

Conjecture 1 *There exists a natural identification*

$$H^*_{int}(\overline{S}, j_*L_\rho) \cong H^*_{(2)}(\overline{S}, L_\rho).$$

Canonically, the proof of Theorem 1 is reduced to prove

Theorem 2 *(The L^2-Poincaré lemma) The complex $\{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}$ is a resolution of j_*L_ρ. This is equivalent to saying that*
1) $j_*L_\rho = \{\eta \in \mathcal{A}^0_{(2)}(L_\rho) \mid D\eta = 0\}$;
2) the differential D satisfies the Poincaré lemma, i.e., if an i-form η in $\mathcal{A}^i_{(2)}(L_\rho)$ is D-closed, then there exists an $i-1$-form $\sigma \in \mathcal{A}^{i-1}_{(2)}(L_\rho)$ satisfying $D\sigma = \eta$, for $i = 1, 2$.

In order to prove the above theorem, we need the following lemma, which contains the basic analysis needed in the proof of Theorem 2.

Lemma 1 *Let V be a constant one dimensional local system over Δ^*, with generator v, and assume that the corresponding line bundle has a Hermitian metric with $\| v \|^2 \sim |\log r|^k$, where r is the Euclidean radius. Then the coho-*

mology sheaves for $\mathcal{A}^{\cdot}_{(2)}(V)$ have stalks at the origin,

$$\mathcal{H}^0\big(\mathcal{A}^{\cdot}_{(2)}(V)\big) = \begin{cases} V & \text{if } k \leq 0 \\ 0 & \text{if } k > 0, \end{cases}$$

$$\mathcal{H}^1\big(\mathcal{A}^{\cdot}_{(2)}(V)\big) = \begin{cases} \frac{dt}{t} \otimes V & \text{if } k \leq -2 \\ 0 & \text{if } k \geq -1, k \neq 1 \\ \mathcal{M}_1 dr \otimes v & \text{if } k = 1, \end{cases}$$

$$\mathcal{H}^2\big(\mathcal{A}^{\cdot}_{(2)}(V)\big) = \begin{cases} 0 & \text{if } k \neq -1 \\ \mathcal{M}_1 dr \wedge \frac{dt}{t} \otimes v & \text{if } k = -1, \end{cases}$$

where $\mathcal{M}_1$ is defined as

$$\frac{\{\textit{measurable functions } f : \int_0^A |f(r)|^2 |\log r| (rdr) < \infty \textit{ for some } A < 1\}}{\{f : f = u' \textit{ weakly with } \int_0^A |u(r)|^2 |\log r|^{-1} (r^{-1}dr) < \infty \textit{ for some } A < 1\}}.$$

Proof. cf. [24], Proposition 6.6.

Proof of Theorem 2. On S, the exactness is standard. So the trouble comes from the punctures $p_1, \cdots, p_s$, and since we are working with sheaves, from now on we just localize the problem to a small punctured disk $\Delta^* \subset \Delta$ around the puncture p_i.

The proof of 1). This is equivalent to showing that an L^2 flat section should be a section of $j_* L_\rho$—an invariant section of γ, which equivalently lies in the kernel of N. To this end, denote the image of the generator of $\pi_1(\Delta^*)$ under the representation ρ by γ, which, by the assumption, is unipotent, $\log \gamma$ by N, which is nilpotent. For a (multi-valued) flat section v of L_ρ, setting

$$\tilde{v} = \exp(\frac{1}{2\pi\sqrt{-1}} N \log t) v, \tag{20}$$

which is single-valued and d''-holomorphic ($D = d' + d''$, the $(1,0)$ and $(0,1)$-part respectively), one then has the canonical extension $\overline{L}_\rho$ of L_ρ to $\overline{S}$ ($= \Delta$ locally) when L_ρ is considered as a d''-holomorphic bundle: sections of $\overline{L}_\rho$ at the origin are generated by sections of the form $\tilde{v}$. On the other hand, as seen in §2.4, since N is nilpotent, each fiber of L_ρ canonically has a weight filtration $\{W_l\}_{l=-k}^k$ (k is the weight of N) which is D-invariant and which therefore determines a filtration of L_ρ by some local subsystems, denoted by $\mathbf{W}_l$ and the corresponding extension by $\overline{\mathbf{W}}_l$.

Let v be a flat section of $\mathbf{W}_l$, but not one of $\mathbf{W}_{l-1}$, briefly denoted by $v \in \mathbf{W}_l - \mathbf{W}_{l-1}$, so that $Nv \in \mathbf{W}_{l-2} - \mathbf{W}_{l-3}$ if Nv does not vanish. By the norm estimates (10) of flat sections, we have

$$\|v\|_h^2 \sim |\log r|^l \quad \text{and} \quad \|Nv\|_h^2 \sim |\log r|^{l-2}.$$

Since N is nilpotent of weight k, we have

$$\begin{aligned}\tilde{v} &= \exp(\frac{1}{2\pi\sqrt{-1}}N\log t)v \\ &= v + (\frac{\log t}{2\pi\sqrt{-1}})Nv + \frac{1}{2!}(\frac{\log t}{2\pi\sqrt{-1}})^2N^2v + \cdots + \frac{1}{k!}(\frac{\log t}{2\pi\sqrt{-1}})^kN^kv;\end{aligned}$$

so, we have the following estimate

$$\|\tilde{v}\|_h^2 \sim |\log r|^l. \tag{21}$$

We now claim that a d''-holomorphic section of $\overline{L}_\rho$ on Δ is L^2 iff it lies in $\overline{\mathbf{W}}_0 + t\overline{L}_\rho$. This can be proved as follows, by using the above estimate (21). We first show when the sections are L^2 of the form $\sigma = t^m\tilde{v}$ for $v \in \mathbf{W}_l - \mathbf{W}_{l-1}$, $m \in \mathbb{Z}$. By the above estimate, the convergence or divergence of the L^2-norm of σ is equivalent to that of the following integral $(0 < A < 1)$

$$\int_0^{2\pi}\int_0^A r^{2m}|\log r|^l \frac{rdrd\theta}{r^2\log^2 r}.$$

Clearly, the integral is convergent precisely when $l \leq 0$ and $m = 0$, or $m \geq 1$; this implies that sections of $\overline{\mathbf{W}}_0 + t\overline{L}_\rho$ are L^2.

The remained is to show that any section σ of $\overline{\mathbf{W}}_l - \overline{\mathbf{W}}_0$, $l \geq 1$ is not L^2 if $\sigma \notin t\overline{L}_\rho$. To this end, we first assume that N has only one Jordan block. Then a section σ of $\overline{\mathbf{W}}_l - \overline{\mathbf{W}}_0$ can be written as

$$\sigma = f(t)\tilde{v} + \tilde{v}_{l-2},$$

where $f(t)$ is a holomorphic function on Δ, $v \in \mathbf{W}_l - \mathbf{W}_{l-2}$, $\tilde{v}$ is defined by (20), and $\tilde{v}_{l-2} \in \overline{\mathbf{W}}_{l-2}$. Since we assume that $\sigma \notin t\overline{L}_\rho$, without loss of generality, we can assume $f(0) \neq 0$; since if not, $f(t)\tilde{v} \in t\overline{L}_\rho$ and hence is L^2; thus the problem is reduced to consider if $\tilde{v}_{l-2}$ is L^2. Now we want to show that $|\log r|^{-l+1}\|f(t)\tilde{v} + \tilde{v}_{l-2}\|_h^2$ is not integrable on Δ under the Poincaré-like metric, and hence σ is not L^2. This is easily obtained by using the estimate (21) and noting

$$\|f(t)\tilde{v} + \tilde{v}_{l-2}\|_h^2 \geq \|f(t)\tilde{v}\|_h^2 - \|\tilde{v}_{l-2}\|_h^2 \sim |\log r|^l - |\log r|^{l-2}.$$

In general, if N has at least two Jordan blocks, σ is of the form

$$\sigma = \sum_j f_j(t)\tilde{v}_j + \tilde{v}'_{l-1},$$

where each $\tilde{v}_j$ belongs to $\overline{\mathbf{W}}_l - \overline{\mathbf{W}}_{l-1}$ and corresponds to a unique Jordan block of N as in the above case of one Jordan block, f_j's are holomorphic on Δ, and

$\tilde{v}'_{l-1} \in \overline{\mathbf{W}}_{l-1}$. Again, we can assume that all $f_j(0) \neq 0$. By using the same argument as above, we only need to show $\| \sum_j f_j(t)\tilde{v}_j\|^2_h \sim |\log r|^l$. To this end, it is sufficient to show that

$$\| \sum_j f_j(t)\tilde{v}_j\|^2_h \sim \sum_j \|f_j(t)\tilde{v}_j\|^2_h;$$

this can be obtained by means of the construction of the initial metric h_0 (cf. (6) and (8)) and that h has the same asymptotic behavior as h_0; more precisely, since $< f_j(t)\tilde{v}_j, f_k(t)\tilde{v}_k >_{h_0} = 0, j \neq k$, the cross-terms $| < f_j(t)\tilde{v}_j, f_k(t)\tilde{v}_k >_h |$ can be controlled by

$$\epsilon\|f_j(t)\tilde{v}_j\|_h\|f_k(t)\tilde{v}_k\|_h \leq \frac{\epsilon}{2}(\|f_j(t)\tilde{v}_j\|^2_h + \|f_k(t)\tilde{v}_k\|^2_h)$$

for some sufficiently small ϵ as $|t| << 1$. (A similar discussion will also be used in the following proof of 2), where, for simplicity of discussion, we want to reduce the discussion to the case that N has only one Jordan block.) Thus, we finish the proof of the claim above.

In order to finish the proof of 1), we next need to show that d'-closed sections of $\overline{\mathbf{W}}_0 + t\overline{L}_\rho$ are generated by sections of the form $\tilde{v}$ satisfying $Nv = 0$, which is just the first part of Proposition 4.1 of [24], where ∇ is just our d'. We remark that although L_ρ of Proposition 4.1 in [24] comes from a VHS, the discussion still works in the present setting. Thus, we finish the proof of 1).

The proof of 2). In order to make a reduction of the problem to the case that N has only one block, we first do some discussion for L^2 i-forms valued in L_ρ in $\mathcal{A}^i_{(2)}(L_\rho)$ (for the case of $i = 0$, this in fact have done in the proof of 1)). As mentioned before, we restrict ourself to a small punctured disk. Let N have the Jordan blocks $N_1, N_2, \cdots, N_p, p > 1$. Clearly, corresponding to the these blocks, $\mathbb{C}^n$ can be decomposed into the direct sum $\mathbb{C}^{n_1} \bigoplus \mathbb{C}^{n_2} \bigoplus \cdots \bigoplus \mathbb{C}^{n_p}$, satisfying that N_j acts trivially on $\mathbb{C}^{n_k}, j \neq k$, and acts on $\mathbb{C}^{n_j}$ with the graded pieces of the corresponding weight filtration being one-dimensional. Correspondingly, L_ρ can be decomposed into the direct sum $L_1 \bigoplus L_2 \bigoplus \cdots \bigoplus L_p$ of some D-invariant subsystems. Thus, an i-form ϕ valued in L_ρ can be decomposed into the sum $\phi = \phi_1 + \phi_2 \cdots + \phi_p$ of some i-forms ϕ_j valued in $L_j, 1 \leq j \leq p$, and ϕ is D-closed iff each ϕ_j is D-closed. In order to make the reduction mentioned above, we only need to show that if ϕ is L^2, then so is each component ϕ_j. This again is obtained by means of the construction of h_0 and the same asymptotic behavior of h as h_0 in §2.4, where the block M_j in (6) can be considered as a metric on L_j. More precisely, one can choose a basis $\{e_1, e_2, \cdots, e_n\}$ of L_ρ, which is compatible with the decomposition $L_\rho = L_1 \bigoplus L_2 \bigoplus \cdots \bigoplus L_p$ and weight filtrations on L_j w.r.t. $N_j, 1 \leq j \leq p$, and satisfies $< e_j, e_k >_{h_0} = 0, j \neq k$; so the cross-terms

$| < e_j, e_k >_h |$, as in the proof of 1), can be controlled by $\epsilon(\|e_j\|_h^2 + \|e_k\|_h^2)$ for some sufficiently small ϵ. Thus, when the form ϕ valued in L_ρ is written as $\psi_1 + \psi_2 + \cdot + \psi_n$ with ψ_j valued in the (smooth) line bundle generated by e_j, $\|\phi\|_{\omega,h}^2 \sim \|\psi_1\|_{\omega,h}^2 + \cdots + \|\psi_n\|_{\omega,h}^2$; in particular, $\|\phi\|_{\omega,h}^2 \sim \|\phi_1\|_{\omega,h}^2 + \cdots + \|\phi_p\|_{\omega,h}^2$, and hence each ϕ_j is L^2.

By the above argument, without loss of generality, we now assume that N, acting on $\mathbb{C}^{m+1}$ $(n = m+1)$, has only one Jordan block. As before, one then has a weight filtration of $\mathbb{C}^{m+1}$:

$$0 \subset W_{-m} \subset W_{-(m-2)} \subset \cdots \subset W_{m-2} \subset W_m = \mathbb{C}^{m+1},$$

which satisfies that the quotient $\mathrm{Gr}_i^W = W_i / W_{i-2}$ is of dimension 1 and $NW_i = W_{i-2}$. Correspondingly, one has the invariant subbundles of L_ρ,

$$0 \subset \mathbf{W}_{-m} \subset \mathbf{W}_{-(m-2)} \subset \cdots \subset \mathbf{W}_{m-2} \subset \mathbf{W}_m = L_\rho,$$

and hence the corrsponding filtration of $\mathcal{A}_{(2)}^{\cdot}(L_\rho)$

$$0 \subset \mathcal{A}_{(2)}^{\cdot}(\mathbf{W}_{-m}) \subset \mathcal{A}_{(2)}^{\cdot}(\mathbf{W}_{-(m-2)}) \subset \cdots \subset \mathcal{A}_{(2)}^{\cdot}(\mathbf{W}_{m-2}) \subset \mathcal{A}_{(2)}^{\cdot}(L_\rho),$$

which is clearly D-invariant. For simplicity, we denote $\mathcal{A}_{(2)}^{\cdot}(\mathbf{W}_{-i})$ by K_i; one has then a filtered complex

$$K_{-m} \supset K_{-m+2} \supset \cdots \supset K_{m-2} \supset K_m \supset 0.$$

Consider the quotient K_i / K_{i+1}, which is clearly $\mathcal{A}_{(2)}^{\cdot}(\mathrm{Gr}_{-i}^W(L_\rho))$. We now have the spectral sequence $(E_r, d_r)_{r \geq 1}$ of the filtered complex $\{K_i, D\}_{i=-m}^m$, which, by the theory of spectral sequences, converges to the cohomology of K_{-m}, namely the sheaf cohomology of $\{\mathcal{A}_{(2)}^{\cdot}(L_\rho), D\}$. To prove 2), we need to analyze the sequence $(E_r, d_r)_{r \geq 1}$.

By the definition of spectral sequences,

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}^p K^*) = \mathcal{H}^{p+q}\big(\mathcal{A}_{(2)}^{\cdot}(\mathrm{Gr}_{-p}^W(L_\rho))\big),$$

where $\mathrm{Gr}^p K_* = K_p / K_{p+1}$. We will show that the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is trivial, as can be observed as follows. For convenience of the following discussion, we write down the following diagram (note that by "diagram" we do not mean that the horizontal sequences are exact, we only want to show the meanings of the maps in question)

$$\begin{array}{ccccc}
\mathcal{A}_{(2)}^{p+q}(\mathbf{W}_{-p-1}) & \xrightarrow{i} & \mathcal{A}_{(2)}^{p+q}(\mathbf{W}_{-p}) & \xrightarrow{Proj} & \mathcal{A}_{(2)}^{p+q}(\mathrm{Gr}_{-p}^W(L_\rho)) \\
\downarrow D & & \downarrow D & & \downarrow D \\
\mathcal{A}_{(2)}^{p+q+1}(\mathbf{W}_{-p-1}) & \xrightarrow{i} & \mathcal{A}_{(2)}^{p+q+1}(\mathbf{W}_{-p}) & \xrightarrow{Proj} & \mathcal{A}_{(2)}^{p+q+1}(\mathrm{Gr}_{-p}^W(L_\rho))
\end{array}$$

where i and $Proj$ are the inclusion and the projection respectively, the third D is the induced differential from the previous two D's by taking the quotients. Let $\phi\otimes\tilde{v}\in\mathcal{A}^{p+q}_{(2)}(\mathbf{W}_{-p})$ represent a cohomology class of $\mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$. (Since we assumed that N has only one Jordan block, a cohomology class of $\mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$ can always be represented as such a single tensor.) In particular, since $D\tilde{v}\in\Omega^1(\mathbf{W}_{-p-2})$, $d\phi=0$, here the d is the differential on regular functions over Δ^*. So $D(\phi\otimes\tilde{v})\in\mathcal{A}^{p+q+1}_{(2)}(\mathbf{W}_{-p-2})$, which represents a trivial cohomology class in $\mathcal{H}^{p+q+1}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p-1}(L_\rho))\big)(=E_1^{p+1,q})$, namely $d_1([\phi\otimes\tilde{v}])=0$.

Next, let us consider d_2. Since $d_1=0$, $E_2^{p,q}=E_1^{p,q}$. Similar to the above argument, taking a form $\phi\otimes\tilde{v}\in\mathcal{A}^{p+q}_{(2)}(\mathbf{W}_{-p})$, which represents a cohomology class of $\mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$, $D(\phi\otimes\tilde{v})=(-1)^{p+q}\frac{1}{2\pi\sqrt{-1}}\big((\phi\wedge\frac{dt}{t})\otimes N\tilde{v}\big)$ and then lies in $\mathcal{A}^{p+q+1}_{(2)}(\mathbf{W}_{-p-2})$, which represents a cohomological class of $E_2^{p+2,q-1}=\mathcal{H}^{p+q+1}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p-2}(L_\rho))\big)$. So, d_2 is induced by $\frac{1}{2\pi\sqrt{-1}}\frac{dt}{t}N$.

Let us now consider the kernel and image of d_2. Applying Lemma 1 to $E_2^{p,q}=\mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$, we get that the only (possibly) nontrivial terms at E_2 are

$$\{E_2^{p,-p}\}_{p\geq 0},\quad \{E_2^{p+2,-p-1}\}_{p\geq 0},\quad E_2^{-1,2},\quad \text{and } E_2^{1,1}.$$

Furthermore, from the above argument together with Lemma 1, we obtain that $d_2: E_2^{p,-p}\to E_2^{p+2,-p-1}, p\geq 0$ (if $E_2^{p+2,-p-1}$ is nontrivial, i.e. $p\leq m-2$) and $d_2: E_2^{-1,2}\to E_2^{1,1}$ are isomorphisms and that $d(E_2^{p+2,-p-1})=0, p\geq 0$ and $d_2(E_2^{1,1})=0$.

Summing all the above up, the only possible nontrivial terms at E_3 are $E_3^{m-1,-m+1}, E_3^{m,-m}$. Thus the spectral sequence $\{E_r,d_r\}_{r\geq 1}$ degenerates at E_3 and the only possible terms are $E_3^{m-1,-m+1}, E_3^{m,-m}$. Therefore, by the theory of spectral sequences of filtered complexes, $\mathcal{H}^i(\mathcal{A}^{\cdot}_{(2)}(L_\rho))=0, i=1,2$; namely, if an i-form η in $\mathcal{A}^i_{(2)}(L_\rho)$ is D-closed, then there exists an $i-1$-form $\sigma\in\mathcal{A}^{i-1}_{(2)}(L_\rho)$ satisfying $D\sigma=\eta$, for $i=1,2$. The proof of 2) is finished. □

3.2 The L^2-Higgs cohomology: The L^2-$\overline{\partial}$-Poincaré Lemma

As seen in §2.4, the harmonic metric h (the ρ-equivariant harmonic map) on L_ρ induces the structure of a Higgs bundle on L_ρ: $(E, D''=\overline{\partial}+\theta)$, satisfying $D=D'+D''$ with $D'=\partial+\overline{\theta}$; moreover, the Hermitian connection $\partial+\overline{\partial}$ w.r.t. the metric h has bounded curvature under the metric h and the Poincaré-like metric so that E can be analytically extended to $\overline{S}$, denoted by j_*E as usual, which is especially coherent. Furthermore, θ has a log-singularity, i.e. $\theta\sim\frac{dt}{t}N$. It is especially worth pointing out that by an argument of Simpson (cf. [21]), *the residue N of θ here coincides with the logarithmic monodromy N in the local system L_ρ*; so although under different bundle structures, we have the same weight filtration under certain suitable identification. Throughout this subsection, we consider the Higgs bundle $(E, D'' = \overline{\partial} + \theta)$ together with the harmonic metric h, satisfying that the meromorphic sections of E at the punctures have the norm estimates (17), just forgetting that it comes from the local system corresponding to the representation ρ.

As in the previous subsection, using the Poincaré-like metric ω on S and the harmonic matric h on (E, D''), one can similarly define a complex $\{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$ of fine sheaves on $\overline{S}$. Let U be an open subset of $\overline{S}$. Then $\mathcal{A}^i_{(2)}(E)(U)$ is defined as the set of

j_*E-valued i-forms η on $U \cap S$, with measurable coeffcients and measurable exterior derivative $\overline{\partial}\eta$, such that η and $D''\eta$ have finite L^2 norm on $K \cap S$, for any compact subset K of U.

Remarks. 1) The Higgs condition $(D'')^2 = 0$ makes $\{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$ a complex, which is actually a complex of certain differential forms valued in j_*E, and hence fine. 2) Due to following lemma, it is actually sufficient to assume that η and $\overline{\partial}\eta$ are L^2 in the above definition. 3) Since the sheaves are fine, so again the hypercohomology $\mathbb{H}^*(\{\mathcal{A}^{\cdot}_{(2)}(E), D''\})$ is computed by the cohomology $H^*(\{\Gamma(\mathcal{A}^{\cdot}_{(2)}(E)), D''\})$ of the complex of global sections; we call it the L^2-*Higgs cohomology* of $\overline{S}$ valued in the Higgs bundle $(E, D = \overline{\partial} + \theta)$, denoted by $H^*_{(2)}(\overline{S}, E)$.

Lemma 2 *θ is an L^2-bounded operator.*

Proof. As mentioned before, $\theta \sim \frac{dt}{t}N$ near the punctures. So it suffices to show that $\frac{dt}{t}N$ is L^2-bounded near the punctures. To this end, as done in the proof of Theorem 2, we first do some discussion for L^2-forms valued in E, as will also be used in the proof of the following theorem 3. As seen in §2.4, we can choose some holomorphic sections of $E^i_0 \subset (j_*E)_{p_i}$, the projections of which on $\mathrm{Gr}^i_0(E)$ form a basis, still denoted by $\{e_1, e_2, \cdots, e_n\}$, of $\mathrm{Gr}^i_0(E)$ near a puncture p_i, which is compatible with the Jordan blocks of N and the filtration $\{W_l\}^k_{l=-k}$ of N, in particular $Ne_i \in \{e_1, e_2, \cdots, e_n\}$. Moreover, one has

$$\|e_i\|^2_h \sim |\log r|^l, \text{ if } e_i \in W_l - W_{l-1};$$

and as $r << 1$,

$$| < e_i, e_j > |_h \leq \epsilon(\|e_i\|_h \cdot \|e_j\|_h),$$

for some sufficiently small $\epsilon > 0$. Thus, a form ϕ valued in E can written as the sum $\phi_1 + \phi_2 + \cdots + \phi_n$ of some forms ϕ_i valued in the invertible sheaf generated e_i; especially, by the estimate for the cross-terms, one has $\|\phi\|^2_h \sim \|\phi_1\|^2_h + \cdots + \|\phi_n\|^2_h$. So, if ϕ is L^2, each component ϕ_i is also L^2. On the other hand, since N lowers weights by 2, we have that $\|Ne_i\|^2_h \sim |\log r|^{-2}\|e_i\|^2_h$ if $Ne_i \neq 0$; while

$\|\frac{dt}{t}\|_\omega^2 = |\log r|^2$. So, $\|\frac{dt}{t}Ne_i\|_h^2 \sim \|e_i\|^2$, and hence $\|\frac{dt}{t}N\phi_i\|_h^2 \sim \|\phi_i\|^2$. The proof of the lemma is finished. $\square$

On the other hand, based on the above lemma, one can also define a subcomplex of $\{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$—the L^2-*holomorphic Dolbeault complex* $\{\Omega^{\cdot}_{(2)}(E), \theta\}$ as follows: $\Omega^i_{(2)}(E)(U)$ is defined as the set of

j_*E-valued meromorphic i-forms η on $U \cap S$ such that η has finite L^2 norm on $K \cap S$, for any compact subset K of U.

$\theta \wedge \theta = 0$ makes $\{\Omega^{\cdot}_{(2)}(E), \theta\}$ again a complex, which is actually a complex of certain meromorphic differential forms valued in j_*E, and hence coherent. We call the hypercohomology $\mathbb{H}^*(\{\Omega^{\cdot}_{(2)}(E), \theta\})$ the L^2-*Dolbeault cohomology* of $\overline{S}$ valued in the Higgs bundle $(E, D = \overline{\partial} + \theta)$.

The purpose of this subsection is then to show that the above two complexes have the same hypercohomologies; more precisely

Theorem 3 *(L^2-$\overline{\partial}$-Poincaré lemma) The inclusion*

$$i : \{\Omega^{\cdot}_{(2)}(E), \theta\} \hookrightarrow \{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$$

is a quasi-isomorphism; and hence one has

$$\mathbb{H}^*(\{\Omega^{\cdot}_{(2)}(E), \theta\}) \cong H^*_{(2)}(\overline{S}, E).$$

Remarks. In the case when E comes from a VHS, the theorem was showed by S. Zucker [24] (for the case of curves) and Jost-Yang-Zuo [14] (for the general case).

In order to prove the theorem, we need the following lemma, which has proved in [24], Proposition 6.4.

Lemma 3 *Let V be a holomorphic line bundle on Δ^* with generating section σ, and with a Hermitian metric satisfying*

$$\|\sigma\|^2 \sim |\log r|^k, \quad k \in \mathbb{Z}, \ k \neq 1.$$

Then for every germ of an L^2 $(0,1)$-form $\phi = f d\bar{t} \otimes \sigma$ at the puncture, there exists an L^2 section $u \otimes \sigma$ with $\overline{\partial} u = f d\bar{t}$.

Proof of Theorem 3. Similar to the proof of Theorem 2, the difficulty again comes from the punctures $\{p_j\}$, so we again restrict ourself to a small enough punctured disk Δ^* at p_j. In order to prove the theorem, we only need to show that the inclusion i induces an isomorphism between the corresponding cohomology sheaves at the puncture; by the argument of standard homological algebra, this is equivalent to showing that *for any D''-closed r-form*

$\phi \in \mathcal{A}^r_{(2)}(E)$ on a neighborhood $U \subset \Delta$ of the puncture p_j, there is a θ-closed r-form $\eta \in \Omega^r_{(2)}(E)$ and an $r-1$-form $\psi \in \mathcal{A}^{r-1}_{(2)}(E)$ on (a possibly smaller) U satisfying $\phi = \eta + D''\psi$, $r = 0, 1, 2$. In the following proof, we continue to use the notation and the discussions in the proof of Lemma 2.

Case of $r = 0$. Clearly, a D''-closed 0-form ϕ is exactly a meromorphic section of j_*E and also θ-closed, so the case of $r = 0$ is obtained.

Case of $r = 2$. In this case, a form $\phi \in \mathcal{A}^2_{(2)}(E)$ can be written as the sum of some forms $\phi_i = (\phi'_i \wedge \frac{dt}{t}) \otimes e_i$, here ϕ'_i is a complex-value $(0,1)$-form. On other hand, by the discussion in the proof of Lemma 2, each component ϕ_i is also L^2. Considering $\frac{dt}{t} \otimes e_i$ as a generator of a holomorphic line bundle, its norm satisfies, under the harmonic metric and the Poincaré-like metric,

$$\|\frac{dt}{t} \otimes e_i\|^2_{\omega,h} \sim |\log r|^k,$$

if $e_i \in W_{k-2} - W_{k-3}$.

When $k \neq 1$, by the above lemma, there exists an L^2-form $u\frac{dt}{t} \otimes e_i$ satisfying

$$\overline{\partial}(u\frac{dt}{t} \otimes e_i) = \phi' \wedge \frac{dt}{t} \otimes e_i,$$

and hence $D''(u\frac{dt}{t} \otimes e_i) = (\phi' \wedge \frac{dt}{t}) \otimes e_i = \phi_i$, i.e., ϕ_i is D''-coclosed.

If $k = 1$, i.e., $\|e_i\|^2_h \sim |\log r|^{-1}$, equivalently, $e_i \in W_{-1} - W_{-2}$; by means of the compatibility of the weight filtration of N and $\{e_1, \cdots, e_n\}$, there exists an $e_j \in \{e_1, \cdots, e_n\}$ satisfying $Ne_j = e_i$ ($e_j \in W_1 - W_0$), and hence $\frac{dt}{t}N(e_j) = \frac{dt}{t} \otimes e_i$. Since $\theta \sim \frac{dt}{t}N$, an approximating discussion also shows that there exists a holomorphic section $e'_i \in W_1 - W_0$ with $\theta(e'_i) = \frac{dt}{t} \otimes e_i$; so $D''(\phi'_i \otimes e'_i) = (\phi'_i \wedge \frac{dt}{t}) \otimes e_i = \phi_i$. Finally, that $(\phi'_i \wedge \frac{dt}{t}) \otimes e_i = \phi_i$ is L^2 implies that the $(0,1)$-form $\phi'_i \otimes e'_i$ is L^2, and hence $\in \mathcal{A}^1_{(2)}(E)$. Summing all the above, we have that any form of $\mathcal{A}^2_{(2)}(E)$ is D''-coclosed. The case of $r = 2$ is obtained.

Case of $r = 1$. We now turn to the case of $r = 1$. In order to make the proof clearer, we can refer from time to time to the following diagram (Here, again, by "diagram" we do not mean that its horizontal sequences are exact.)

$$\begin{array}{ccccc}
\Omega^0_{(2)}(E) & \xrightarrow{i} & \mathcal{A}^{0,0}_{(2)}(E) & \xrightarrow{\overline{\partial}} & \mathcal{A}^{0,1}_{(2)}(E) \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\
\Omega^1_{(2)}(E) & \xrightarrow{i} & \mathcal{A}^{1,0}_{(2)}(E) & \xrightarrow{\overline{\partial}} & \mathcal{A}^{1,1}_{(2)}(E).
\end{array}$$

Write $\phi = \phi^{1,0} + \phi^{0,1}$, $\phi^{1,0}$ (resp. $\phi^{0,1}$) being the part of type $(1,0)$ (resp. $(0,1)$). The D''-closedness of ϕ is equivalent to

$$\overline{\partial}\phi^{1,0} + \theta\phi^{0,1} = 0. \tag{22}$$

Write $\phi^{0,1}$ as the sum of forms $\phi_i^{0,1} \otimes e_i$, $i = 1, \cdots, n$, which, again by the same discussion as in the proof of Lemma 2, are L^2. Assume that $\|e_{i_0}\|_h \sim |\log r|$ (note that the number of such e_{i_0}'s may > 1) and for $i \neq i_0$, $\|e_i\|_h \sim |\log r|^k, k \neq 1$. By the above lemma, for $i \neq i_0$, there exists an L^2 section $u_i e_i$ (and hence $\in \mathcal{A}^0_{(2)}(E)$) satisfying

$$\overline{\partial} u_i \otimes e_i = \phi_i^{0,1} \otimes e_i.$$

Thus, the part of type $(0,1)$ of the form $\phi - D''(u_i e_i)$ contains no longer any term of the form $\phi_i^{0,1} \otimes e_i, i \neq i_0$.

By the discussion above, we may assume that

$$\phi^{0,1} = \sum_i \phi_i^{0,1} \otimes e_i,$$

each e_i satisfying $\|e_i\|_h \sim |\log r|$ and each term $\phi_i^{0,1} \otimes e_i$ being L^2. In order to deal with such terms $\phi_i^{0,1} \otimes e_i$, we now use the D''-closedness (22) of ϕ, i.e. $\overline{\partial}\phi^{1,0} + \theta\phi^{0,1} = 0$. Due to the D''-closedness of $\phi = \phi^{1,0} + \phi^{0,1}$, one can write part of $\phi^{1,0}$ as the following sum

$$\phi'^{1,0} = \sum_i u_i \theta e_i$$

with each e_i satisfying $\|e_i\|_h \sim |\log r|$ and $\overline{\partial} u_i = \phi_i^{0,1}$. Note that the remaining part of $\phi^{1,0}$ is cancelled by $\overline{\partial}$; but θ does not cancel any part of $\phi^{0,1} = \sum_i \phi_i^{0,1} \otimes e_i$, since $\|e_i\|_h \sim |\log r|$ (i.e., $Ne_i \neq 0$) for each e_i. In order of the following discussion, we need to show that $\phi'^{1,0}$ is also L^2. To this end, considering the holomorphic vector bundle $\frac{dt}{t} \otimes E$ and the basis $\{\frac{dt}{t} \otimes e_1, \cdots, \frac{dt}{t} \otimes e_n\}$ and writing $\phi^{1,0}$ as the sum $\sum_j v_j \frac{dt}{t} \otimes e_j$, the same discussion of Lemma 2 shows that each component is L^2; on the other hand, since $\|\theta e_j\|_{h,\omega} \sim \|\frac{dt}{t} \otimes Ne_j\|_{h,\omega} \sim \|e_j\|_h$ if $Ne_j \neq 0$, an approximating discussion then shows that $\phi'^{1,0}$ is L^2.

Now, we consider the sum $\sum_i u_i e_i$. Clearly,

$$\theta(\sum_i u_i e_i) = \phi'^{1,0} \quad \text{and} \quad \overline{\partial}(\sum_i u_i e_i) = \phi^{0,1}.$$

On the other hand, since $\|e_i\| \sim |\log r|$ for each e_i, so $Ne_i \neq 0$ and $\|\theta(e_i)\|_h \sim |\log r|$. Thus, the same discussion as in the proof of Lemma again shows that $\phi'^{1,0}$ being L^2 implies that $\sum_i u_i e_i$ is also L^2, namely, $\sum_i u_i e_i \in \mathcal{A}^0_{(2)}(E)$. Finally, a simple computation shows that $\phi - D''(\sum_i u_i e_i)$ contains no longer the part of type $(0,1)$.

Thus, we may continue to reduce the problem to the assumption that ϕ is D''-closed and contains only the part of type $(1,0)$. Clearly, a D''-closed L^2-form of type $(1,0)$ is holomophic and θ-closed.

Summing all the argument above up, we have that for any D''-closed form $\phi \in \mathcal{A}^1_{(2)}(E)$ on a neighborhood U of the origin, there is a θ-closed form $\eta \in \Omega^1_{(2)}(E)$ and a section $\psi \in \mathcal{A}^0_{(2)}(E)$ on (a possibly smaller) U satisfying $\phi = \eta + D''\psi$. This finishes the proof of the theorem. $\square$

Using the Kähler identity for harmonic bundles (cf. §2), $H^*_{(2)}(\overline{S}, L_\rho)$ can be identified with $H^*_{(2)}(\overline{S}, E)$, and hence we have the following

Corollary 1

$$H^*(\overline{S}, j_* L_\rho) \cong \mathbb{H}^*(\{\Omega^{\cdot}_{(2)}(E), \theta\}).$$

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