Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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(revised version: August 2006)

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Preprint no.: 72 2006

1

AN L^p TWO WELL LIOUVILLE THEOREM

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Abstract. We provide a different approach to and prove a (partial) generalisation of a recent theorem on the structure of low energy solutions of the compatible two well problem in two dimensions [Lor 05], [Co-Sc 06]. More specifically we will show that a "quantitative" two well Liouville theorem holds for the set of matrices $K = SO(2) \cup SO(2)H$ where $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ under a constraint on the L^p norm of the second derivative. Our theorem is the following.

Let $p \geq 1$, $q > 1$. Let $u \in W^{2,p}(B_1(0)) \cap W^{1,q}(B_1(0))$. There exists positive constants $\mathcal{C}_1 << 1, \mathcal{C}_2 >> 1$ depending only on σ, p, q such that if u satisfies the following inequalities

$$\int_{B_1(0)} d^q(Du(z), K)\, dL^2 z \leq \mathcal{C}_1 \varepsilon, \quad \int_{B_1(0)} |D^2 u(z)|^p\, dL^2 z \leq \mathcal{C}_1 \varepsilon^{1-p}$$

then there exist $A \in K$ such that

$$\int_{B_{\frac{1}{2}}(0)} |Du(z) - A|^q\, dL^2 z \leq \mathcal{C}_2 \varepsilon^{\frac{1}{2q}}. \tag{2}$$

We provide a proof of this result by use of a theorem related to the isoperimetric inequality, the approach is conceptually simpler than those previously used in [Lor 05], [Co-Sc 06], however it does not given the optimal $c\varepsilon^{\frac{1}{q}}$ bound for (2) that has been proved (for the $p = 1$ case) in [Co-Sc 06].

In 1850 Liouville [Lio 50] proved the following classic theorem: given domain $\Omega \subset \mathbb{R}^3$ and function $u \in C^4(\Omega : \mathbb{R}^3)$ with the property $Du(x) = \lambda(x) O(x)$ where $\lambda(x) \in \mathbb{R}$ and $O(x)$ is an orthogonal $n \times n$ matrix, then u is a Mobius transformation.

There are many works generalising this theorem, an incomplete list is Gehring [Ge 62], Resnetnak [Re 67], Bojarski and Iwaniec [Bo-Iw 82]. A corollary to Liouville's Theorem is that a function whose gradient is in $SO(n)$ is an affine mapping. Recently Friesecke, James and Müller [Fr-Ja-Mu 02] have proved an optimal quantitative version of this corollary.

Theorem 1 (Friesecke, James, Müller)**.** *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $q > 1$. There exists a constant $C(U, q)$ with the following property. For each $v \in W^{1,q}(U, \mathbb{R}^n)$ there exists an associated rotation $R \in SO(n)$ such that*

$$\|Dv - R\|_{L^q(U)} \leq C(U, q)\, \|\mathrm{dist}(Dv, SO(n))\|_{L^q(U)}. \tag{3}$$

This theorem has already had important applications [Fr-Ja-Mu 02], [Fr-Ja-Mu 06] and there have been a number of generalisations of it [Cha-Mu 03], [Fa-Zh 05], [De-Se 06]. However the corresponding statement for $SO(n)$ replaced by a set of matrices $L \subset M^{m \times n}$ which contains rank-1 connections (i.e. there exists $A, B \in L$ such that $\mathrm{rank}(A - B) = 1$) is trivially false.

However recently a version of Theorem 1 has been proved in two dimensions for the set of matrices $K = SO(2) A \cup SO(2) B$ where the matrix AB^{-1} is rank-1 connected to some matrix in $SO(2)$. The first result was by the author [Lor 05] for invertible bilipschitz mappings with control in inequality (2) of order $\varepsilon^{\frac{1}{800}}$. This was greatly generalised by Conti, Schweitzer [Co-Sc 06], Theorem 2.1, Corollary 2.5. Our current theorem is:

Theorem 2. *Let $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ for $\sigma > 0$. Let $p \geq 1$, $q \geq 1$. Let $K = SO(2) \cup SO(2) H$. Let $u \in W^{2,p}(B_1(0) : \mathbb{R}^2) \cap W^{1,q}(B_1(0) : \mathbb{R}^2)$.*

[1] MSC 74N15, keywords Two Wells, Liouville

Date: 31,8,2006.

There exists positive constants $\mathcal{C}_1 << 1, \mathcal{C}_2 >> 1$ depending only on σ, p, q such that if u satisfies the following inequalities

$$\int_{B_1(0)} d^q\left(Du\left(z\right), K\right) dL^2 z \leq \mathcal{C}_1 \varepsilon \tag{4}$$

$$\int_{B_1(0)} \left|D^2 u\left(z\right)\right|^p dL^2 z \leq \mathcal{C}_1 \varepsilon^{1-p}, \tag{5}$$

then there exists $J \in \{Id, H\}$ such that $\int_{B_1(0)} d^q\left(Du\left(z\right), SO\left(2\right) J\right) dL^2 z \leq \mathcal{C}_2 \varepsilon^{\frac{1}{2q}}$ and consequently (by application of Theorem 1) for the case $q > 1$, for some $R \in SO\left(2\right)$ we have

$$\int_{B_{\frac{1}{2}}(0)} \left|Du\left(z\right) - RJ\right|^q dL^2 z \leq \mathcal{C}_2 \varepsilon^{\frac{1}{2q}}. \tag{6}$$

In [Co-Sc 06] the hypotheses were that u satisfies (4) and (5) for the case $p = 1$, (i.e. the L^1 version of this theorem) however their theorem states the optimal inequality, namely that (6) holds for $\varepsilon^{\frac{1}{q}}$, they also established the theorem for the more general sets of matrices $SO\left(2\right) A \cup SO\left(2\right) B$ and stated it for Lipschitz domains.

Our approach differs from that of [Lor 05], [Co-Sc 06] in two ways. Firstly we will use the hypotheses to reduce the situation to one in which we can apply a theorem related to the isoperimetric inequality, this will allow us to gain control of our function in a central sub-ball. Though this method does not produce optimal results, it is conceptually simpler in that it is the fastest way to see why this initially surprising result should be true.

Secondly and more importantly we provide a different approach than [Co-Sc 06] to proving the result for non-invertible mappings, specifically our argument does not require the use of the embedding $W^{1,1}\left(B_1\left(0\right)\right) \hookrightarrow L^2\left(B_1\left(0\right)\right)$. This embedding together with degree arguments were used in a essential way in [Co-Sc 06] to prove the first result for non invertible mappings, the main reason these methods can not be extended to higher dimensions is to do with the failure of this embedding for dimension $n \geq 3$. Using our methods we hope to prove Theorem 2 for $n \geq 3$ in a forth coming paper [Lor pr2]. Our basic idea is to use the fact that on a large subset $A \subset B_{\frac{1}{2}}\left(0\right)$ the function $w := u_{\lfloor A}$ forms a quasi-regular mapping and we obtain partial invertibility properties of u inside $w\left(A\right)$.

Our main motivation for proving Theorem 2 is to use it to establish a sharp reduction of the problem of calculating the energy of the functional $I_\varepsilon^p\left(u\right) = \int_\Omega d\left(Du, K\right) + \varepsilon\left|D^2 u\right|^p$ over $W^{2,p}\left(\Omega\right) \cap A_F$ (where A_F is the space of functions with affine boundary condition F) to the problem of calculating the energy of I_0^1 over the space of functions that are affine on a piecewise affine triangular grid, for results in this direction see [Lor pr1].

One of the main tools we will use to prove Theorem 2 is a theorem charactering the case of equality in the isoperimetric inequality. More specifically, it is well known that amongst all bodies B of volume 1 in $\mathbb{R}^n$, the ball minimises $H^{n-1}\left(\partial B\right)$, i.e. the ball gives the case of equality of the isoperimetric inequality. A quantitative statement of this kind is given by the following theorem of Hall, Haymann, Weitsman [Ha-Ha-We 91].

Theorem 3 (Hall et al.). *Let E be a set of finite perimeter [1] in $\mathbb{R}^2$, $R := \left(\frac{L^2(E)}{\pi}\right)^{\frac{1}{2}}$ and let the Fraenkel asymmetry $\lambda\left(E\right)$ be defined by*

$$\lambda\left(E\right) := \inf_{a \in \mathbb{R}^2} \frac{L^2\left(E \backslash B_R\left(a\right)\right)}{\pi R^2}. \tag{7}$$

[1] Hall et al. state their Lemma for sets with smooth boundaries. By Theorem 3.41 [Am-Fu-Pa 00] we can approximate any set A of finite perimeter with a sequence of sets $\left(A_n\right)$ that converge in measure to A which have smooth boundaries and for which $\mathrm{Per}\left(A_n\right) \to \mathrm{Per}\left(A\right)$ as $n \to \infty$, hence its easy to see the lemma holds for sets of finite perimeter.

Then

$$(\mathrm{Per}\,(E))^2 \geq 4\pi \left(1 + \frac{(\lambda\,(E))^2}{4}\right) L^2\,(E)\,. \tag{8}$$

The starting idea of the proof of the Theorem 2 is the same starting idea as that of Theorem 1 of [Lor 05] and that of Theorem 2.1 of [Co-Sc 06]. This idea is to surround a central sub-ball with a lower dimensional set on which u is close to affine. In [Lor 05] the set was the boundary of a diamond, in [Co-Sc 06] the corners of a triangle. In both papers the lower dimensional set is found using that fact that hypotheses (4), (5) (for $p = 1$) forces the perimeter of the set

$$\mathcal{W} = \{x \in B_1\,(0) : d\,(Du\,(x)\,, SO\,(2)) < d\,(Du\,(x)\,, SO\,(2)\,H)\} \tag{9}$$

to be less that $\mathcal{C}_1$, for example since $H^1\,(\partial\mathcal{W}) \leq \mathcal{C}_1$ it is easy to find (by Fubini's Theorem) many intervals $[a,b] \subset B_1\,(0)$ for which $[a,b] \cap \partial\mathcal{W} = \emptyset$ so (possibly after a change of variables) $[a,b] \subset \mathcal{W}$ and then the full force of hypothesis (4) goes to show that for "most" intervals the gradient of Du stays close to $SO\,(2)$ and hence there is no *stretching* of $u\,([a,b])$ in the sense that we have the inequality $|u\,(a) - u\,(b)| \leq H^1\,(u\,([a,b])) \leq |a-b| + c\varepsilon^{\frac{1}{q}}$. To begin to establish affine type properties we would like to show an inequality of the form

$$|u\,(a) - u\,(b)| \geq |a-b| - c\varepsilon^{\frac{1}{q}}. \tag{10}$$

In [Lor 05] it was established that there exists two "special directions" $\eta_1, \eta_2 \in S^1$ (defined by $\left|H^{-1}\eta_i\right| = 1$ for $i = 1,2$) for which (10) holds true for intervals parallel to η_1 and η_2 and for which $\int_{[a,b]} d\,(Du\,(z)\,, K)\,dH^1 z \leq c\varepsilon^{\frac{1}{q}}$. Hence it was possible to show u is close to affine on the boundary of a diamond.

In [Co-Sc 06], (10) was established using the fact that the inverse map u^{-1} satisfies an inequality of the form (4) and "in some sense" an inequality of the form (5) in the image $u\,(B_1\,(0))$, so assuming that intervals $[a,b]$ and $[u\,(a)\,, u\,(b)]$ satisfy the appropriate inequalities both in the reference configuration and the image, the non-stretching argument can be carried out on $[u\,(a)\,, u\,(b)]$ and on $[a,b]$ to establish

$$|a-b| \approx |u\,(a) - u\,(b)| \pm c\varepsilon^{\frac{1}{q}}. \tag{11}$$

With this approach it is only necessary to control three points $\{a,b,c\}$ that form the corners of an equilateral triangle because (11) shows that the distances of the set $\{u\,(a)\,, u\,(b)\,, u\,(c)\}$ are (almost) preserved, and hence $\{u\,(a)\,, u\,(b)\,, u\,(c)\}$ comes close to forming the corners of an equilateral triangle. With one further geometric idea (the "two triangles" argument of [Co-Sc 06], p847, p848) this can be used to show that in ball $B_{r_0}\,(0)$ contained in the triangle, $L^2\,(B_{r_0}\,(0) \backslash \mathcal{W}) \leq \varepsilon^{\frac{1}{q}}$, the theorem then follows by an application of Theorem 1, the main gain in control comes from this strategy, i.e. to reduce the situation to a point where we have the hypotheses to apply Theorem 1 .

In the proof of Theorem 2 we exploit the bound $H^1\,(\partial\mathcal{W}) \leq \mathcal{C}_1$ a bit differently. This time instead of lines we consider the boundary of balls, we can chose $r_0 \in \left(\frac{1}{4}, \frac{3}{4}\right)$ so that $\partial B_{r_0}\,(0) \subset \mathcal{W}$ and $\int_{\partial B_{r_0}(0)} d^p\,(Du\,(z)\,, K)\,dH^1 z \leq \varepsilon$, and hence we have (possibly after change of variables) $H^1\,(u\,(\partial B_{r_0}\,(0))) \leq 2\pi r_0 + c\varepsilon^{\frac{1}{q}}$. Assuming u is an open mapping (which it almost is since inequality (4) implies there is a set Z with $L^2\,(B_1\,(0) \backslash Z) \leq c\varepsilon^{\frac{1}{q}}$ for which $u_{\lfloor Z}$ is a quai-regular mapping) we have $H^1\,(\partial u\,(B_{r_0}\,(0))) \leq H^1\,(u\,(\partial B_{r_0}\,(0))) \leq 2\pi r_0 + c\varepsilon^{\frac{1}{q}}$. And since by some degree arguments it is not hard to show $L^2\,(u\,(B_{r_0}\,(0))) \approx \int_{B_{r_0}(0)} \det\,(Du\,(z))\,dL^2 z \geq \pi r^2 - c\varepsilon^{\frac{1}{q}}$ we have that the set $u\,(B_{r_0}\,(0))$ comes very close to optimising the constants in the isoperimetric inequality so applying Theorem 3 we have that the Fraenkel asymmetry of $u\,(B_{r_0}\,(0))$ satisfies

$$\lambda\,(u\,(B_{r_0}\,(0))) \leq c\varepsilon^{\frac{1}{2q}}. \tag{12}$$

The loss of a factor 2 in control comes from using Theorem 3, as Theorem 3 is optimal this is a feature of the approach. However having (12) it is not hard to show $L^2\left(B_{r_0}(0)\backslash\mathcal{W}\right)\leq c\varepsilon^{\frac{1}{2q}}$, (6) then follows by application of Theorem 1. Conceptually this approach is simpler in that it avoids many of the quite delicate issues of finding substitutes for invertibility of u and controlling lines simultaneously in the reference configuration and in the image, however only suboptimal bounds can be established with the "isoperimetric method". For optimal bounds the "non stretching in lines" method of [Co-Sc 06] is best.

We would like to acknowledge that in the overall strategy (i.e. getting to the point of being able to apply Theorem 1 as soon as possible) and in the technical details (the use of degree theory, co-area argument along rays) we use many ideas of [Co-Sc 06].

Definition 1. *Given a connected open set $\Omega\subset\mathbb{R}^n$. A function $f\in W^{1,2}(\Omega:\mathbb{R}^n)$ with the property that* $\det(Du(z))\geq 0$ *for a.e. $x\in\Omega$ is said to be of* **finite dilation** *if and only if $\|Df(x)\|^n\leq K(x)\left|\det(Df(x))\right|$ a.e. where $1\leq K(x)<\infty$. The function f is said to have* **integrable dilation** *if and only if $\int_\Omega K(x)\,dL^n x<\infty$.*

We will need the following theorem [Iw-Sv 93].

Theorem 4 (Iwaniec, Šverák)**.** *Let $\Omega\subset\mathbb{R}^2$ be a connected open set. Given function $f:\Omega\to\mathbb{R}^2$, $f\in W^{1,2}(\Omega)$ which has integrable dilation then f is open and discrete.*

It is also well known that functions of finite dilation are continuous [Vo-Go 76].

Lemma 1. *Let*

$$d_0:=\min\left\{d\left(SO(2),SO(2)H\right),d\left(SO(2),\{P:\det(P)\leq 0\}\right)\right\}\tag{13}$$

and let $X\subset\mathbb{R}^2$ be an open bounded connected set. Suppose $f:\overline{X}\to\mathbb{R}^2$ is C^1 with the property that $\sup\left\{d\left(Df(z),SO(2)\right):z\in\overline{X}\right\}\leq\frac{9d_0}{10}$ *then for any open subset $Y\subset X$ we have*

$$\partial f(Y)\subset f(\partial Y).\tag{14}$$

Proof. Since $\|Df\|_{L^\infty(\overline{X})}<\infty$ we know for some constant c, $\|Df(z)\|^2\leq c\det(Df(z))$ for all $z\in\overline{X}$ and hence f is a function of integrable dilation. Thus by Theorem 4 we know it is an open map and it is well known (see Exercise 9.12 [Vu 88]) that (14) follows for any open $Y\subset X$.

Definition 2. *For C^1 function $w:\Omega\to\mathbb{R}^n$ and subset $B\subset\Omega$ we can define the Brouwder degree $d(y,w,B)$ via Definition 1.9* [Fo-Ga 95]*, note that for y such that*

$$w^{-1}(y)\subset\{x\in\Omega:\det(Dw(x))\neq 0\},$$

we have

$$d(y,w,B)=\sum_{x\in w^{-1}(y)\cap B}\operatorname{sgn}\left(\operatorname{Det}(Dw(x))\right)\tag{15}$$

where $\operatorname{sgn}(t)=1$ *for $t>0$ and* $\operatorname{sgn}(t)=-1$ *for $t<0$. We define*

$$N(y,w,B):=\operatorname{Card}\left(\left\{x\in w^{-1}(y)\cap B\right\}\right).\tag{16}$$

We will repeatedly use the following change of variable formula Theorem 5.27 from [Fo-Ga 95], we will state it in less generality than in [Fo-Ga 95]

Theorem 5. *Let $D\subset\mathbb{R}^n$ be an open, bounded set and let $w:D\to\mathbb{R}^n$ be a C^1 function. Let $\phi\in L^\infty(\mathbb{R}^n)$, then for every open subset $G\subset D$*

$$\int_G\phi(w(x))\det(Dw(x))\,dL^n x=\int_{\mathbb{R}^n}\phi(y)\,d(w,G,y)\,dL^n y.\tag{17}$$

1. Proof of Theorem 2

1.1. **Reduction.** Given $u \in W^{2,p}(B_1(0)) \cap W^{1,q}(B_1(0))$ we can convolve u with a standard convolution kernel ϕ to form $u_\rho := \phi_\rho * u$. Since we know $u_\rho \overset{W^{1,q}(B_1(0))}{\rightarrow} u$ and $u_\rho \overset{W^{2,p}(B_1(0))}{\rightarrow} u$ as $\rho \to 0$ (see for example Section 4.2 [Ev-Ga 92]). So for small enough ρ_0 we have a smooth function $\psi := u_{\rho_0}$ which satisfies

$$\int_{B_1(0)} d^q(D\psi(z), K)\, dL^2 z \leq 2\mathcal{C}_1\varepsilon \tag{18}$$

$$\int_{B_1(0)} \left|D^2\psi(z)\right|^p dL^2 z \leq 2\mathcal{C}_1\varepsilon^{1-p}, \tag{19}$$

and

$$\|u-\psi\|_{W^{1,q}(B_1(0))} \leq \varepsilon. \tag{20}$$

Let $\epsilon = \varepsilon^{\frac{1}{q}}$. By Holder's inequality (18) implies

$$\int_{B_1(0)} d(D\psi(z), K)\, dL^2 z \leq 2\pi\mathcal{C}_1^{\frac{1}{q}}\epsilon. \tag{21}$$

We will argue our main lemmas for function ψ.

2. Main lemmas

In the coming lemma we establish the basic consequences of $\mathcal{W}$ (see (9)) having small perimeter. By the relative isoperimetric inequality we have

$$\min\left\{L^2(\mathcal{W}), L^2(B_1(0)\backslash\mathcal{W})\right\} \leq c\mathcal{C}_1^2,$$

depending on which is the minimum we make a changes of variables to obtain a function v with the property $\int d(Dv, SO(2)) \leq c\mathcal{C}_1^2$ and has all the important properties of ψ. Throughout our proof c will denote any constant depending only on matrix H, note that c may be used repeatedly inside a proof denoting different constants on each occasion.

Lemma 2. *Let $p \geq 1$. Let p^* be the Holder conjugate of p, i.e. $\frac{1}{p}+\frac{1}{p^*}=1$. Suppose $\psi \in C^1(B_1(0))$ satisfies (18), (19) and (21). Define*

$$L(\psi) := \int_{B_1(0)} d(D\psi(z), SO(2)) - d(D\psi(z), SO(2)H)\, dL^2 z. \tag{22}$$

Let l_H be an affine function with the property that $l_H(0)=0$ and $Dl_H = H$. Let us define $v : B_{\frac{1}{2}}(0) \to \mathbb{R}^2$ by

$$v(z) := \begin{cases} \psi(l_H(\sigma z))\,\sigma^{-1}, & \text{if } L(\psi) \geq 0 \\ \psi(z), & \text{if } L(\psi) < 0. \end{cases} \tag{23}$$

We will show there exists positive constant $c_2 = c_2(\sigma) > 1$ such that v has the following properties.

- *For the set of matrices $\widetilde{K} := SO(2) \cup SO(2)J$ (where J is a diagonal matrix with eigenvalues σ, σ^{-1}) we have*

$$\int_{B_{\frac{1}{2}}(0)} d\left(Dv(z), \widetilde{K}\right) dL^2 z \leq c_2\mathcal{C}_1\epsilon. \tag{24}$$

and

$$\int_{B_{\frac{1}{2}}(0)} d^q\left(Dv(z), \widetilde{K}\right) dL^2 z \leq 3\mathcal{C}_1\varepsilon. \tag{25}$$

- $$\int_{B_{\frac{1}{2}}(0)} d^{\frac{q}{p^*}}\left(Dv(z), \widetilde{K}\right)\left|D^2 v(z)\right| dL^2 z \leq c_2 \mathcal{C}_1. \tag{26}$$
- $$\int_{B_{\frac{1}{2}}(0)} d\left(Dv(z), SO(2)\right) dL^2 z \leq c_2 \mathcal{C}_1^2. \tag{27}$$
- *Let* $\beta := \frac{1}{2\left(1+\frac{q}{p^*}\right)}$, *for any* $b \in B_{\frac{1}{4}}(0)$ *there exists a set* $K_b \subset \left(0, \frac{1}{2}\right)$ *with* $L^1\left(\left(0, \frac{1}{2}\right) \backslash K_b\right) \leq 8c_2\sqrt{\mathcal{C}_1}$ *and the properties*
$$\int_{B_{\frac{1}{2}}(0)\cap \partial B_r(b)} d\left(Dv(z), SO(2)\right) dH^1 z \leq c\epsilon \text{ for each } r \in K_b. \tag{28}$$
And
$$\sup\left\{d\left(Dv(z), SO(2)\right) : z \in \partial B_r(b) \cap B_{\frac{1}{2}}(0)\right\} \leq \mathcal{C}_1^{\beta}. \tag{29}$$

Proof.
Step 1. We will show we can find $a_1 \in \left[\frac{9d_0}{10}, d_0\right]$ such that
$$H^1\left(\left\{x \in B_{\frac{1}{2}}(0) : d\left(D\psi(x), SO(2)\right) = a_1\right\}\right) < c\mathcal{C}_1. \tag{30}$$
Let
$$G_{a_1} = \left\{x \in B_{\frac{1}{2}}(0) : d\left(D\psi(x), SO(2)\right) < a_1\right\}$$
and let
$$B_{a_1} = \left\{x \in B_{\frac{1}{2}}(0) : d\left(D\psi(x), SO(2)H\right) < a_1\right\}.$$
We will also show
$$\min\left\{L^2\left(B_{\frac{1}{2}}(0)\backslash G_{a_1}\right), L^2\left(B_{\frac{1}{2}}(0)\backslash B_{a_1}\right)\right\} \leq c\mathcal{C}_1^2. \tag{31}$$
Proof of Step 1. Let p^* be the Holder conjugate of p. By Young's inequality
$$\begin{aligned}\int_{B_{\frac{1}{2}}(0)} \varepsilon d^{\frac{q}{p^*}}\left(D\psi(x), K\right)\left|D^2\psi(x)\right| dL^2 x &\leq \int_{B_{\frac{1}{2}}(0)} d^q\left(D\psi(x), K\right) + \varepsilon^p\left|D^2\psi(x)\right|^p dL^2 x\\ &\overset{(18),(19)}{\leq} 4\mathcal{C}_1\varepsilon,\end{aligned}$$
which gives
$$\int_{B_{\frac{1}{2}}(0)} d^{\frac{q}{p^*}}\left(D\psi(x), K\right)\left|D^2\psi(x)\right| dL^2 x \leq 4\mathcal{C}_1. \tag{32}$$
Let $S(x) = d\left(D\psi(x), SO(2)\right)$. By the Co-area formula
$$\begin{aligned}\int_{\frac{9d_0}{10}}^{d_0} H^1\left(S^{-1}(h)\right) dL^1 h &= \int_{\left\{x\in B_{\frac{1}{2}}(0): \frac{9d_0}{10} < d(D\psi(x), SO(2)) < d_0\right\}} \left|DS(x)\right| dL^2 x\\ &\leq c\int_{B_{\frac{1}{2}}(0)} d^{\frac{q}{p^*}}\left(D\psi(x), K\right)\left|D^2\psi(x)\right| dL^2 x\\ &\overset{(32)}{\leq} c\mathcal{C}_1.\end{aligned}$$
So we can find $a_1 \in \left(\frac{9d_0}{10}, d_0\right)$ such that $H^1\left(S^{-1}(a_1)\right) \leq c\mathcal{C}_1$. By the relative isoperimetric inequality [Am-Fu-Pa 00] Remark 3.49, 3.43 we have
$$\begin{aligned}\min\left\{L^2\left(G_{a_1} \cap B_{\frac{1}{2}}(0)\right)^{\frac{1}{2}}, L^2\left(B_{\frac{1}{2}}(0)\backslash G_{a_1}\right)^{\frac{1}{2}}\right\} &\leq cH^1\left(S^{-1}(a_1)\right)\\ &\leq c\mathcal{C}_1.\end{aligned}$$

If $L(\psi) < 0$ then we must have $L^2\left(B_{\frac{1}{2}}(0)\backslash G_{a_1}\right) \leq c\mathcal{C}_1^2$ and if $L(\psi) \geq 0$ we must have $L^2\left(B_{\frac{1}{2}}(0)\cap G_{a_1}\right) \leq c\mathcal{C}_1^2$. Now

$$\begin{aligned} L^2\left(B_{\frac{1}{2}}(0)\backslash B_{a_1}\right) &= L^2\left(B_{\frac{1}{2}}(0)\cap G_{a_1}\right) \\ &\quad +L^2\left(\left\{x\in B_{\frac{1}{2}}(0):\min\left\{d\left(D\psi(x),SO(2)\right),d\left(D\psi(x),SO(2)H\right)\right\}>a_1\right\}\right) \\ &\leq c\mathcal{C}_1^2+a_1^{-1}\int_{B_{\frac{1}{2}}(0)} d\left(D\psi(x),K\right)dL^2x \\ &\overset{(18)}{\leq} c\mathcal{C}_1^2. \end{aligned}$$

This completes the proof of Step 1.

Step 2. Defining v by (23) we will show v satisfies (24), (25), (26), (27).
Proof of Step 2. In the case where $L(\psi) < 0$, (24) follows by Holder's inequality

$$\begin{aligned} \int_{B_{\frac{1}{2}}(0)} d\left(Dv(z),K\right)dL^2z &\leq \left(\int_{B_{\frac{1}{2}}(0)} d^q\left(Dv(z),K\right)dL^2z\right)^{\frac{1}{q}} \\ &\leq 2\mathcal{C}_1\epsilon. \end{aligned}$$

Inequality (27) follows because if $x\notin G_{a_1}$ then $d\left(Dv(z),SO(2)\right)\leq cd\left(Dv(z),K\right)+c$ so

$$\begin{aligned} \int_{B_{\frac{1}{2}}(0)} d\left(Dv(z),SO(2)\right)dL^2z &\leq \int_{B_{\frac{1}{2}}(0)\cap G_{a_1}} d\left(Dv(z),K\right)dL^2z \\ &\quad +c\int_{B_{\frac{1}{2}}(0)\backslash G_{a_1}} d\left(Dv(z),K\right)dL^2z \\ &\quad +cL^2\left(B_{\frac{1}{2}}(0)\backslash G_{a_1}\right) \\ &\overset{(24)}{\leq} c\mathcal{C}_1^2. \end{aligned} \tag{33}$$

Finally (26) is immediate from (32).

In the case where $L(\psi)\geq 0$ for $\widetilde{K}=SO(2)\cup SO(2)H^{-1}$, (24) follows from (18) by change of variables. We can also show $\int_{B_{\frac{1}{2}}(0)} d\left(Dv(z),SO(2)H\right)dL^2z\leq c\mathcal{C}_1^2$ by an identical argument to (33), inequality (27) then follows by a change of variables.

Inequality (26) follows from (32) in the following way

$$\begin{aligned} &\int_{B_{\frac{1}{2}}(0)} d^{\frac{q}{p^*}}\left(Dv(z),\widetilde{K}\right)\left|D^2v(z)\right|dL^2z \\ &\quad =\int_{B_{\frac{1}{2}}(0)} d^{\frac{q}{p^*}}\left(D\psi\left(l_H(\sigma z)\right)H,\widetilde{K}\right)\left|D\left[D\psi\left(l_H(\sigma z)\right)H\right]\right|dL^2z \\ &\quad \leq c\int_{B_{\frac{1}{2}}(0)} d^{\frac{q}{p^*}}\left(D\psi\left(l_H(\sigma z)\right),K\right)\left|D^2\psi\left(l_H(\sigma z)\right)\right|dL^2z \\ &\quad \overset{(32)}{\leq} c\mathcal{C}_1. \end{aligned}$$

Step 3. We will show v satisfies (28), (29).

Proof of Step 3. Let

$$K_b^1 = \left\{ h \in \left(0, \frac{1}{2}\right) : \int_{\partial B_h(0)} d^{\frac{q}{p^*}} \left(Dv(z), \widetilde{K} \right) \left| D^2 v(z) \right| dH^1 z \leq \frac{\sqrt{\mathcal{C}_1}}{2^{2\beta+1}} \right\}.$$

So by (26) $L^1\left(\left(0, \frac{1}{2}\right) \backslash K_b^1\right) \leq 8c_2\sqrt{\mathcal{C}_1}$.

$$K_b^2 = \left\{ h \in \left(0, \frac{1}{2}\right) : \int_{\partial B_h(0)} d\left(Dv(z), SO(2)\right) dH^1 z \leq \mathcal{C}_1 \right\}.$$

By (27) $L^1\left(\left(0, \frac{1}{2}\right) \backslash K_b^2\right) \leq c_2 \mathcal{C}_1$. We claim that for any $h \in K_b^1 \cap K_b^2$ we have

$$\sup\left\{ d\left(Dv(z), SO(2)\right) : z \in \partial B_h(0) \right\} < \mathcal{C}_1^{\beta}. \tag{34}$$

Suppose (34) is false, then we must be able to find $a_1, a_2 \in \partial B_h(0)$ with the following properties

- $d\left(Dv(a_1), SO(2)\right) = \frac{\mathcal{C}_1^{\beta}}{2}$, $d\left(Dv(a_2), SO(2)\right) = \mathcal{C}_1^{\beta}$.
- We can find a connected component of $\partial B_h(0) \setminus \{a_1, a_2\}$ which we will denote by T with the property that

$$d\left(Dv(x), SO(2)\right) \in \left[\frac{\mathcal{C}_1^{\beta}}{2}, \mathcal{C}_1^{\beta}\right] \text{ for all } x \in T. \tag{35}$$

Thus

$$\begin{aligned} \int_T d^{\frac{q}{p^*}} \left(Dv(z), \widetilde{K} \right) \left| D^2 v(z) \right| dH^1 z &\geq \left(\frac{\mathcal{C}_1^{\beta}}{2} \right)^{\frac{q}{p^*}} \int_T \left| D^2 v(z) \right| dH^1 z \\ &\geq \frac{\mathcal{C}_1^{\beta\left(\frac{q}{p^*}+1\right)}}{2^{\frac{q}{p^*}+1}} \\ &\geq \frac{\sqrt{\mathcal{C}_1}}{2^{2\beta}} \end{aligned}$$

and this contradicts the fact that $h \in K_b^1$. Let

$$K_b^3 = \left\{ h \in \left(0, \frac{1}{2}\right) : \int_{\partial B_h(0)} d\left(Dv(z), \widetilde{K} \right) dH^1 z \leq c_2 \sqrt{\mathcal{C}_1} \epsilon \right\}.$$

By (24) we know $L^1\left(\left(0, \frac{1}{2}\right) \backslash K_b^3\right) \leq \sqrt{\mathcal{C}_1}$. For any $h \in K_b^1 \cap K_b^2 \cap K_b^3$ we have that if $z \in \partial B_h(0)$ then $d\left(Dv(z), \widetilde{K}\right) = d\left(Dv(z), SO(2)\right)$ so defining $K_b := K_b^1 \cap K_b^2 \cap K_b^3$ the set K_b satisfies (28) and (29) and this completes the proof.

2.1. **Introduction to Lemma 3.** *In the introduction we mapped a ball into the image, for reasons to do with lack of invertibility it will turn out to be more convenient to "pull back" a ball $B_h(y)$ from the image, this is essentially because in this way we can guarantee that $L^2\left(v^{-1}\left(B_h(y)\right)\right)$ is "almost" greater or equal to πh^2. If we can show $v^{-1}\left(\partial B_h(y)\right)$ is well defined and forms a Jordan curve and $H^1\left(v^{-1}\left(\partial B_h(y)\right)\right) \leq 2\pi h + c\varepsilon^{\frac{1}{q}}$ then we can apply Theorem 3. However to carry this out we need to establish some limited form of invertibility of v, specifically we need $v^{-1}\left(\partial B_h(y)\right)$ to form a Jordan curve.*

2.1.1. *Motivation for Step 4. To establish the invertibility properties described in (2.1) we need to consider a function w defined on a subset $A \subset B_{\frac{1}{2}}(0)$ for which $\det(Dv) > c$. In addition we need to show that the degree of w is 1 on the boundaries of many balls in the image of w. This can be done by establishing $L^2(w(A)) \approx \frac{\pi}{4}$, which we will show via truncation arguments and the use of the lower bound (47).*

2.1.2. *Motivation for Step 5.* *Having shown that $w^{-1}\left(\partial B_h\left(y\right)\right)$ is a Jordan curve, let $\mathcal{I}_y$ denote its interior. We now need to show $L^2\left(\mathcal{I}_y\right) \geq \pi h^2 - c\varepsilon^{\frac{1}{q}}$, this could be established if we know every point in $\mathcal{I}_y \cap A$ is mapped into the ball $B_h\left(y\right)$. Step 5 shows this via the following argument, since some of the points of $\mathcal{I}_y \cap A$ must be mapped inside $B_h\left(y\right)$, if $w\left(\mathcal{I}_y \cap A\right)$ spreads outside $B_h\left(y\right)$ we must have $w\left(\mathcal{I}_y \cap A\right) \cap \partial B_h\left(y\right) \neq \emptyset$ however this implies there exists $z \in \partial B_h\left(y\right)$ such that* Card $\left(w^{-1}\left(z\right)\right) \geq 2$ *because $w\left(\partial \mathcal{I}_y\right) = \partial B_h\left(y\right)$ and this contradicts the fact w has degree 1 on $\partial B_h\left(y\right)$.*

2.1.3. *Motivation for Step 6.* *Having established that $\mathcal{I}_y$ has the property $L^2\left(\mathcal{I}_y\right) \geq \pi h^2 - c\varepsilon^{\frac{1}{q}}$ and $H^1\left(\partial \mathcal{I}_y\right) \leq 2\pi h + c\varepsilon^{\frac{1}{q}}$ we can apply Theorem 3 to show there exists ω_b such that $L^2\left(\mathcal{I}_y \triangle B_{p_h}\left(\omega_b\right)\right) \leq c\varepsilon^{\frac{1}{2q}}$ (where $p_h = \sqrt{\frac{L^2(\mathcal{I}_y)}{\pi}}$). In some sense this implies $\partial \mathcal{I}_y$ is "close" to a circle. We would like to use this to show $L^2\left(\mathcal{I}_y \backslash \mathcal{W}\right)$ is small. To do this we will use the fact J has "shrink directions", by this we mean there exists $\theta_1, \theta_2 \in S^1$ such that $|J\theta_i| = 1$ for $i = 1, 2$ and denoting by $\mathcal{S}$ the set of ψ "between" θ_1 and θ_2 we have $|J\psi| < 1$ for all $\psi \in \mathcal{S}$. The argument will be that if $L^2\left(\mathcal{W}^c \cap \mathcal{I}_y\right)$ is large then we must be able to find many lines (parallel to the shrink directions) starting from the ω_y and going to the boundary $\partial \mathcal{I}_y$ which has large intersection with $\mathcal{W}^c$ hence the image of the path will be less than h so (assuming ω_y is mapped close to y and $p_h \leq h + c\varepsilon^{\frac{1}{2q}}$) this will be a contradiction. This argument will only work if for "most" $\psi \in \mathcal{S}$, the line starting from ω_y, parallel to ψ and ending in $\partial \mathcal{I}_y$ (denoted l_ψ) has the property that $\int_{l_\psi} d\left(Dv, \widetilde{K}\right)$ is small. Formally we need $\int_{\psi \in \mathcal{S}} \int_{l_\psi} d\left(Dv, \widetilde{K}\right) < c\varepsilon^{\frac{1}{q}}$. To find this we need to use the Co-area formula with a function Ψ_y defined by $|x - \omega_y| e^{i\Psi_y(x)} = x - \omega_y$ (identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the obvious way) and since $|D\Psi_y\left(z\right)| \approx \frac{1}{|z - \omega_y|}$ we need to have $\int d\left(Dv\left(z\right), K\right) |z - \omega_y|^{-1} dL^2 z \leq c\varepsilon^{\frac{1}{q}}$. Let c_0 denote the "centre" of $v\left(B_{\frac{1}{2}}\left(0\right)\right)$, assuming the set of points $\left\{\omega_y : y \in B_{\frac{1}{8}}\left(c_0\right)\right\}$ has positive measure, by a Fubini trick learnt from* [Co-Sc 06] *we can find a ω_y for which this holds. The point of Step 6 is to establish the existence of such a large set of $\left\{\omega_y : y \in B_{\frac{1}{8}}\left(c_0\right)\right\}$. Specifically we show there is a large set $\Upsilon_0 \subset B_{\frac{1}{8}}\left(0\right)$ such that for every $x \in \Upsilon_0$, the point $y := v\left(x\right)$ has the properties we want (i.e. invertibility of w on $\partial B_h\left(y\right)$). Since (as we will later show) $x \approx \omega_{v(x)}$ the set Υ_0 provide us with the large set points we require.*

2.1.4. *Motivation for Step 7.* *As mentioned in 2.1.3, in order for our arguments with the "shrink directions" to work we need that $p_h \leq h + c\varepsilon^{\frac{1}{2q}}$ and $|w\left(\omega_y\right) - y| \leq \varepsilon^{\frac{1}{2q}}$ since otherwise the image of lines from ω_y to $\partial \mathcal{I}_y$ can indeed have non-trivial intersection with $\mathcal{W}^c$ and they could still reach $\partial B_h\left(y\right)$. To establish these two things we will pull back lines of the form $[y, t_\theta]$ where $t_\theta \in \partial B_h\left(y\right)$. If we find three such points t_{θ_1}, t_{θ_2} and t_{θ_3} where the angle between any two of them is close to $\frac{2\pi}{3}$ and we can show $H^1\left(u^{-1}\left([y, t_{\theta_i}]\right)\right) \leq h + c\varepsilon^{\frac{1}{2q}}$ for $i = 1, 2, 3$ then since this implies $\omega_h \in \bigcap_{i=1}^3 B_{h + c\varepsilon^{\frac{1}{2q}}}\left(w^{-1}\left(t_{\theta_i}\right)\right)$ it follows $\left|\omega_h - w^{-1}\left(b\right)\right| \leq c\varepsilon^{\frac{1}{2q}}$, from this it is easy to show $p_h \leq h + c\varepsilon^{\frac{1}{2q}}$. The purpose of Step 7 is to show we can find such lines.*

Lemma 3. *Given a function $v \in C^4\left(B_{\frac{1}{2}}\left(0\right)\right)$ satisfying properties (24), (26), (27), (28) and (29) of Lemma 2. We will show there exists a set $\Lambda_0 \subset B_{\frac{1}{8}}\left(0\right)$ with $L^2\left(B_{\frac{1}{8}}\left(0\right) \backslash \Lambda_0\right) \leq c\mathcal{C}_1^{\frac{1}{4q}}$ such that for any $b \in \Lambda_0$ we can find a set $D_b \subset \left(\frac{1}{8}, \frac{5}{16}\right)$ with $L^1\left(\left(\frac{1}{8}, \frac{5}{16}\right) \backslash D_b\right) \leq c\mathcal{C}^{\frac{1}{32q}}$ and for any $h \in D_b$ there exists a connected open set I_b with the following properties*

$$v\left(\partial I_b\right) = \partial B_h\left(v\left(b\right)\right), \tag{36}$$

$$\partial I_b \subset N_{c\mathcal{C}_1^{\frac{1}{16}}}\left(\partial B_h\left(b\right)\right). \tag{37}$$

And

$$L^2\left(I_b\backslash B_h\left(b\right)\right)\leq c\sqrt{\epsilon}. \tag{38}$$

Proof.

Step 1. We will show that for any $b\in B_{\frac{1}{4}}\left(0\right)$ there exists a set $\mathcal{Y}_b\subset\left(0,\frac{1}{2}\right)$ with $L^1\left(\left(0,\frac{1}{2}\right)\backslash\mathcal{Y}_b\right)\leq c\sqrt{\mathcal{C}_1}$ affine function l_R with derivative $R\in SO\left(2\right)$ such that

$$\left\|v-l_R\right\|_{L^\infty\left(\partial B_r\left(b\right)\right)}\leq c\sqrt{\mathcal{C}_1}\text{ for each }r\in\mathcal{Y}_b. \tag{39}$$

Proof of Step 1. By applying Proposition A1 of [Fr-Ja-Mu 02] (and taking $\lambda=10\sigma^{-1}$) we have a c-Lipschitz function $\tilde{v}$ and

$$\begin{aligned}L^2\left(\left\{x\in B_{\frac{1}{2}}\left(0\right):\tilde{v}\left(x\right)\neq v\left(x\right)\right\}\right) &\leq \frac{c}{10\sigma^{-1}}\int_{\left\{x\in B_{\frac{1}{2}}\left(0\right):\left|Dv\left(x\right)\right|>10\sigma^{-1}\right\}}\left|Dv\left(z\right)\right|dL^2z\\ &\leq \frac{c}{10\sigma^{-1}}\int_{\left\{x\in B_{\frac{1}{2}}\left(0\right):d\left(Dv\left(x\right),\widetilde{K}\right)>5\sigma^{-1}\right\}}d\left(Dv\left(z\right),\widetilde{K}\right)dL^2z\\ &\overset{(24)}{\leq} c\epsilon. \end{aligned}\tag{40}$$

And in the same way

$$\begin{aligned}\left\|Dv-D\tilde{v}\right\|_{L^1\left(B_{\frac{1}{2}}\left(0\right)\right)} &\leq \frac{c}{10\sigma^{-1}}\int_{\left\{x\in B_{\frac{1}{2}}\left(0\right):\left|Dv\left(x\right)\right|>10\sigma^{-1}\right\}}\left|Dv\left(z\right)\right|dL^2z\\ &\leq c\epsilon. \end{aligned}\tag{41}$$

Thus

$$\begin{aligned}\int_{B_{\frac{1}{2}}\left(0\right)}d^2\left(D\tilde{v}\left(z\right),SO\left(2\right)\right)dL^2z &\leq c\int_{B_{\frac{1}{2}}\left(0\right)}d\left(D\tilde{v}\left(z\right),SO\left(2\right)\right)dL^2z\\ &\overset{(27),(41)}{\leq} c\mathcal{C}_1^2. \end{aligned}\tag{42}$$

Thus applying Theorem 1 there exists $R\in SO\left(2\right)$ such that

$$\begin{aligned}\int_{B_{\frac{1}{2}}\left(0\right)}\left|D\tilde{v}\left(z\right)-R\right|dL^2z &\leq c\left(\int_{B_{\frac{1}{2}}\left(0\right)}\left|D\tilde{v}\left(z\right)-R\right|^2dL^2z\right)^{\frac{1}{2}}\\ &\leq c\left(\int_{B_{\frac{1}{2}}\left(0\right)}d^2\left(D\tilde{v}\left(z\right),SO\left(2\right)\right)dL^2z\right)^{\frac{1}{2}}\\ &\overset{(42)}{\leq} c\mathcal{C}_1. \end{aligned}$$

And by (41) we have $\int_{B_{\frac{1}{2}}\left(0\right)}\left|Dv\left(z\right)-R\right|dL^2z\leq c\mathcal{C}_1$. By Poincaré's inequality there exists and affine map l_R with $Dl_R=R$ such that

$$\int_{B_{\frac{1}{2}}\left(0\right)}\left|v\left(z\right)-l_R\left(z\right)\right|dL^2z\leq c\mathcal{C}_1. \tag{43}$$

So by the co-area formula there exists a set $\mathcal{Y}_b\subset\left(0,\frac{1}{2}\right)$ with $L^1\left(\left(0,\frac{1}{2}\right)\backslash\mathcal{Y}_b\right)\leq c\sqrt{\mathcal{C}_1}$ such that for each $r\in\mathcal{Y}_b$ we have

$$\int_{\partial B_r\left(b\right)}\left|v\left(z\right)-l_R\left(z\right)\right|+\left|Dv\left(z\right)-R\right|dH^1z\leq c\sqrt{\mathcal{C}_1}. \tag{44}$$

By the fundamental theorem of Calculus any $r \in \mathcal{Y}_b$ satisfies (39) so this completes the proof of Step 1.

Step 2. Let $c_0 = l_R(0)$. We will show there exists $l_0 \in \mathcal{Y}_0 \cap K_0 \cap \left(\frac{1}{2} - c\sqrt{\mathcal{C}_1}, \frac{1}{2}\right)$ such that the Brouwder degree of v and $\tilde{v}$ satisfy

$$d\left(v, B_{l_0}(0), z\right) = 1 \text{ for any } z \in B_{l_0 - c\sqrt{\mathcal{C}_1}}(c_0) \tag{45}$$

and

$$d\left(\tilde{v}, B_{l_0}(0), z\right) = 1 \text{ for any } z \in B_{l_0 - c\sqrt{\mathcal{C}_1}}(c_0). \tag{46}$$

Hence

$$L^2\left(\tilde{v}\left(B_{l_0}(0)\right) \cap B_{\frac{1}{2}}(c_0)\right) \geq \frac{\pi}{4} - c\sqrt{\mathcal{C}_1}. \tag{47}$$

Proof of Step 2. Let

$$F_0 := \left\{h \in \left(0, \frac{1}{2}\right) : H^1\left(\left\{x \in B_{\frac{1}{2}}(0) : \tilde{v}(x) \neq v(x)\right\} \cap \partial B_h(0)\right) \leq c\sqrt{\epsilon}\right\}. \tag{48}$$

From (40) we know $L^1\left(\left(0, \frac{1}{2}\right) \backslash F_0\right) \leq c\sqrt{\epsilon}$. Pick $l_0 \in \mathcal{Y}_0 \cap F_0 \cap \left(\frac{1}{2} - c\sqrt{\mathcal{C}_1}, \frac{1}{2}\right)$. By (39) we know

$$v\left(\partial B_{l_0}(0)\right) \subset N_{c\sqrt{\mathcal{C}_1}}\left(\partial B_{l_0}(c_0)\right). \tag{49}$$

In addition since $\tilde{v}$ is Lipschitz using (49) and the fact that $l_0 \in F_0$ we must have

$$\tilde{v}\left(\partial B_{l_0}(0)\right) \subset N_{c\sqrt{\mathcal{C}_1}}\left(\partial B_{l_0}(c_0)\right). \tag{50}$$

Now let us define the Homopoty $H(x,t) = (1-t)v(x) + tl_R(x)$. And define $h_t(x) := H(x,t)$. Note that $B_{l_0 - c\sqrt{\mathcal{C}_1}}(c_0) \cap h_t\left(\partial B_{l_0}(0)\right) = \emptyset$ for any $t \in [0,1]$ and hence by Theorem 2.3 [Fo-Ga 95] we have

$$d\left(v, B_{l_0}(0), p\right) = d\left(l_R, B_{l_0}(0), p\right) = 1 \text{ for any } p \in B_{l_0 - c\sqrt{\mathcal{C}_1}}(c_0)$$

and thus establishes (45). Using (50), (46) follows via an identical argument. By Theorem 2.1 [Fo-Ga 95] (46) implies $B_{l_0 - c\sqrt{\mathcal{C}_1}}(c_0) \subset \tilde{v}\left(B_{l_0}(0)\right)$ hence (47) follows.

Step 3. Let $Q : \mathbb{R} \to \mathbb{R}_+$ be defined by $Q(t) = t - 4\epsilon$ if $t \geq 4\epsilon$ and $Q(t) = 0$ if $t < 4\epsilon$. Let $Q_\epsilon := Q * \phi_\epsilon$ where ϕ_ϵ is the standard rescaled convolution kernel on $\mathbb{R}$ (i.e. $\mathrm{Spt}\phi_\epsilon \subset [-\epsilon, \epsilon]$). Let $J(M) := d\left(M, \widetilde{K}\right)$. Finally we define $L_\epsilon(z) = Q_\epsilon\left(J\left(Dv(z)\right)\right)$. Note $L_\epsilon \in C^3\left(B_{\frac{1}{2}}(0)\right)$. It could be that $\left\{z \in B_{\frac{1}{2}}(0) : \left|DL_\epsilon(z)\right| = 0\right\}$ is uncountable. However by the Area formula

$$\begin{aligned}
&\int_{B_\epsilon(0) \cap DL_\epsilon\left(B_{l_0}(0)\right)} \mathrm{Card}\left(\left\{z \in \overline{B_{l_0}(0)} : DL_\epsilon(z) = P\right\}\right) dL^2 P \\
&\leq \int_{\overline{B_{l_0}(0)}} \det\left(D^2 L_\epsilon(z)\right) dL^2 z \\
&< \infty.
\end{aligned} \tag{51}$$

So we must be able to find $P_0 \in B_\epsilon(0)$ such that

$$\mathrm{Card}\left(\left\{z \in B_{l_0}(0) : DL_\epsilon(z) = P_0\right\}\right) < \infty. \tag{52}$$

Defined $\mathcal{L}(z) := L_\epsilon(z) - P_0 \cdot z$, so

$$\mathrm{Card}\left(\left\{z \in \overline{B_{l_0}(0)} : \left|D\mathcal{L}(z)\right| = 0\right\}\right) = \mathrm{Card}\left(\left\{z \in \overline{B_{l_0}(0)} : DL_\epsilon(z) = P_0\right\}\right) < \infty. \tag{53}$$

Let $\beta = \frac{1}{2\left(1 + \frac{q}{p^*}\right)}$. We will assume $\mathcal{C}_1$ is small enough so that $8\mathcal{C}_1^\beta < d_0$ (recall Definition (13)). We will show we can find $H \subset \left(2\mathcal{C}_1^\beta, 4\mathcal{C}_1^\beta\right)$ with $L^1(H) \geq \frac{19}{10}\mathcal{C}_1^\beta$ such that for any $a \in H$

$$H^1\left(\mathcal{L}^{-1}(a)\right) \leq c\sqrt{\mathcal{C}_1}. \tag{54}$$

Proof of Step 3. We know

$$\begin{aligned}\left|D\mathcal{L}\left(z\right)\right| &\leq \left|DL_{\epsilon}\left(z\right)\right|+\epsilon\\ &\leq \left|DQ_{\epsilon}\left(J\left(Dv\left(z\right)\right)\right)\right|\left|D^{2}v\left(z\right)\right|+\epsilon\\ &\leq \left|D^{2}v\left(z\right)\right|+\epsilon.\end{aligned}$$

By the Co-area formula

$$\begin{aligned}\int_{2\mathcal{C}_1^{\beta}}^{4\mathcal{C}_1^{\beta}} H^1\left(\mathcal{L}^{-1}\left(a\right)\right)dL^1a &= \int_{\left\{z\in B_{\frac{1}{2}}(0):2\mathcal{C}_1^{\beta}\leq\mathcal{L}(z)\leq 4\mathcal{C}_1^{\beta}\right\}}\left|D\mathcal{L}\left(z\right)\right|dL^2z\\ &\leq \int_{\left\{z\in B_{\frac{1}{2}}(0):2\mathcal{C}_1^{\beta}\leq\mathcal{L}(z)\leq 4\mathcal{C}_1^{\beta}\right\}}\left|D^2v\left(z\right)\right|dL^2z+c\epsilon\\ &\overset{(26)}{\leq} c\mathcal{C}_1^{1-\frac{\beta q}{p^*}}.\end{aligned}$$

As $1-\left(\frac{q}{p^*}+1\right)\beta=\frac{1}{2}$, the set

$$H:=\left\{a\in\left[2\mathcal{C}_1^{\beta},4\mathcal{C}_1^{\beta}\right]:H^1\left(\mathcal{L}^{-1}\left(a\right)\right)\leq c\sqrt{\mathcal{C}_1}\right\} \tag{55}$$

has the property that $L^1\left(H\right)\geq\frac{19}{10}\mathcal{C}_1^{\beta}$. This completes the proof of Step 3.

Step 4. Let $a_1\in H\cap\left[3\mathcal{C}_1^{\beta},4\mathcal{C}_1^{\beta}\right]$. Let

$$\Psi_{a_1}=\left\{x\in B_{\frac{1}{2}}\left(0\right):d\left(Dv\left(x\right),\widetilde{K}\right)<a_1\right\}. \tag{56}$$

Let $l_0\in\left(\frac{1}{2}-c\sqrt{\mathcal{C}_1},\frac{1}{2}\right)\cap\mathcal{Y}_0\cap K_0$ be the number satisfying (45) and (46) from Step 2. We will show there exists open subset $A\subset B_{l_0}\left(0\right)\cap\Psi_{a_1}$ with the properties

- $$L^2\left(B_{l_0}\left(0\right)\backslash A\right)\leq c\epsilon \text{ and } \partial B_{l_0}\left(0\right)\subset\overline{A}. \tag{57}$$
- There exists $a_2\in\left[2\mathcal{C}_1^{\beta},3\mathcal{C}_1^{\beta}\right]$ such that defining
$$W_{a_2}:=\left\{x\in B_{\frac{1}{2}}\left(0\right):\mathcal{L}\left(z\right)=a_2\right\} \tag{58}$$
we have
$$\partial A\subset\partial B_{l_0}\left(0\right)\cup W_{a_2}. \tag{59}$$
- Also
$$B_{l_0}\left(0\right)\backslash\overline{A}=\bigcup_{k=1}^{m_0}D_k \text{ where } \left\{D_1,D_2,\dots D_{m_0}\right\} \text{ are connected open sets.} \tag{60}$$

In addition defining $w:\overline{A}\to\mathbb{R}^2$ by $w\left(x\right):=v\left(x\right)$ for $x\in A$ we will show w satisfies

- $$L^2\left(w\left(A\right)\cap B_{\frac{1}{2}}\left(c_0\right)\right)\geq\frac{\pi}{4}-c\sqrt{\mathcal{C}_1}. \tag{61}$$
- $$\partial w\left(A\right)\subset w\left(\partial A\right). \tag{62}$$

Finally for any $y\in B_{\frac{1}{4}}\left(c_0\right)$ there exists a set $L_y\subset\left(0,\frac{1}{2}+\left|y-c_0\right|\right)$ with the property that

$$L^1\left(\left(\left(0,\frac{1}{2}+\left|y-c_0\right|\right)\backslash L_y\right)\leq c\mathcal{C}_1^{\frac{1}{16}} \tag{63}$$

and denoting $l_1 := l_0 - c\sqrt{\mathcal{C}_1}$, $U_y := \left(\bigcup_{h\in L_y} \partial B_h(y)\right) \cap B_{l_1}(c_0)$ we have

$$U_y \subset w(A) \text{ and } d(w,A,z) = 1 \text{ for all } z \in U_y. \tag{64}$$

Proof of Step 4. Let

$$a_2 \in \left[2\mathcal{C}_1^{\beta}, \frac{5}{2}\mathcal{C}_1^{\beta}\right] \cap H \tag{65}$$

and define

$$\mathcal{B} = \{x \in B_{l_0}(0) : \mathcal{L}(x) > a_2\}. \tag{66}$$

Since $l_0 \in K_0$ from (29) (assuming ϵ is small enough) we know

$$\partial B_{l_0}(0) \cap \overline{\mathcal{B}} = \emptyset \text{ hence } d\left(\partial B_{l_0}(0), \overline{\mathcal{B}}\right) > 0. \tag{67}$$

Now since $\mathcal{B}$ is open we can find countably many open connected sets $D_1, D_2, \dots$ such that $\mathcal{B} = \bigcup_{k=1}^{\infty} D_k$. However by continuity of Dv we know that

$$\mathcal{L}(z) = Q_\epsilon(J(Dv(z))) - P_0 \cdot Dv(z) = a_2 \text{ for any } z \in \partial\mathcal{B}. \tag{68}$$

Since from (53) we know $|D\mathcal{L}(z)| \neq 0$ except for finitely many points, for any k the boundary ∂D_k forms a piecewise smooth set of finite H^1 measure. In addition for any $k_1 \neq k_2$ if $z_0 \in \partial D_{k_1} \cap \partial D_{k_2}$ as $\frac{DL_\delta(z_0)}{|DL_\delta(z_0)|}$ has to be the inward pointing unit normal to both ∂D_{k_1} and ∂D_{k_2} at z_0 and this is only possible if $|D\mathcal{L}(z_0)| = 0$. Thus $\mathrm{Card}(\partial D_{k_1} \cap \partial D_{k_2}) < \infty$ for any $k_1 \neq k_2$. Thus

$$\begin{aligned} \sum_{k=1}^{\infty} H^1(\partial D_k) &= H^1\left(\bigcup_{k=1}^{\infty} D_k\right) \\ &\leq c\sqrt{\mathcal{C}_1}. \end{aligned} \tag{69}$$

As $\mathrm{diam}(D_k) \leq H^1(\partial D_k)$ we know $\mathrm{diam}(D_k) \to 0$ as $k \to \infty$. Now recall v is C^4, so $\mathcal{L}$ is Lipschitz on any compact subset of $B_{\frac{1}{2}}(0)$ and as (68) holds for $z \in \partial D_k$, there exists $m_0 \in \mathbb{N}$ such that for any $k > m_0$, if $z \in D_k$

$$\begin{aligned} \mathcal{L}(z) &\leq c\,\mathrm{diam}(D_k) + a_2 \\ &\leq \frac{11}{4}\mathcal{C}_1^{\beta}. \end{aligned}$$

Hence defining $A := B_{l_0}(0) \setminus (\bigcup_{k=1}^{m_0} D_k)$ we have that $A \subset \Psi_{a_1}$, A satisfies (60) and it is clear from continuity of Dv that (59) is satisfied. Now note

$$\begin{aligned} L^2\left(\bigcup_{k=1}^{m_0} D_k\right) &\leq L^2(\mathcal{B}) \\ &\overset{(65),(66)}{\leq} L^2\left(\left\{x \in B_{\frac{1}{2}}(0) : d\left(Dv(x), \widetilde{K}\right) > \mathcal{C}_1^{\beta}\right\}\right) \\ &\overset{(24)}{\leq} c\epsilon. \end{aligned} \tag{70}$$

As $B_{l_0}(0) \backslash \overline{\mathcal{B}} \subset A$, (67) together with (70) implies (57). Let

$$N := \left\{x \in B_{\frac{1}{2}}(0) : \tilde{v}(x) = v(x)\right\} \tag{71}$$

so by (40)

$$L^2(A \backslash N) \leq c\epsilon. \tag{72}$$

Now

$$\begin{aligned}\tilde{v}\left(B_{l_0}(0)\setminus(N\cap A)\right) &\leq \int_{B_{l_0}(0)\setminus(N\cap A)} \det\left(D\tilde{v}(z)\right)dL^2 z\\ &\overset{(57),(72)}{\leq} c\epsilon. \end{aligned} \tag{73}$$

And as

$$\tilde{v}\left(B_{l_0}(0)\right)\cap B_{\frac{1}{2}}(c_0)\subset\left(\tilde{v}\left(B_{l_0}(0)\setminus(N\cap A)\right)\cup\tilde{v}(N\cap A)\right)\cap B_{\frac{1}{2}}(c_0), \tag{74}$$

we know

$$\begin{aligned}L^2\left(v(N\cap A)\cap B_{\frac{1}{2}}(c_0)\right) &= L^2\left(\tilde{v}(N\cap A)\cap B_{\frac{1}{2}}(c_0)\right)\\ &\overset{(73),(74)}{\geq} L^2\left(\tilde{v}\left(B_{l_0}(0)\right)\cap B_{\frac{1}{2}}(c_0)\right)-c\epsilon\\ &\overset{(47)}{\geq} \frac{\pi}{4}-c\sqrt{\mathcal{C}_1}.\end{aligned} \tag{75}$$

Hence (61) follows. Let $\mathcal{U}_1,\mathcal{U}_2,\dots\mathcal{U}_{m_1}$ denote the connected components of A, by (60) we know there are only finitely many such components. Finally by Lemma 1 we know that for any $i\in\{1,2,\dots m_1\}$ we have $\partial w(\mathcal{U}_i)\subset w(\partial\mathcal{U}_i)$ and this establishes (62).

We will assume a_2 was chosen to be one of the a.e. numbers such that W_{a_2} (as the level set of a Lipschitz function [Fed 69] 3,3.2.15) forms a rectifiable set.

By (39) we know $w\left(\partial B_{l_0}(0)\right)\subset N_{c\sqrt{\mathcal{C}_1}}\left(l_R\left(\partial B_{l_0}(0)\right)\right)=N_{c\sqrt{\mathcal{C}_1}}\left(\partial B_{l_0}(c_0)\right)$. So for $l_1:=l_0-c\sqrt{\mathcal{C}_1}$

$$\begin{aligned}\partial w(A)\cap B_{l_1}(c_0) &\overset{(62)}{\subset} w(\partial A)\cap B_{l_1}(c_0)\\ &\overset{(59)}{\subset} w\left(W_{a_2}\right).\end{aligned} \tag{76}$$

So as $a_2\in H$

$$\begin{aligned}H^1\left(w\left(W_{a_2}\right)\right) &\leq \int_{W_{a_2}}\left|Dw(z)\,t_z\right|dH^1 z\\ &\leq cH^1\left(W_{a_2}\right)\\ &\overset{(55)}{\leq} c\sqrt{\mathcal{C}_1}.\end{aligned} \tag{77}$$

Let

$$T_y:=\left\{h\in\left(0,\frac{1}{2}+|y-c_0|\right):\partial B_h(y)\cap w\left(W_{a_2}\right)\neq\emptyset\right\}. \tag{78}$$

Let $X_0:\mathbb{R}^2\to\mathbb{R}$ be defined by $X_0(z)=|z-y|$ so $T_y\subset X_0\left(w\left(W_{a_2}\right)\right)$ and as X_0 is 1-Lipschitz so $L^1\left(X_0\left(w\left(W_{a_2}\right)\right)\right)\leq c\sqrt{\mathcal{C}_1}$. Hence

$$L^1\left(T_y\right)\leq c\sqrt{\mathcal{C}_1}. \tag{79}$$

Let

$$Y_0=\left\{h\in\left(0,\frac{1}{2}+|y-c_0|-2\mathcal{C}_1^{\frac{1}{4}}\right)\setminus T_y:\partial B_h(y)\cap w(A)\cap B_{l_1}(c_0)=\emptyset\right\}.$$

See figure 1

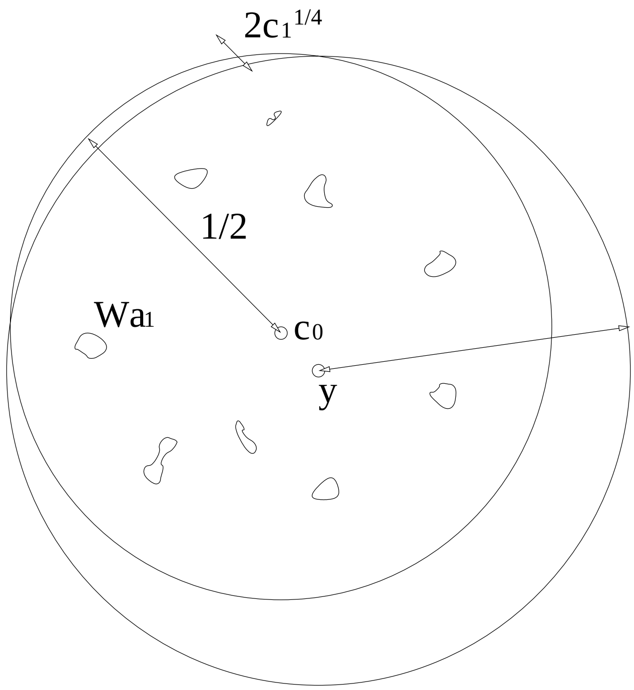

FIGURE 1

$$\begin{aligned} L^2\left(\left(\bigcup_{h\in Y_0}\partial B_h(y)\right)\cap B_{l_1}(c_0)\right) &\geq \mathcal{C}_1^{\frac{1}{4}}\int_{Y_0} h dL^1 h \\ &\geq \mathcal{C}_1^{\frac{1}{4}}\int_0^{L^1(Y_0)} h\, dL^1 h \\ &\geq \mathcal{C}_1^{\frac{1}{4}}\frac{\left(L^1(Y_0)\right)^2}{2}. \end{aligned}$$

And as $\left(\bigcup_{h\in Y_0}\partial B_h(y)\right)\cap B_{l_1}(c_0)\subset B_{\frac{1}{2}}(c_0)\backslash w(A)$ and from (61) we have

$$L^1(Y_0)\leq c\mathcal{C}_1^{\frac{1}{8}}. \tag{80}$$

Let $Y_1 := \left(0,\frac{1}{2}+|y-c_0|\right)\setminus(T_y\cup Y_0)$. Let

$$E_0 := \left(\bigcup_{h\in Y_1}\partial B_h(y)\right)\cap B_{l_1}(c_0). \tag{81}$$

For $h\in Y_1$, as $h\notin Y_0$ there exists $z_0\in\partial B_h(y)\cap w(A)\cap B_{l_1}(c_0)$, now suppose $\partial B_h(y)\cap B_{l_1}(c_0)\not\subset w(A)$ then we must have $\partial B_h(y)\cap B_{l_1}(c_0)\cap\partial w(A)\neq\emptyset$ and from (76) this implies $B_h(y)\cap B_{l_1}(c_0)\cap w(W_{a_2})\neq\emptyset$ which by (78) is a contradiction. Thus $E_0\subset w(A)\backslash w(\partial A)$.

Now for any $h\in Y_1$ as $\partial B_h(y)\cap B_{l_1}(c_0)$ is a connected set it must belong to a connected component of $\mathbb{R}^2\backslash w(\partial A)$ and hence by Theorem 2.3 [Fo-Ga 95] there exists a function $N: Y_1\to\mathbb{N}$ such that $d(z,w,A)=N(h)$ for any $z\in\partial B_h(y)\cap B_{l_1}(c_0)$. Let $Y_2=\{h\in Y_1 : N(h)\geq 2\}$

and define $E_1 = \bigcup_{h\in Y_2} \partial B_h(y) \cap B_{l_0}(c_0)$. So

$$\begin{aligned}\int_{E_0} d(w,A,y)\,dL^2 y &= \int_{E_0\backslash E_1} d(w,A,y)\,dL^2 y + \int_{E_1} d(w,A,y)\,dL^2 y \\ &\geq L^2(E_1) + L^2(E_0). \end{aligned} \tag{82}$$

So using Theorem 5 (taking $\phi = \chi_{w(A)}$) recalling that $A \subset \Psi_{a_1}$

$$\begin{aligned}\int_{E_0} d(w,A,y)\,dL^2 y &\leq \int_{w(A)} d(w,A,y)\,dL^2 y \\ &= \int_A \det(Dw(z))\,dL^2 z \\ &\overset{(24)}{\leq} L^2(A) + c\epsilon. \end{aligned} \tag{83}$$

Thus we have

$$\begin{aligned}\frac{\pi}{4} + c\epsilon &\geq L^2(A) + c\epsilon \\ &\overset{(83),(82)}{\geq} L^2(E_1) + L^2(E_0). \end{aligned} \tag{84}$$

Now

$$\begin{aligned}L^2(E_0) &\overset{(81)}{\geq} L^2(B_{l_1}(c_0)) - L^2\left(\bigcup_{\left(0,\frac{1}{2}+|y-c_0|\right)\backslash Y_1} \partial B_h(y)\right) \\ &\geq \frac{\pi}{4} - c\sqrt{\mathcal{C}_1} - cL^1(T_y \cap Y_0) \\ &\overset{(80),(79)}{\geq} \frac{\pi}{4} - c\mathcal{C}_1^{\frac{1}{8}}. \end{aligned} \tag{85}$$

Thus $L^2(E_1) \overset{(84),(85)}{\leq} c\mathcal{C}_1^{\frac{1}{8}}$ since

$$\begin{aligned}L^2(E_1) &\geq 2\pi \int_{Y_2} r\; dL^1 r \\ &\geq 2\pi \int_0^{L^1(Y_2)} r\; dL^1 r \\ &= 2\pi \left(L^1(Y_2)\right)^2, \end{aligned}$$

and as $c\sqrt{L^2(E_1)} \geq L^1(Y_2)$ this implies $L^2(Y_2) \leq c\mathcal{C}_1^{\frac{1}{16}}$. Let $L_y = Y_1 \backslash Y_2$, so L_y satisfies all the properties of Step 4.

Step 5. Let $y_0 \in B_{\frac{1}{8}}(c_0)$, let L_{y_0} be as defined in Step 4. For any $h \in L_{y_0} \cap \left(0, \frac{1}{8}\right)$ we will show $w^{-1}(\partial B_h(y_0))$ is a Jordan curve. Let $\mathcal{I}_{y_0}$ denote the interior of the curve we will prove

$$w(\partial \mathcal{I}_{y_0}) = \partial B_h(y_0), \quad w(\mathcal{I}_{y_0} \cap A) \subset B_h(y_0). \tag{86}$$

And

$$w\left(\left(B_{l_0}(0) \backslash \overline{\mathcal{I}_{y_0}}\right) \cap A\right) \subset \overline{B_h(y_0)}^c. \tag{87}$$

Proof of Step 5. Since $A \subset \Psi_{a_1}$ we know for every $x \in \mathbb{R}^2$

$$\begin{aligned}d(w,A,x) &= \sum_{z\in w^{-1}(x)} \operatorname{sgn}(\det(Dw(z))) \\ &= \operatorname{Card}\left(w^{-1}(x)\right) \end{aligned} \tag{88}$$

so from (64) we know

$$\mathrm{Card}\left(w^{-1}(x)\right)=1 \text{ for any } x\in\partial B_h(y_0). \tag{89}$$

So $w^{-1}(\partial B_h(y_0))$ is a closed curve with no intersections, i.e. $w^{-1}(\partial B_h(y_0))$ forms a Jordan curve. Thus $\mathbb{R}^2\backslash w^{-1}(\partial B_h(y_0))$ has two connected components, let $\mathcal{I}_{y_0}$ denote the interior component. Recall (60) on the structure of the set A. Since $\partial\mathcal{I}_{y_0}$ is a compact set contained in open set A so

$$\begin{aligned} d\left(\partial\mathcal{I}_{y_0},\{D_1,D_2,\dots D_{m_0}\}\right) &> d\left(\partial\mathcal{I}_{y_0},\partial A\right)\\ &> 0. \end{aligned} \tag{90}$$

We will show that

$$w\left(\mathcal{I}_{y_0}\backslash\left(\bigcup_{k=1}^{m_0}D_k\right)\right)\subset B_h(y_0) \tag{91}$$

and

$$w\left(\left(B_{l_0}(0)\backslash\overline{\mathcal{I}_{y_0}}\right)\backslash\left(\bigcup_{k=1}^{m_0}D_k\right)\right)\subset\overline{B_h(y_0)}^c. \tag{92}$$

As $A\cap\mathcal{I}_{y_0}\subset\mathcal{I}_{y_0}\backslash\left(\bigcup_{k=1}^{m_0}D_k\right)$ thus (91) implies the second part of (86). And similarly (92) implies (87). First we will establish (91). Let $x_0\in\partial\mathcal{I}_{y_0}$ since we know $\det(Dw(x_0))>c$ it is easy to see that for small enough α, $w(B_\alpha(x_0)\cap\mathcal{I}_{y_0})\subset B_h(y_0)$. For any $z_1\in\mathcal{I}_{y_0}\backslash\left(\bigcup_{k=1}^{m_0}\overline{D_k}\right)$, as $\mathcal{I}_{y_0}$ is connected we must be able to find a path in $\mathcal{I}_{y_0}$ starting from $z_0\in B_\alpha(x_0)\cap\mathcal{I}_{y_0}$ and ending in z_1. Formally, there exists a function $P:[0,\gamma]\to\mathcal{I}_{y_0}$ with $P(0)=z_0$, $P(\gamma)=z_1$ and $P([0,\gamma])\subset\mathcal{I}_{y_0}$.

Let $J=P^{-1}\left(P([0,\gamma])\cap\left(\bigcup_{k=1}^{m_0}\overline{D_k}\right)\right)$ let $I_1,I_2,\dots I_{m_1}$ denote the connected components of $[0,\gamma]\backslash J$ labelled so that $\sup I_i\leq\inf I_{i+j}$. Let a_i, b_i be the endpoints of I_i, i.e. $[a_i,b_i]=\overline{I_i}$. Now $P(a_1)=P(0)=z_0$ but $P(b_1)\in\bigcup_{k=1}^{m_0}\partial D_k$. As $P((a_1,b_1))$ is connected we claim we must have

$$w(P((a_1,b_1)))\subset B_h(y_0) \tag{93}$$

since otherwise there exists $y\in w(P((a_1,b_1)))\cap\partial B_h(y_0)$ and so there must be $x_1\in P((a_1,b_1))\subset\mathcal{I}_{y_0}\cap A$ and $x_2\in w^{-1}(\partial B_h(y_0))=\partial\mathcal{I}_{y_0}$ with $w(x_1)=w(x_2)=y$ and thus

$$d(w,A,y)=\sum_{x\in w^{-1}(y)}\mathrm{sgn}\left(\det(Dw(x))\right)\geq 2, \tag{94}$$

which contradicts (89) thus (93) is established. Now

$$\exists\ k_1\in\{1,2,\dots m_0\} \text{ such that } P(b_1)\in\partial D_{k_1} \text{ and also } P(a_2)\in\partial D_{k_1} \tag{95}$$

so we have

$$w(P(b_1)),w(P(a_2))\in w(\partial D_{k_1}). \tag{96}$$

From (93) we have $w(P(b_1))\in\overline{B_h(y_0)}$ and we claim must have

$$w(\partial D_{k_1})\subset B_h(y_0) \tag{97}$$

since otherwise there must exist $y\in w(\partial D_{k_1})\cap\partial B_h(y_0)$ and in the same way we establish (94) (using the fact $D_{k_1}\subset\mathcal{I}_{y_0}$) this implies $d(w,A,y)\geq 2$. So as $P(a_2)\in\partial D_{k_1}$ we know $w(P(a_2))\in B_h(y_0)$. In the same way as before we have $P((a_2,b_2))\subset B_h(y_0)$ and again $P(b_2)\in D_{k_2}$ for some $k_2\in\{1,2,\dots m_0\}$, we can then repeat the argument to show $w(\partial D_{k_2})\subset B_h(y_0)$. So continuing in this way we have $v(P((a_{m_0},b_{m_0})))\subset B_h(y_0)$ and as this means $v(z_1)=v(P(\gamma))=v(P(b_{m_0}))\in B_h(y_0)$ we have established (91). The proof of (92) is identical. This completes the proof of Step 5.

Step 6. We will show we can find a set $\Upsilon_0 \subset B_{\frac{1}{8}}(0) \cap A$ such that

$$L^2\left(B_{\frac{1}{8}}(0)\backslash\Upsilon_0\right) \le c\sqrt{\mathcal{C}_1} \tag{98}$$

and Υ_0 has the property that for any $b \in \Upsilon_0$ there exists a set $D_b \subset L_{v(b)} \cap \left(\frac{1}{8}, \frac{5}{16}\right)$ such that

$$L^1\left(\left(\frac{1}{8}, \frac{5}{16}\right)\backslash D_b\right) \le c\mathcal{C}_1^{\frac{1}{32q}} \tag{99}$$

and any $h \in D_b$ has the property that

$$w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right) \subset N_{c\mathcal{C}_1^{\frac{1}{32}}}\left(\partial B_h\left(b\right)\right). \tag{100}$$

$$\int_{\partial B_h(v(b))} d\left(Dw^{-1}\left(z\right), SO\left(2\right)\right) dH^1 z \le c\epsilon. \tag{101}$$

In addition Υ_0 has the properties

$$v\left(x\right) \in B_{\sqrt{\mathcal{C}_1}}\left(l_R\left(x\right)\right) \subset B_{\frac{1}{8}}\left(c_0\right) \text{ for any } x \in \Upsilon_0, \tag{102}$$

$$d\left(w, A, v\left(x\right)\right) = 1 \text{ for each } x \in \Upsilon_0. \tag{103}$$

Proof of Step 6. Recall $c_0 = l_R(0)$, let U_{c_0} be defined as in Step 4. Let $E_0 := w^{-1}(U_{c_0})$. Now for any $x \in U_{c_0}$, $Dw^{-1}(x) = \left[Dw\left(w^{-1}(x)\right)\right]^{-1}$ and as $w^{-1}(x) \in A$ we have

$$d\left(Dw\left(w^{-1}\left(x\right)\right), \widetilde{K}\right) \le 4\mathcal{C}_1^{\beta} \text{ where } \beta = \frac{1}{2\left(1 + \frac{q}{p^*}\right)}.$$

This implies $d\left(\left[Dw\left(w^{-1}\left(x\right)\right)\right]^{-1}, SO\left(2\right) \cup SO\left(2\right) J^{-1}\right) \le 32\mathcal{C}_1^{\beta}$. Hence

$$\begin{aligned} L^2\left(E_0\right) &= \int_{U_{c_0}} \det\left(Dw^{-1}\left(z\right)\right) dL^2 z \\ &\ge \left(1 - c\mathcal{C}_1^{\beta}\right) L^2\left(U_{c_0}\right) \\ &\overset{(63)}{\ge} \left(1 - c\mathcal{C}_1^{\frac{1}{16q}}\right)\frac{\pi}{4}. \end{aligned} \tag{104}$$

Note that since for any $x \in E_0$ we have $v(x) \in U_{c_0}$ and hence by (64) we know

$$d\left(w, A, v\left(x\right)\right) = 1 \text{ for } x \in E_0. \tag{105}$$

Let

$$E_1 := \left\{x \in A : \left|l_R\left(x\right) - v\left(x\right)\right| \le \sqrt{\mathcal{C}_1}\right\} \tag{106}$$

we know from (43) that

$$L^2\left(A\backslash E_1\right) \le c\sqrt{\mathcal{C}_1}. \tag{107}$$

Now for any $b \in E_0 \cap E_1 \cap B_{\frac{1}{8}}(0)$ let $\mathcal{A}_b = \bigcup_{h \in \left(\frac{1}{4}, \frac{5}{16}\right) \cap L_{v(b)}} \partial B_h\left(v\left(b\right)\right)$. So note since $b \in E_1$

$$\begin{aligned} \mathcal{A}_b &\subset B_{\frac{5}{16}}\left(v\left(b\right)\right) \\ &\overset{(106)}{\subset} B_{\frac{5}{16}+\sqrt{\mathcal{C}_1}}\left(l_R\left(b\right)\right) \\ &\subset B_{\frac{15}{32}}\left(c_0\right). \end{aligned}$$

Note

$$\begin{aligned} L^2\left(v\left(E_0\cap E_1\cap N\right)\right) &= L^2\left(\tilde{v}\left(E_0\cap E_1\cap N\right)\right)\\ &\geq L^2\left(\tilde{v}\left(A\cap N\right)\right)-L^2\left(\tilde{v}\left(A\backslash\left(E_1\cup E_2\right)\right)\right)\\ &\overset{(104),(107)}{\geq} L^2\left(\tilde{v}\left(A\cap N\right)\right)-c\mathcal{C}_1^{\frac{1}{16q}}\\ &\overset{(75)}{\geq} \frac{\pi}{4}-c\mathcal{C}_1^{\frac{1}{16q}}. \end{aligned} \tag{108}$$

Now by Step 5 for any $h\in\left(\frac{1}{4},\frac{5}{16}\right)\cap L_{v(b)}$ we know $w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)$ is a Jordan curve and $w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)\subset\Psi_{a_1}$, by continuity of Dv and since $a_1<\frac{d_0}{8}$ (see Definition (13)) we know either

$$\left\{Dv\left(z\right):z\in w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)\right\}\subset N_{2a_1}\left(SO\left(2\right)\right) \tag{109}$$

or

$$\left\{Dv\left(z\right):z\in w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)\right\}\subset N_{2a_1}\left(SO\left(2\right)J\right). \tag{110}$$

Let

$$S_b^1=\left\{h\in\left(\frac{1}{4},\frac{5}{16}\right)\cap L_{v(b)}:(109)\text{ holds true}\right\},$$

$$S_b^2=\left\{h\in\left(\frac{1}{4},\frac{5}{16}\right)\cap L_{v(b)}:(110)\text{ holds true}\right\}.$$

Thus

$$S_b^1\cup S_b^2=\left(\frac{1}{4},\frac{5}{16}\right)\cap L_{v(b)}. \tag{111}$$

Now

$$\begin{aligned} \int_{\mathcal{A}_b} d\left(Dw^{-1}\left(z\right),SO\left(2\right)\right)dL^2z &= \int_{\mathcal{A}_b} d\left(\left[Dw\left(w^{-1}\left(z\right)\right)\right]^{-1},SO\left(2\right)\right)dL^2z\\ &= \int_{w^{-1}(\mathcal{A}_b)} d\left(\left[Dw\left(y\right)\right]^{-1},SO\left(2\right)\right)\left(\det\left(\left[Dw\left(y\right)\right]^{-1}\right)\right)^{-1}dL^2y. \end{aligned}$$

And since $w^{-1}\left(\mathcal{A}_b\right)\subset A$ so for any $y\in w^{-1}\left(\mathcal{A}_b\right)$ we have $d\left(Dw\left(y\right),\widetilde{K}\right)\leq 4\mathcal{C}_1^{\beta}$ which implies $d\left(\left[Dw\left(y\right)\right]^{-1},SO\left(2\right)\cup SO\left(2\right)J^{-1}\right)\leq 16\mathcal{C}_1^{\beta}$ and hence

$$\begin{aligned} \int_{\mathcal{A}_b} d\left(Dw^{-1}\left(z\right),SO\left(2\right)\right)dL^2z &\leq c\int_{w^{-1}(\mathcal{A}_b)} d\left(\left[Dw\left(y\right)\right]^{-1},SO\left(2\right)\right)dL^2y\\ &\leq c\int_{B_{\frac{1}{2}}(0)} d\left(Dw\left(y\right),SO\left(2\right)\right)dL^2y\\ &\overset{(27)}{\leq} c\mathcal{C}_1^2. \end{aligned} \tag{112}$$

Now let $W_b^2=\bigcup_{h\in S_b^2}\partial B_h\left(v\left(b\right)\right)$

$$\begin{aligned} \int_{W_b^2} d\left(Dw^{-1}\left(z\right),SO\left(2\right)\right)dL^2z &= \int_{W_b^2} d\left(\left[Dw\left(w^{-1}\left(z\right)\right)\right]^{-1},SO\left(2\right)\right)dL^2z\\ &\geq \frac{d_0}{2}L^2\left(W_b^2\right) \end{aligned}$$

so from (112) we have

$$L^1\left(S_b^2\right)\leq cL^2\left(W_b^2\right)\leq c\mathcal{C}_1^2. \tag{113}$$

Let $W_b^1 := \bigcup_{h\in S_b^1} \partial B_h(v(b))$ so arguing as before there exists a positive constant $c_3 = c_3(\sigma)$

$$\begin{aligned}
&\int_{W_b^1} d\left(Dw^{-1}(z), SO(2)\right) dL^2 z \\
&\quad= \int_{W_b^1} d\left(\left[Dw\left(w^{-1}(z)\right)\right]^{-1}, SO(2)\right) dL^2 z \\
&\quad= \int_{w^{-1}\left(W_b^1\right)} d\left(\left[Dw(y)\right]^{-1}, SO(2)\right)\left(\det\left(\left[Dw(y)\right]^{-1}\right)\right)^{-1} dL^2 y \\
&\quad\leq c\int_{w^{-1}\left(W_b^1\right)} d\left(Dw(y), SO(2)\right) dL^2 y \\
&\quad\overset{(24),(109)}{\leq} c_3\epsilon.
\end{aligned} \tag{114}$$

Let

$$P_b = \left\{h \in S_b^1 : \int_{\partial B_h(v(b))} d\left(Dw^{-1}(z), SO(2)\right) dH^1 z \leq \mathcal{C}_1^{-1} c_3 \epsilon\right\} \tag{115}$$

so from (114) we have $L^1\left(S_b^1 \backslash P_b\right) \leq \mathcal{C}_1$ and from this and (111), (113) and (63) we have

$$L^1\left(\left(\frac{1}{8}, \frac{5}{16}\right)\backslash P_b\right) \leq c\mathcal{C}_1^{\frac{1}{16}}. \tag{116}$$

Let

$$D_b = \left\{h \in P_b : H^1\left(\partial B_h(v(b))\backslash v(E_0 \cup E_1)\right) \leq \mathcal{C}_1^{\frac{1}{32}}\right\}. \tag{117}$$

So

$$\begin{aligned}
c\mathcal{C}_1^{\frac{1}{16q}} &\overset{(108)}{\geq} L^2\left(A\left(v(b), \frac{1}{4}, \frac{5}{16}\right)\backslash v(E_0 \cup E_1)\right) \\
&= \int_{\frac{1}{8}}^{\frac{5}{16}} H^1\left(\partial B_h(v(b))\backslash v(E_0 \cup E_1)\right) dL^1 h \\
&\overset{(117)}{\geq} \mathcal{C}_1^{\frac{1}{32}} L^1\left(P_b \backslash D_b\right)
\end{aligned}$$

and thus we have

$$L^1(D_b) \geq \frac{3}{16} - c\mathcal{C}_1^{\frac{1}{32q}}. \tag{118}$$

Let $h \in D_b$. Let $z_0 \in \partial B_h(v(b)) \cap w(E_0 \cap E_1) \subset U_{c_0}$ thus $d(w, A, z_0) = 1$ and hence $\mathrm{Card}\left(w^{-1}(z_0)\right) = 1$. Thus as $w^{-1}(z_0) \in E_1$ we have

$$\begin{aligned}
\left|z_0 - l_R\left(w^{-1}(z_0)\right)\right| &= \left|w\left(w^{-1}(z_0)\right) - l_R\left(w^{-1}(z_0)\right)\right| \\
&\overset{(106)}{\leq} \sqrt{\mathcal{C}_1}.
\end{aligned} \tag{119}$$

Thus as $b \in E_1$ and $z_0 \in \partial B_h(v(b))$ we have

$$w^{-1}(z_0) \overset{(119)}{\in} B_{\sqrt{\mathcal{C}_1}}\left(l_R^{-1}(z_0)\right) \subset N_{\sqrt{\mathcal{C}_1}}\left(\partial B_h\left(l_R^{-1}(v(b))\right)\right) \overset{(106)}{\subset} N_{2\sqrt{\mathcal{C}_1}}\left(\partial B_h(b)\right). \tag{120}$$

And for any $z_1 \in \partial B_h(v(b))\backslash v(E_0 \cap E_1)$ from (117) we can find a point $z_2 \in \partial B_h(v(b)) \cap v(E_0 \cap E_1)$ such that if W denote the short connected component of $\partial B_h(v(b)) \backslash \{z_1, z_2\}$ then

$H^1(W) \leq \mathcal{C}_1^{\frac{1}{32}}$. So

$$\begin{aligned} \left|w^{-1}(z_1) - w^{-1}(z_2)\right| &= \left|\int_W Dw^{-1}(z)\, t_z dH^1 z\right| \\ &\leq H^1(W) + \int_{\partial B_h(v(b))} d\left(Dw^{-1}(z), SO(2)\right) dH^1 z \\ &\overset{(115)}{\leq} c\mathcal{C}_1^{\frac{1}{32}}. \end{aligned}$$

Hence

$$w^{-1}(\partial B_h(v(b))) \subset N_{c\mathcal{C}_1^{\frac{1}{32}}}(\partial B_h(b)). \tag{121}$$

Letting $\Upsilon_0 = E_1 \cap E_2 \cap B_{\frac{1}{8}-c\sqrt{\mathcal{C}_1}}(0)$, by (105), (106), (115), (118) and (120) Υ_0 satisfies (99), (100), (101), (102) and (103) and this completes the proof of Step 6.

*Step 7.*We will show there exists a set $\Xi_0 \subset B_{\frac{1}{8}}(c_0) \cap w(A)$ such that

$$L^2(\Xi_0) \geq \frac{\pi}{64} - c\mathcal{C}_1^{\frac{1}{4q}} \tag{122}$$

and for any $a \in \Xi_0$ there exists $\Theta_a \subset S^1$ with the following properties

- $$H^1\left(S^1 \backslash \Theta_a\right) \leq c\mathcal{C}_1^{\frac{1}{8}}. \tag{123}$$
- For each $\theta \in \Theta_a$ let $t(\theta) \in \mathbb{R}_+$ be the smallest number such that $a+\theta t(\theta) \in v(\partial B_{l_0}(0))$, we will show $[a, a+\theta t(\theta)) \subset w(A)$ and
$$d(w, A, y) = 1 \text{ for any } y \in [a, a+\theta t(\theta)). \tag{124}$$
- For any $\theta \in \Theta_a$
$$\int_{[a,a+\theta t(\theta))} d\left(Dw^{-1}(z), SO(2)\right) dL^1 z \leq c\epsilon. \tag{125}$$

Proof Step 7. Recall inclusion (59) $\partial A \subset \partial B_{l_0}(0) \cup W_{a_2}$ (where W_{a_2} is defined by (58) and recall $a_2 \in H \subset \left[2\mathcal{C}_1^{\beta}, 3\mathcal{C}_1^{\beta}\right]$) and as $l_0 \in K_0$ from (29) we have $\partial B_{l_0}(0) \cap W_{a_2} = \emptyset$. Let $\Gamma = w(W_{a_2})$, since Γ is the Lipschitz image of a rectifiable set it is rectifiable and from (77) we have $H^1(\Gamma) \leq c\sqrt{\mathcal{C}_1}$. Define measure μ by $\mu(B) = H^1(B \cap \Gamma)$. So $\mu\left(\mathbb{R}^2\right) \leq c\sqrt{\mathcal{C}_1}$. By Fubini's Theorem

$$\begin{aligned} \int_{B_2(c_0)} \int_{B_2(c_0)} \frac{1}{|z-y|} d\mu z dL^2 y &= \int_{B_2(c_0)} \int_{B_2(c_0)} \frac{1}{|z-y|} dL^2 y d\mu z \\ &\leq c\mu(B_2(c_0)) \\ &\leq c\sqrt{\mathcal{C}_1}. \end{aligned} \tag{126}$$

Let

$$\Xi_1 := \left\{ y \in B_{\frac{1}{8}}(c_0) : \int_{B_2(c_0)} \frac{1}{|z-y|} d\mu z \leq \mathcal{C}_1^{\frac{1}{4}} \right\}. \tag{127}$$

So from (126)

$$L^2\left(B_{\frac{1}{8}}(0) \backslash \Xi_1\right) \leq c\mathcal{C}_1^{\frac{1}{4}}. \tag{128}$$

Let $E_a(z) : \Gamma \to S^1$ be defined by $E_a(z) := \frac{(z-a)}{|z-a|}$, so $|DE_a(z)| = \frac{1}{|z-a|}$. Now using the Co-area formula for rectifiable sets, Theorem 3.2.22 [Fed 69] we have that for any $a \in \mathbb{R}^2$

$$\begin{aligned}\int_\Gamma \frac{\chi_{B_2(c_0)}(z)}{|z-a|} dH^1 z &\geq \int_{S^1}\int_{E_a^{-1}(\theta)\cap\Gamma} \chi_{B_2(c_0)}(x)\, dH^0 x dH^1\theta \\ &= \int_{S^1} \mathrm{Card}\left(E_a^{-1}(\theta)\cap\Gamma\cap B_2(c_0)\right) dH^1\theta. \end{aligned} \tag{129}$$

So if $a \in \Xi_1$ we have

$$\int_{S^1} \mathrm{Card}\left(E_a^{-1}(\theta)\cap\Gamma\cap B_2(c_0)\right) dH^1\theta \overset{(129),(127)}{\leq} \mathcal{C}_1^{\frac{1}{4}}.$$

Thus each $a \in \Xi_1$ we can find a set $\Sigma_a^1 \subset S^1$ such that

$$H^1\left(S^1\backslash\Sigma_a^1\right) \leq \mathcal{C}_1^{\frac{1}{8}} \tag{131}$$

and for every $\theta \in \Sigma_a^1$ we have $\mathrm{Card}\left(E_a^{-1}(\theta)\cap\Gamma\cap B_2(c_0)\right) = 0$. Since $l_0 \in \mathcal{Y}_0$ we know

$$v\left(\partial B_{l_0}(0)\right) \overset{(39)}{\subset} N_{c\sqrt{\mathcal{C}_1}}\left(\partial B_{l_0}(c_0)\right). \tag{132}$$

Given $b \in B_{\frac{1}{8}}(c_0)$, for each $\theta \in S^1$ we define $t_b(\theta) \in \mathbb{R}_+$ to be the smallest number such that $[b+\theta t_b(\theta)] \cap v\left(\partial B_{l_0}(0)\right) \neq \emptyset$. Thus for $a \in \Xi_1 \cap w(A)$, $\theta \in \Sigma_a^1$ as $w(\partial A) \overset{(59)}{\subset} \Gamma \cup v\left(\partial B_{l_0}(0)\right)$ we have

$$\begin{aligned}[a, a+\theta t_a(\theta)) \cap \partial w(A) &\overset{(62)}{\subset} [a, a+\theta t_a(\theta)) \cap w(\partial A) \\ &\subset [a, a+\theta t_a(\theta)) \cap \Gamma \\ &= \emptyset\end{aligned}$$

and this implies

$$\bigcup_{\theta\in\Sigma_a^1} [a, a+\theta t_a(\theta)) \subset w(A)\backslash w(\partial A) \text{ for any } a \in \Xi_1 \cap w(A). \tag{133}$$

Hence as $d(w, A, y)$ is constant on the connected components of $\mathbb{R}^2\backslash w(\partial A)$ and $[a, \theta t_a(\theta))$ must belong to one such connected component there exists, $N(\theta) \geq 1$ such that $d(w, A, y) = N(\theta)$ for any $y \in [a, a+\theta t_a(\theta))$. Let

$$\mathbb{H}_a := \left\{\theta \in \Sigma_a^1 : N(\theta) \geq 2\right\}. \tag{134}$$

Arguing as we did in Step 4

$$\begin{aligned}\int_{\bigcup_{\theta\in\Sigma_a^1}[a,a+\theta t_a(\theta))} d(w,A,y)\, dL^2 y &= \int_{\bigcup_{\theta\in\mathbb{H}_a}[a,a+\theta t_a(\theta))} d(w,A,y)\, dL^2 y \\ &\quad + \int_{\bigcup_{\theta\in\Sigma_a^1\backslash\mathbb{H}_a}[a,a+\theta t_a(\theta))} d(w,A,y)\, dL^2 y \\ &\geq L^2\left(\bigcup_{\theta\in\Sigma_a^1}[0,\theta t_a(\theta))\right) + L^2\left(\bigcup_{\theta\in\mathbb{H}_a}[0,\theta t_a(\theta))\right).\end{aligned}$$

As by Theorem 5 (again taking $\phi = \chi_{w(A)}$)

$$\begin{aligned}
\int_{\bigcup_{\theta\in\Sigma_a^1}[a,a+\theta t_a(\theta))} d(w,A,y)\,dL^2y &\leq \int_{w(A)} d(w,A,y)\,dL^2y \\
&= \int_A \det(Dw(y))\,dL^2y \\
&\leq L^2(A) + c\epsilon \\
&\overset{(24)}{\leq} \frac{\pi}{4} + c\epsilon
\end{aligned}$$

and as from (132) we know $t_a(\theta) \geq \frac{1}{16}$ for every $\theta \in S^1$ thus

$$\frac{H^1(\mathbb{H}_a)}{16} + L^2\left(\bigcup_{\theta\in\Sigma_a^1}[0,\theta t_a(\theta))\right) \leq \frac{\pi}{4} + c\epsilon. \tag{135}$$

However

$$\begin{aligned}
L^2\left(\bigcup_{\theta\in\Sigma_a^1}[0,\theta t_a(\theta))\right) &\geq L^2\left(\bigcup_{\theta\in S^1}[0,\theta t_a(\theta))\right) - cH^1\left(S^1\backslash\Sigma_a^1\right) \\
&\overset{(132),(131)}{\geq} L^2\left(B_{\frac{1}{2}-c\sqrt{\mathcal{C}_1}}(c_0)\right) - c\mathcal{C}_1^{\frac{1}{8}} \\
&\geq \frac{\pi}{4} - c\mathcal{C}_1^{\frac{1}{8}}
\end{aligned} \tag{136}$$

so from (136), (135) we have

$$H^1(\mathbb{H}_a) \leq c\mathcal{C}_1^{\frac{1}{8}}. \tag{137}$$

Let

$$\Sigma_a^2 := \Sigma_a^1\backslash\mathbb{H}_a \text{ and } \mathbb{S}_a^1 := \bigcup_{\theta\in\Sigma_a^2}[a, a+\theta t_a(\theta)). \tag{138}$$

Let $\mathbb{W} := \bigcup_{a\in\Xi_1\cap w(A)}\mathbb{S}_a^1$. From (133) we know $\mathbb{W} \subset w(A)$ from the definition of Σ_a^2 (see (138), (134)) we know for any $y \in \mathbb{W}$, we have Card $\left(w^{-1}(y)\right) = d(w,A,y) = 1$ and hence the inverse of w is well defined on $\mathbb{W}$.

It will simplify the notation to define $Q : \left\{M \in \mathbb{M}^{2\times 2} : \det(M) > 0\right\} \to \mathbb{M}^{2\times 2}$ by $Q(M) = M^{-1}$, let $\mathbb{K} := SO(2) \cup SO(2)J^{-1}$ so as $w^{-1}(\mathbb{W}) \subset A \overset{(56)}{\subset} \left\{x \in B_{\frac{1}{2}}(0) : d\left(Dv(x), \widetilde{K}\right) \leq 5\mathcal{C}^{\beta}\right\}$ and as $Dw^{-1}(y) = \left[Dw\left(w^{-1}(y)\right)\right]^{-1}$

$$\begin{aligned}
&\int_{\mathbb{W}} \left|D^2w^{-1}(y)\right| \left|d^{\frac{q}{p^*}}\left(Dw^{-1}(y),\mathbb{K}\right)\right| dL^2y \\
&= \int_{\mathbb{W}} \left|D\left(Q\left(Dw\left(w^{-1}(y)\right)\right)\right)\right| \left|d^{\frac{q}{p^*}}\left(\left[Dw\left(w^{-1}(y)\right)\right]^{-1},\mathbb{K}\right)\right| dL^2y \\
&\leq c\int_{\mathbb{W}} \left|DQ\left(Dw\left(w^{-1}(y)\right)\right)\right| \left|D^2w\left(w^{-1}(y)\right)\right| d^{\frac{q}{p^*}}\left(\left[Dw\left(w^{-1}(y)\right)\right]^{-1},\mathbb{K}\right) dL^2y \\
&\leq c\int_{w^{-1}(\mathbb{W})} \left|DQ\left(Dw(z)\right)\right| \left|D^2w(z)\right| d^{\frac{q}{p^*}}\left([Dw(z)]^{-1},\mathbb{K}\right)\left(\det\left([Dw(z)]^{-1}\right)\right) dL^2z \\
&\leq c\int_{B_{\frac{1}{2}}(0)} \left|D^2v(z)\right| d^{\frac{q}{p^*}}\left(Dv(z),\widetilde{K}\right) dL^2z \\
&\overset{(26)}{\leq} c\mathcal{C}_1.
\end{aligned} \tag{139}$$

Similarly

$$\begin{aligned}\int_{\mathbb{W}} d\left(Dw^{-1}(y), \mathbb{K}\right) dL^2 y &= \int_{\mathbb{W}} d\left(\left[Dw\left(w^{-1}(y)\right)\right]^{-1}, \mathbb{K}\right) dL^2 y \\ &\leq c \int_{w^{-1}(\mathbb{W})} d\left(\left[Dw(z)\right]^{-1}, \mathbb{K}\right) dL^2 z \\ &\overset{(24)}{\leq} c\epsilon. \end{aligned} \tag{140}$$

Finally

$$\begin{aligned}\int_{\mathbb{W}} d\left(Dw^{-1}(y), SO(2)\right) dL^2 y &= \int_{\mathbb{W}} d\left(\left[Dw\left(w^{-1}(y)\right)\right]^{-1}, SO(2)\right) dL^2 y \\ &\leq c \int_{w^{-1}(\mathbb{W})} d\left(\left[Dw(z)\right]^{-1}, SO(2)\right) dL^2 z \\ &\overset{(27)}{\leq} c\mathcal{C}_1^2. \end{aligned}$$

Now by Theorem 5 and (103), since $\Upsilon_0 \subset \Psi_{a_1}$

$$\begin{aligned} L^2\left(w\left(\Upsilon_0\right)\right) &= \int_{\Upsilon_0} \det\left(Dw(z)\right) dL^2 z \\ &\geq \left(1 - c\mathcal{C}_1^{\beta}\right) L^2\left(\Upsilon_0\right) \\ &\overset{(98)}{\geq} \frac{\pi}{64} - c\mathcal{C}_1^{\frac{1}{2q}}. \end{aligned}$$

And as by (102) $w(\Upsilon_0) \subset B_{\frac{1}{8}}(0)$ it is clear from (128) that $L^2\left(w\left(\Upsilon_0\right) \cap \Xi_1\right) \geq \frac{\pi}{64} - c\mathcal{C}_1^{\frac{1}{4q}}$. Now by the same Fubini argument we used to established (127), (128) we can find a set $\Xi_0 \subset \Xi_1 \cap w(\Upsilon_0)$ with

$$\begin{aligned} L^2\left(\Xi_0\right) &\geq L^2\left(\Xi_1 \cap w\left(\Upsilon_0\right)\right) - c\sqrt{\mathcal{C}_1} \\ &\geq \frac{\pi}{64} - c\mathcal{C}_1^{\frac{1}{4q}} \end{aligned} \tag{141}$$

and for any $a \in \Xi_0$ we have

$$\int_{\mathbb{S}_a^1} \left|D^2 w^{-1}(y)\right| d^{\frac{q}{p^*}}\left(Dw^{-1}(y), \mathbb{K}\right) |y-a|^{-1} dL^2 y \leq c\sqrt{\mathcal{C}_1}, \tag{142}$$

$$\int_{\mathbb{S}_a^1} d\left(Dw^{-1}(y), \mathbb{K}\right) |y-a|^{-1} dL^2 y \leq c\epsilon \tag{143}$$

and

$$\int_{\mathbb{S}_a^1} d\left(Dw^{-1}(y), SO(2)\right) |y-a|^{-1} dL^2 y \leq c\mathcal{C}_1^{\frac{3}{2}}. \tag{144}$$

By the Co-area formula for by $a \in \Xi_0$ we can find $\Theta_a \subset \Sigma_a^2$ with

$$H^1\left(\Sigma_a^2 \backslash \Theta_a\right) \leq \mathcal{C}_1^{\frac{1}{4}} \tag{145}$$

and any $\theta \in \Theta_a$ has the property

$$\int_{[a, a+\theta t_a(\theta))} \left|D^2 w^{-1}(y)\right| d^{\frac{q}{p^*}}\left(Dw^{-1}(y), \mathbb{K}\right) dH^1 y \leq c\mathcal{C}_1^{\frac{1}{4}}, \tag{146}$$

$$\int_{[a, a+\theta t_a(\theta))} d\left(Dw^{-1}(y), \mathbb{K}\right) dH^1 y \leq c\epsilon \tag{147}$$

and
$$\int_{[a,a+\theta t_a(\theta))} d\left(Dw^{-1}(y), SO(2)\right) dH^1 y \leq c\mathcal{C}_1^{\frac{5}{4}}. \tag{148}$$
And as we have seen before in (34) of Lemma 2, inequalities (146) and (148) imply
$$d\left(Dw^{-1}(z), SO(2)\right) < d\left(Dw^{-1}(z), SO(2)J\right) \text{ for any } z \in [a, a+\theta t_a(\theta))$$
and thus (147) gives
$$\int_{[a,a+\theta t_a(\theta))} d\left(Dw^{-1}(y), SO(2)\right) dH^1 z \leq c\epsilon. \tag{149}$$
From (131), (137), (145) Θ_a satisfies (123). By (134), (138) it satisfies (124), from (149) it satisfies (125) and finally from (141) it satisfies (122). This completes the proof of Step 7.

Step 8. Recall the definition of set Υ_0, from Step 6. We will show that for any $b \in \Upsilon_0$ and any $h \in D_b$
$$H^1\left(w^{-1}\left(\partial B_h\left(v(b)\right)\right)\right) \leq 2\pi h + c\epsilon \tag{150}$$
and denoting the interior of $w^{-1}\left(\partial B_h\left(v(b)\right)\right)$ by I_b (i.e. $I_b := \mathcal{I}_{v(b)}$ of Step 5) we have
$$L^2\left(I_b \cap A\right) \geq \pi h^2 - c\epsilon. \tag{151}$$

Proof of Step 8. As $b \in \Upsilon_0$, $v(b) \in B_{\frac{1}{8}}(c_0)$ and so
$$B_h(v(b)) \subset B_{\frac{15}{32}}(c_0) \subset B_{l_1}(c_0). \tag{152}$$
From Step 4 (64) we know that for $h \in D_h$ we have $\partial B_h(v(b)) \subset w(A)$ and $d(w, A, z) = 1$ for $z \in \partial B_h(v(b))$ thus it makes sense to consider the inverse of w on $\partial B_h(v(b))$, we also know $w^{-1}\left(\partial B_h(v(b))\right)$ is a Jordan curve and recall N is the set of points at which v and $\tilde{v}$ agree (see (71)) and from (40) we know that $L^2\left(B_{l_0}(0)\backslash N\right) \leq c\epsilon$. We will show
$$L^2\left(B_h(v(b)) \backslash v\left(I_b \cap A \cap N\right)\right) \leq c\epsilon. \tag{153}$$
Let $O = B_{l_0}(0) \backslash \overline{I_b}$. By (87)
$$\tilde{v}\left(N \cap A \cap O\right) \cap B_h(v(b)) = \emptyset. \tag{154}$$
So as from (152), (47)
$$B_h(v(b)) \subset \tilde{v}\left(N \cap A \cap O\right) \cup \tilde{v}\left(N \cap A \cap \overline{I_b}\right) \cup \tilde{v}\left(B_{l_0}(0) \setminus (N \cap A)\right) \tag{155}$$
and as
$$\begin{aligned} L^2\left(\tilde{v}\left(B_{l_0}(0) \setminus (N \cap A)\right)\right) &\leq cL^2\left(B_{l_0}(0) \setminus (N \cap A)\right) \\ &\overset{(57),(40)}{\leq} c\epsilon \end{aligned}$$
together with (154), (155) this implies (153). By Theorem 5 (taking $\phi = \chi_{v(I_b \cap A \cap N)}$)
$$\begin{aligned} \int_{I_b \cap A \cap N} \det\left(Dv(x)\right) dL^2 x &= \int_{v(I_b \cap A \cap N)} N\left(v, I_b \cap A \cap N, z\right) dL^2 z \\ &\overset{(153)}{\geq} \pi h^2 - c\epsilon. \end{aligned} \tag{156}$$
And as
$$\begin{aligned} \int_{I_b \cap A \cap N} \det\left(Dv(x)\right) dL^2 x &\leq \int_{I_b \cap A \cap N} 1 + cd\left(Dv(x), \widetilde{K}\right) dL^2 x \\ &\overset{(24)}{\leq} L^2\left(I_b \cap A \cap N\right) + c\epsilon. \end{aligned}$$
Together with (156) this gives
$$L^2\left(I_b \cap A \cap N\right) \geq \pi h^2 - c\epsilon$$

which establishes (151). By Step 6, (101)

$$\begin{aligned} H^1\left(w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)\right) &= \int_{\partial B_h(v(b))} \left|Dw^{-1}\left(z\right)t_z\right| dH^1 z \\ &\leq 2\pi h + c\epsilon \end{aligned}$$

which establishes (150) and completes the proof of Step 8.

Step 9. Let $b \in \Upsilon_0$, $h \in D_b$ for $p_h := \sqrt{\frac{L^2(I_b)}{\pi}}$ there exists $\omega_b \in B_{\frac{1}{2}}\left(0\right)$ such that

$$L^2\left(I_b \backslash B_{p_h}\left(\omega_b\right)\right) \leq c\sqrt{\epsilon}. \tag{157}$$

Proof of Step 9. Recall from Step 5 $\partial I_b = w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)$ and

$$\begin{aligned} H^1\left(\partial I_b\right) &= H^1\left(w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right)\right) \\ &\overset{(150)}{\leq} 2\pi h + c\epsilon \end{aligned} \tag{158}$$

and since by (151) we know $L^2\left(I_b\right) \geq \pi h^2 - c\epsilon$ by Theorem 3 the Fraenkel asymmetry $\lambda\left(I_b\right)$ satisfies

$$\begin{aligned} \left(2\pi h + c\epsilon\right)^2 &\overset{(8),(158)}{\geq} 4\pi\left(1 + \frac{\left(\lambda\left(I_b\right)\right)^2}{4}\right) L^2\left(I_b\right) \\ &\geq 4\pi\left(1 + \frac{\left(\lambda\left(I_b\right)\right)^2}{4}\right)\left(\pi h^2 - c\epsilon\right) \end{aligned}$$

thus $4\pi^2 h^2 + c\epsilon \geq 4\pi^2 h^2 + \pi^2 h^2 \left(\lambda\left(I_b\right)\right)^2$ thus $\lambda\left(I_b\right) \leq c\sqrt{\epsilon}$. Thus there exists $\omega_b \in \mathbb{R}^2$ such that (157) is satisfied.

Step 10. Let $b \in \Upsilon_0$ be such that $v\left(b\right) \in \Xi_0$, for any $h \in D_b$ we will show

$$L^2\left(I_b \backslash B_h\left(b\right)\right) \leq c\sqrt{\epsilon}. \tag{159}$$

Proof of Step 10. Let $\omega_b \in \mathbb{R}^2$ satisfy (157) for $p_h = \sqrt{\frac{L^2(I_b)}{\pi}}$. First note (157) implies $L^2\left(I_b \cap B_{p_h}\left(\omega_b\right)\right) \geq \pi p_h^2 - c\sqrt{\epsilon}$ and thus

$$L^2\left(B_{p_h}\left(\omega_b\right) \backslash I_b\right) \leq c\sqrt{\epsilon}. \tag{160}$$

Since $\partial I_b = w^{-1}\left(\partial B_h\left(v\left(b\right)\right)\right) \overset{(100)}{\subset} N_{c\mathcal{C}_1^{\frac{1}{32}}}\left(\partial B_h\left(b\right)\right)$ it is easy to see

$$\omega_b \in B_{c\mathcal{C}_1^{\frac{1}{32}}}\left(b\right) \text{ and } \left|p_h - h\right| \leq c\mathcal{C}_1^{\frac{1}{32}}. \tag{161}$$

For each $\theta \in S^1$ let $E\left(\theta\right) > 0$ be the largest number such that

$$\left(\left(\left(p_h + E\left(\theta\right)\right)\theta, \left(p_h - E\left(\theta\right)\right)\theta\right) + \omega_b\right) \cap \partial I_b = \emptyset.$$

Let

$$\mathcal{X}_1 := \left\{\theta \in S^1 : \left(\left(p_h + E\left(\theta\right)\right)\theta, \left(p_h - E\left(\theta\right)\right)\theta + \omega_b\right) \subset I_b\right\}$$

and let

$$\mathcal{X}_2 := \left\{\theta \in S^1 : \left(\left(p_h + E\left(\theta\right)\right)\theta, \left(p_h - E\left(\theta\right)\right)\theta + \omega_b\right) \cap I_b = \emptyset\right\}.$$

For any $\theta \in \mathcal{X}_1$ we know

$$\left(\left(p_h + E\left(\theta\right)\right)\theta, p_h\theta\right) + \omega_b \subset \left(I_b \backslash B_{p_h}\left(\omega_b\right)\right).$$

So there exists constant $c_4 = c_4(\sigma) > 0$ such that

$$\begin{aligned}
\int_{\mathcal{X}_1} E(\theta)\, dH^1\theta &\leq \int_{\mathcal{X}_1} H^1\left(\left(I_b \backslash B_{p_h}(\omega_b)\right) \cap \{\omega_b + \theta \mathbb{R}_+\}\right) dH^1\theta \\
&= \int_{I_b \backslash B_{p_h}(\omega_b)} |z - \omega_b|^{-1}\, dL^2 z \\
&\leq cL^2\left(I_b \backslash B_{p_h}(\omega_b)\right) \\
&\overset{(157)}{\leq} c_4\sqrt{\epsilon}. \qquad (162)
\end{aligned}$$

In the same way if $\theta \in \mathcal{X}_2$ then we know

$$(p_h\theta, (p_h - E(\theta))\,\theta) + \omega_b \subset \left(B_{p_h}(\omega_b) \backslash I_b\right) \cap \{\omega_b + \theta\mathbb{R}_+\}$$

and arguing in exactly the same way as (162) we get

$$\begin{aligned}
\int_{\mathcal{X}_2} E(\theta)\, dH^1\theta &\leq cL^2\left(B_{p_h}(\omega_b) \backslash I_b\right) \\
&\overset{(160)}{\leq} c_4\sqrt{\epsilon}. \qquad (163)
\end{aligned}$$

Let $\mathbb{U} = \left\{\theta \in S^1 : E(\theta) < 2c_4\mathcal{C}_1^{-1}\sqrt{\epsilon}\right\}$ so from (162), (163) we have

$$H^1\left(S^1 \backslash \mathbb{U}\right) \leq \mathcal{C}_1. \qquad (164)$$

For any $\theta \in \mathbb{U}$ we can find

$$Q(\theta) \in \{\omega_b + \theta\mathbb{R}_+\} \cap N_{2E(\theta)}\left(\partial B_{p_h}(\omega_b)\right) \cap \partial I_b.$$

Let $\mathbb{D}_0 := \bigcup_{\theta \in \mathbb{U}} Q(\theta)$, note

$$\mathbb{D}_0 \subset N_{c\sqrt{\epsilon}}\left(\partial B_{p_h}(\omega_b)\right) \qquad (165)$$

and as $\mathbb{D}_0 \subset \partial I_b$, $\mathbb{D}_0$ is rectifiable.

Define $\mathbb{P} : \mathbb{R}^2 \to p_h S^1$ by $\mathbb{P}(z) = p_h \frac{z - \omega_b}{|z - \omega_b|}$, so $|D\mathbb{P}(z)| = \frac{p_h}{|z - \omega_b|}$. Now $\mathbb{P}(\mathbb{D}_0) = p_h\mathbb{U}$ and from (164) we have

$$H^1\left(\mathbb{P}(\mathbb{D}_0)\right) \geq 2\pi p_h - \mathcal{C}_1. \qquad (166)$$

As $\mathbb{D}_0$ is a rectifiable set we know

$$\begin{aligned}
H^1\left(\mathbb{P}(\mathbb{D}_0)\right) &\leq \int_{\mathbb{D}_0} |D\mathbb{P}(z)\, t_z|\, dH^1 z \\
&\leq \left(1 + c\sqrt{\epsilon}\right) H^1(\mathbb{D}_0). \qquad (167)
\end{aligned}$$

Which implies

$$H^1(\mathbb{D}_0) \overset{(166),(167)}{\geq} 2\pi p_h - c\mathcal{C}_1. \qquad (168)$$

Define $\mathbb{M}_b := \partial B_h(v(b)) \setminus \left(h\Theta_{v(b)} + v(b)\right)$ (see figure 2), as $v(b) \in \Xi_0$ (recall this is one of the hypotheses of Step 10) we know

$$H^1(\mathbb{M}_b) \overset{(123)}{\leq} c\mathcal{C}_1^{\frac{1}{8}}. \qquad (169)$$

And as $h \in D_b$ we have that

$$\begin{aligned}
H^1\left(w^{-1}(\mathbb{M}_b)\right) &= \int_{\mathbb{M}_b} \left|Dw^{-1}(z)\, t_z\right| dH^1 z \\
&\overset{(101)}{\leq} H^1(\mathbb{M}_b) + c\epsilon \\
&\overset{(169)}{\leq} c\mathcal{C}_1^{\frac{1}{8}}. \qquad (170)
\end{aligned}$$

Note

$$H^1\left(\mathbb{P}\left(\mathbb{D}_0 \backslash w^{-1}(\mathbb{M}_b)\right)\right) \geq H^1\left(\mathbb{P}(\mathbb{D}_0)\right) - H^1\left(\mathbb{P}\left(w^{-1}(\mathbb{M}_b)\right)\right) \qquad (171)$$

and from (100), (161) we have $w^{-1}(\mathbb{M}_b) \subset N_{c\mathcal{C}_1^{\frac{1}{32}}}(\partial B_{p_h}(\omega_b))$ and so

$$\begin{aligned} H^1\left(\mathbb{P}\left(w^{-1}(\mathbb{M}_b)\right)\right) &= \int_{w^{-1}(\mathbb{M}_b)} |D\mathbb{P}(z)\,t_z|\,dH^1 z \\ &\leq \left(1+c\mathcal{C}_1^{\frac{1}{32}}\right) H^1\left(w^{-1}(\mathbb{M}_b)\right) \\ &\overset{(170)}{\leq} c\mathcal{C}_1^{\frac{1}{8}}. \end{aligned} \tag{172}$$

Let $\mathbb{D}_1 = \mathbb{D}_0 \backslash w^{-1}(\mathbb{M}_b)$, so from (168), (170) we know $H^1(\mathbb{D}_1) \geq 2\pi p_h - c\mathcal{C}_1^{\frac{1}{8}}$. From (166), (171), (172) there must exists constant $c_5 = c_5(\sigma) > 0$ such that we can pick points $p_1, p_2, p_3 \in \mathbb{D}_1$ for which the angle between any two of them is (roughly) $\frac{2\pi}{3}$, formally

$$\left|\frac{p_{i_1}}{|p_{i_1}|} \cdot \frac{p_{i_2}}{|p_{i_2}|} + \frac{1}{2}\right| < c\mathcal{C}_1^{\frac{1}{8}} \text{ for } i_1, i_2 \in \{1,2,3\}. \tag{173}$$

And by definition of $\mathbb{D}_1$ we know $\frac{v(p_i)-v(b)}{|v(p_i)-v(b)|} \in \Theta_{v(b)}$ for $i = 1,2,3$. Again see figure 2.

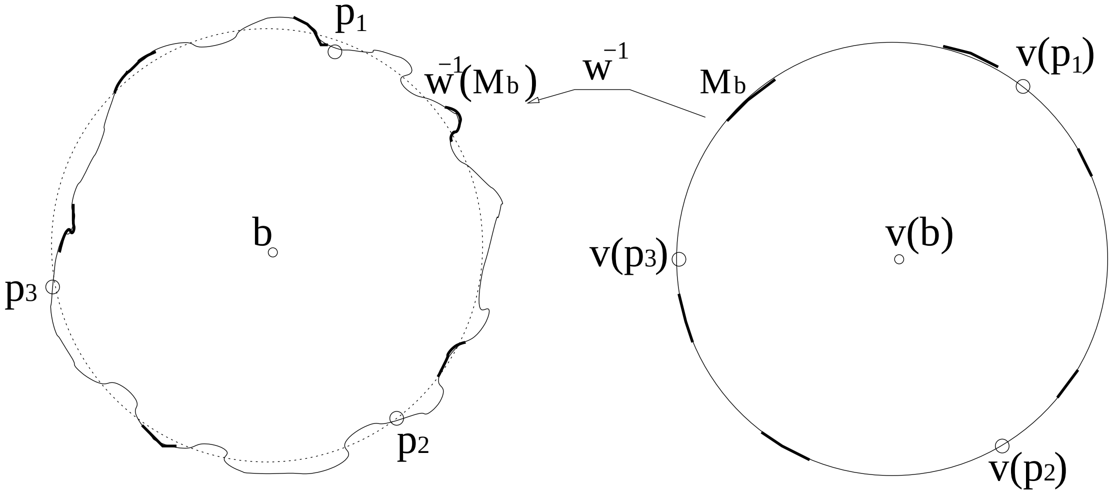

FIGURE 2

Let $\theta_i := \frac{v(p_i)-v(b)}{|v(p_i)-v(b)|}$ and let $t(\theta_i) \geq 0$ be the smallest number such that $v(b) + \theta_i t(\theta_i) \in v(\partial B_{l_0}(0))$, from (124) the path $w^{-1} : [v(b), v(b) + \theta_i t(\theta_i)) \to A$ is well defined, since $p_i \in \partial I_b$ so $v(p_i) \in \partial B_h(v(b)) \subset B_{\frac{15}{32}}(c_0) \subset v(B_{l_0}(0))$ hence $[v(b), v(p_i)] \subset [v(b), v(b) + \theta_i t(\theta_i))$ thus the path $w^{-1}([v(b), v(p_i)])$ is also well defined and so as $v(p_i) \in \partial B_h(v(b))$ we have

$$\begin{aligned} |b - p_i| &\leq H^1\left(w^{-1}([v(b), v(p_i)])\right) \\ &= \int_{[v(b),v(p_i)]} \left|Dw^{-1}(z)\,t_z\right| dH^1 z \\ &\overset{(125)}{\leq} h + c\epsilon. \end{aligned} \tag{174}$$

Note

$$p_h = \sqrt{\frac{L^2(I_b)}{\pi}} \overset{(151)}{\geq} h - c\epsilon. \tag{175}$$

Define the half-plane

$$\mathcal{H}(x,v) := \left\{z \in \mathbb{R}^2 : (z-x)\cdot v \geq 0\right\}. \tag{176}$$

Let $\mathbb{W}_i := \frac{p_i - \omega_b}{|p_i - \omega_b|}$ for $i = 1, 2, 3$. So using the fact $p_1, p_2, p_3 \overset{(165)}{\in} N_{c\sqrt{\epsilon}}\left(\partial B_{p_h}\left(\omega_b\right)\right)$ for the last inclusion (see figure 3)

$$
\begin{aligned}
b &\overset{(174)}{\in} B_{h+c\epsilon}\left(p_i\right) \\
&\subset \mathcal{H}\left(p_i + \left(h + c\epsilon\right)\mathbb{W}_i, -\mathbb{W}_i\right) \\
&\overset{(175)}{\subset} \mathcal{H}\left(p_i + \left(p_h + c\epsilon\right)\mathbb{W}_i, -\mathbb{W}_i\right) \\
&\subset \mathcal{H}\left(\omega_b + c\sqrt{\epsilon}\mathbb{W}_i, -\mathbb{W}_i\right).
\end{aligned}
$$

Thus

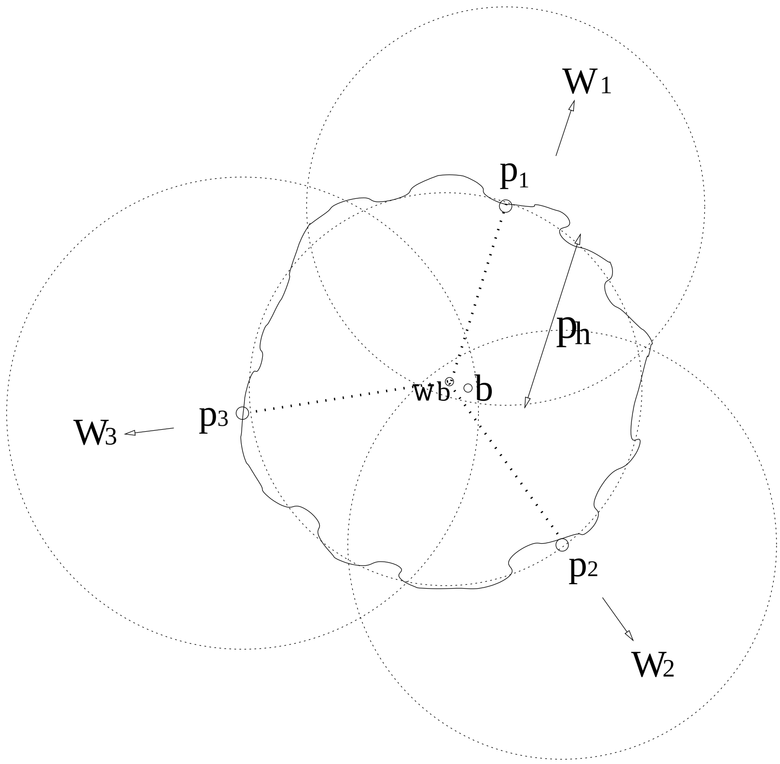

FIGURE 3

$$
\begin{aligned}
b &\in \bigcap_{i=1}^{3} \mathcal{H}\left(\omega_b + c\sqrt{\epsilon}\mathbb{W}_i, -\mathbb{W}_i\right) \\
&\overset{(173)}{\subset} B_{c\sqrt{\epsilon}}\left(\omega_b\right).
\end{aligned}
\tag{177}
$$

Again since $p_i \overset{(165)}{\in} N_{c\sqrt{\epsilon}}\left(\partial B_{p_h}\left(\omega_b\right)\right)$ and as $|p_i - \omega_b| \overset{(177)}{\leq} |p_i - b| + c\sqrt{\epsilon} \overset{(174)}{\leq} h + c\sqrt{\epsilon}$ thus $p_h - c\sqrt{\epsilon} \leq h + c\sqrt{\epsilon}$ this together with (175) gives $|p_h - h| \leq c\sqrt{\epsilon}$, this completes the proof of Step 10.

Proof of Lemma 3 completed. Note by Theorem 5

$$\begin{aligned} L^2(w(\Upsilon_0)) &\overset{(103)}{=} \int_{w(\Upsilon_0)} d(w,A,y)\,dL^2y \\ &= \int_{\Upsilon_0} \det(Dw(z))\,dL^2z \\ &\geq \left(1-c\mathcal{C}_1^{\beta}\right)L^2(\Upsilon_0) \\ &\overset{(98)}{\geq} \left(1-c\mathcal{C}_1^{\frac{1}{2q}}\right)\frac{\pi}{64}. \end{aligned}$$

So by (122) we know $L^2(w(\Upsilon_0)\cap\Xi_0)\geq\left(1-c\mathcal{C}_1^{\frac{1}{4q}}\right)\frac{\pi}{64}$, let $\Lambda_0:=w^{-1}(w(\Upsilon_0)\cap\Xi_0)$, note

$$\begin{aligned} L^2(\Lambda_0) &\geq \int_{w(\Upsilon_0)\cap\Xi_0} \det\left(Dw^{-1}(y)\right)dL^2y \\ &= \int_{w(\Upsilon_0)\cap\Xi_0} \det\left(\left[Dw\left(w^{-1}(y)\right)\right]^{-1}\right)dL^2y \\ &\geq \left(1-c\mathcal{C}_1^{\beta}\right)L^2(w(\Upsilon_0)\cap\Xi_0) \\ &\geq \left(1-c\mathcal{C}_1^{\frac{1}{4q}}\right)\frac{\pi}{64}. \end{aligned}$$

For any $b\in\Lambda_0$ by Step 9 (159) I_b satisfies (38). In addition by (86), (99), (100) there exists $D_b\subset\left(\frac{1}{8},\frac{5}{16}\right)$ with $L^1\left(\left(\frac{1}{8},\frac{5}{16}\right)\backslash D_b\right)\leq c\mathcal{C}_1^{\frac{1}{32q}}$ such that inequalities (36) and (37) of the statement of the lemma are satisfied. This completes the proof of Lemma 3.

Having established in Lemma 3 there is a large set of points Λ_0 with the property that for any $b\in\Lambda_0$, for many radii $h\in\left(\frac{1}{8},\frac{5}{16}\right)$ we have a connected set I_b with $L^2(I_b\triangle B_h(b))\leq\varepsilon^{\frac{1}{2q}}$ and with the property that v maps ∂I_b onto $\partial B_h(v(b))$. We will use the "shrink directions" argument described in (2.1.3) to prove that in a central sub-ball the gradient stays close to $SO(2)$.

Lemma 4. *Given a function $v\in C^4\left(B_{\frac{1}{2}}(0)\right)$ satisfying properties (24), (26), (27), (28) and (29) of Lemma 2, define*

$$\mathbb{B}:=\left\{x\in B_{\frac{1}{2}}(0):d(Dv(x),SO(2)J)<d(Dv(x),SO(2))\right\} \tag{178}$$

we will show there exists constant $\mathcal{C}_3=\mathcal{C}_3(\sigma)>0$ such that

$$L^2(B_{\mathcal{C}_3}(0)\cap\mathbb{B})\leq c\sqrt{\epsilon}. \tag{179}$$

Proof of Lemma 4. From Lemma 3 we know there exists a set $\Lambda_0\subset B_{\frac{1}{8}}(0)$ with $L^2\left(B_{\frac{1}{8}}(0)\backslash\Lambda_0\right)\leq c\mathcal{C}_1^{\frac{1}{4q}}$ such that for $b\in\Lambda_0$ we have set $D_b\subset\left(\frac{1}{8},\frac{5}{16}\right)$ with $L^1\left(\left(\frac{1}{8},\frac{5}{16}\right)\backslash D_b\right)\leq c\mathcal{C}_1^{\frac{1}{32q}}$ and for any $h\in D_b$ there is a connected open set I_b satisfying (36), (37), (38). Note

$$\begin{aligned} \int_{\Lambda_0}\int_{B_{\frac{1}{2}}(0)} d\left(Dv(z),\widetilde{K}\right)|z-x|^{-1}dL^2z\,dL^2x &= \int_{B_{\frac{1}{2}}(0)} d\left(Dv(z),\widetilde{K}\right)\int_{\Lambda_0}|z-x|^{-1}dL^2x\,dL^2z \\ &\leq c\int_{B_{\frac{1}{2}}(0)} d\left(Dv(z),\widetilde{K}\right)dL^2z \\ &\overset{(24)}{\leq} c\epsilon. \end{aligned}$$

So we can find a set $\Lambda_1 \subset \Lambda_0$ with $L^2(\Lambda_1) \geq \frac{L^2(\Lambda_0)}{2}$ such that every $x \in \Lambda_1$ has the property

$$\int_{B_{\frac{1}{2}}(0)} d\left(Dv(z), \widetilde{K}\right) |z-x|^{-1} dL^2 z \leq c\epsilon. \tag{180}$$

Let $b \in \Lambda_1$ and $h \in D_b \cap \left(\frac{5}{16}, \frac{6}{16}\right)$.

As in Step 10 of Lemma 3 for $\theta \in [0, 2\pi)$ define $E(\theta) > 0$ to be the largest number so that $(((h - E(\theta))\theta, (h + E(\theta))\theta) + b) \cap \partial I_b = \emptyset$. Note that from (37) we know $E(\theta) < c\mathcal{C}_1^{\frac{1}{16}}$. In exactly the same way as we established (162), (163) of Lemma 3 we can show

$$\int_{S^1} E(\theta)\, dH^1\theta \leq c\sqrt{\epsilon}. \tag{181}$$

Since J is a diagonal matrix with eigenvalues σ, σ^{-1} we must be able to find $\theta_1, \theta_2 \in S^1$ with the following properties

- $|J\theta_i| = 1$ for $i = 1, 2$.
- Letting $\mathcal{H}_0$ denote the "short" connected component of S^1 between θ_1, θ_2 we have $|J\eta| < 1$ for any $\eta \in \mathcal{H}_0$.

If we divide $\mathcal{H}_0$ into three equal sized sub-arcs, let $\mathcal{H}_1$ denote the central sub-arc, then there exists constant $c_6 = c_6(\sigma) > 0$ such that $|H\eta| < 1 - c_6$ for any $\eta \in \mathcal{H}_1$. Let $V_\alpha(0) := \left(B_\alpha(0) \backslash (B_{\frac{\alpha}{2}}(0)\right) \cap \{\mathbb{R}\eta : \eta \in \mathcal{H}_1\}$ and let $V_\alpha(x) := V_\alpha(0) + x$.

Step 1. We will show that

$$L^2(V_h(b) \cap \mathbb{B}) \leq c\sqrt{\epsilon}. \tag{182}$$

Proof of Step 1. For each $\theta \in \mathcal{H}_1$ we can find $a_\theta \in (((h - 2E(\theta))\theta, (h + 2E(\theta))\theta) + b) \cap \partial I_b$ and by (36) we know $v(a_\theta) \in \partial B_h(v(b))$ so letting $e_\theta := \int_{[b,a_\theta]} d\left(Dv(z), \widetilde{K}\right) dH^1 z$ we have

$$\begin{aligned}
h &= |v(a_\theta) - v(b)| \\
&\leq \int_{[b,a_\theta]} |Dv(x)\theta|\, dH^1 x \\
&= \int_{[b,a_\theta]\cap\mathbb{B}} |Dv(x)\theta|\, dH^1 x + \int_{[b,a_\theta]\backslash\mathbb{B}} |Dv(x)\theta|\, dH^1 x \\
&\leq (1 - c_6 + e_\theta) L^1([b,a_\theta] \cap \mathbb{B}) + (1 + e_\theta) L^1([b,a_\theta] \backslash \mathbb{B}) \\
&\leq |b - a_\theta| - c_6 L^1([b,a_\theta] \cap \mathbb{B}) + ce_\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
L^1\left(\left[b, \frac{4}{16}\theta\right] \cap \mathbb{B}\right) &\leq L^1([b,a_\theta] \cap \mathbb{B}) \\
&\leq c\,|h - |b - a_\theta|| + ce_\theta \\
&\leq 2E(\theta) + ce_\theta.
\end{aligned}$$

And note that by the co-area formula

$$\begin{aligned}
\int_0^{2\pi} e_\theta dH^1\theta &= \int_{B_{\frac{1}{2}}(0)} d\left(Dv(z), \widetilde{K}\right) |z-b|^{-1} dL^2 z \\
&\leq c\epsilon.
\end{aligned} \tag{183}$$

So again by the Co-area formula (see figure 4)

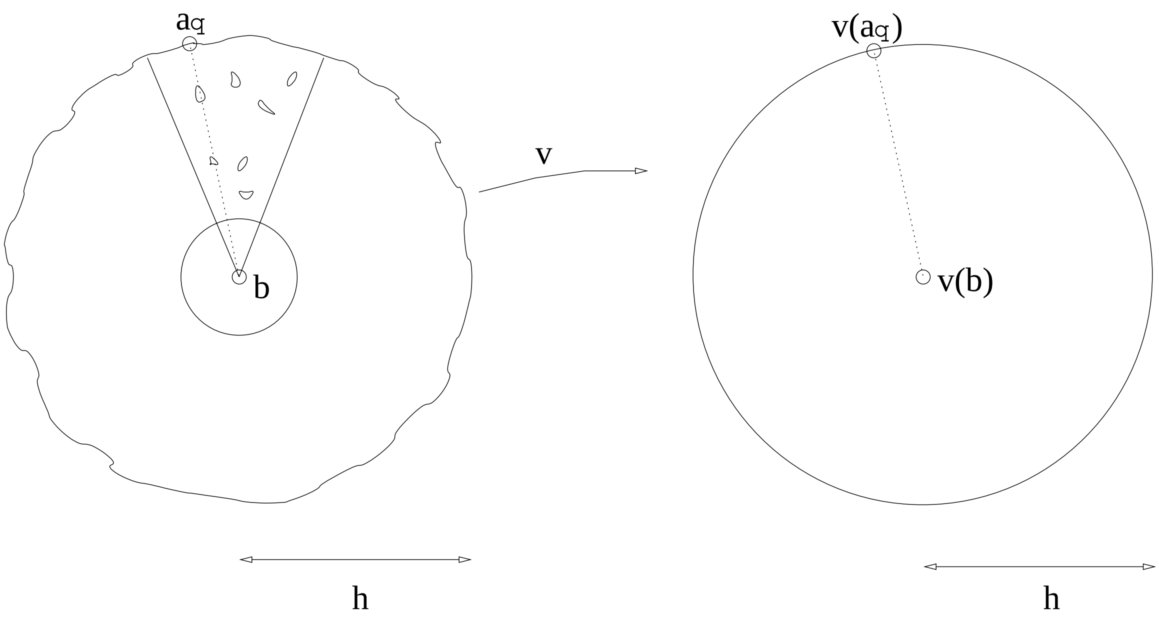

FIGURE 4

$$L^2\left(V_{\frac{1}{4}}(b)\cap\mathbb{B}\right) \leq c\int_{\mathbb{H}_1} L^1\left(\left[b,\frac{4}{16}\theta\right]\cap\mathbb{B}\backslash B_{\frac{1}{8}}(b)\right)dH^1\theta \overset{(183)}{\leq} c\int_{\mathbb{H}_1} E(\theta)+ce_\theta dH^1\theta \overset{(181),(183)}{\leq} c\sqrt{\epsilon}. \quad (184)$$

Proof of Lemma completed. Assuming $\mathcal{C}_1$ is small enough we must be able to find $b\in\Lambda_1\cap V_{\frac{1}{4}}(0)\backslash B_{\frac{3}{16}}(0)$. So pick $h\in D_b\cap\left(\frac{4}{16},\frac{5}{16}\right)$ then we have for some constant $\mathcal{C}_3=\mathcal{C}_3(\sigma)>0$ that $B_{\mathcal{C}_3}(0)\subset V_h(b)$, then inequality (179) follows from (184).

3. Proof of Theorem 2 completed

Recall we have convolved u to form a smooth function $\psi:=u_{\rho_0}$ that satisfies (18), (19), (20) and (21). By applying Lemma 2 function v defined by (23) satisfies (24), (25), (26), (27), (28) and (29) and has all the necessary hypotheses to apply Lemma 4. So

$$\int_{B_{\mathcal{C}_3}} d(Dv(x),SO(2))\,dL^2x = \int_{B_{\mathcal{C}_3}\backslash\mathbb{B}} d(Dv(x),SO(2))\,dL^2x + \int_{\mathbb{B}} d(Dv(x),SO(2))\,dL^2x \overset{(24)}{\leq} \epsilon + c\int_{\mathbb{B}} d\left(Dv(x),\widetilde{K}\right)dL^2x + cL^2(\mathbb{B}) \overset{(24),(179)}{\leq} c\sqrt{\epsilon}. \quad (185)$$

Since $d^q(Dv(x),SO(2))\leq cd(Dv(x),SO(2))+cd^q(Dv(x),K)$ this gives

$$\int_{B_{\mathcal{C}_3}(0)} d^q(Dv(x),SO(2))\,dL^2x \overset{(185),(25)}{\leq} c\sqrt{\epsilon}.$$

From the definition of v this implies there exists $J \in \{Id, H\}$ such that

$$\int_{B_{\mathcal{C}_3}(0)} d^q \left(Dv\left(z\right), SO\left(2\right) J\right) dL^2 z \leq c\varepsilon^{\frac{1}{2q}}.$$

Assuming $\mathcal{C}_1$ is chosen small enough we can apply the same argument to show that for each $x_0 \in B_{\frac{1}{2}}(0)$ there exists $J_{x_0} \in \{Id, H\}$ such that

$$\int_{B_{\frac{\mathcal{C}_3}{2}}(x_0)} d^q \left(Du\left(z\right), SO\left(2\right) J_{x_0}\right) dL^2 z \leq c\varepsilon^{\frac{1}{2q}}. \tag{187}$$

By Besicovitch covering Theorem we can find a finite collection of points $\{x_1, x_2, \ldots x_{m_0}\}$ with the properties that $B_{\frac{1}{2}}(0) \subset \bigcup_{i=1}^{m_0} B_{\frac{\mathcal{C}_3}{8}}(x_i)$ and $\|\sum_{i=1}^{m_0} \chi_{B_{\frac{\mathcal{C}_3}{8}}(x_i)}\|_\infty \leq 5$. Now if for some $i_1, i_2 \in \{1, 2, \ldots m_0\}$ we have $x_{i_1} \in B_{\frac{\mathcal{C}_3}{4}}(x_{i_2})$ then

$$\left(\int_{B_{\frac{\mathcal{C}_3}{8}}\left(\frac{x_{i_1}+x_{i_2}}{2}\right)} d^q \left(Dv\left(z\right), SO\left(2\right) J_{x_a}\right) dL^2 z\right)^{\frac{1}{q}} \leq c\varepsilon^{\frac{1}{2q^2}} \text{ for } a = 1, 2.$$

And this implies $J_{x_{i_1}} = J_{x_{i_2}}$ and hence we can find $J \in \{Id, H\}$ such that

$$J_{x_i} = J \text{ for } i = 1, 2, \ldots m_0. \tag{188}$$

Thus $\int_{B_{\frac{\mathcal{C}_3}{2}}(x_i)} d^q \left(Du\left(z\right), SO\left(2\right) J\right) dL^2 z \leq c\varepsilon^{\frac{1}{2q}}$ for $i = 1, 2, \ldots m_0$. Hence

$$\begin{aligned} \int_{B_{\frac{1}{2}}(0)} d^q \left(Du\left(z\right), SO\left(2\right) J\right) dL^2 z &\leq c\sum_{k=1}^{m_0} \int_{B_{\frac{\mathcal{C}_3}{4}}(x_i)} d^q \left(Du\left(z\right), SO\left(2\right) J\right) dL^2 z \\ &\leq c\varepsilon^{\frac{1}{2q}} \end{aligned}$$

thus establishes the first part of the conclusion of Theorem 2.

Now consider the case $q > 1$. If $J = Id$ we can then apply Theorem 1 to conclude there exists $R \in SO(2)$ such that (6) holds true. If $J = H$ we define $w = u \cdot l_{H^{-1}}$ where $l_{H^{-1}}$ is an affine functions with derivative H^{-1}, then

$$\int_{l_{H^{-1}}^{-1}\left(B_{\frac{1}{2}}(0)\right)} d^q \left(Dw\left(z\right), SO\left(2\right)\right) dL^2 z \leq c\epsilon^{\frac{1}{2q}}.$$

Applying Theorem 1 again allows us to conclude there exists R such that

$$\int_{l_{H^{-1}}^{-1}\left(B_{\frac{1}{2}}(0)\right)} \left|Dw\left(z\right) - R\right|^q dL^2 z \leq c\epsilon^{\frac{1}{2q}},$$

changing varibles then allows to conclude (6). □

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