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by

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In this note we consider the existence problem for symplectic 3-forms on 7-manifolds. We find a first example of a closed 3-form of $\tilde{G}_2$-type on $S^3 \times S^4$. We also prove that any integral symplectic 3-forms on a 7-manifold M^7 can be obtained by embedding M^7 to a universal space (W^N, h), where $N = 3(81 + 8.C_8^3)$.

MSC: 53C10, 53C42
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1 Introduction.

Let $\Lambda^k V^n$ be the space of k-linear anti-symmetric forms on a given linear space V^n. For each $\omega \in \Lambda^k(V^n)$ we denote by I_ω the linear map

$$I_\omega : V^n \to \Lambda^{k-1}(V^n),\ x \mapsto (x \rfloor \omega) := \omega(x, \cdots).$$

A k-form ω is called **multi-symplectic**, if I_ω is a monomorphism.

The classification (under the action of $Gl(V^n)$) of multi-symplectic 3-forms in dimension 7 has been done by Bures and Vanzura [B-V2002]. There are together 8 types of these forms, among them there two generic classes of G_2-form ω_1^3 and $\tilde{G}_2$-form ω_2^3. They are generic in the sense of $Gl(V^7)$-action, more precisely the orbits $Gl(V^7)(\omega_i^3), i = 1, 2$, are open sets in $\Lambda^3(V^7)$. The corresponding isotropy groups are the compact group G_2 and its dual non-compact group $\tilde{G}_2$.

We shall write here a canonical expression of the G_2-form ω_1^3 and $\tilde{G}_2$-form ω_2^3 (see e.g. [Bryant1987], [B-V2002])

$$\omega_1^3 = \theta_1 \wedge \theta_2 \wedge \theta_3 + \alpha_1 \wedge \theta_1 + \alpha_2 \wedge \theta_2 - \alpha_3 \wedge \theta_3. \tag{1.1}$$

$$\omega_2^3 = \theta_1 \wedge \theta_2 \wedge \theta_3 + \alpha_1 \wedge \theta_1 + \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_3. \tag{1.2}$$

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Here α_i are 2-forms on V^7 which can be written as

$$\alpha_1 = y_1 \wedge y_2 + y_3 \wedge y_4,\ \alpha_2 = y_1 \wedge y_3 - y_2 \wedge y_4,\ \alpha_3 = y_1 \wedge y_4 + y_2 \wedge y_3$$

and $(\theta_1, \theta_2, \theta_3, y_1, y_2, y_3, y_4)$ is an oriented basis of $(V^7)^*$.

A 7-dimensional manifold M^7 is said to be provided with **a G_2-structure**, ($\tilde{G}_2$, resp.) if there is given differential 3-form ϕ^3 on it such that at every point $x \in M^7$ the form $\phi^3(x)$ is of G_2-type ($\tilde{G}_2$ resp.).

- A G-structure ϕ is called **closed**, $G = G_2$ or $\tilde{G}_2$, if $d\phi = 0$.

Using existing terminology, we shall also call a closed G_2-form ($\tilde{G}_2$-form, resp.) **symplectic form of G_2-type ($\tilde{G}_2$-type** resp.). If a G_2-structure ϕ is closed and coclosed then the G_2 structure is torsion free, i.e. the Ricci curvature of the associated Riemannian metric $g(\phi)$ (via the canonical embedding $G_2 \to SO(7)$) vanishes [F-G1982] (see also [Salamon1989, Lemma 11.5]). An analogous statement is also valid for $\tilde{G}_2$-structure by using the same argument in [Salamon1989]. We notice that the first examples of a Riemannian metric with G_2 holonomy has been constructed by Joyce [Joyce1996] by deforming certain closed G_2-structures. Closed 3-forms have been also used by Severa and Weinstein to deform Poisson structures [V-W2001].

- We shall call that a closed structure G **integral**, $G = G_2$ or $\tilde{G}_2$, if the cohomology of the G-form ϕ is an integral class in $H^3(M^7, \mathbb{Z}) \subset H^3(M^7, \mathbb{R})$.

Without additional conditions the existence of a G_2-structure is a purely topological question (see [Gray1969]). The same can be proved for the existence of a $\tilde{G}_2$-structure (see Proposition 2.3). On the other hand the existence of a torsion free G_2-structure (as well as of $\tilde{G}_2$-structure) is really "exceptional" in the sense that this structure is a solution to an overdetermined PDE (see e.g. [Bryant1986]). The intermediate class of closed G_2-structures (and $\tilde{G}_2$-structures resp.) is nevertheless has not been investigated in deep. We know only few examples of these structures on compact homogeneous spaces [Fernandez1987], [Bryant2005] and their local geometry [C-I2003]. The existence of local metrics with G_2-holonomy ($\tilde{G}_2$-holonomy resp.) has been proved by Bryant in 1984, see [Bryant1987]. The examples of torsion free G_2-structures on M^7 obtained by Joyce [Joyce1996] and Kovalev [Kovalev2001] have a common geometrical flavor, that they begin with M^7 with simple (or well understood) holonomy and then modify topologically these manifolds.

In this note we propose to construct a symplectic 3-form by embedding a closed manifold M^7 into a semi-simple group G. The motivation for this construction is the fact that there exists a closed multi-symplectic bi-invariant 3-form on G, so "generically" the restriction of this 3-form to any 7-manifold in G must be a G_2-form or $\tilde{G}_2$-form. We shall show different ways to get a closed $\tilde{G}_2$-structure on $S^3 \times S^4$ by this method (Theorem 2.5 and Theorem 2.13). Bryant informed me, that we cannot find any G_2-submanifold in $SU(3)$ by this method, since the restriction of the form ϕ_0^3 to any hyperplane in $SU(3)$ is never of G_2-type. In Theorem 3.6 we prove that any integral closed 3-form ϕ on a compact M^7 can be immersed in a smooth manifold W^N provided with a universal closed 3-form h such that the pull-back of h is equal to ϕ. This immersion can be chosen as an embedding, if ϕ is symplectic. Our theorem is close to the Tischler theorem on the embedding of compact integral symplectic manifold to $\mathbb{C}P^n$. We prove theorem 3.6 by using Gromov H-principle. We also showed in Theorem-Remark 3.17 that the existence of a symplectic 3-form of G_2-type or of $\tilde{G}_2$-type on an open manifold M^7 is purely a topological question. This can be done in the same way as Gromov proved the analogous theorem for open symplectic manifolds. Theorem 3.17 is also called a remark, because it is a direct consequence of the Eliashberg-Mishachev holonomy appoximation theorem.

This note also contains an Appendix written in communication with Kaoru Ono which contains a new "soft" proof of a version Theorem 3.6 on the existence of a universal space for closed 3-forms.

2 Closed multi-symplectic 3-forms of type $\tilde{G}_2$.

In this section we show a necessary and sufficient condition for the existence of a $\tilde{G}_2$-structure on a 7-manifold. We also construct a symplectic 3-form of $\tilde{G}_2$-type on $S^3 \times S^4$.

We follow ideas of Bryant [Bryant1987] and Hitchin [Hitchin2000] to associate each non-degenerate 3-form on V^7 a bilinear form.

2.1. Associated pseudo Riemannian metric to 3-form of type $\tilde{G}_2$.
We put

$$g_{\omega_2^3}(x, y) = \frac{1}{6}\omega_2^3 \wedge (x\rfloor\omega_2^3) \wedge (y\rfloor\omega_2^3).$$

Instead of using the basis (θ_i, y_i) as in (1.2) we shall use a vector basis $(e_1, \cdots, e_6)$ for $\mathbb{R}^7$ and $(e^1, \cdots, e^7)$ for $(\mathbb{R}^7)^*$. Further we denote by $e^{e_{i_1}\cdots e_{i_k}}$ the exterior form $e^{i_1} \wedge \cdots \wedge e^{i_k}$. Using the following expression for $x = x^i e_i$

$$x\rfloor\omega_2^3 = x^1(e^{23} + e^{34} + e^{67}) + x^2(-e^{13} + e^{46} - e^{57}) + x^3(e^{12} + e^{47} + e^{56})$$

$$+x^4(-e^{15}-e^{26}-e^{37})+x^5(e^{14}+e^{27}-e^{36})+x^6(-e^{17}+e^{24}+e^{35})+x^7(e^{16}-e^{25}+e^{34}),$$

we easily get

$$g_{\omega_2^3}(x, y) = (x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4 - x^5y^5 - x^6y^6 - x^7y^7) \cdot \omega^{1234567}.$$

The bilinear form g corresponds to the linear map $K_g : \mathbb{R}^7 \to (\mathbb{R}^7)^* \otimes \Lambda^7(R^7)$ by the formula $< K_g(a), b >= g(a, b)$. Thus

$$L(e_i) = \varepsilon_i(e^i)e^{1234567}$$

where $\varepsilon_i = +$, if $1 \le i \le 3$ and $\varepsilon_i = -$, if $4 \le i \le 7$. Hence

$$\det K_g(e_1, e_2, \cdots, e_7) = (e^{1234567})^8.$$

$$\Longleftrightarrow \det K_g = (e^{1234567})^9.$$

Thus the bilinear form

$$B_{\omega_2^3} = g_{\omega_2^3} \cdot (\det K_g)^{-1/9} = (e^1)^2 + (e^2)^2 + (e^3)^2 - (e^4)^2 - (e^5)^2 - (e^6)^2 - (e^7)^2$$

is the associated to ω_2^3 bilinear form as in [Hitchin2000]. In particular the isotropy group $\tilde{G}_2$ also preserves the form $B_{\omega_2^3}$.

2.2. Proposition. [Bryant1987, Theorem 2] *The isotropy group of ω_2^3 is $\tilde{G}_2$. This group is the non-compact dual of G_2. It is connected, of dimension 14 and satisfies $\pi_1(\tilde{G}_2) = \mathbb{Z}_2$. Morover $\tilde{G}_2$ acts transitively on the spaces of positive lines in V, null lines in V negative lines in V and the space of 2-planes in V of a fixed signature and rank with respect to $B_{\omega_2^3}$.*

In fact in [Bryant1987] Bryant used another form $\tilde{\phi}$ of ω_2^3, namely

$$\tilde{\phi} = \omega^{123} - \omega^{145} - \omega^{167} - \omega^{246} + \omega^{257} + \omega^{347} + \omega^{356}$$

which can be reduced to ω_2^3 by changing $e_1 \mapsto -e_1$, $e_2 \mapsto -e_2$.

It also follows from the dimension count that the space of $\tilde{G}_2$ forms is open in $\Lambda^3(\mathbb{R}^7)$.

2.3. Proposition. (Existence of $\tilde{G}_2$-structure.) *A manifold M^7 admits a $\tilde{G}_2$-structure, if and only if it is orientable with vanishing Euler class and vanishing Stiefel-Whitney classes w_5, w_6.*

Proof. Since the maximal compact group of $\tilde{G}_2$ is $SO(4)$, a manifold M^7 admits a $\tilde{G}_2$-structures, if and only if it admits $S0(4)$ structures. From the obstruction theory (see [Steenrod1951, §39]), it follows that, M^7 admits a $SO(4)$-structure, if and only if it is orientable with vanishing Euler class and vanishing Stiefel-Witney classes w_5, w_6. □

In the remained part of this note we shall construct examples of symplectic 3-form of $\tilde{G}_2$-type on $S^3 \times S^4$.

Our examples (Theorem 2.5 and Theorem 2.13) are closed submanifolds $S^3 \times S^4$ in semi-simple Lie groups $SU(3)$ and $G \times (SU(2))^N$, $N = \max(N_0, 80 + 8.C_8^3)$, where N_0 is a finite number defined after Corollary 3.2. On each semi-simple Lie group G there exists a natural bi-invariant 3-form ϕ_0^3 which is defined at the Lie algebra $g = T_e G$ as follows

$$\phi_0^3(X, Y, Z) = < X, [Y, Z] >,$$

where $<,>$ denotes the Killing form on g.

2.4. Lemma. *The form ϕ_0^3 is multi-symplectic.*

Proof. We need to show that $I_{\phi_0^3}$ is monomorphism. We notice that if $X \in \ker I_{\phi_0^3}$, then

$$< X, [Y, Z] >= 0 \text{ for all } Y, Z \in g.$$

But this condition contradicts the semi-simplicity of g. $\square$

Let us consider the group $G = SU(3)$. For each $1 \le i \le j \le 3$ let $g_{ij}(g)$ be the complex function on $SU(3)$ induced from the standard unitary representation ρ of $SU(3)$ on $\mathbb{C}^3$: $g_{ij}(g) :=< \rho(g) \circ e_i, \bar{e}_j >$. Here $\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$ is a unitary basis of $\mathbb{C}^3$. Now we denote by X^7 the co-dimension 1 subset in $SU(3)$ which is defined by the equation $Im(g_{11}(g)) = 0$.

2.5. Theorem. *The subset X^7 is diffeomorphic to the manifold $S^3 \times S^4$. Moreover X^7 is provided with a closed G_2-form ω^3 which is the restriction of ϕ_0^3 to X^7.*

Proof. Let $SU(2)$ be the subgroup in $SU(3)$ consisting of all $g \in SU(3)$ such that $\rho(g) \circ e_1 = e_1$. We denote by π the natural projection

$$\pi : SU(3) \to SU(3)/SU(2).$$

We identify $SU(3)/SU(2)$ with the sphere $S^5 \subset \mathbb{C}^3$ via the standard representation ρ of $SU(3)$ on $\mathbb{C}^3$. This identification denoted by $\tilde{\rho}$ is expressed as follows.

$$\tilde{\rho}(g \cdot SU(2)) = g \circ e_1.$$

We denote by Π the composition $\tilde{\rho} \circ \pi : SU(3) \to SU(3)/SU(2) \to S^5$. Let $S^4 \subset S^5$ be the geodesic sphere which consists of points $v \in S^5$ such that $Im\, e^1(v) = 0$. Here $\{e^i, i = 1,2,3\}$ are the complex 1-forms on $\mathbb{C}^3$ which are dual to $\{e_i\}$. The pre-image $\Pi^{-1}(S^4)$ consists of all $g \in SU(3)$ such that

$$Im\, e^1(g \circ e_1) = 0.$$

$$\Longleftrightarrow Im\,(g_{11}) = 0.$$

So X^7 is $SU(2)$-fibration over S^4. But this fibration is the restriction of the $SU(2)$-fibration $\Pi^{-1}(D^5)$ over the half-sphere D^5 to the boundary $\partial D^5 = S^4$. So it is a trivial fibration. This proves the first statement of Theorem 2.2.

We fix now a subgroup $SO(2)^1$ in $SU(3)$ where $SO(2)^1$ is the orthogonal group of the real space $\mathbb{R}^2 \subset \mathbb{C}^3$ such that $\mathbb{R}^2$ is the span of e_1 and e_2 over $\mathbb{R}$.

We denote by $m_L(g)$ (resp. $m_R(g)$) the left multiplication (resp. the right multiplication) by an element $g \in SU(3)$.

2.6. Lemma. *X^7 is invariant under the action of $m_L(SU(2)) \cdot m_R(SU(2))$. For each $v \in S^4$ there exist an element $\alpha \in SO(2)^1$ and an element $g \in SU(2)$ such that $\Pi(g \cdot \alpha) = v$. Consequently for any point $x \in X^7$ there are $g_1, g_2 \in SU(2)$ and $\alpha \in SO(2)^1$ such that*

(2.6.1) $$x = g_1 \cdot \alpha \cdot g_2,$$

Proof. The first statement follows from straightforward calculations, (our realization that $X^7 = \Pi^{-1}(S^4)$ implies that the orbit of $m_R(SU(2))$-action on

X^7 are the fiber $\Pi^{-1}(v)$). Let $v = (\cos\alpha, z_2, z_3) \in S^4$, where $z_i \in \mathbb{C}$. We choose $\alpha \in SO(2)^1$ so that

$$\rho(\alpha) \circ e_1 = (\cos\alpha, \sin\alpha) \in \mathbb{R}^2. \tag{2.7}$$

Clearly α is defined by v uniquely up to sign $\pm$. We set

$$w := (\sin\alpha, 0) \in \mathbb{C}^2 =< e_2, e_3 >_{\otimes\mathbb{C}} .$$

We notice that

$$|z_2|^2 + |z_3|^2 = \sin^2\alpha.$$

Since $SU(2)$ acts transitively on the sphere S^3 of radius $|\sin\alpha|$ in $\mathbb{C}^2 =< e_2, e_3 >_{\otimes\mathbb{C}}$, there exists an element $g \in SU(2)$ such that $\rho(g) \circ w = (z_2, z_3)$. Clearly

$$\Pi(g \cdot \alpha) = v.$$

The last statement of Lemma 2.6 follows from the second statement and the fact that $X^7 = \Pi^{-1}(S^4)$ □

Using (2.6.1) to complete the proof of Theorem 2.5 it suffices to check that the value of ω at any $\alpha \in SO(2)^1 \subset X^7$ is a G_2-form, since ϕ_0^3 is a bi-invariant form on $SU(3)$. We divide the remaining part of the proof of Theorem 2.5 into two steps. In the first step we shall compute that value ω^3 at $\alpha = e$ and in the second step we shall compute the value ω^3 at any $\alpha \in SO(2)^1$.

Step 1. Let us first compute the value $\omega^3(e) \in X^7$. We shall use the Killing metric to identify the Lie algebra $su(3)$ with its co-algebra g. Thus in what follows we shall not distinguish co-vectors and vectors, poly-vector and exterior forms on $su(3)$. Clearly we have

$$T_eX^7 = \{v \in su(3) : Im\, g_{11}(v) = 0\}.$$

Now we identify $gl(\mathbb{C}^3)$ with $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ and we denote by e_{ij} the element of $gl(\mathbb{C}^3)$ of the form $e_i \otimes (e_j)^*$.

A straightforward calculation gives us

$$\omega^3(x_0) = \sqrt{2}\delta_1 \wedge \delta_2 \wedge \delta_3 + \frac{1}{\sqrt{2}}\omega_1 \wedge \delta_1 + \frac{1}{\sqrt{2}}\omega_2 \wedge \delta_2 + \frac{1}{\sqrt{2}}\omega_3 \wedge \delta_3, \tag{2.8}$$

where δ_i are 1-forms in T_eX^7 which are defined as follows:

$$\delta_1 = \frac{i}{\sqrt{2}}(e_{22} - e_{33}), \delta_2 = \frac{1}{\sqrt{2}}(e_{23} - e_{32}), \delta_3 = \frac{i}{\sqrt{2}}(e_{23} + e_{32}).$$

Furthermore, ω_i are 2-forms on T_eX^7 which have the following expressions:

$$2\omega_1 = -(e_{12} - e_{21}) \wedge i(e_{12} + e_{21}) + (e_{13} - e_{31}) \wedge i(e_{13} + e_{31}),$$

$$2\omega_2 = -(e_{12} - e_{21}) \wedge (e_{13} - e_{31}) - i(e_{12} + e_{21}) \wedge i(e_{13} + e_{31}),$$

$$2\omega_3 = -(e_{12} - e_{21}) \wedge i(e_{13} + e_{31}) + i(e_{12} + e_{21}) \wedge (e_{13} - e_{31}).$$

Now compare (2.8) with (1.2) we observe that these two 3-forms are $Gl(\mathbb{R}^7)$ equivalent (e.g. by rescaling δ_i with factor (1/2)). This proves that $\omega^3(x_0)$ is a $\tilde{G}_2$-form. This completes the step 1.

Step 2. Using step 1 it suffices to show that

$$D\, m_L(\alpha^{-1})(T_\alpha X^7) = T_e X^7 \tag{2.9}$$

for any $\alpha \in SO(2)^1 \subset X^7$, $\alpha \neq e$.

Since $X^7 \supset \alpha \cdot SU(2)$, we have

$$su(2) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.10}$$

Denote by $SO(3)$ the standard orthogonal group of $\mathbb{R}^3 \subset \mathbb{C}^3$. Since $\alpha \in SO(3) \subset X^7$, we have $D\, m_L(\alpha^{-1})(T_\alpha SO(3)) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7)$. In particular we have

$$< (e_{12} - e_{21}), (e_{13} - e_{31}) >_{\otimes \mathbb{R}} \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.11}$$

Since $SU(2) \cdot \alpha \subset X^7$, we have

$$Ad(\alpha^{-1}) su(2) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.12}$$

Using the formula

$$Ad(\alpha^{-1}) = \exp(-ad(t \cdot \frac{e_{12} - e_{21}}{\sqrt{2}})),\ t \not\equiv 0$$

we get immediately from (2.10), (2.11), (2.12) the following inclusion

$$< i(e_{12} + e_{21}), i(e_{13} + e_{31}) >_{\otimes \mathbb{R}} \subset D\, m_L(\alpha^{-1})(T_\alpha X^7))$$

which together with (2.10), (2.11) imply the desired equality (2.6). □

This completes the proof of Theorem 2.5. □

Theorem 2.13. *For any given simply-connected compact semi-simple Lie group G, and any given integral closed 3-form ϕ on $S^3 \times S^4$ (e.g. that from Theorem 2.5) there exists an embedding $f : S^3 \times S^4 \to G' = G \times (SU(2))^{80+4\cdot C_8^3}$ such that the restriction of form $(\sqrt{2}\pi)^{-1}\phi_0^3$ from G' to $f(S^3 \times S^4)$ is equal to ϕ. Moreover we can require that the pull-back (via the projection) of a given non-decomposable element $\alpha \in H^3(M, \mathbb{Z})$ to the image $f(S^3 \times S^4)$ is equal to $[\phi] \in H^3(M, \mathbb{Z})$.*

Proof. Using the fact that $H^3(S^3 \times S^4, \mathbb{Z}) = \pi_3(S^3) = \mathbb{Z}$, and taking into account for a Lie group G as in Theorem 2.13 the following identity: $H^3(G, \mathbb{Z}) = \pi_3(G)$ we can find a map $f_1 : M^7 \to G$ such that the second condition in Theorem 2.13 holds. Now I shall modify this map f_1 to the required embedding f by using the same H-principle as in our proof of Theorem 3.6. The only thing we can improve in this proof is the dimension of the target manifold. Instead of number 8 of special coverings on M^7 (using in the step 2 of the proof of Theorem 3.6) we can chose 4 open disks which cover $S^3 \times S^4$. □

3 Universal space for integral closed 3-forms on compact 7-manifolds.

In this section we shall show that any integral closed 3-form ϕ on a compact 7-dimensional smooth manifold M^7 can be induced from an immersion M^7 to a universal space (W^N, h) (Theorem 3.6). This immersion can be chosen as an embedding, if ϕ is symplectic.

Our definition of the universal space (W^N, h) is based on the work of Dold and Thom [D-T1958] as well as an idea of Gromov [Gromov2006].

Let $SP^q(X)$ be the q-fold symmetric product of a locally compact, para-compact Hausdorff pointed space $(X, 0)$, i.e. $SP^q(X)$ is the quotient space of the q-fold Cartesian $(X^q, 0)$ over the permutation group σ_q. We shall denote by $SP(X, 0)$ the inductive limit of $SP^q(X)$ with the inclusion

$$X = SP^1(X) \xrightarrow{i_1} SP^2(X) \xrightarrow{i_2} \cdots \to SP^q(X) \xrightarrow{i_q} \cdots ,$$

where

$$SP^q(X) \xrightarrow{i_q} SP^{q+1}(X) : \ [x_1, x_2, \cdots, x_q] \mapsto [0, x_1, x_2, \dots, x_q].$$

Equivalently we can write

$$SP(X, 0) = \sum_q SP^q(X)/([x_1, x_2, \cdots, x_q] \sim [0, x_1, x_2, \cdots, x_q]).$$

So we shall also denote by i_q the canonical inclusion $SP^q(X) \to SP(X, 0)$.

3.1. Theorem (see [D-T1958, Satz 6.10]). *There exist natural isomorphisms $j : H_q(X, \mathbb{Z}) \to \pi_q(SP(X, 0))$ for $q > 0$.*

3.2. Corollary. ([D-T1958]) *The space $SP(S^n, 0)$ is the Eilenberg-McLane complex $K(\mathbb{Z}, n)$.*

Denote by N_0 the minimal number such that the inclusion $i : SP^2(S^3) \to SP^{N_0}(S^3)$ induces the trivial map: $i_*(\pi_6(SP^2(S^3)) = 0 \in \pi_6(SP^{N_0})$. From [D-P1961, (12.12)] which says that

$$\pi_i(SP^n(X)) = H_n(X) \text{ for } i < k + 2n - 1, \ n > 1,$$

if X is connected and $H_i(X) = 0$ for $0 < i < k$, we get immediately that $\pi_6(SP^3(S^3)) = 0$. Thus $N_0 = 3$. (To see that $\pi_6(SP^2(S^3)) = \mathbb{Z}_3 \neq 0$ we apply the exact sequence in [D-P1961, (12.13)]

$$H_{k+2n}(X) \to H_k(X) \otimes \mathbb{Z}_p \to \pi_{k+2n-1}(SP^n(X)) \to H_{k+2n-1}(X) \to 0$$

for a connected X with $H_i(X) = 0$ for $0 < i < k$, $k > 2$, and $n + 1 = p^r$, p is prime and $r > 0$.)

Now let τ be the generator of $H^3(SP(S^3,0),\mathbb{Z})$ and by abusing notations we also denote by τ the restriction of the generator τ to any subspace $SP^q(S^3) \subset SP(S^3,0)$. The following Lemma shows that we can replace a classifying map from (M^7,α) to $SP(S^3,0)$ by a map from (M^7,α) to $SP^3(S^3)$.

3.3. Lemma. *Let $\alpha \in H^3(M^7,\mathbb{Z})$. Then there exists a continuous map f from M^7 to $(SP^3(S^3))$ such that $f^*(\tau) = \alpha$.*

Proof. Let f_0 be a classifying map from M^7 to $SP(S^3,0)$ such that $f_0^*(\tau) = \alpha$. Denote by K^i the i-dimensional skeleton of $SP(S^3,0)$ and by $\bar\tau$ the restriction of τ to K^7. Then we know that f_0 is homotopic equivalent to a continuous map $f_1 : M^7 \to K^7$ such that $f_1^*(\bar\tau) = \alpha$. To prove Lemma 3.3 it suffices to find a map $g : K^7 \to SP^3(S^3)$ such that $g^*(\tau) = \bar\tau$. Then the map $f = g \circ f_1$ satisfies the condition of Lemma 3.3.

We observe that K^3 consists of the sphere S^3 and therefore there is a map $f_2 : K^3 \to SP^2(S^3)$ such that

$$f_2(\tau) = \bar\tau,$$

where we also denote by $\bar\tau$ the restriction of $\bar\tau$ to K^3. We denote by $\tilde f_2$ a composition $i \circ f_2$, where i is the inclusion of S^3 to $SP^2(S^3)$. Next we note that $K^5 = K^3 \cup_{f_i} D^5_i$, where f_i denote a non-trivial element of $\pi_4(S^3)$. Using the obstruction theorem we see easily that there is an extension f_3 of map $\tilde f_2$ to a map $f_3 : K^6 \to SP^2(S^3)$ since Liao [Liao1953, 13.3, 13.6] showed that $\pi_4(SP^2(S^3)) = 0 = \pi_5(SP^2(S^3))$. Let i_2 be the canonical embedding $SP^2(S^3) \to SP^3(S^3))$. Then there is a map $g : K^7 \to SP^3(S^3)$ extending the map $i_2 \circ f_3$ because of our choice of $N_0 = 3$. Clearly the map g satisfies the required property that $g^*(\tau) = \bar\tau$. □

Since $SP^2(S^3)$ has a finite simplicial decomposition it is easy to get the following Lemma (see e.g. [Thom1954, III.2])

3.4. Lemma. *The space $SP^3(S^3)$ can be embedded into a smooth manifold $\mathcal{M}^{19}$ such that (the image of) $SP^3(S^3)$ is a retract of $\mathcal{M}^{19}$.*

Denote also by τ the pull back of the universal class τ from $SP^3(S^3)$ to $\mathcal{M}^{19}$ and let α be any differential form representing τ on $\mathcal{M}^{19}$.

Let β_k be the following 3-form on $\mathbb{R}^{3k}$:

$$\beta_k = dx_1 \wedge dy_1 \wedge dz_1 + \cdots dx_k \wedge dy_k \wedge dz_k.$$

Set $N_2 = 80 + 8C_8^3$.

Now we state the main theorem of this section. Set $(W^N, h) = (\mathcal{M}^{19} \times \mathbb{R}^{3N_2}, \alpha \oplus \beta_{N_2})$.

3.6. Theorem. *Suppose that ϕ is a closed integral 3-form on a smooth manifold M^7. Then there exists an immersion $f : M^7 \to (W^N, h)$ such that $f * (h) = \phi$. Moreover for any given map $\tilde f : M^7 \to (W^N, h)$ such that $\tilde f^*[h] = [\phi]$ there exists a C^0-close to $\tilde f$ immersion $f : M^7 \to W^N$ such that $f^*(h) = \phi$. If ϕ is symplectic, then we can require f to be an embedding.*

Proof of Theorem 3.6. Using Lemma 3.3 and Lemma 3.4 we see that the first statement of Theorem 3.6 follows from the second statement of Theorem 3.6. Furthermore we shall reduce the second statement to an immersion problem for exact 3-forms as follows. Denote by $f_1 : M^7 \to \mathcal{M}^{19}$ the projection of $\tilde{f}$ to the first factor. Then we have $f_1^*(\tau) = [\phi] \in H^3(M, \mathbb{Z})$. Let

$$g = \phi - f_1^*(\alpha).$$

Clearly g is an exact 3-form on M^7. Thus the first statement of Theorem 3.6 is a corollary of the following Proposition (compare with [Gromov1986, 3.4.1.B']).

3.7. Proposition. *For any given map $f_0 : M^7 \to \mathbb{R}^{N_2}$ there is a C^0-close to f_0 immersion $f_3 : M^7 \to (\mathbb{R}^{3N_2}, \beta_{N_2})$ such that $f_3^*(\beta_{N_2}) = g$.*

Proof. Proposition 3.7 can be obtained directly from the Gromov H-principle[1] for differential forms in [Gromov1986, 3.4.1]. For sake of completeness we shall present a detailed proof here. Let us quickly recall several notions introduced by Gromov in [Gromov1986].

Let V and W be smooth manifolds. We denote by $(V, W)^{(r)}$, $r \geq 0$, the space of r-jets of smooth mappings from V to W. We shall think of each map $f : V \to W$ as a section of the fibration $V \times W = (V, W)^{(0)}$ over V. Thus $(V, W)^{(r)}$ is a fibration over V, and we shall denote by p^r the canonical projection $(V, W)^{(r)}$ to V, and by p^s_r the canonical projection $(V, W)^{(s)} \to (V, W)^{(r)}$.

We also say that a differential relation $\mathcal{R} \subset (V, W)^{(r)}$ satisfies the **H-principle C^0-near a map** $f_0 : V \to W$, if every continuous section $\phi_0 : V \to \mathcal{R}$ which lies over f_0, (i.e. $p^r_0 \circ \phi_0 = f_0$) can be brought to a holonomic section ϕ_1 by a homotopy of sections $\phi_t : V \to \mathcal{R}_U$, $t \in [0, 1]$, for an arbitrary small neighborhood U of $f_0(V)$ in $V \times W$ [Gromov1986, 1.2.2]. Here for an open set $U \subset V \times W$, we write

$$\mathcal{R}_U := (p^r_0)^{-1}(U) \cap \mathcal{R} \subset (V, W)^r.$$

The H-principle is called **C^0-dense**, if it holds true C^0-near every map $f : V \to W$.

Let h be a smooth differential k-form on W. A subspace $T \subset T_w W$ is called **$h(w)$-regular**, if the composition of $I_{h(w)}$ with the restriction homomorphism $\Lambda^{k-1} T_w W \to \Lambda^{k-1} T$ sends $T_w W$ onto $\Lambda^{k-1} T$.

An immersion $f : V \to W$ is called **h-regular**, if for all $v \in V$ the subspace $Df(T_v V)$ is $h(f(v))$-regular.

We also use the notions of a flexible sheaf and a microflexible sheaf introduced by Gromov in order to study the H-principle.

Suppose we are given a differential relation $\mathcal{R} \subset (V, W)^{(r)}$. Fix an integer $k \geq r$ and denote by $\Phi(U)$ the space of C^k-solution of $\mathcal{R}$ over U for all open $U \subset V$. This set equipped with the natural restriction $\Phi(U) \to \Phi(U')$ for all

[1] to avoid confusing between the original notion h-principle of Gromov and his notion of h as a differential form, we decide to use the capital H for H-principle.

$U' \subset U$ makes Φ a sheaf. We shall say that Φ **satisfies the H-principle**, if $\mathcal{R}$ satisfies the H-principle. [2]

A sheaf Φ is called **flexible (microflexible)**, if the restriction map $\Phi(C) \to \Phi(C')$ is a fibration (microfibration) for all pair of compact subsets C and $C' \subset C$ in M. We recall that the map $\alpha : A \to A'$ is called **microfibration**, if the lifting homotopy property for a homotopy $\psi : P \times [0,1] \to A'$ is valid only "micro", e.g. there exists $\varepsilon > 0$ such that ψ can lift to a $\bar{\psi} : P \times [0, \varepsilon] \to A$.

Plan of the proof of Proposition 3.7. Roughly speaking, we add the β_{N_2}-regularity to the isometry property (i.e. $f_3^*(\beta_{N_2}) = g$) and extend this equation for mappings also denoted by f_3 from the manifold $M^8 = M^7 \times (-1,1)$ provided with a form $g \oplus 0$ which we shall also denote by g to the space $(\mathbb{R}^{3N_2}, \beta_{N_2})$. We shall prove that the solution sheaf restricted to $M^7 \subset M^8$ satisfies the H-principle (Proposition 3.10). So to prove the existence of a β_{N_2}-regular isometric immersion f_3 which is C^0-close to a given map f_0, it suffices to find a section of this extended differential relation which lies over f_0 (Proposition 3.12). If ϕ is symplectic, then we can perturb an isometric immersion f_3 to get an isometric embedding.

Now we are going to define our extended differential relation. Let f_0 be a map $M^8 \to (\mathbb{R}^{3N_2}, \beta_{N_2})$. We denote by F_0 the corresponding section of the bundle $M^8 \times \mathbb{R}^{N_2} \to M^8$, i.e. $F_0(v) = (v, f_0(v))$. Denote by $\Gamma_0 \subset M^8 \times \mathbb{R}^{3N_2}$ the graph of f_0 (i.e. it is the image of F_0), and let $p^*(g)$ and $p^*(\beta_{N_2})$ be the pull-back of the forms g and β_{N_2} to $M^8 \times \mathbb{R}^{3N_2}$ under the obvious projection. Take a small neighborhood $Y \supset \Gamma_0$ in $M^8 \times \mathbb{R}^{3N_2}$. Since β_{N_2} and g are exact forms we get

$$p^*(\beta_{N_2}) - p^*(g) = d\hat{\beta}$$

for some smooth 2-form $\hat{\beta}$ on Y.

Our next observation is

3.8. Lemma. *Suppose that a map $F : M^8 \to Y$ corresponds to a β_{N_2}-regular immersion $f : M^8 \to \mathbb{R}^{3N_2}$. Then F is a $d\hat{\beta}_{N_2}$-regular immersion.*

Proof. We need to show that for all $y = F(z) \in Y$, $z \in M^8$, the composition ρ of the maps

$$T_yY \overset{I_{p^*(\beta_{N_2})-p^*(g)}}{\to} \Lambda^2 T_yY \to \Lambda^2(dF(T_{(z)}(M^8))$$

is onto. This follows from the consideration of the restriction of ρ to the subspace $S \subset T_yY$ which is tangent to the fiber $\mathbb{R}^{3N_2}$ in $M^8 \times \mathbb{R}^{3N_2} \supset Y$. □

Now for a map $d\hat{\beta}_{N_2}$-regular map $F : M^8 \to Y$ and a 1-form ϕ on M^8 we set

$$\mathcal{D}(F, \phi) := F^*(\hat{\beta}) + d\phi \tag{3.9}$$

[2] The reader can look at [Gromov 1986, 2.2.1] for a more general definition.

With this notation the maps $f : M^8 \to \mathbb{R}^{3N_2}$ corresponding to $F : M^8 \to Y$ satisfy

$$f^*(\beta_{N_2}) = F^*(p^*(\beta_{N_2})) = g + F^*(d\hat{\beta}) = g + d\mathcal{D}(F, \phi),$$

for any ϕ. Since the space of 1-forms ϕ is contractible, it follows that the space of $d\hat{\beta}_{N_2}$-regular sections $F : M^8 \to Y$ for which

$$f^*(\beta_{N_2}) = g + dg_1 \tag{3.9.1}$$

for a given 2-form g_1 has the same homotopy type as the space of solutions to the equation

$$\mathcal{D}(F, \phi) = g_1.$$

In particular the equation $f_3^*(\beta_{N_2}) = g$ reduces to the equation $\mathcal{D}(F, \phi) = 0$ in so far as the unknown map f_3 is C^0-close to f_0 (so that its graph lies inside Y).

We define by $\tilde{\Phi}_{reg}$ the solution sheaf of the equation (3.9) whose component F is $d\hat{\beta}_{N_2}$-regular.

3.10. Proposition. *The restriction of $\tilde{\Phi}_{reg}$ to M^7 satisfies the H-principle. Hence the solution sheaf of β_{N_2}-regular isometric immersion $f : (M^8, g) \to (\mathbb{R}^{3N_2}, \beta_{N_2})$ such that $F(M^8) \subset Y$ restricted to M^7 satisfies the H-principle.*

Proof of Proposition 3.10. First we shall prove the following

3.11. Lemma. *The differential operator $\mathcal{D}$ is infinitesimal invertible at those pairs (F, ϕ) for which the underlying map f is a β-regular immersion.*

Proof. By Lemma 3.8 the map F_0 is a $d\hat{\beta}$-regular immersion. Hence the system

$$F^*(\partial\rfloor d\hat{\beta}_{N_2}) = \tilde{g}, \tag{3.11.1}$$

$$F^*(\partial\rfloor\hat{\beta}_{N_2}) + \tilde{\phi} = 0 \tag{3.11.2}$$

is solvable for all 2-form $\tilde{g}$ on M^8. Clearly every solution $(\partial, \tilde{\phi})$ of (3.11.2) and (3.11.1) satisfies (3.11). □

Now using Lemma 3.11 and A.3' (Nash implicit function theorem), A.4 (Nash implicit function theorem implies the microflexibility) and get the microflexibility of $\tilde{\Phi}_{reg}$. Next we use the Gromov observation [Gromov1986, 3.4.1.B'] that M^7 is a sharply movable submanifold by strictly exact diffeotopies in M^8, taking into account A.2 (movability +microflexibility implies H-principle) we get the first statement Proposition 3.10 immediately. The second one follows by a remark above relating (3.9) and (3.9.1). □

Completion of the proof of Proposition 3.7.

Suppose we are given a map $f_0 : M^7 \to \mathbb{R}^{3N_2}$. Since M^7 is a deformation retract of M^8 the map f_0 extends to a map $f : M^8 \to \mathbb{R}^{3N_2}$.

For each $z \in M^8$ we denote by $Mono((T_zM^8, g), (T_{f(z)}\mathbb{R}^{3N_2}, \beta_{N_2}))$ the set of all monomorphisms $\rho : T_zM^8 \to T_{f(z)}\mathbb{R}^{3N_2}$ such that the restriction of $h(f(z))$ to $df(T_zM^8)$ is equal to $(df^{-1})^*g$. To save the notation, whenever we consider the restriction of the form g to an open subset $U \subset M^8$ we shall denote also by g this restriction.

3.12. Proposition. *There exists a section s of the fibration $Mono((TM^8, g), (f^*(T\mathbb{R}^{3N_2}, \beta_{N_2}))$ such that $s(z)(T_zM^8)$ is β_{N_2}-regular subspace for all $z \in M^8$.*

Proof of Proposition 3.12. The proof of Proposition 3.12 consists of 3 steps.

Step 1. In the first step we show the existence of a section $s_1 \in Mono(TM^8, M \times \mathbb{R}^{3N_0})$ such that the image of s_1 is β_{N_2}-regular sub-bundle of dimension 8 in $M \times \mathbb{R}^{3N_2}$. To save notation we also denote by β the following 3-form on $\mathbb{R}^{3N_0}$

$$\beta = \sum_{j=1}^{N_2} dx_j^1 \wedge dx_j^2 \wedge dx_j^3.$$

It is easy to see that β is multi-symplectic. Furthermore we shall assume that $(w_j^i), 1 \le i \le 3$, is some fixed vector basis in $\mathbb{R}^3_j$.

3.13. Lemma. *For each given $k \ge 3$ there there exists a k-dimensional subspace V^k in $\mathbb{R}^{3N_0}$ such that V^k is β-regular subspace, provided that $N_0 \ge 5 + (k/2 - 2)(3 + k/2)$, if k is even, and $N_0 \ge 6 + ([k/2] - 2)(3 + [k/2]) + [k/2]$, if k is odd.*

Proof. We shall construct a linear embedding $f : V^k \to \mathbb{R}^{3N_2}$ whose image satisfies the condition of Lemma 3.12. Each linear map f can be written as

$$f = (f_1, f_2, \cdots, f_{N_0}),\ f_i : V^k \to \mathbb{R}^3_i, i = \overline{1, N_0}.$$

Now we can assume that $V^3 \subset V^4 \subset \cdots \subset V^k$ is a chain of subspaces in V^k which is generated by some vector basis $(e_1, \cdots e_k)$ in V^k. We denote by $(e_1^*, \cdots, e_k^*)$ the dual basis of $(V^k)^*$. By construction, the restriction of $(e_1^*, \cdots, e_i^*)$ to V^i is the dual basis of $(e_1, \cdot s, e_i) \in V^i$. For the simplicity we shall denote the restriction of any v_j^* to these subspaces also by v_j^* (if the restriction is not zero). We shall construct f_i inductively on the dimension k of V^k such that the following condition holds for all $3 \le i \le k$

$$< f_1^*(\Lambda^2(\mathbb{R}^3_1)), f_2^*(\Lambda^2(\mathbb{R}^3_2)), \cdots, f_{\delta(i)}^*(\Lambda^2(\mathbb{R}^3_{\delta(i)}) >_{\otimes\mathbb{R}} = \Lambda^2(V^i). \tag{3.14}$$

The condition (3.14) implies that $f(V^i)$ is β-regular, since the image

$$I_h(\mathbb{R}^3_1 \times \cdots \times \mathbb{R}^3_{\delta(i)}) = \oplus_{j=1}^{\delta(i)} \Lambda^2(\mathbb{R}^3_j).$$

For $i = 3$ we can take $f_1 = Id$, and $\delta(1) = 1$. Suppose that $f_{\delta(i)}$ is already constructed. To find f_j, $\delta(i) + 1 \le j \le \delta(i+1)$, so that (3.14) holds, it suffices to find linear embeddings $f_{\delta(i)+1}, \cdots, f_{\delta(i+1)}$ with the following property

$$< f_{\delta(i)+1}^*\Lambda^2(\mathbb{R}^3_{\delta(i)+1}), \cdots, f_{\delta(i+1)}^*\Lambda^2(\mathbb{R}^3_{\delta(i+1)}) >_{\otimes\mathbb{R}} \supset e_{i+1}^* \wedge \Lambda^1(V^i). \tag{3.15}$$

We can proceed as follows. We let

$$f_j(e_{i+1}) = w_j^1 \in \mathbb{R}_j^3, \text{ if } j \geq \delta(i)+1, \ f_j(e_{i+1}) = 0, \text{ if } j \leq \delta(i).$$

To complete the construction of f_j we need to specify $f_j(e_l)$, for $1 \leq l \leq i$ and $j \geq \delta(i)+1$. For such l and j we shall define $f_j(e_l) = 0$ or $f_j(e_l) = w_j^2$ or $f_j(e_l) = w_j^3$ so that (3.14) holds. A simple combinatoric calculation shows that the most economic " distribution" of $f_j(e_l)$ satisfies the estimate for $\delta(i)$ as in Lemma 3.13. □

Now once we have chosen a h-regular subspace V^{17} in $\mathbb{R}^{80}$ by Lemma 3.13, we shall find a section s_1 for the step 1 by requiring that s_1 is a section of $Mono(TM^8, M \times V^{17})$. This section exists, since the fiber $Mono(T_xM^8, \mathbb{R}^{17})$ is homotopic equivalent to $SO(17)/SO(9)$ which has all homotopy groups π_j vanishing, if $j \leq 8$. This completes the step 1.

Step 2. Once a section s_1 in Step 1 is specified we put the following form g_1 on TM^8:

$$g_1 = g - s_1^*(\beta).$$

In this step we show the existence of a section s_2 of the fibration $End((TM^8, g_1), (M^8 \times \mathbb{R}^{3N_1} \to M^8, \beta))$ (we do not require that s_2 is a monomorphism).

Using the Nash trick [Nash1956] we can find a finite number of open coverings U_i^j, $j = \overline{1,8}$ of M^8 which satisfy the following properties:

$$N_i^j \cap N_k^j = \emptyset, \ \forall j = \overline{1,8} \text{ and } i \neq k, \tag{3.16}$$

and moreover U_i^j is diffeomorphic to an open ball for all i, j. Since U_i^j satisfy the condition (3.16), for a fixed j we can embed the union $\hat{U}^j = \cup_i U_i^j$ into $\mathbb{R}^8$. Thus for each j on the union $\hat{U}^j$ we have local coordinates x_j^r, $r = \overline{1,8}$, $j = \overline{1,8}$. Using partition of unity functions $f_j(z)$ corresponding to $\hat{U}^j$ we can write

$$g_1(z) = \sum_{j=1}^{8} f_j(z) \cdot \sum_{1 \leq r_1 < r_2 < r_3 \leq 8} \mu_j^{r_1 r_2 r_3}(z) \cdot dx_j^{r_1} \wedge dx_j^{r_2} \wedge dx_j^{r_3}.$$

We numerate (i.e. find a function θ with values in $\mathbb{N}^+$) on the set $\{(j, r_1 r_2 r_3)\}$ of $N_1 = 8 \cdot C_8^3$ elements. Next we find a section s_2 of the form

$$s_2(z) = (s_1(z), \cdots, s_{N_1}(z)), \ s_q(z) \in End(T_z U^j, \mathbb{R}_q^3)$$

such that

$$s_{\theta(j, r_1 r_2 r_3)}(z) = f_j(z) \cdot \mu_j^{r_1 r_2 r_3}(z) \cdot A_{r_1, r_2, r_3},$$

$$\text{where } A_{r_1, r_2, r_3}(\partial x_{r_l} \in T_z M^8) = \delta_l^i e_i \in \mathbb{R}_q^3.$$

Here (e_1, e_2, e_3) is a vector basis in $\mathbb{R}_q^3$ for $q = \theta(j, r_1 r_2 r_3)$ and ∂x_{r_l} is the coordinate vector field on $\hat{U}^j$. Clearly the section s_2 satisfies the condition $s_2^*(h(z)) = g_1(z)$ for all $z \in M^8$. This completes the second step.

Step 3. We put

$$s = (s_1, s_2),$$

where s_1 is the constructed section in Step 1 and s_2 is the constructed section in Step 2. Clearly s satisfies the condition of Lemma 3.13. $\square$

Proposition 3.7 now follows from Proposition 3.10 and Proposition 3.12. $\square$

3.17. Theorem-Remark. It follows directly from the Eliashberg-Mishachev Theorem on the approximation of a given differential form by closed forms [E-M2002,10.2.1] and from the openess and and from the $Gl(\mathbb{R}^7)$-invariance of the space of G_2-forms ($\tilde{G}_2$-form, resp.) that any G_2-form ($\tilde{G}_2$-form resp.) on an open manifold M^7 is homotopic to a closed G_2-form ($\tilde{G}_2$-closed form resp.) on M^7.

3.18. Further remark. By the same argument we can find universal space for closed k-forms on manifold of fixed dimension.

4 Appendix: Flexibility, microflexibility and Nash-Gromov implicit function theorem.

In this appendix we recall Gromov theorems on the relation between flexibility as well as microflexibility and H-principle.

A.1. H-principle and flexibility[Gromov1986, 2.2.1.B]. *If V is a locally compact countable polyhedron (e.g. manifold), then every flexible sheaf over V satisfies the H-principle.* (Actually the parametric H-principle which implies the H-principle.)

To formulate the relation between the flexibility and microflexibility (of solution sheafs) under certain conditions in [A2] we need to describe these conditions with the notion of acting in a solution sheaf diffeotopies, which move sharply a set.

Suppose that $U \subset U' \subset V$ are open subsets in V. We say that diffeotopies $\delta_t : U \to U', t \in [0,1], \delta_0 = Id$, **act in a sheaf Φ on subset $\Phi' \subset \Phi(U')$**, if δ_t assigns each section $\phi \in \Phi'$ a homotopy of sections in $\Phi(U)$ which we shall call $\delta_t^*\phi$ such that the following conditions hold

- $\delta_0^*\phi = \phi_{|U}$
- If two sections $\phi_1, \phi_2 \in \Phi'$ coincide at some point $u_0' \in U'$ and if $\delta_{t_0}(u_0) = u_0'$ for some $u_0 \in U$ and $t_0 \in [0,1]$, then $(\delta_{t_0}^*\phi_1)(u) = (\delta_{t_0}^*\phi_2)(u_0)$. This allows us to write $\phi(\delta_t(u))$ instead of $(\delta_t^*\phi)(u), u \in U$.
- Let $U_0 \subset U$ be the maximal open subset where $(\delta_t)_{|U} = Id$. Then the homotopy $\delta_t^*(\phi)$ is constant in t over U_0.

- If the diffeotopy δ_t is constant in t for $t \geq t_0$ over all U, then the homotopy $\delta_t^* \phi$ is also constant in t for $t \geq t_0$.
- If $\phi_p \in \Phi'$, $p \in P$ is a continuous family of sections, then the family $\delta_t^* \phi_p$ is jointly continuous in t and p.

Let V_0 be a closed subset of the above $U' \subset V$. Suppose that V is provided with some metric. Let $\mathcal{A}$ be a set of diffeotopies $\delta_t : U' \supset \mathcal{O}pV_0 \to U'$. We call $\mathcal{A}$ **strictly moving a given subset** $S \subset V_0$, if $dist(\delta_t(S), V_0) \geq \mu > 0$ for $t \geq 1/2$ and for all $\delta_t \subset \mathcal{A}$.

Further we call $\mathcal{A}$ **sharp at** S, if for every $\mu > 0$ there exists $\delta_t \in \mathcal{A}$ such that

- $(\delta_t)_{|\mathcal{O}p(v)} = Id, t \in [0,1]$ for all points $v \in V_0$ such that $dist(v, S) \geq \mu$, where $\mathcal{O}p(v)$ is an (arbitrary) small neighborhood of v,
- $\delta_t = \delta_{1/2}$ for $t \geq 1/2$.

For a given sheaf Φ on V and for a given action of the set $\tilde{\mathcal{A}}$ of diffeotopies δ_t on subset $\Phi'_{\delta_t} \subset \Phi(U')$, we say that acting diffeotopies **sharply move** V_0 **at** $S \subset V_0$, if for each compact family of sections $\Phi_p \in \Phi(U')$ there exists a subset $\mathcal{A} \subset \tilde{\mathcal{A}}$ which is strictly moving S and sharp at S such that $\phi_p \in \Phi'_{\delta_t}$ for all $\delta_t \in \mathcal{A}$.

We say that acting in Φ diffeotopies **sharply moves a submanifold** $V_0 \subset V$, if each point $v \in V_0$ admits a neighborhood $U' \subset V$ such that acting diffeotopies $\delta_t : V_0' = V_0 \cap U' \to U'$ sharply move V_0' at any given closed hypersurface $S \subset V_0'$.

A.2. A criterion on flexibility. [Gromov1986, 2.2.3.C"] *Let Φ be a microflexible sheaf over V and let a submanifold $V_0 \subset V$ be sharply movable by acting in Φ diffeotopies. Then the sheaf $\Phi_0 = \Phi_{|V_0}$ is flexible and hence it satisfies the h-principle.*

Before stating the Nash-Gromov implicit Function Theorem in A.3 and A.3' we need to introduce several new notions. Let X be a C^∞-fibration over an n-dimensional manifold V and let $G \to V$ be a smooth vector bundle. We denote by $\mathcal{X}^\alpha$ and $\mathcal{G}^\alpha$ respectively the spaces of C^α-sections of the fibrations X and G for all $\alpha = 0, 1, \cdots, \infty$. Let $\mathcal{D} : \mathcal{X}^r \to \mathcal{G}^0$ be a differential operator of order r. In other words the operator $\mathcal{D}$ is given by a map $\triangle : X^{(r)} \to G$, namely $\mathcal{D}(x) = \triangle \circ J_x^r$, where $J_x^r(v)$ denotes the r-jet of x at $v \in V$. We assume below that $\mathcal{D}$ is a C^∞-operator and so we have continuous maps $\mathcal{D} : \mathcal{X}^{\alpha + r} \to \mathcal{G}^\alpha$ for all $\alpha = 0, 1, \cdots, \infty$.

We say that the operator $\mathcal{D}$ is **infinitesimal invertible over a subset** $\mathcal{A}$ in the space of sections $x : V \to X$ if there exists a family of linear differential operators of certain order s, namely $M_x : \mathcal{G}^s \to \mathcal{Y}_x^0$, for $x \in \mathcal{A}$, such that the following three properties are satisfied.

1. There is an integer $d \geq r$, called **the defect of the infinitesimal inversion** M, such that $\mathcal{A}$ is contained in $\mathcal{X}^d$, and furthermore, $\mathcal{A} = \mathcal{A}^d$ consists (exactly and only) of C^d-solutions of an open differential relation $A \subset X^{(d)}$. In particular, the sets $\mathcal{A}^{\alpha+d} = \mathcal{A} \cap \mathcal{X}^{\alpha+d}$ are open in $\mathcal{X}^{\alpha+d}$ in the respective fine $C^\alpha + d$-topology for all $\alpha = 0, 1, \cdots, \infty$.

2. The operator $M_x(g) = M(x, g)$ is a (non-linear) differential operator in x of order d. Moreover the global operator

$$M : \mathcal{A}^d \times \mathcal{G}^s \to \mathcal{J}^0 = T(\mathcal{X}^0)$$

is a differential operator, that is given by a C^∞-map $A \oplus G^{(s)} \to T_{vert}(X)$.

3. $L_x \circ M_x = Id$ that is

$$L(x, M(x, g)) = g \text{ for all } x \in \mathcal{A}^{d+r} \text{ and } g \in \mathcal{G}^{r+s}.$$

Now let $\mathcal{D}$ admit over an open set $\mathcal{A} = \mathcal{A}^d \subset \mathcal{X}^d$ an infinitesimal inversion M of order s and of defect d. For a subset $\mathcal{B} \subset \mathcal{X}^0 \times \mathcal{G}^0$ we put $\mathcal{B}^{\alpha,\beta} := \mathcal{B} \cap (\mathcal{X}^\alpha \times \mathcal{G}^\beta)$. Let us fix an integer σ_0 which satisfies the following inequality

$$(*) \qquad \sigma_0 > \bar{s} = \max(d, 2r + s).$$

Finally we fix an arbitrary Riemannian metric in the underlying manifold V.

A.3. Nash-Gromov implicit function theorem. [Gromov1986, 2.3.2]. *There exists a family of sets $\mathcal{B}_x \subset \mathcal{G}^{\sigma_0+s}$ for all $x \in \mathcal{A}^{\sigma_0+r+s}$, and a family of operators $\mathcal{D}_x^{-1} : \mathcal{B}_x \to \mathcal{A}$ with the following five properties.*

1. *Neighborhood property: Each set $\mathcal{B}_x$ contains a neighborhood of zero in the space $\mathcal{G}^{\sigma_0+s}$. Furthermore, the union $\mathcal{B} = \{x\} \times \mathcal{B}_x$ where x runs over $\mathcal{A}^{\sigma_0+r+s}$, is an open subset in the space $\mathcal{A}^{\sigma_0+r+s} \times \mathcal{G}^{\sigma_0+s}$.*

2. *Normalization Property: $\mathcal{D}_x^{-1}(0) = x$ for all $x \in \mathcal{A}^{\sigma_0+r+s}$.*

3. *Inversion Property: $\mathcal{D} \circ \mathcal{D}_x^{-1} - \mathcal{D}(x) = Id$, for all $x \in \mathcal{A}^{\sigma_0+r+s}$, that is*

$$\mathcal{D}(\mathcal{D}_x^{-1}(g)) = \mathcal{D}(x) + g,$$

for all pairs $(x, g) \in \mathcal{B}$.

4. *Regularity and Continuity: If the section $x \in \mathcal{A}$ is C^{η_1+r+s}-smooth and if $g \in \mathcal{B}_x$ is C^{σ_1+s}-smooth for $\sigma_0 \leq \sigma_1 \leq \eta_1$, then the section $\mathcal{D}_x^{-1}(g)$ is C^σ-smooth for all $\sigma < \sigma_1$. Moreover the operator $\mathcal{D}^{-1} : \mathcal{B}^{\eta_1+r+s,\sigma_1+s} \to \mathcal{A}^\sigma$, $\mathcal{D}^{-1}(x, g) = \mathcal{D}_x^{-1}(g)$, is jointly continuous in the variables x and g. Furthermore, for $\eta_1 > \sigma_1$, the section $\mathcal{D}^{-1} : \mathcal{B}^{\eta_1+r+s,\sigma_1+s} \to \mathcal{A}^{\sigma_1}$ is continuous.*

5. *Locality: The value of the section $\mathcal{D}_x^{-1}(g) : V \to X$ at any given point $v \in V$ does not depend on the behavior of x and g outside the unit ball $B_v(1)$ in V with center v, and so the equality $(x,g)_{|B_v(1)} = (x',g')_{|B_v(1)}$ implies $\mathcal{D}_x^{-1}(g))(v) = (\mathcal{D}_{x'}^{-1}(g'))(v)$.*

A.3'. Corollary. Implicit Funtion Theorem. *For every $x_0 \in \mathcal{A}^\infty$ there exists fine $C^{\bar{s}+s+1}$-neighborhood $\mathcal{B}_0$ of zero in the space of $\mathcal{G}\bar{s}+s+1$, where $\bar{s} = \max(d, 2r+s)$, such that for each $C^{\sigma+s}$-section $g \in \mathcal{B}_0$, $\sigma \geq \bar{s}+1$, the equation $\mathcal{D}(x) = \mathcal{D}(x_0) + g$ has a C^σ-solution.*

Finally we shall show a big class of microflexible solution sheafs Φ by using the Nash-Gromov implicit function theorem.

Let us fix a C^∞-section $g : V \to G$ and call a C^∞-germ $x : \mathcal{O}p(v) \to X$, $v \in V$, **an infinitesimal solution of order** α **of the equation** $\mathcal{D}(x) = g$, if at the point v the germ $g' = g - \mathcal{D}(x)$ has zero α-jet , i.e. $J^\alpha_{g'}(v) = 0$. We denote by $\mathcal{R}^\alpha(\mathcal{D}, g) \subset X^{(r+\alpha)}$ the set of all jets represented by these infinitesimal solutions of order α over all points $v \in V$. Now we recall the open set $A \subset X^{(d)}$ defining the set $\mathcal{A} \subset X^{(d)}$, and for $\alpha \geq d - r$ we put

$$\mathcal{R}_\alpha = \mathcal{R}_\alpha(A, \mathcal{D}, g) = A^{r+\alpha-d} \cap \mathcal{R}^\alpha(\mathcal{D}, g) \subset X^{(r+\alpha)},$$

where $A^{r+\alpha-d} = (p_d^{r+\alpha})^{-1}(A)$ for $p_d^{r+\alpha} : X^{r+\alpha} \to X^d$.

A $C^{r+\alpha}$-section $x : V \to X$ satisfies $\mathcal{R}_\alpha$, iff $\mathcal{D}(x) = g$ and $x \in \mathcal{A}$.

Now we set $\mathcal{R} = \mathcal{R}_{d-r}$ and denote by $\Phi = \Phi(\mathcal{R}) = \Phi(A, \mathcal{D}, g)$ the sheaf of C^∞-solutions of $\mathcal{R}$.

A.4. Microflexibility of the sheaf of solutions and Nash-Gromov implicit functions.[Gromov1986 2.3.2.D"] *The sheaf Φ is microflexible.*

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Appendix.

in communication with Kaoru Ono

A soft proof of the existence of a universal space.

For readers' convenience, we present here an elementary proof of the following

B.1. Theorem. *For any given positive integers n, k there exists a smooth manifold $\mathcal{M}$ of dimension $N(n,k)$ and a closed differential k-form α on it with the following property. For any closed differential k-form ω on a smooth manifold M^n such that $[\omega] \in H^k(M^n, \mathbb{Z})$ there is a smooth map $f : M^n \to \mathcal{M}^{N(n,k)}$ such that $f^*(\alpha) = \omega$.*

Proof. As in the proof of Theorem 3.6 we reduce this problem to the existence of an embedding of M^n to the space $\mathbb{R}^{\overline{N_1}}$ with the constant k-form $\beta_{\overline{N_1}}$ such that the pull-back of $\beta_{\overline{N_1}}$ is equal to a given exact k-form g. Since g is an exact form there exists a $(k-1)$-form ϕ on M^n such that $d\phi = \omega$.

Next we use the Nash trick of a construction of an open covering A_i on M^n

$$M^n = \cup_{i=0}^n A_i, \tag{B.2}$$

such that each A_i is the union of disjoint open balls $D_{i,j}$, $j = 1, \ldots, J(i)$ on M^n. (Pick a simplicial decomposition of M^n and construct A_i by the induction on i. Let $D_{0,j}$ be a small coordinate neighborhood of the j-th vertex. We may assume that they are mutually disjoint. Set $A_0 = \cup_{j=1}^{J(0)} D_{0,j}$. Suppose that $A_0, \ldots, A_i$ are defined. Let $D_{i+1,j}$ be a small coordinate neighborhood, which contains $S_j^{i+1} \setminus \cup_{\ell=0}^i A_\ell$, where S_j^{i+1} is the j-th $i+1$-dimensional simplex. We may assume that they are mutually disjoint. Set $A_{i+1} = \cup_{j=1}^{J(i+1)} D_{i+1,j}$. Hence we obtain desired open sets $A_0, \ldots, A_n$.)

Let $\{\rho_i\}$ be the partition of unity on M subordinte to the covering $\{A_i\}$. We write $\phi(x) = \sum_{i=0}^n \rho_i(x) \cdot \phi$. Note that $\omega = d\phi = \sum d\phi_i$. Clearly the form $\phi_i = \rho_i(x) \cdot \phi$ has support on A_i.)

Let $N_1 = \binom{n}{k-1}$ and

$$\gamma = \sum_{j=1}^{N_1} x_j^1 dx_j^2 \wedge \cdots \wedge dx_j^k.$$

Note that $j = 1, \ldots, \binom{n}{k-1}$ are in one-to-one correspondence with the sequences $1 \le i_2 < \cdots < i_k \le n$.

B.3. Proposition. *There is an embedding* $f_i : A_i \to (\mathbb{R}^{N_1 k}, \gamma)$ *such that* $f_i^*(\gamma) = \phi_i$. *In particular,* $f_i^* d\gamma = d\phi_i$.

Proof of Proposition B.3. Since A_i is a union of the disjoint balls $D_{i,j}$ it suffices to prove the existence of map f_i on each ball $D = D_{i,j}$. Take some coordinate $(x_1, \cdots, x_n)$ on the ball D. We can write the restriction of the $k-1$-form ϕ_i to D as ϕ, where

$$\phi(x) = \sum_{1 \le i_2 < \cdots i_k \le n} \lambda_{i_2 \cdots i_k} dx^{i_2} \wedge \cdots dx^{i_k}.$$

We construct map f_i as follows

$$f_i(x) = (\dots, f_{i;i_2\cdots i_k}(x), \dots)_{1 \le i_2 < \cdots < i_k \le n}$$

where for $x = (x_1, x_2, \cdots, x_n)$ we put

$$f_{i;i_2\cdots i_k}(x) : D \to \mathbb{R}^k_{i_2\cdots i_k}(x^1, x^2, \cdots, x^k),$$

$$(x_1, \cdots, x_n) \mapsto (x^1 = \lambda_{i_2\cdots i_k}(x), x^2 = x^{i_2}, \cdots x^k = x^{i_k}).$$

Clearly we have $f_i^*(\omega) = \phi$. It is easy to check that f_i is an embedding on D.□

We shall use cut-off functions χ_i with support contained in $A_i \subset M$ such that $\chi_i = 1$ on the support of ρ_i. Then $\widetilde{f_i} = \chi_i \cdot f_i$ can be extended to the whole M^n.

Now we construct an embedding $f : M^n \to \mathbb{R}^{\overline{N_1}} = \mathbb{R}^{N_1 k(n+1)}$ by setting

$$f(x) = (\tilde{f}_0, \cdots, \tilde{f}_n).$$

Clearly f is an embedding such that $f^*\alpha = \omega$.

Finally, we note that we can choose $f : M \to \mathbf{R}^{\overline{N_1}}$ such that its image is contained in an arbitrary small neighborhood of the origin.

Choose $R > 0$ such that the image of f is contained in the R-ball centered at the origin $O \in \mathbf{R}^{\overline{N_1}}$. For a given integer m, we pick a refinement $\{V_p\}$ of the covering $\{D_{i,j}\}_{i,j}$ such that $f_{i(p)}(V_p)$ is contained in a ball of radius $1/m^2$. (The center of the ball may not be the origin.) Here $i(p)$ is chosen so that $V_p \subset A_{i(p)}$, i.e., there is $j(p)$ such that $V_p \subset D_{i(p),j(p)}$. Applying the Nash trick again to refine $\{V_p\}$ so that there is an open covering $\{A'_\ell\}$ of M such that each of A'_i is a union of some mutually disjoint family of V_p's. Denote by ρ'_i, χ'_i a partition of unity and cut-off functions for the covering $\{A'_i\}$, respectively. On $V_p \subset A'_i$ we modify the construction of the mapping $f_{i;i_2,\dots,i_k}$ as follows. Using the translation in $x_2, \dots x_k$-coordinates in each $\mathbf{R}^k_{i_2,\dots,i_k}$, we may assume that

$$f_{i;i_2,\dots i_k}(V_p) \subset [-R, R] \times [-1/m^2, 1/m^2] \times \cdots \times [-1/m^2, 1/m^2].$$

Now we consider the mapping

$$\Phi_m : (x_1, x_2, \dots, x_k) \mapsto (\frac{x_1}{m}, m \cdot x_2, x_3, \dots, x_k).$$

Then we find that $\Phi_m^* dx^1 \wedge \cdots \wedge dx^k = dx^1 \wedge \cdots \wedge dx^k$ and

$$\Phi_m \circ f_{i;i_2,\dots,i_k}(V_j) \subset [-R/m, R/m] \times [-1/m, 1/m] \times [-1/m^2, 1/m^2] \times \cdots \times [-1/m^2, 1/m^2].$$

Multiplying $\chi_i' \cdot \rho_i'$ to the first factor and χ_i' to the rest, we obtain f_i' on $V_p \subset A_i'$. Using f_i' instead of f_i in the previous argument and taking m large, we can make the image of $f : M \to \mathbf{R}^{N_1}$ arbitrary small around the origin.