Cohomologies of unipotent harmonic bundles over noncompact curves

by

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1 Introduction

Let \( \mathbb{S} \) be a compact Riemann surface (holomorphic curve) of genus \( g \). Let \( p_1, p_2, \ldots, p_s \) be \( s > 0 \) points on it; these points define a divisor, and we denote the open Riemann surface \( \mathbb{S} \setminus \{ p_1, \ldots, p_s \} \) by \( S \). When \( 3g - 3 + s > 0 \), it carries a complete hyperbolic metric of finite volume, the so-called Poincaré metric; the points \( p_1, p_2, \ldots, p_s \) then become cusps at infinity. Even in the remaining cases, that is, for a once or twice punctured sphere, we can equip \( S \) with a metric that is hyperbolic in the vicinity of the cusp(s), and for our purposes, the behavior of the metric there is all that counts, and we call such a metric Poincaré-like. In any case, our metric on \( S \) is denoted by \( \omega \). Denote the inclusion map of \( S \) in \( \mathbb{S} \) by \( j \). Let \( \rho : \pi_1(S) \to GL(n, \mathbb{C}) \) be a semisimple linear representation of \( \pi_1(S) \) which is unipotent near the cusps (for the precise definition, cf. \( \S 2.4 \)). Corresponding to such a representation \( \rho \), one has a local system \( L_\rho \) over \( S \) and a \( \rho \)-equivariant harmonic map \( h : S \to GL(n, \mathbb{C})/U(n) \) with a certain special growth condition near the divisor, which is especially of finite energy (for details, see \( \S 2.4 \)). For the present case of complex dimension 1, this is actually elementary; it also follows from the general result of [11]. We also remark that if not imposing any growth condition, \( \rho \)-equivariant harmonic maps are not unique in general, even of infinite energy (cf. [13]). In order to see explicitly the behavior at the cusps of the harmonic map \( h \) used in this note, we explicitly construct an initial map, which has the required asymptotic behavior, and show why it has finite energy. Then the existence of the harmonic map \( h \) and its behavior are obtained in a standard manner presently. It should be pointed out that the construction here is essentially the same as that in [11]; but the target manifolds of [11] are very general, so the construction there was getting very complicated. The idea of the explicit construction here will also be used in the case of higher dimension. The harmonic map \( h \) obtained above

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can be considered as a Hermitian metric on $L_\rho$—harmonic metric—so that we have a so-called harmonic bundle $(L_\rho, h)$ [21]. Such a bundle carries interesting structures, e.g. a Higgs bundle structure $(E, \theta)$, where $\theta = \partial h$, and it has a log-singularity at the divisor.

The purpose of this note is to investigate various cohomologies of $S$ with degenerating coefficients $L_\rho$ (considered as a local system — a flat vector bundle, a Higgs bundle, or a $\mathcal{D}$-module, depending on the context): the Čech cohomology of $j_*L_\rho$ (note that in the higher dimensional case, one needs to consider the corresponding intersection cohomology [4]), the $L^2$-cohomology, the $L^2$-Dolbeault cohomology, and the $L^2$-Higgs cohomology, and the relationships between them. Here, $L^2$ is defined by using the Poincaré(-like) metric $\omega$ and the harmonic metric $h$. We want to generalize the results [24] valid for the case of variations of Hodge structures (VHS) to the case of harmonic bundles, as was suggested by Simpson [21]; in principle, in view of our assumption on the representations in question being unipotent and the growth property of the harmonic metric, the situation should be similar to the case of VHS.

Remark. We remark that as mentioned before, the harmonic metric used here satisfies a special growth condition (cf. §2.4); naturally one should ask what the situation is when one uses some other harmonic metrics, e.g. general tame harmonic metrics defined by Simpson [21].

This paper is meant to be a part of the general program of studying cohomologies with degenerating coefficients on quasi-projective varieties and their Kählerian generalizations. The general aim here is not restricted to the case of curves nor to the one of representations that are unipotent near the divisor. One of the purposes of this note therefore is to illuminate at this particular case where many of the analytic and geometric difficulties of the general case are not present what differences will appear when we consider unipotent harmonic bundles instead of VHSs. In the future, we wish to consider the case of higher dimensional quasi-projective varieties, but still assuming that the representations of $\pi_1$ are unipotent. Geometrically, the unipotent condition is very natural. It is well-known that a rational VHS is unipotent up to a lifting; more generally, if the representation $\rho$ of $\pi_1$ is into $Sl(n, \mathbb{Z})$, then a variant of Borel’s lemma (cf. e.g. [19], Lemma 4.5) implies that $\rho$ is unipotent near the divisors after a lifting, assuming the existence of an $\rho$-equivariant pluriharmonic map. For the case of VHSs, the various cohomologies have been considered by various authors [1, 16, 17, 23, 14] and we should say that the question have been well understood by now.

After submitting a revised version of the paper, we were informed by the
referee about related work by Sabbah. In his paper, Sabbah is generalizing Saito’s mixed Hodge modules to the case of twistor $\mathcal{D}$-module structures, and he wants to prove a decomposition theorem for the higher direct image. Thus, he has to treat some kind of Zucker-like result on curves in the case of polarizable regular twistor $\mathcal{D}$-modules instead of VHSs. As he remarks in Remark 6.2.6 of his paper, he does not follow the proof given by Zucker, namely, he does not prove the corresponding $\overline{\partial}$-Poincaré/Poincaré lemma near punctures. In contrast, in the present paper, we are generalizing Zucker’s approach; technically, this means proving the corresponding $\overline{\partial}$-Poincaré/Poincaré lemma near punctures in the case of harmonic bundles, which is itself of interest in its own. So our argument here is a very direct one. We expect that the idea here is also useful for a direct treatment of the case of higher dimension.

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2 The geometry associated with representations of fundamental groups

2.1 The decomposition of a flat connection

In order to make this note an introduction into our general program, we describe some background material in this §§ and the next two §§. The knowledgable reader may skip these §§. Since the discussion below is local, we do not specify whether the domain manifolds $X$ are compact or noncompact, unless stated otherwise. But when the question involves the existence of harmonic metrics (maps), such an assumption will be essential, as will be seen in the following.

Let $X$ be a Riemannian manifold, $V$ a $\text{Gl}(n, \mathbb{C})$ bundle on $X$ with a flat connection $D$ or, equivalently, a representation

$$\rho : \pi_1(X) \to \text{Gl}(n, \mathbb{C}).$$

A metric $\langle \cdot, \cdot \rangle$ on $V$ is equivalent to a $\rho$-equivariant map

$$h : \tilde{X} \to \text{Gl}(n, \mathbb{C})/U(n) =: \tilde{Y},$$

---

1Claude Sabbah, Polarizable twistor $\mathcal{D}$-modules, arXiv: math.AG 0503038.
where $h$ is defined by setting $\langle v, v \rangle = h_{ij}v^iv^j$, and $\tilde{X}$ is the universal covering of $X$. Afterwards, we use $h$ to mean either a metric or a map, as will be clear from the context. When the map $h$ is harmonic, we call the flat bundle $V$ together with the corresponding metric a harmonic bundle, and the metric a harmonic metric. The simplest case is, of course, $n = 1$. Then $V$ is a line bundle, and

$$GL(1, \mathbb{C})/U(1) = \mathbb{C}^*/S^1 \simeq \mathbb{R}^+.$$  

Using the isomorphism

$$\log : \mathbb{R}^+ \to \mathbb{R},$$

a metric can then be written locally as

$$h = e^\lambda,$$

for $\lambda : \tilde{X} \to \mathbb{R}$,

and so $h^{-1}dh = d\lambda.^2$

Given any metric $h$ on $V$, the flat connection $D$ will not preserve it in general, and so, we split $D$ by means of the Cartan decomposition as

$$D = D_h + \Theta,$$

where $D_h$ preserves $h$, i.e.,

$$d\langle v, v \rangle = \langle D_hv, v \rangle + \langle v, D_hv \rangle.$$  

We then have $\Theta = 0$ iff $D$ is unitary iff $h$ is constant. The derivative $dh$ measures the deviation of $D$ from being unitary, i.e., $dh = \Theta$. Thus, the energy of the map $h$ is defined by $\int \|\Theta\|^2$, and $h$ is harmonic iff

$$D_h^\ast \Theta = 0.$$  

(1)

For $n = 1$, reverting to our $h^{-1}dh$ notation, we have

$$D_h = d - h^{-1}dh,$$  

i.e. $\Theta = h^{-1}dh = d\lambda$.

The harmonic map equation (1) becomes

$$0 = (d^\ast - h^{-1}dh)d\lambda = d^\ast d\lambda - d\lambda \wedge d\lambda = d^\ast d\lambda,$$

and so $\lambda$ is a harmonic function on $\tilde{X}$. It should be pointed out that $\lambda$ is not well defined as a function on $X$; $d\lambda$, however, is well defined from the above argument, and a harmonic 1-form when the metric $h$ is harmonic.

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2$h^{-1}dh$ is of course the derivative of the map $h : \tilde{X} \to \tilde{Y}$; usually, we should write $dh$, but the problem is that $d$ has two meanings, namely on one hand, the exterior derivative $d$, leading to the notation $h^{-1}dh$, and on the other hand, the differential $d$ of a map between Riemannian manifolds, suggesting to write $dh$. In the sequel, we use the notation $h^{-1}dh$ only for the case $n = 1$ and write $dh$ else, except for the energy estimate of the initial map in §2.4; also, see the footnote there.
2.2 Variations of Hodge structures

Let us just consider the following geometric setting. Let $Z$ and $X$ be Kähler manifolds, with $\dim Z > \dim X$, $f : Z \to X$ a proper holomorphic map with smooth fibers. Canonically, one has a flat vector bundle $H^k$ over $X$ with the flat connection $D$ and the fibre over $x \in X$ being the $k$-th cohomology $H^k(Z_x; \mathbb{C})$ of $Z_x$ (cf. e.g. [5]). For each fibre $H^k(Z_x; \mathbb{C})$ over $x \in X$, we have the Hodge decomposition

$$H^k(Z_x; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Z_x).$$

This induces a filtration of $H^k(Z_x; \mathbb{C})$ (resp. $H^k$)

$$0 \subset F^k_x \subset \cdots \subset F^{p+1}_x \subset F^p_x \subset \cdots \subset F^0_x = H^k(Z_x),$$

(resp. $0 \subset F^k \subset \cdots \subset F^{p+1} \subset F^p \subset \cdots \subset F^0 = H^k$)

with $F^p_x = \bigoplus_{i \geq p} H^{i,k-i}(Z_x)$. This filtration defines an element of a subdomain of a Grassmannian of flags. From this, one obtains the Griffiths period map

$$X \to \mathcal{D}/\Gamma$$

into the quotient $\mathcal{D}/\Gamma$ of the Griffiths period domain $\mathcal{D}$ by a certain discrete group $\Gamma$ (ref. e.g. [5]). In particular, it should be pointed out that the period map can be equivalently explained as a metric on the bundle $H^k$; this idea will be used in a more general setting, i.e. that of harmonic bundles, as will be seen in the following.

The Griffiths period domain $\mathcal{D}$ is a homogeneous complex manifold, but generally not a Hermitian symmetric space. If $\mathcal{D}$ is not Hermitian symmetric, it is a Riemannian submersion over a (non-Hermitian) symmetric space of noncompact type under the canonical invariant metrics and the corresponding horizontal distribution is holomorphic but not integrable; more importantly, the image of the period map is tangential to this distribution. Furthermore, $\mathcal{D}$ has nonpositive curvature in the horizontal directions (for details, ref. [7], also [9]).

Decompose the flat connection $D$ into the $(1,0)$-part $d'$ and $(0,1)$-part $d''$. For the above filtration $\{F^p\}$ of $H^k$, Griffiths showed (cf. e.g. [5]) that $F^p$ is a holomorphic subbundle of $H^k$ under the holomorphic structure $d''$ and the following transversal property

$$d' : F^p \to \Omega^{1,0}(F^{p-1}).$$

Taking the quotient $F^p/F^{p-1}$, denoted by $E^{p,q}$, $(p + q = k)$, which is holomorphic under the induced holomorphic structure, $d'$ then induces a map from
\( E^{p,q} \) into \( \Omega^{1,0}(E^{-1,q+1}) \), denoted by \( \theta \). Consequently, one gets a formal decomposition \( D = \partial + \bar{\partial} + \theta + \bar{\theta} \) such that
\[
\begin{align*}
\partial : E^{p,q} &\to \Omega^{1,0}(E^{p,q}), \\
\bar{\partial} : E^{p,q} &\to \Omega^{0,1}(E^{p,q}), \\
\theta : E^{p,q} &\to \Omega^{1,0}(E^{-1,q+1}), \\
\bar{\theta} : E^{p,q} &\to \Omega^{0,1}(E^{+1,q-1}).
\end{align*}
\]

Since \( D^2 = 0 \), in particular, one has \( (\bar{\partial} + \partial)^2 = 0 \). So \( E = \bigoplus_{p+q=k} E^{p,q} \) together with \( \bar{\partial} + \partial \) is a special Higgs bundle (for its definition, see the next §§) satisfying
\[
\theta : E^{p,q} \to E^{-1,q+1} \otimes \Omega^1(X).
\]

Using the idea of the above geometric setting, one can also give the definition of an abstract (complex) variation of Hodge structures, and from such a variation one can formally get a Higgs bundle; since this is not directly related to our present purpose, we omit these; for details, one can refer to e.g. [22].

### 2.3 Harmonic bundles and Higgs bundles

A **Higgs bundle** \((E, \theta)\) over a complex manifold \( X \) consists of a holomorphic vector bundle \( E \) and a holomorphic morphism \( \theta : E \to E \otimes \Omega^1(X) \) satisfying \( \theta \wedge \theta = 0 \). It was then Simpson’s fundamental idea ([20]) to revert the construction in the previous §§ by using the Hermitian-Yang-Mills equation, namely, under certain suitable geometric conditions, to construct a harmonic bundle from a Higgs bundle. The understanding of the relationship between harmonic bundles and Higgs bundles initially comes from Hitchin’s important paper [8], which considered the above problem in the case of smooth curves.

From now on, we always assume \( X \) is Kähler. When \( X \) is compact, by means of a theorem by Narasimhan-Seshadri, Hitchin for curves and by Donaldson, Uhlenbeck-Yau, Simpson for higher dimensional varieties, for a stable Higgs bundle \((E, \theta)\), one can construct a Hermitian Yang-Mills connection (resp. metric) \( D_0 \) (resp. \( h \)), which furthermore satisfies that \( D := D_0 + \theta + \bar{\theta} \) is flat if all \( c_i(E) = 0 \), where \( \bar{\theta} \) is defined by setting
\[
< \bar{\theta}u, v >_h = < u, \theta v >_h .
\]

Such a metric \( h \) is then a harmonic metric on \( E \) when \( E \) is considered as a flat bundle with the flat connection \( D \), i.e., a harmonic map from \( \tilde{X} \) into the symmetric space \( \text{Gl}(n, \mathbb{C})/U(n) \), \( n = \text{Rank}E \).

Conversely, letting \( \rho : \pi_1(X) \to \text{Gl}(n, \mathbb{C}) \) be a semisimple representation, \( L_\rho \) the corresponding flat bundle, Corlette [2] then obtained a \( \rho \)-equivariant harmonic map
\[
h : \tilde{X} \to \text{Gl}(n, \mathbb{C})/U(n);
\]
as shown before, this is equivalently a metric on $L_\rho$ so that $L_\rho$ is a harmonic bundle. Furthermore, the harmonic metric $h$ induces a Higgs bundle structure $(E, \theta)$ on $L_\rho$ with $\theta = \partial h$. Note that $\theta \wedge \theta = 0$ follows from the pluriharmonicity of $h$ as originally discovered in a somewhat different context by Jost-Yau [10]; on the other hand, since we will assume that dim$X = 1$ in the present note, $\theta \wedge \theta = 0$ is automatic because $\theta$ is a $(1, 0)$-form. It should however be pointed out that when $X$ is noncompact Kähler and dim$X > 1$, the pluriharmonicity of $h$ is a subtle matter.

For sake of convenience later on, let us now make some formal derivations, which are again local, from a harmonic bundle to the corresponding Higgs bundle, provided that $h$ be pluriharmonic (such an assumption is automatic if dim$X = 1$). As before, decompose our flat connection $D = d' + d''$ into operators of type $(1, 0)$ and $(0, 1)$ respectively. Let $\delta'$ and $\delta''$ be the unique operators of type $(1, 0)$ and $(0, 1)$ such that the connections $\delta' + d''$ and $\delta'' + d'$ preserve the metric $h$. Set

$$\partial = (d' + \delta')/2, \quad \bar{\partial} = (d'' + \delta'')/2,$$

$$\theta = (d' - \delta')/2, \quad \bar{\theta} = (d'' - \delta'')/2.$$ 

It is clear [21] that the connection $\mu = \partial + \bar{\partial}$ also preserves the metric, $\theta = \partial h$ (more precisely, $\theta = h^{-1}\partial h$, see the footnotes 1 and 2), and $\bar{\theta}$ is the conjugate adjoint of $\theta$ w.r.t. $h$ (cf. (4)). Also, one has $(\bar{\partial} + \theta)^2 = 0$. Thus we obtain a structure of Higgs bundle on the bundle $L_\rho$ with $\bar{\partial}$ being the holomorphic structure and $\theta$ the Higgs field; later on we denote it by $(E, \bar{\partial} + \theta)$. From the above argument, we also have $D = \partial + \bar{\partial} + \theta + \bar{\theta}$, as have seen in the case of VHS.

Correspondingly, one also has the Kähler identities, which was observed by Deligne in the case of VHS. Set

$$D'_h = \bar{\partial} + \theta, \quad D''_h = \partial + \bar{\theta}, \quad D'_h = D''_h - D'_h.$$ 

Note that $D = D' + D''$ and $D''_h = (D + D'_h)/2$. Let $\Lambda$ be the adjoint of the operation of wedging with the Kähler form $\omega$. Then one has the first order Kähler identities

$$(D'_h)^* = \sqrt{-1}[\Lambda, D''_h], \quad (D''_h)^* = -\sqrt{-1}[\Lambda, D'_h]$$

$$(D'_h)^* = -\sqrt{-1}[\Lambda, D], \quad (D)^* = \sqrt{-1}[\Lambda, D'_h],$$

where $^*$ represents the adjoint of the respective operator. Set $\Delta = DD^* + D^*D$ and $\Delta'' = D''_h(D''_h)^* + (D''_h)^*D''_h$. Using the above first order identities, one then has

$$\Delta = 2\Delta''.$$ 

This shows that spaces of $\Delta$-harmonic forms valued in the local system $L_\rho$ can be identified with that of $\Delta''$-harmonic forms valued in the Higgs bundle $E$. 
2.4 Harmonic metrics in the case of noncompact curves

When $X$ is no longer compact, the local geometry works as before, but there arise difficulties with the existence of the harmonic map (metric) $h$ and its pluriharmonicity (in the higher dimensional case). On the other hand, the geometry of the related bundles near a compactifying divisor (provided that the harmonic metric exist) leads to very interesting structures which are, in fact, our main interest. As in the Introduction, we from now on assume that $X$ is a noncompact curve, i.e. a compact Riemannian surface deleting finitely many points, and change the symbol $X$ into $S$, the compactification of which is denoted by $\overline{S}$; $S = \overline{S} \setminus \{p_1, \ldots, p_s\}$ and $j : S \to \overline{S}$ is the inclusion map. Under such a case together with the following assumption about the representation $\rho$ of $\pi_1(S)$, one can easily get a $\rho$-equivariant harmonic map of finite energy and its behavior near the punctures $\{p_1, \ldots, p_s\}$; the purpose of this § will then be to treat these problems, which will be essential for the later development. It is worth pointing out that in [11], the authors considered these problems without the restriction of dimension. Since there the target manifolds are very general, the construction of the initial metrics are getting very complicated. In our present case, since the representation is linear, we will give an explicit construction though the basic idea is the same as [11]; more importantly, we can see explicitly why the initial map is of finite energy, and hence the analysis of [11] is applicable. The idea of the construction for the initial map will be used in the most general case, where representations need not be unipotent and also the energy of maps need not be finite; here, we should point out that the construction in [12] is also a very special one (cf. [13]).

Throughout this note, we take the Poincaré-like metric on the base manifolds, namely, on punctured disks $\Delta^*$ near the divisor (puncture) the metric is isometric to

$$\frac{dt \wedge d\bar{t}}{|t|^2(\log |t|)^2}.$$  \hspace{1cm} (5)

Where $t$ is the Euclidean complex coordinate of $\Delta^*$. Such a metric is complete, of finite volume and bounded geometry.

Let $\rho : \pi_1(S) \to GL(n, \mathbb{C})$ be a semisimple linear representation, and restrict $\rho$ to a neighborhood of $p_i$, say a small punctured disk $\Delta^*$ around $p_i$, which we call the boundary representation of $\rho$, denoted by $\rho_i$. Throughout this note, we assume that all the boundary representations of $\rho$ are unipotent. A result of Borel tells us that this is the case for VHS up to a suitable lifting (cf. e.g. [19], Lemma 4.5). Such an assumption means that if denoting the image under $\rho_i$ of the generator of $\pi_1(\Delta^*)$ by the matrix $\gamma$, then $\gamma$, under a suitable basis, can be represented by a upper-triangle matrix with all the diagonal entries being 1. As usual, $\gamma$ is called the monodromy transformation of $L_\rho$ at $p_i$ under the standard flat connection $D$. Take the logarithm of $\gamma$, denoted
by \( N \), which is also upper-triangle and all the diagonal entries of which are 0.

We now proceed to construct an initial metric on \( L \rho \), equivalently, an initial \( \rho \)-equivariant map from \( \hat{S} \) to \( \text{Gl}(n, \mathbb{C})/U(n) \). To this end, let us first give some preliminary. Let \( \mathcal{P}_n \) be the set of all positive definite hermitian symmetric matrices of order \( n \). \( \text{Gl}(n, \mathbb{C}) \) acts transitively on \( \mathcal{P}_n \) by \( g \circ H = gH^t \bar{g}, \ H \in \mathcal{P}_n, \ g \in \text{GL}(n, \mathbb{C}) \).

Obviously, the action has the isotropic subgroup \( U(n) \) at the identity \( I_n \). Thus \( \mathcal{P}_n \) can be identified with the coset space \( \text{Gl}(n, \mathbb{C})/U(n) \), and can be uniquely endowed an invariant metric\(^3\) up to some constants. In particular, under such a metric, the geodesics through the identity \( I_n \) are of the form \( \exp(tA), \ t \in \mathbb{R} \), \( A \) being a hermitian matrix.

Let the Jordan normal form of \( N \) have \( p \) Jordan blocks, denoted by \( N_j, 1 \leq j \leq p \). By the Jacobson-Morosov theorem, one can expand each block \( N_j \) into an \( \mathfrak{sl}_2 \)-triple \( \{ Y_j, N_j, N_j^{-} \} \), i.e., \( [Y_j, N_j] = 2N_j, [Y_j, N_j^{-}] = -2N_j^{-} \) and \( [N_j, N_j^{-}] = Y_j \); a theorem of Kostant tells us that such a triple, up to conjugations, is unique (cf. e.g. [15]). Take the Euclidean coordinate \( t \) on \( \Delta^* \) with \( t(p_i) = 0 \) and \( t = \text{re}^{-\sqrt{-1} i} \); also take the universal covering \( H_\alpha = \{ z = x + \sqrt{-1} y \mid x \in \mathbb{R}, \ y > -\log \alpha \} \) of \( \Delta^* \) with \( y = -\log r \), for a positive number \( \alpha < 1 \). Corresponding to a (once and for all) fixed flat sections basis of \( L \rho \), we construct the required Hermitian metric of \( L \rho \) on \( \Delta^* \) as

\[
  h_i = \begin{pmatrix}
    M_1 & 0 & & \\
    & \ddots & & \\
    0 & & \ddots & \\
    & & & M_p
  \end{pmatrix},
\]

(6)

where \( M_j = \exp(xN_j) \circ \exp((\frac{1}{2} \log |\log r|)Y_j) \). In the following, \( h_i \) is considered as both a metric and the above matrix under the fixed basis. Clearly, \( h_i \), as a map from \( H_\alpha \) to \( \mathcal{P}_n \), is \( \rho_i \)-equivariant when changing \( \log r \) into \( y \). Geometrically, this is a geodesic \( \rho_i \)-equivariant embedding of \( \Delta^* \) into \( \mathcal{P}_n \) which maps the puncture to the infinity of \( \mathcal{P}_n \), as can be explicitly seen when one embeds geodesically the upper-half plane into the real hyperbolic 3-space \( \mathbb{H}^3 \) by using the matrix model of \( \mathbb{H}^3 \) (for more details, cf. [13]). We also remark that each \( h_i \) is actually harmonic. Now one can easily extend the metrics \( \{ h_i \} \) to

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\(^3\)In terms of matrices, such an invariant metric can be defined as follows. At the identity \( I_n \), the tangent elements just are hermitian matrices; let \( A, B \) be such matrices, then the Riemannian inner product \( \langle A, B \rangle_{\mathcal{P}_n} \) is defined by \( \text{tr}(AB) \). In general, let \( H \in \mathcal{P}_n, \ A, B \) two tangent elements at \( H \), then the Riemannian inner product \( \langle A, B \rangle_{\mathcal{P}_n} \) is defined by \( \text{tr}(H^{-1}AH^{-1}B) \).
a global metric on \(L_\rho\), denoted by \(h_0\). Naturally, the corresponding map is \(\rho\)-equivariant.

We now want to show that each \(h_i\), and hence \(h_0\), is of finite energy. For simplicity, we may assume here that the Jordan normal form of \(N\) has only one Jordan block. Let \(\{Y, N, N^{-}\}\) be the corresponding \(\mathfrak{sl}_2\)-triple. The semisimple element \(Y\) can actually be described as follows. Canonically, \(\mathbb{C}^n\) has a filtration

\[
0 \subset W_{-(n-1)} \subset W_{-(n-3)} \subset \cdots \subset W_{n-3} \subset W_{n-1} = \mathbb{C}^n, \tag{7}
\]

satisfying that \(N(W_i) \subset W_{i-2}, Y\) preserves each \(W_i\), and all the quotients \(W_i/W_{i-2}\) are 1-dimensional. Then the (induced) action of \(Y\) on \(W_i/W_{i-2}\) is multiplying by \(i\). Actually, one can also choose a basis \(\{e_{-(n-1)}, e_{-(n-3)}, \ldots, e_{n-3}, e_{n-1}\}\) of \(\mathbb{C}^n\), which is compatible with the above filtration (i.e., \(Ne_j = e_{j-2}\) and \(\{e_j\}_{j \leq i}\) generates \(W_i\) and satisfies \(Ye_i = ie_i\). exp((\(\frac{1}{2}\log |\log r|)Y)), under the above basis, can then be written as

\[
\begin{pmatrix}
|\log r|^{-(n-1)} & 0 & \cdots & 0 & 0 \\
0 & |\log r|^{-(n-3)} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & |\log r|^{n-3} & 0 \\
0 & 0 & \cdots & 0 & |\log r|^{n-1}
\end{pmatrix}. \tag{8}
\]

Then, using the invariant metric of \(P_n\), a simple computation shows that the energy of \(h_i\) satisfies

\[
E(h_i) = \int_{\Delta^*} |h_i^{-1}dh_i| * 1 \leq C \int_0^\alpha |\log r|^{-2}r^{-1}dr < \infty.
\]

The above argument gives an important by-product, namely, the norm estimates near the punctures of flat sections of \(L_\rho\) under the metric \(h_0\). We continue to restrict ourselves to a small punctured disk \(\Delta^*\) near \(p_i\). As before, the fiber \((L_\rho)_x\) over \(x \in \Delta^*\) canonically has a weight filtration \(\{W_l\}_{l=-k}^k\) (\(k\) is the weight of \(N\)) arising from \(N\) and satisfying \(N(W_l) \subset W_{l-2}\); this filtration is moreover invariant w.r.t. the flat connection \(D\) of \(L_\rho\) and hence determines a filtration of \(L_\rho\) by some local subsystems, denoted by \(W_l, -k \leq l \leq k\). We remark that \(\{W_l\}_{l=-k}^k\) can be decomposed into the direct sum of some subfiltrations, each of which corresponds to a unique Jordan block of \(N\) and is of the form as in (7), if the number of the Jordan blocks of \(N\) is \(> 1\). By the construction of \(h_i\) and (8), one now has that a flat section \(v\) of \(W_l\), if not lying in \(W_{l-1}\), has the following norm estimate\(^4\)

\[
\|v\|^2_{h_0} \sim |\log r|^l, \tag{9}
\]

\(^4\)Here and afterwards, we use the notation \(\sim\) to mean "is within a bounded multiple of"
on any ray from the puncture of $\Delta^*$. 

As mentioned before, since the initial map $h_0$ is of finite energy, the analysis of [11] works; especially, one has the following

Proposition 1 Let $\rho : \pi_1(S) \to GL(n, \mathbb{C})$ be a semisimple representation all the boundary representations of which are unipotent, $h_0$ be the $\rho$-equivariant map (initial metric) constructed above. Then there exists a $\rho$-equivariant harmonic map (harmonic metric) of finite energy

$$h : \tilde{S} \to GL(n, \mathbb{C})/U(n),$$

which has the same asymptotic behavior as $h_0$ near the punctures, where $\tilde{S}$ is the universal covering of $S$; moreover, the norm of the derivative $dh$ of $h$ satisfies, when going down to $S$ and measured near the divisor with respect to the Poincaré-like metric (5) and the standard Riemannian symmetric metric on $GL(n, \mathbb{C})/U(n)$,

$$|dh|^2 \leq C|\log r|^2$$

for some constant $C > 0$, where $r$ is the radial Euclidean coordinate of $\Delta^*$.

A harmonic bundle over $S$ is said to be tame (cf. [21]) if, on any ray from a puncture of $S$, the norm under the harmonic metric of any flat section grows at most polynomially in $r$. Let $(L_\rho, h)$ be the harmonic bundle with the harmonic metric $h$ in the Proposition 1 and $D$ the flat connection. Since the harmonic metric $h$ has the same behavior as $h_0$ near the punctures, especially, the harmonic bundle is tame, our discussions in the remaining part of this subsection therefore lie in the framework of Simpson [21](except for the choice of the metric of the base manifold $S$) and are even simpler, since, as seen from the above construction, we have no the factors like $r^b$ ($b \in \mathbb{R} \setminus \{0\}$) in the norm estimates at the punctures; indeed, it is such a factor that produces infinite energy of maps. On the other hand, the behavior of $h$ also shows that the flat sections of $L_\rho$ have the same estimates as in (9) under the harmonic metric $h$. For convenience, we here write down again that for a flat section $v$ of $W_l$, if not lying in $W_l - W_{l-1}$, one has

$$\|v\|_h^2 \sim |\log r|^l,$$

on any ray from the puncture of $\Delta^*$. Furthermore, from the above construction of $h_0$, we can choose a flat basis of $L_\rho$, $\{e_1, e_2, \ldots, e_n\}$, which is compatible with the Jordan blocks of $N$ and the corresponding filtrations (in particular $Ne_j \in \{e_1, e_2, \ldots, e_n\}$), and satisfies that if $e_j \in W_l - W_{l-1}$, $\|e_j\|_{h_0} = |\log r|^l$ and that $< e_j, e_k >_{h_0} = 0, j \neq k$. Thus, under the harmonic metric $h$, we have that if $e_j \in W_l - W_{l-1}$, $\|e_j\|_h^2 \sim |\log r|^l$, and that as $r << 1$,

$$< e_j, e_k >_h \leq \epsilon(\|e_j\|_h \cdot \|e_k\|_h), \ j \neq k,$$

(11)
for some sufficiently small $\epsilon$.

As explained in §2.3, from the harmonic bundle $(L_\rho, h)$, one can derive a structure of Higgs bundle $(E, \overline{\partial} + \theta)$ on $L_\rho$ with $\theta = \partial h$. Again since $(E, \overline{\partial} + \theta)$ comes from a tame harmonic bundle, Theorem 1 of [21] works; in particular, the curvature $R_\mu$ of the connection $\mu = \partial + \overline{\partial}$ is bounded under the Poincaré-like metric (5) and the harmonic metric $h$, and the Higgs field $\theta$ is of the log-singularity. If using the Euclidean metric of $\Delta^*$ as in [21], the estimate of $R_\mu$ is read as

$$|R_\mu| \leq \frac{C}{r^2 \ln r^2},$$ \hfill (12)

Therefore, by the theory of Griffiths-Cornalba [3] (also cf. [20], §10), one can algebraically extend $E$ across the punctures, denoted by $j_! E$ as usual; in particular, $j_! E$ is coherent. Using the log-singularity of $\theta$, one also has, at $p_i$,

$$\theta : j_! E \rightarrow j_! E \otimes \Omega^1_S(\log p_i),$$ \hfill (13)

where $\Omega^1_S(\log p_i)$ is the sheaf of logarithmic differentials at $p_i$, i.e. that generated by $\frac{dt}{t}$. We call $(j_! E, \overline{\partial} + \theta)$ a logarithmic Higgs bundle.

Now, we want to give the norm estimates near the punctures of meromorphic sections of the logarithmic Higgs bundle $(j_! E, \overline{\partial} + \theta)$ under the harmonic metric $h$, by using the theory of harmonic bundles of Simpson over noncompact curves [21] together with the above norm estimates (10) for flat sections of $(L_\rho, h)$. To this end, we need some preliminary (also cf. §3 of [21]).

A filtered vector bundle over $S$ is defined as a locally free sheaf $V$ together with filtrations \( \{ V_\alpha \}_{\alpha \in \mathbb{R}} \) of $(j_! V)_{p_i}$ at the punctures, satisfying that each $V_\alpha$ is coherent, $E_\alpha \subset E_\beta$ whenever $\alpha \geq \beta$, $E_{\alpha - \epsilon} = E_\alpha$ for sufficiently $\epsilon > 0$ (left continuity), and $E_{\alpha + 1} = tE_\alpha$. For a holomorphic vector bundle $(E, \overline{\partial})$ with a Hermitian metric $K$ over $S$, one can construct a filtration $\{ E_{\alpha} \}$ of $(j_! E)_{p_i}$ for each puncture $p_i$ as follows. The germs of sections of $E_{\alpha}$ at $p_i$ are the holomorphic sections $e$ of $E$ in a small punctured disk around $p_i$, satisfying $\|e\|_K \leq C r^{\alpha - \epsilon}$ for every $\epsilon > 0$. Then, a argument of Simpson (cf. [20], §10; also [21], Proposition 3.1), by using the theory of Griffiths-Cornalba [3], tells us that if the curvature of the Hermitian connection $\partial + \overline{\partial}$ of $E$ w.r.t. $K$ satisfies the same estimate as in (12), the above construction forms a filtered vector bundle; in particular, each $E_\alpha$ is coherent.

Due to the curvature estimate of the connection $\mu = \partial + \overline{\partial}$ in (12), one can apply the above construction to our Higgs bundle $(E, \overline{\partial} + \theta)$ together with the harmonic metric $h$. Thus, we obtain the corresponding filtered vector bundle $(E, \{ E_{\alpha}^i \}_{\alpha \in \mathbb{R}, i = 1, 2, \ldots, s})$, satisfying

$$\theta : E^i_{\alpha} \rightarrow E^i_{\alpha} \otimes \Omega^1_S(\log p_i),$$ \hfill (14)
where by \( \{E^i_\alpha\}_{\alpha \in \mathbb{R}} \) we mean the filtration of \((j_*E)_p\). Furthermore, we can see that these filtrations actually satisfy

\[
E^i_\alpha = E^i_1, \quad \text{for } 0 < \alpha < 1,
\]
and hence the filtrations are \( \{E^i_q\}_{q \in \mathbb{Z}} \). This is in fact a direct consequence of the table in p.720 of [21] together with the assumption of the representation \( \rho \) being unipotent (which implies that the eigenvalues of monodromy transformations of \( L_\rho \) are always 1).

Take the quotients \( \text{Gr}^i_q(E) = E^i_q/E^i_{q+1}, \ q \in \mathbb{Z} \). We then define the residue \( \text{Res}_{p_i}E \) of \( j_*E \) at the puncture \( p_i \) as the direct sum \( \oplus_{q \in \mathbb{Z}} \text{Gr}^i_q(E) \). On the other hand, by means of (14), from \( \theta \) one has a induced homomorphism

\[
\text{Gr}^i_q(\theta) : \text{Gr}^i_q(E) \rightarrow \text{Gr}^i_q(E) \otimes \Omega^1_{\mathbb{C}}(\log p_i).
\]

Since \( E^i_{q+1} = tE^i_q \), one can in fact identify all \( \text{Gr}^i_q(E) \) (resp. \( \text{Gr}^i_q(\theta) \)), \( q \neq 0 \), with \( \text{Gr}^i_0(E) \) (resp. \( \text{Gr}^i_0(\theta) \)). Take the residue of \( \text{Gr}^i_0(\theta) \) at puncture \( p_i \) on \( \text{Gr}^i_0(E) \), denoted by \( N' \). Again, from the table of [21], we have that \( N' \) is nilpotent, i.e. all the eigenvalues of \( N' \) are 0. Moreover, an argument of Simpson (cf. [21], pp719-721; also, §5.6.7) tells us that \( N' \) can be identified with \( N \); equivalently, the corresponding weight filtrations on the corresponding residues can be identified with (for the definition of the residue of \( L_\rho \) at the punctures, cf. [21], p719). We just write \( N' \) as \( N \) afterwards.

Using this identification, we can now give the following precise description of the norm estimates near the punctures of meromorphic sections of \( j_*E \) under the harmonic metric \( h \). From the above discussion, we see that one needs only to consider those sections which lie in \( E^i_0 \) and the projections of which on \( \text{Gr}^i_0(E) \) do not vanish. Let us first assume that the residue \( N \) of \( \text{Gr}^i_0(\theta) \) on \( \text{Gr}^i_0(E) \) has only one Jordan block. One can then choose some holomorphic sections \( \{e_{-(n-1)}, e_{-(n-3)}, \ldots, e_{n-3}, e_{n-1}\} \) of \( E^i_0 \), the projections of which on \( \text{Gr}^i_0(E) \), still denoted by \( \{e_{-(n-1)}, e_{-(n-3)}, \ldots, e_{n-3}, e_{n-1}\} \), form a basis of \( \text{Gr}^i_0(E) \); moreover, as a basis of \( \text{Gr}^i_0(E) \), it is compatible with the weight filtration \( W_{-(n-1)} \subset W_{-(n-3)} \subset \cdots \subset W_{n-3} \subset W_{n-1} \) of \( N \) on \( \text{Gr}^i_0(E) \) (i.e., \( Ne_j = e_{j-2} \) and \( \{e_j\}_{j \leq l} \) generates \( W_l \)). Then, one has, for \( e_l \),

\[
\|e_l\|_h^2 \sim |\log r|^l.
\]

Furthermore, the cross-terms can be controlled very well, i.e., as \( r << 1 \),

\[
< e_j, e_k >_h \leq \epsilon (\|e_j\|_h \cdot \|e_k\|_h), \quad j \neq k,
\]

for some sufficiently small \( \epsilon \), as can be obtained from the control (11) of the corresponding cross-terms for \( L_\rho \) by means of the above identification and the
same asymptotic behavior of \( h \) as \( h_0 \). When \( N \) has at least two Jordan blocks, we can do the same discussion as before (11).

In general, let \( \{W_i\}_{i=-k}^k \) be the weight filtration of \( N \) on \( \text{Gr}^i_0(E) \), \( e \) a holomorphic section of \( E_0^i \), the projections of which on \( \text{Gr}^i_0(E) \) lies in \( W_i \) but not in \( W_{i-1} \). Then one has

\[
\|e\|_h^2 \leq C|\log r|^l.
\]

(19)

Here, we use \( \leq \) instead of \( \sim \), since a holomorphic section can have zero-points in general.

3 The \( L^2 \)-cohomology and the \( L^2 \)-Higgs cohomology

In this section, we continue to assume that the representation \( \rho \) is semisimple and unipotent at infinity, i.e. monodromy transformations of \( L_\rho \) around the punctures are unipotent. Therefore we have the corresponding harmonic bundle \((L_\rho, h)\) and the Higgs bundle \((E, \theta)\) with \( \theta = \partial h \) of the log-singularity at the punctures (Proposition 1); we also have various norm estimates: (10), (17) (19) at the punctures w.r.t. the metric \( h \) for flat sections of \( L_\rho \) or holomorphic sections of \( E \); the most important thing is that one can identify the logarithmic monodromy transformations \( N \) with the residues of \( \theta \) at the punctures [21]. The purpose of this section is then consider the \( L^2 \)-cohomologies with coefficients in \( L_\rho \) or \( E \) and their relationship. The problem in the case of VHS was considered by S. Zucker [24].

3.1 The \( L^2 \)-cohomology: The \( L^2 \)-Poincaré Lemma

As before, denoting the inclusion map of \( S \) in \( \overline{S} \) by \( j \), one has the direct image sheaf \( j_*L_\rho \) of the local system \( L_\rho \) on \( S \) and then the Čech cohomology \( H^*(\overline{S}, j_*L_\rho) \). On the other hand, using the Poincaré-like metric \( \omega \) (5) on \( S \) and the harmonic metric \( h \) on \( L_\rho \), one can define a complex \( \mathcal{A}_{(2)}(L_\rho, D) \) of sheaves over \( \overline{S} \) as follows. Let \( U \) be an open subset of \( \overline{S} \). Then \( \mathcal{A}_{(2)}(L_\rho)(U) \) is defined as the set of

\( L_\rho \)-valued \( i \)-forms \( \eta \) on \( U \cap S \), with measurable coefficients and measurable exterior derivative \( D\eta \), such that \( \eta \) and \( D\eta \) have finite \( L^2 \) norm on \( K \cap S \), for any compact subset \( K \) of \( U \), where \( D \) is the canonical flat connection of \( L_\rho \).

Since the sheaves \( \mathcal{A}_{(2)}(L_\rho) \) can regarded as a module over the sheaf of germs of local complex valued smooth functions on \( \overline{S} \), using the partition of unit on \( \overline{S} \), it is easy to show that these sheaves are fine. So the cohomology of the
complex of global sections computes the hypercohomology of \( \{\mathcal{A}_{(2)}(L_\rho), D\} \) (cf. e.g. [6]), namely
\[
H^*(\mathcal{S}, \{\mathcal{A}_{(2)}(L_\rho), D\}) \cong H^*(\{\Gamma(\mathcal{A}_{(2)}(L_\rho)), D\}).
\]

We call this cohomology the \( L^2 \)-cohomology of \( \mathcal{S} \) with values in \( L_\rho \), denoted by \( H^*_{(2)}(\mathcal{S}, L_\rho) \). The purpose of this subsection is then to establish the following identification

**Theorem 1** There exists a natural identification

\[
H^*(\mathcal{S}, j_* L_\rho) \cong H^*_{(2)}(\mathcal{S}, L_\rho).
\]

**Remarks.** If \( L_\rho \) comes from a variation of Hodge structure (VHS), the identification was proved by S. Zucker [24]. In the higher dimensional case, instead of the Čech cohomology, one needs to consider the intersection cohomology [4]; the identification in the case of VHS was proved by Cattani-Kaplan-Schmid [1] and Kashiwara-Kawai [16] independently. In general, one has the following

**Conjecture 1** There exists a natural identification

\[
H^*_{\text{int}}(\mathcal{S}, j_* L_\rho) \cong H^*_{(2)}(\mathcal{S}, L_\rho).
\]

Canonically, the proof of Theorem 1 is reduced to prove

**Theorem 2** (The \( L^2 \)-Poincaré lemma) The complex \( \{\mathcal{A}_{(2)}(L_\rho), D\} \) is a resolution of \( j_* L_\rho \). This is equivalent to saying that

1) \( j_* L_\rho = \{ \eta \in \mathcal{A}_{(2)}^0(L_\rho) \mid D\eta = 0 \} \);

2) the differential \( D \) satisfies the Poincaré lemma, i.e., if an \( i \)-form \( \eta \) in \( \mathcal{A}_{(2)}^i(L_\rho) \) is \( D \)-closed, then there exists an \( i-1 \)-form \( \sigma \in \mathcal{A}_{(2)}^{i-1}(L_\rho) \) satisfying \( D\sigma = \eta \), for \( i=1,2 \).

In order to prove the above theorem, we need the following lemma, which contains the basic analysis needed in the proof of Theorem 2.

**Lemma 1** Let \( V \) be a constant one dimensional local system over \( \Delta^* \), with generator \( v \), and assume that the corresponding line bundle has a Hermitian metric with \( ||v||^2 \sim |\log r|^k \), where \( r \) is the Euclidean radius. Then the coho-
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-modology sheaves for $A_{(2)}(V)$ have stalks at the origin,

$$
\begin{align*}
\mathcal{H}^0( A_{(2)}(V) ) &= \begin{cases} 
 V & \text{if } k \leq 0 \\
 0 & \text{if } k > 0,
\end{cases} \\
\mathcal{H}^1( A_{(2)}(V) ) &= \begin{cases} 
 \frac{dt}{t} \otimes V & \text{if } k \leq -2 \\
 0 & \text{if } k \geq -1, k \neq 1 \\
 \mathcal{M}_1 dr \otimes v & \text{if } k = 1,
\end{cases} \\
\mathcal{H}^2( A_{(2)}(V) ) &= \begin{cases} 
 0 & \text{if } k \neq -1 \\
 \mathcal{M}_1 dr \wedge \frac{dt}{t} \otimes v & \text{if } k = -1,
\end{cases}
\end{align*}
$$

where $\mathcal{M}_1$ is defined as

\[
\{ \text{measurable functions } f : f^A \left| f(r) \right|^2 \left| \log r \right| (rdr) < \infty \text{ for some } A < 1 \} \\
\{ f : f = u' \text{ weakly with } \int_0^A |u(r)|^2 \left| \log r \right|^{-1} (r^{-1}dr) < \infty \text{ for some } A < 1 \}.
\]

Proof. cf. [24], Proposition 6.6.

Proof of Theorem 2. On $S$, the exactness is standard. So the trouble comes from the punctures $p_1, \ldots, p_s$, and since we are working with sheaves, from now on we just localize the problem to a small punctured disk $\Delta^* \subset \Delta$ around the puncture $p_i$.

The proof of 1). This is equivalent to showing that an $L^2$ flat section should be a section of $j_* \mathcal{L}_\rho$—an invariant section of $\gamma$, which equivalently lies in the kernel of $N$. To this end, denote the image of the generator of $\pi_1(\Delta^*)$ under the representation $\rho$ by $\gamma$, which, by the assumption, is unipotent, log $\gamma$ by $N$, which is nilpotent. For a (multi-valued) flat section $v$ of $\mathcal{L}_\rho$, setting

$$
\tilde{v} = \exp \left( \frac{1}{2\pi \sqrt{-1}} N \log t \right) v,
$$

which is single-valued and $d''$-holomorphic ($D = d' + d''$, the (1, 0) and (0, 1)-part respectively), one then has the canonical extension $\mathcal{L}_\rho$ of $L_\rho$ to $\mathcal{S}$ ($= \Delta$ locally) when $L_\rho$ is considered as a $d''$-holomorphic bundle: sections of $\mathcal{L}_\rho$ at the origin are generated by sections of the form $\tilde{v}$. On the other hand, as seen in §2.4, since $N$ is nilpotent, each fiber of $L_\rho$ canonically has a weight filtration $\{ W_i \}_{i=-k}^k$ ($k$ is the weight of $N$) which is $D$-invariant and which therefore determines a filtration of $L_\rho$ by some local subsystems, denoted by $\mathcal{W}_i$ and the corresponding extension by $\mathcal{W}_i$.

Let $v$ be a flat section of $\mathcal{W}_i$, but not one of $\mathcal{W}_{i-1}$, briefly denoted by $v \in \mathcal{W}_i - \mathcal{W}_{i-1}$, so that $Nv \in \mathcal{W}_{i-2} - \mathcal{W}_{i-3}$ if $Nv$ does not vanish. By the norm estimates (10) of flat sections, we have

$$
\|v\|_h^2 \sim |\log r|^i \text{ and } \|Nv\|_h^2 \sim |\log r|^{i-2}.
$$
Since $N$ is nilpotent of weight $k$, we have
\[
\tilde{v} = \exp\left(\frac{1}{2\pi\sqrt{-1}}N\log t\right)v \\
= v + \left(\frac{\log t}{2\pi\sqrt{-1}}\right)Nv + \frac{1}{2!}\left(\frac{\log t}{2\pi\sqrt{-1}}\right)^2N^2v + \cdots + \frac{1}{k!}\left(\frac{\log t}{2\pi\sqrt{-1}}\right)^kN^kv;
\]
so, we have the following estimate
\[
\|\tilde{v}\|_h^2 \sim |\log r|^l. \quad (21)
\]
We now claim that a $d''$-holomorphic section of $L_{\rho}$ on $\Delta$ is $L^2$ iff it lies in $W_0 + tL_{\rho}$. This can be proved as follows, by using the above estimate (21). We first show when the sections are $L^2$ of the form $\sigma = t^m\tilde{v}$ for $v \in W_l - W_{l-1}$, $m \in \mathbb{Z}$. By the above estimate, the convergence or divergence of the $L^2$-norm of $\sigma$ is equivalent to that of the following integral ($0 < A < 1$)
\[
\int_0^{2\pi} \int_0^A r^{2m}|\log r|^l \frac{r dr d\theta}{r^2 \log^2 r}.
\]
Clearly, the integral is convergent precisely when $l \leq 0$ and $m = 0$, or $m \geq 1$; this implies that sections of $W_0 + tL_{\rho}$ are $L^2$.

The remained is to show that any section $\sigma$ of $W_l - W_0$, $l \geq 1$ is not $L^2$ if $\sigma \notin tL_{\rho}$. To this end, we first assume that $N$ has only one Jordan block. Then a section $\sigma$ of $W_l - W_0$ can be written as
\[
\sigma = f(t)\tilde{v} + \tilde{v}_{l-2},
\]
where $f(t)$ is a holomorphic function on $\Delta$, $v \in W_l - W_{l-2}$, $\tilde{v}$ is defined by (20), and $\tilde{v}_{l-2} \in W_{l-2}$. Since we assume that $\sigma \notin tL_{\rho}$, without loss of generality, we can assume $f(0) \neq 0$; since if not, $f(t)\tilde{v} \in tL_{\rho}$ and hence is $L^2$; thus the problem is reduced to consider if $\tilde{v}_{l-2}$ is $L^2$. Now we want to show that $|\log r|^{-l+1}\|f(t)\tilde{v} + \tilde{v}_{l-2}\|_h^2$ is not integrable on $\Delta$ under the Poincaré-like metric, and hence $\sigma$ is not $L^2$. This is easily obtained by using the estimate (21) and noting
\[
\|f(t)\tilde{v} + \tilde{v}_{l-2}\|_h^2 \geq \|f(t)\tilde{v}\|_h^2 - \|\tilde{v}_{l-2}\|_h^2 \sim |\log r|^l - |\log r|^{l-2}.
\]
In general, if $N$ has at least two Jordan blocks, $\sigma$ is of the form
\[
\sigma = \sum_j f_j(t)\tilde{v}_j + \tilde{v}'_{j-1},
\]
where each $\tilde{v}_j$ belongs to $W_l - W_{l-1}$ and corresponds to a unique Jordan block of $N$ as in the above case of one Jordan block, $f_j$'s are holomorphic on $\Delta$, and
\( \tilde{v}'_{i-1} \in \mathfrak{W}_{i-1} \). Again, we can assume that all \( f_j(0) \neq 0 \). By using the same argument as above, we only need to show \( \| \sum_j f_j(t) \tilde{v}_j \|_h^2 \sim | \log r |^{2} \). To this end, it is sufficient to show that

\[
\| \sum_j f_j(t) \tilde{v}_j \|_h^2 \sim \sum_j \| f_j(t) \tilde{v}_j \|_h^2 ;
\]

this can be obtained by means of the construction of the initial metric \( h_0 \) (cf. (6) and (8)) and that \( h \) has the same asymptotic behavior as \( h_0 \); more precisely, since \( < f_j(t) \tilde{v}_j, f_k(t) \tilde{v}_k >_{h_0} = 0, j \neq k \), the cross-terms \( | < f_j(t) \tilde{v}_j, f_k(t) \tilde{v}_k >_h | \) can be controlled by

\[
\epsilon \| f_j(t) \tilde{v}_j \|_h \| f_k(t) \tilde{v}_k \|_h \leq \frac{\epsilon}{2} \left( \| f_j(t) \tilde{v}_j \|_h^2 + \| f_k(t) \tilde{v}_k \|_h^2 \right)
\]

for some sufficiently small \( \epsilon \) as \( |t| << 1 \). (A similar discussion will also be used in the following proof of 2), where, for simplicity of discussion, we want to reduce the discussion to the case that \( N \) has only one Jordan block.) Thus, we finish the proof of the claim above.

In order to finish the proof of 1), we next need to show that \( d' \)-closed sections of \( \mathfrak{W}_0 + i \mathcal{L}_p \) are generated by sections of the form \( \tilde{v} \) satisfying \( N \tilde{v} = 0 \), which is just the first part of Proposition 4.1 of [24], where \( \nabla \) is just our \( d' \). We remark that although \( L_p \) of Proposition 4.1 in [24] comes from a VHS, the discussion still works in the present setting. Thus, we finish the proof of 1).

The proof of 2). In order to make a reduction of the problem to the case that \( N \) has only one block, we first do some discussion for \( L^2 \)-closed \( i \)-forms valued in \( \mathfrak{W}_p \) in \( \mathcal{A}'_{(2)}(L_p) \) (for the case of \( i = 0 \), this in fact have done in the proof of 1)). As mentioned before, we restrict ourself to a small punctured disk. Let \( N \) have the Jordan blocks \( N_1, N_2, \ldots, N_p, p > 1 \). Clearly, corresponding to the these blocks, \( \mathbb{C}^n \) can be decomposed into the direct sum \( \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \oplus \cdots \oplus \mathbb{C}^{n_p} \), satisfying that \( N_j \) acts trivially on \( \mathbb{C}^{n_k}, j \neq k \), and acts on \( \mathbb{C}^{n_j} \) with the graded pieces of the corresponding weight filtration being one-dimensional. Correspondingly, \( L_p \) can be decomposed into the direct sum \( L_1 \oplus L_2 \oplus \cdots \oplus L_p \) of some \( D \)-invariant subsystems. Thus, an \( i \)-form \( \phi \) valued in \( L_p \) can be decomposed into the sum \( \phi = \phi_1 + \phi_2 + \cdots + \phi_p \) of some \( i \)-forms \( \phi_j \) valued in \( L_j, 1 \leq j \leq p \), and \( \phi \) is \( D \)-closed iff each \( \phi_j \) is \( D \)-closed. In order to make the reduction mentioned above, we only need to show that if \( \phi \) is \( L^2 \), then so is each component \( \phi_j \). This again is obtained by means of the construction of \( h_0 \) and the same asymptotic behavior of \( h \) as \( h_0 \) in §2.4, where the block \( M_j \) in (6) can be considered as a metric on \( L_j \). More precisely, one can choose a basis \( \{ e_1, e_2, \cdots, e_p \} \) of \( L_p \), which is compatible with the decomposition \( L_p = L_1 \oplus L_2 \oplus \cdots \oplus L_p \) and weight filtrations on \( L_j \) w.r.t. \( N_j, 1 \leq j \leq p \), and satisfies \( < e_j, e_k >_{h_0} = 0, j \neq k \); so the cross-terms
| < \epsilon_j, \epsilon_k > h |, as in the proof of 1), can be controlled by \( \epsilon(\sum_{j,k} \epsilon_j^2 + \epsilon_k^2) \) for some sufficiently small \( \epsilon \). Thus, when the form \( \phi \) valued in \( L_p \) is written as \( \psi_1 + \psi_2 + \cdots + \psi_n \) with \( \psi_j \) valued in the (smooth) line bundle generated by \( e_j \), \( \| \phi \|_{\omega,h} \) is given by \( \| \psi_1 \|_{\omega,h}^2 + \cdots + \| \psi_n \|_{\omega,h}^2 \); in particular, \( \| \phi \|_{\omega,h}^2 \) is given by \( \| \psi_1 \|_{\omega,h}^2 + \cdots + \| \psi_n \|_{\omega,h}^2 \), and hence each \( \phi_j \) is \( L^2 \).

By the above argument, without loss of generality, we now assume that \( \alpha \), acting on \( \mathbb{C}^{m+1} (n = m + 1) \), has only one Jordan block. As before, one then has a weight filtration of \( \mathbb{C}^{m+1} \):

\[
0 \subset W_{-m} \subset W_{-m-2} \subset \cdots \subset W_{-2} \subset W = \mathbb{C}^{m+1},
\]

which satisfies that the quotient \( \text{Gr}_W^W = W_i/W_{i-2} \) is of dimension 1 and \( NW_i = W_{i-2} \). Correspondingly, one has the invariant subbundles of \( L_{\rho} \),

\[
0 \subset W_{-m} \subset W_{-m-2} \subset \cdots \subset W_{-m-2} \subset W = L_{\rho},
\]

and hence the corresponding filtration of \( A_{(2)}(L_{\rho}) \)

\[
0 \subset A_{(2)}(W_{-m}) \subset A_{(2)}(W_{-m-2}) \subset \cdots \subset A_{(2)}(W_{-m-2}) \subset A_{(2)}(L_{\rho}),
\]

which is clearly \( D \)-invariant. For simplicity, we denote \( A_{(2)}(W_{-i}) \) by \( K_i \); one has then a filtered complex

\[
K_{-m} \supset K_{-m+2} \supset \cdots \supset K_{m-2} \supset K_m \supset 0.
\]

Consider the quotient \( K_i/K_{i+1} \), which is clearly \( A_{(2)}(\text{Gr}_W^W(L_{\rho})) \). We now have the spectral sequence \( (E_r, d_r)_{r \geq 1} \) at the filtered complex \( \{ K_i, D \}_{i \geq -m} \), which, by the theory of spectral sequences, converges to the cohomology of \( K_{-m} \), namely the sheaf cohomology of \( \{ A_{(2)}(L_{\rho}), D \} \). To prove 2), we need to analyze the sequence \( (E_r, d_r)_{r \geq 1} \).

By the definition of spectral sequences,

\[
E_1^{p,q} = H^{p+q}(\text{Gr}_p K^*) = \mathcal{H}^{p+q}(A_{(2)}(\text{Gr}_p^W(L_{\rho}))),
\]

where \( \text{Gr}_p K^* = K_p/K_{p+1} \). We will show that the differential \( d_1 : E_1^{p,q} \to E_1^{p+1,q} \) is trivial, as can be observed as follows. For convenience of the following discussion, we write down the following diagram (note that by "diagram" we do not mean that the horizontal sequences are exact, we only want to show the meanings of the maps in question)

\[
\begin{array}{ccc}
A_{(2)}^{p+q}(W_{-p-1}) & \xrightarrow{i} & A_{(2)}^{p+q}(W_{-p}) \xrightarrow{\text{Proj}} A_{(2)}^{p+q}(\text{Gr}_p^W(L_{\rho})) \\
\downarrow D & & \downarrow D \\
A_{(2)}^{p+q+1}(W_{-p-1}) & \xrightarrow{i} & A_{(2)}^{p+q+1}(W_{-p}) \xrightarrow{\text{Proj}} A_{(2)}^{p+q+1}(\text{Gr}_p^W(L_{\rho}))
\end{array}
\]
where $i$ and $\text{Proj}$ are the inclusion and the projection respectively, the third $D$ is the induced differential from the previous two $D$’s by taking the quotients. Let $\phi \otimes \tilde{v} \in \mathcal{A}_{(2)}^{p,q}(W_{-p})$ represent a cohomology class of $\mathcal{H}^{p+q}(\mathcal{A}_{(2)}(\text{Gr}_{-p}^W(L_\rho)))$. (Since we assumed that $N$ has only one Jordan block, a cohomology class of $\mathcal{H}^{p+q}(\mathcal{A}_{(2)}(\text{Gr}_{-p}^W(L_\rho)))$ can always be represented as such a single tensor.) In particular, since $D\tilde{v} \in \Omega^1(W_{-p-2})$, $d\phi = 0$, here the $d$ is the differential on regular functions over $\Delta^*$. So $D(\phi \otimes \tilde{v}) \in \mathcal{A}_{(2)}^{p+q+1}(W_{-p-2})$, which represents a trivial cohomology class in $\mathcal{H}^{p+q+1}(\mathcal{A}_{(2)}(\text{Gr}_{-p-2}^W(L_\rho))) = E_1^{p+1,q}$, namely $d_1(\phi \otimes \tilde{v}) = 0$.

Next, let us consider $d_2$. Since $d_1 = 0$, $E_2^{p,q} = E_1^{p,q}$. Similar to the above argument, taking a form $\phi \otimes \tilde{v} \in \mathcal{A}_{(2)}^{p,q}(W_{-p})$, which represents a cohomology class of $\mathcal{H}^{p+q}(\mathcal{A}_{(2)}(\text{Gr}_{-p}^W(L_\rho)))$, $D(\phi \otimes \tilde{v}) = (-1)^{p+q} \frac{1}{2\pi i} (\phi \wedge \frac{d}{\sqrt{-1}} \otimes N\tilde{v})$ and then lies in $\mathcal{A}_{(2)}^{p+q+1}(W_{-p-2})$, which represents a cohomological class of $E_2^{p+2,q-1} = \mathcal{H}^{p+q+1}(\mathcal{A}_{(2)}(\text{Gr}_{-p-2}^W(L_\rho)))$. So, $d_2$ is induced by $\frac{1}{2\pi i} \frac{dt}{t} N$.

Let us now consider the kernel and image of $d_2$. Applying Lemma 1 to $E_2^{p,q} = \mathcal{H}^{p+q}(\mathcal{A}_{(2)}(\text{Gr}_{-p}^W(L_\rho)))$, we get that the only (possibly) nontrivial terms at $E_2$ are

$$\{E_2^{p,-p}\}_{p \geq 0}, \quad \{E_2^{p+2,-p-1}\}_{p \geq 0}, \quad E_2^{-1,2}, \quad \text{and} \quad E_2^{1,1}.$$ 

Furthermore, from the above argument together with Lemma 1, we obtain that $d_2 : E_2^{p,-p} \to E_2^{p+2,-p-1}, p \geq 0$ if $E_2^{p+2,-p-1}$ is nontrivial, i.e. $p \leq m-2$ and $d_2 : E_2^{-1,2} \to E_2^{1,1}$ are isomorphisms and that $d(E_2^{p+2,-p-1}) = 0, p \geq 0$ and $d_2(E_2^{1,1}) = 0$.

Summing all the above up, the only possible nontrivial terms at $E_3$ are $E_3^{m-1,-m+1}, E_3^{m,-m}$. Thus the spectral sequence $\{E_r,d_r\}_{r \geq 1}$ degenerates at $E_3$ and the only possible terms are $E_3^{m-1,-m+1}, E_3^{m,-m}$. Therefore, by the theory of spectral sequences of filtered complexes, $\mathcal{H}^i(\mathcal{A}_{(2)}(L_\rho)) = 0, i = 1, 2$; namely, if an $i$-form $\eta$ in $\mathcal{A}_{(2)}(L_\rho)$ is $D$-closed, then there exists an $i-1$-form $\sigma \in \mathcal{A}_{(2)}^{i-1}(L_\rho)$ satisfying $D\sigma = \eta$, for $i = 1, 2$. The proof of 2) is finished.

### 3.2 The $L^2$-Higgs cohomology: The $L^2-\overline{\partial}$-Poincaré Lemma

As seen in §2.4, the harmonic metric $h$ (the $\rho$-equivariant harmonic map) on $L_\rho$ induces the structure of a Higgs bundle on $L_\rho$: $(E, D' = \overline{\partial} + \theta)$, satisfying $D = D' + D''$ with $D' = \partial + \overline{\partial}$; moreover, the Hermitian connection $\partial + \overline{\partial}$ w.r.t. the metric $h$ has bounded curvature under the metric $h$ and the Poincaré-like metric so that $E$ can be analytically extended to $\mathcal{S}$, denoted by $j_*, E$ as usual, which is especially coherent. Furthermore, $\theta$ has a log-singularity, i.e. $\theta \sim \frac{dt}{t} N$. It is especially worth pointing out that by an argument of Simpson (cf. [21]), the residue $N$ of $\theta$ here coincides with the logarithmic monodromy $N$ in the local system $L_\rho$; so although under different bundle structures, we have
the same weight filtration under certain suitable identification. Throughout this subsection, we consider the Higgs bundle \((E, D'' = \overline{\partial} + \theta)\) together with the harmonic metric \(h\), satisfying that the meromorphic sections of \(E\) at the punctures have the norm estimates (17), just forgetting that it comes from the local system corresponding to the representation \(\rho\).

As in the previous subsection, using the Poincaré-like metric \(\omega\) on \(S\) and the harmonic metric \(h\) on \((E, D'')\), one can similarly define a complex \(\{\mathcal{A}_{(2)}(E), D''\}\) of fine sheaves on \(\overline{S}\). Let \(U\) be an open subset of \(\overline{S}\). Then \(\mathcal{A}_{(2)}^\ast(E)(U)\) is defined as the set of

\[ j_* E\text{-valued } i\text{-forms } \eta \text{ on } U \cap S, \text{ with measurable coefficients and measurable exterior derivative } \overline{\partial} \eta, \text{ such that } \eta \text{ and } D'' \eta \text{ have finite } L^2 \text{ norm on } K \cap S, \text{ for any compact subset } K \text{ of } U. \]

**Remarks.** 1) The Higgs condition \((D'')^2 = 0\) makes \(\{\mathcal{A}_{(2)}(E), D''\}\) a complex, which is actually a complex of certain differential forms valued in \(j_* E\), and hence fine. 2) Due to following lemma, it is actually sufficient to assume that \(\eta\) and \(\overline{\partial} \eta\) are \(L^2\) in the above definition. 3) Since the sheaves are fine, so again the hypercohomology \(H^\ast(\{\mathcal{A}_{(2)}(E), D''\})\) is computed by the cohomology \(H^\ast(\{\Gamma(\mathcal{A}_{(2)}(E)), D''\})\) of the complex of global sections; we call it the \(L^2\)-Higgs cohomology of \(\overline{S}\) valued in the Higgs bundle \((E, D = \overline{\partial} + \theta)\), denoted by \(H^\ast_{(2)}(\overline{S}, E)\).

**Lemma 2** \(\theta\) is an \(L^2\)-bounded operator.

**Proof.** As mentioned before, \(\theta \sim \frac{d}{dt} N\) near the punctures. So it suffices to show that \(\frac{d}{dt} N\) is \(L^2\)-bounded near the punctures. To this end, as done in the proof of Theorem 2, we first do some discussion for \(L^2\)-forms valued in \(E\), as will also be used in the proof of the following theorem 3. As seen in \(\S 2.4\), we can choose some holomorphic sections of \(E^0_0 \subset (j_* E)_p\), the projections of which on \(Gr^0_0(E)\) form a basis, still denoted by \(\{e_1, e_2, \ldots, e_n\}\), of \(Gr^0_0(E)\) near a puncture \(p_i\), which is compatible with the Jordan blocks of \(N\) and the filtration \(\{W_i\}_{i=1-k}^k\) of \(N\), in particular \(N e_i \in \{e_1, e_2, \ldots, e_n\}\). Moreover, one has

\[\|e_i\|_h^2 \sim |\log r|^l, \text{ if } e_i \in W_i - W_{i-1};\]

and as \(r \ll 1\),

\[|e_i, e_j|_h \leq \epsilon(\|e_i\|_h \cdot \|e_j\|_h),\]

for some sufficiently small \(\epsilon > 0\). Thus, a form \(\phi\) valued in \(E\) can be written as the sum \(\phi_1 + \phi_2 + \cdots + \phi_n\) of some forms \(\phi_i\) valued in the invertible sheaf generated \(e_i\); especially, by the estimate for the cross-terms, one has \(\|\phi\|_h^2 \sim \|\phi_1\|_h^2 + \cdots + \|\phi_n\|_h^2\). So, if \(\phi\) is \(L^2\), each component \(\phi_i\) is also \(L^2\). On the other hand, since \(N\) lowers weights by 2, we have that \(\|Ne_i\|_h^2 \sim |\log r|^{-2}\|e_i\|_h^2\) if \(Ne_i \neq 0\); while
∥dt∥_T^2 = |\log r|^2. So, ∥dt N e_i∥_h^2 ∼ ∥e_i∥^2, and hence ∥dt N φ_i∥_h^2 ∼ ∥φ_i∥^2. The proof of the lemma is finished.

On the other hand, based on the above lemma, one can also define a sub-complex of \{A(2) \leq E, D''\}—the L^2-holomorphic Dolbeault complex \{Ω(2) \leq E, \theta\} as follows: \Ω(2)_i(U) is defined as the set of \j^*E-valued meromorphic i-forms \eta on U \cap S such that \eta has finite L^2 norm on K \cap S, for any compact subset K of U.

θ ∧ θ = 0 makes \{Ω(2) \leq E, \theta\} again a complex, which is actually a complex of certain meromorphic differential forms valued in \j^*E, and hence coherent. We call the hypercohomology \H^*(\{Ω(2) \leq E, \theta\}) the L^2-Dolbeault cohomology of \S valued in the Higgs bundle (E, D = \∂ + \theta).

The purpose of this subsection is then to show that the above two complexes have the same hypercohomologies; more precisely

**Theorem 3** (L^2-\O-Poincaré lemma) The inclusion

\[ i: \{Ω(2) \leq E, \theta\} \hookrightarrow \{A(2) \leq E, D''\} \]

is a quasi-isomorphism; and hence one has

\[ \H^*(\{Ω(2) \leq E, \theta\}) \cong H^*_c(S, E). \]

**Remarks.** In the case when E comes from a VHS, the theorem was showed by S. Zucker [24] (for the case of curves) and Jost-Yang-Zuo [14] (for the general case).

In order to prove the theorem, we need the following lemma, which has proved in [24], Proposition 6.4.

**Lemma 3** Let V be a holomorphic line bundle on Δ^* with generating section \sigma, and with a Hermitian metric satisfying

\[ ∥\sigma∥^2 \sim |\log r|^k, \ k \in \mathbb{Z}, \ k \neq 1. \]

Then for every germ of an L^2 (0, 1)-form \phi = f d\bar{τ} \otimes \sigma at the puncture, there exists an L^2 section u \otimes \sigma with \overline{\partial}u = f d\bar{τ}.

**Proof of Theorem 3.** Similar to the proof of Theorem 2, the difficulty again comes from the punctures \{p_j\}, so we again restrict ourself to a small enough punctured disk Δ^* at p_j. In order to prove the theorem, we only need to show that the inclusion i induces an isomorphism between the corresponding cohomology sheaves at the puncture; by the argument of standard homological algebra, this is equivalent to showing that for any D''-closed r-form
\( \phi \in \mathcal{A}_{(2)}^r(E) \) on a neighborhood \( U \subset \Delta \) of the puncture \( p_j \), there is a \( \theta \)-closed \( r \)-form \( \eta \in \Omega_{(2)}^r(E) \) and an \( r-1 \)-form \( \psi \in \mathcal{A}_{(2)}^{r-1}(E) \) on (a possibly smaller) \( U \) satisfying \( \phi = \eta + D'' \psi \), \( r = 0, 1, 2 \). In the following proof, we continue to use the notation and the discussions in the proof of Lemma 2.

**Case of** \( r = 0 \). Clearly, a \( D'' \)-closed 0-form \( \phi \) is exactly a meromorphic section of \( j_* E \) and also \( \theta \)-closed, so the case of \( r = 0 \) is obtained.

**Case of** \( r = 2 \). In this case, a form \( \phi \in \mathcal{A}_{(2)}^2(E) \) can be written as the sum of some forms \( \phi_i = (\phi_i' \wedge \frac{dt}{T}) \otimes e_i \), here \( \phi_i' \) is a complex-value \((0,1)\)-form. On other hand, by the discussion in the proof of Lemma 2, each component \( \phi_i \) is also \( L^2 \). Considering \( \frac{dt}{T} \otimes e_i \) as a generator of a holomorphic line bundle, its norm satisfies, under the harmonic metric and the Poincaré-like metric,

\[
\| \frac{dt}{T} \otimes e_i \|_{h, \theta}^2 \sim \| \log r \|^k,
\]

if \( e_i \in W_{k-2} - W_{k-3} \).

When \( k \neq 1 \), by the above lemma, there exists an \( L^2 \)-form \( u \frac{dt}{T} \otimes e_i \) satisfying

\[
\bar{\partial}(u \frac{dt}{T} \otimes e_i) = \phi' \wedge \frac{dt}{T} \otimes e_i,
\]

and hence \( D''(u \frac{dt}{T} \otimes e_i) = (\phi' \wedge \frac{dt}{T}) \otimes e_i = \phi_i \), i.e., \( \phi_i \) is \( D'' \)-coclosed.

If \( k = 1 \), i.e., \( \| e_i \|_{h}^2 \sim \| \log r \|^{-1} \), equivalently, \( e_i \in W_{-1} - W_{-2} \); by means of the compatibilty of the weight filtration of \( N \) and \( \{ e_1, \ldots, e_n \} \), there exists an \( e_j \in \{ e_1, \ldots, e_n \} \) satisfying \( Ne_j = e_i \), \( e_j \in W_{1} - W_{0} \), and hence \( \frac{dt}{T} N(e_j) = \frac{dt}{T} e_i \). Since \( \theta \sim \frac{dt}{T} N \), an approximating discussion also shows that there exists a holomorphic section \( e'_i \in W_{1} - W_{0} \) with \( \theta(e'_i) = \frac{dt}{T} \otimes e_i \); so \( D''(\phi'_i \otimes e'_i) = (\phi'_i \wedge \frac{dt}{T}) \otimes e_i = \phi_i \). Finally, that \( (\phi'_i \wedge \frac{dt}{T}) \otimes e_i = \phi_i \) is \( L^2 \) implies that the \((0,1)\)-form \( \phi'_i \otimes e'_i \) is \( L^2 \), and hence \( \phi_i \in \mathcal{A}_{(2)}^1(E) \). Summing all the above, we have that any form of \( \mathcal{A}_{(2)}^2(E) \) is \( D'' \)-coclosed. The case of \( r = 2 \) is obtained.

**Case of** \( r = 1 \). We now turn to the case of \( r = 1 \). In order to make the proof clearer, we can refer from time to time to the following diagram (Here, again, by ”diagram” we do not mean that its horizontal sequences are exact.)

\[
\begin{array}{ccc}
\Omega_{(2)}^0(E) & \overset{i}{\longrightarrow} & \mathcal{A}_{(2)}^{0,0}(E) \\
\downarrow{\theta} & & \downarrow{\theta} \\
\Omega_{(2)}^1(E) & \overset{i}{\longrightarrow} & \mathcal{A}_{(2)}^{1,0}(E)
\end{array}
\]

Write \( \phi = \phi^{1,0} + \phi^{0,1} \) (resp. \( \phi^{0,1} \)) being the part of type \((1,0)\) (resp. \((0,1)\)). The \( D'' \)-closedness of \( \phi \) is equivalent to

\[
\bar{\partial}\phi^{1,0} + \theta^{1,0} = 0.
\]
Write $\phi^{0,1}$ as the sum of forms $\phi_i^{0,1} \otimes e_i$, $i = 1, \cdots, n$, which, again by the same discussion as in the proof of Lemma 2, are $L^2$. Assume that $\|e_i\|_h \sim |\log r|$ (note that the number of such $e_i$’s may $> 1$) and for $i \neq i_0$, $\|e_i\|_h \sim |\log r|^k$, $k \neq 1$. By the above lemma, for $i \neq i_0$, there exists an $L^2$ section $u_i e_i$ (and hence $\in \mathcal{A}_2^0(E)$) satisfying

$$\overline{\partial} u_i \otimes e_i = \phi_i^{0,1} \otimes e_i.$$ 

Thus, the part of type $(0,1)$ of the form $\phi - D''(u_i e_i)$ contains no longer any term of the form $\phi_i^{0,1} \otimes e_i$, $i \neq i_0$.

By the discussion above, we may assume that

$$\phi^{0,1} = \sum_i \phi_i^{0,1} \otimes e_i,$$

each $e_i$ satisfying $\|e_i\|_h \sim |\log r|$ and each term $\phi_i^{0,1} \otimes e_i$ being $L^2$. In order to deal with such terms $\phi_i^{0,1} \otimes e_i$, we now use the $D''$-closedness (22) of $\phi$, i.e. $\overline{\partial} \phi^{1,0} + \theta \phi^{0,1} = 0$. Due to the $D''$-closedness of $\phi \equiv \phi^{1,0} + \phi^{0,1}$, one can write part of $\phi^{1,0}$ as the following sum

$$\phi^{1,0} = \sum_i u_i \theta e_i$$

with each $e_i$ satisfying $\|e_i\|_h \sim |\log r|$ and $\overline{\partial} u_i = \phi_i^{0,1}$. Note that the remaining part of $\phi^{1,0}$ is cancelled by $\overline{\partial}$; but $\theta$ does not cancel any part of $\phi^{0,1} = \sum_i \phi_i^{0,1} \otimes e_i$, since $\|e_i\|_h \sim |\log r|$ (i.e., $N e_i \not\equiv 0$) for each $e_i$. In order of the following discussion, we need to show that $\phi^{1,0}$ is also $L^2$. To this end, considering the holomorphic vector bundle $\mathcal{O}_E$ and the basis $\left\{ \frac{d}{dz} \otimes e_1, \cdots, \frac{d}{dz} \otimes e_n \right\}$ and writing $\phi^{1,0}$ as the sum $\sum_j v_j \frac{d}{dz} \otimes e_j$, the same discussion of Lemma 2 shows that each component is $L^2$; on the other hand, since $\|\theta e_j\|_{h,\omega} \sim \|\frac{d}{dz} \otimes Ne_j\|_{h,\omega} \sim \|e_j\|_h$ if $Ne_j \not\equiv 0$, an approximating discussion then shows that $\phi^{1,0}$ is $L^2$.

Now, we consider the sum $\sum_i u_i e_i$. Clearly,

$$\theta(\sum_i u_i e_i) = \phi^{1,0}$$

and $\overline{\partial}(\sum_i u_i e_i) = \phi^{0,1}$.

On the other hand, since $\|e_i\| \sim |\log r|$ for each $e_i$, so $Ne_i \not\equiv 0$ and $\|\theta(e_i)\|_h \sim |\log r|$. Thus, the same discussion as in the proof of Lemma again shows that $\phi^{1,0}$ being $L^2$ implies that $\sum_i u_i e_i$ is also $L^2$, namely, $\sum_i u_i e_i \in \mathcal{A}_2^0(E)$. Finally, a simple computation shows that $\phi - D''(\sum_i u_i e_i)$ contains no longer the part of type $(0,1)$.

Thus, we may continue to reduce the problem to the assumption that $\phi$ is $D''$-closed and contains only the part of type $(1,0)$. Clearly, a $D''$-closed $L^2$-form of type $(1,0)$ is holomorphic and $\theta$-closed.
Summing all the argument above up, we have that for any $D''$-closed form $\phi \in \mathcal{A}^1_\omega(E)$ on a neighborhood $U$ of the origin, there is a $\theta$-closed form $\eta \in \Omega^1_{\omega}(E)$ and a section $\psi \in \mathcal{A}^0_\omega(E)$ on (a possibly smaller) $U$ satisfying $\phi = \eta + D''\psi$. This finishes the proof of the theorem. \qed

Using the Kähler identity for harmonic bundles (cf. §2), $H^*_\omega(S, L_\rho)$ can be identified with $H^*_\omega(S, E)$, and hence we have the following

**Corollary 1**

$$H^*\left(S, j_*L_\rho\right) \cong \mathbb{H}^*\left(\{\Omega^1_\omega(E), \theta\}\right).$$

**References**


Max-Planck Institute for Mathematics in the Sciences, Leipzig

Department of Applied Mathematics, Tongji University, Shanghai

E-mail: yhyang@tongji.edu.cn

Department of Mathematics, Mainz University, Mainz