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The thermodynamical approach to market

by

Dimitry A. Leites, and Victor Sergeev

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Victor Sergeev

translated from the Russian and edited by
Dimitry Leites

Abstract. The book gives an explanation of several intriguing phenomena, providing new insights and answers to some deeply vexing questions.

Why the economic "shock therapy" implemented in Eastern Europe was doomed to a failure whereas the approach adopted by China and Vietnam should inevitably lead to economic growth (damped, perhaps, by corruption and inconsistencies)?

Why some restrictions imposed on markets are dangerous? Politicians (and laymen) usually believe that the more restrictions you impose on the society, the easier it is to govern. The reality seems to be more subtle. Some restrictions are necessary for the markets to function. However, restrictions on salary, for example, seem to always result in unemployment, as a simple "spin" model shows. Evidently, optimizing freedom is an equilibrium problem, and none of the extrema is devoid of danger.

Why should crooked dealings be prevented? While honesty has a price, the the dishonest market collapses.

What is a reasonable rate of taxation, if any such exists?

Other questions abound, raising many points of interest.

Appendices contain an essay which informally can be entitled "Why mathematics and physics major should study economics" and a deep mathematical paper summarizing a century long study of nonholonomic systems (such as ideal gas or market economy) in the quest for an analog of the curvature tensor in nonholonomic setting.

Contents

Preface

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The most intriguing words in Feynman's *Lectures* were those about the mysterious notion of *force*. At about the same time I read *Lectures*, I learned about Hertz's attempts to exorcize the notion of *force* from the set of basic physical notions. These ideas were related with (ascending to Riemann) notion of "geometrodynamic", namely the idea that there are no *forces*, just the curved space.

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Having realized this much, I was on the look out for examples of nonholonomic manifolds. My interest in the problem of mathematical description of economic systems given in this book is related with a particular way of this description, namely, in terms of nonintegrable distributions[3] or nonholonomic manifolds (a *distribution* is singled out in the tangent bundle to a manifold by a system of Pfaff equations, i.e., is the common set of zeros of differential 1-forms).[4]

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For whom is this book. It was inexplicable to me at first why *Limits* was not a best-seller, at least, among physicists and economists, but also among other people: it does not use mathematics more involved than the notion of partial derivative.

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Economists, on the other hand, were scared by the second half by allegedly difficult notions from physics. Therefore I would like to inform the reader: **no mathematics apart from partial differential equations is needed to understand the main text**. Conclusions do not require any mathematics at all and should be clear even to government officials.

Mathematicians and physicists should, perhaps, begin with Chapter 5 and turn to earlier chapters for motivations addressed, mainly, to experts in economics, as well as for general background and history.

Appendices require rudiments of differential geometry. The fact that the book mainly appeals to common sense (and uses only most basic mathematics) is its extra attraction to me.

The present book is not just a translation into English of its Russian counterpart. It is about twice enlarged version, with elucidations and comments.

The main body of the book is appended with an essay "*Physics as a tool in sociology*" which can be considered as an extended summary with its own message: *it gives an answer to the question "Why mathematics and physics major should study sociology or economy"*. This Appendix can be interpreted as follows: whereas the main body of the book is a paraphrase of the first few chapters of any good text-book on Statistical Physics in terms of market economy, to translate other notions and theorems is an interesting open problem, a challenging problem for Ph.D. **students majoring in theoretical physics and seeking real life applications**.

This book not only indicates a method for answering various vital questions of economics but also widens horizons of applications for **mathematics major students**. Similar miracles (when a branch of mathematics becomes an indispensable tool or language in a branch of another natural science) were known (applications of the theory of Hilbert space operators in quantum theory); Sergeev makes economics a field of application of representation theory of Lie algebras (these are needed, for example, to compute the nonholonomic analog of the curvature tensor, see Vershik's Appendix and ref. [**L**] in it).

Literature on econophysics Since the first pioneer papers by Rozonoer published in 1973 (commented on in the main text) and Sergeev's *Limits* written in 1996 the ideas to apply statistical physics to economy became widespread.

Ten years after that first bold approach, in collaboration with A. Tsyrlin, Rozonoer published another three papers in the same journal. This time no analogies between thermodynamics and economics were mentioned but a very impressive theory of optimal control in thermodynamic processes was suggested. Possibly, the cause of such a turn away from economics was related with general political atmosphere in Soviet science, not encouraging for advanced studies in market economics. These new ideas of optimal control over thermodynamic processes gave a start to a flow of articles in this direction, but this activity was not related directly with studies of economics. Recently S. Amelkin, K. Martinash and A. Tsirlin[5] returned to the idea of applying the methods of optimal control in thermodynamics to economic problems.

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Part 1

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None of the above listed rival works covers the ideas, approaches and results presented in the book you are about to read.

Introduction

This book is a result of my thoughts on problems of construction of models of social phenomena. In the European scientific tradition starting from Descartes and Bacon, a thick line was drawn between the study of Nature and the study of Man. This line became a separating truncheon about a hundred years ago when Dilthey, Windelbandt and Rickert divided the domains of science into the ones where the essence of the study is the search for the laws that connect the notions (physics is a prime example here) and "sciences of the Spirit", i.e., the domains of science where the essence of the study is the description of the individual and which are not reproducible (history is a prime example here).

But even without accepting the neo-Kantian position, it is difficult not to concede that the faceless world of Nature should be essentially different from the world of the Man colored by his personality and possessing at least empirical freedom of choice and abilities to understand the world.

To what extent do this freedom and ability to understand (which an individual is supposed to possess) influence the properties of communities?

What happens when the interaction between individuals becomes so essential that, studying the individual, we cannot actually say anything about the whole?

This borderline situation requires a special consideration. Are the methods of the study developed for the description of Nature applicable in this case? Or the individuality is, in the end, non-removable and one has to take it into account, at least through the values, i.e., through the criteria that determine the choices of individuals? To what extent these criteria themselves are individual?

Basically, the study of human societies in the various incarnations — economic, political, social — is a "no man's land" between the sciences of Nature and sciences of Spirit. It was already in XVIII century starting with F. Quesnay and A. Smith that it became clear that certain domains of this "no man's land" are much better suited to be subjected to a formal "logical analysis" than the other ones and that economics is a most promising domain for the construction of rigorous models thanks to the domination of the interests (in principle, accessible to a logical analysis) in the economic behavior of the people.

In this way, we determine, on the individual level, a force that governs the behavior of society as a whole.

Such arguments range in a simple and neat scheme — the individual choice of behavioral alternatives is determined on the base of a personal interest — (in the mathematical models such a choice is expressed through the maximization of the "utility function") and the collective behavior is considered as the sum of individual behavioral acts. On the choice of a human performance certain auxiliary conditions may be imposed that restrict this choice, and therefore the problem to determine the "conditional maximum" becomes the center around which the mathematical methods of theoretical economics are formed.

The efforts of several generations of theoreticians resulted in an impressive apparatus that enables one to explain the existence of a market equilibrium under the conditions of an ideal competition and to study the properties of an equilibrium state of economy. The successes of mathematical economics far surpass the achievements in the construction of models in the social and political studies. It was already A. Smith who pointed out at the cause of this situation by discovering the existence of a general regulative principle in economics — "the invisible hand" of the market that transforms the individual aspirations into a common order.

The successes of theoretical and mathematical models cannot, however, hide a serious crisis in this domain of science; a crisis that developed slowly and became noticed only recently.

The gap between the mathematical economics, i.e., the discipline with its own quite interesting and meaningful object of the study that develops more and more abstract methods and partly became an esoteric domain of mathematics that studies certain particular types of functionals in partly ordered topological vector spaces and the applied studies whose problems turned out to be weakly related to the existence problems and the properties of the market equilibrium became wider and wider.

In practice, it is much more important to know how the system reaches an equilibrium under various constraints than in the non-constrained situation.

This problem became manifest in the study of "economies in transition", for which the abstract existence theorems concerning equilibrium are of little value. In the study of economies in transition, it is important to understand where this equilibrium will be reached under initial non-equilibrium conditions.

The deepening of the theoretical studies of the market as such also led to serious questions that put the theoretical orthodoxy to doubt. F. Hayek gave serious arguments in favor of the fact that the market as a social institute exists not as a corollary of an unbounded strive for full satisfaction of personal interests but rather thanks to the opposite — thanks to the existence of a system of a rigid moral rules that bound this strive.

In this case, the initial paradigm that pillared the mathematical models — the maximization of the utility function as the base for determining equilibrium — should be questioned.

The purpose of this book is to develop a new approach to the study of Nature and the mathematical apparatus of the theory of economic equilibrium; the point of view conceptually lying in the trend of ideas of F. Hayek and the Austrian school and the evolutionary approach to economics.

The idiosyncrasy towards econometrics shared by representatives of the Austrian school is well known. The purpose of this work is to show that the mathematical apparatus that enables one to rigorously formulate certain basic ideas of C. Menger, L. von Mises and F. Hayek actually exists and, moreover, is well known.

This apparatus is the statistical thermodynamics which, as follows from the studies of C. Shannon, L. Brillouin and N. Wiener and preceded in 1920s by L. Szilard's ideas, has applications far wider than statistical physics.

Statistical thermodynamics and information theory coincide, essentially. And what is the market if not the search for information and the continuous stream of decision-making?

"Information" here is understood in its most formal theoretical meaning as a means to diminish the number of alternatives for choice.

That is precisely how Hayek interpreted the institute of market and it is precisely this interpretation that provides, as we will show, a possibility to construct a qualitative theory of market economy. This theory possesses, we think, far greater possibilities than the orthodox theory based on ideas of Walras and Samuelson.

Several final comments. Our discussion of the existing economic theory is by no means a review or overview. We consider only theories and works which were of interest to us in the context of the formulated problem — the development of a new approach to the study of economic equilibrium.

One more remark concerns the rigor of the mathematical tools applied. We stick to the standards developed in theoretical physics, i.e., consider the functions differentiable as many times as is needed and freely replace the sums by integrals (but doing so we naturally remember the constraints that should be imposed on the mathematical objects involved).

CHAPTER 1

Models and their role in economics

In which sense should one understand models in economics?

Is the economic modelling an uncovering of certain "common laws" of economics or this is a mere instrument that helps to evaluate a concrete economic situation?

To answer these questions, we have to overview the types of models used in economics and, what is most important, establish what type of knowledge these models allow one to deduce from the analysis of economic situations.[1]

The role of economic models in the estimation of practical decisions is obvious. How to make use of resources optimally? How to invest with the least risk or with the highest yield? What assortment of goods should the firm produce at the current market situation?

The answers to the multitude of similar questions constitute the base of management and in this sense the mathematical models are of undoubted value as a means for solving the routine problems of control of economic activity. The main concept in the formulation of such problems and obtaining the answers is the notion of optimization — in the process of the search for solutions a certain essential parameter of the problem is tested for the maximum or minimum value.[2]

A typical problem of this type is the problem of replacement in the neoclassical theory of demand in the style of Slutsky-Hicks: How do variations of the price of one of the goods influence the demand? (See [**AEA**].)

This is an optimization problem with one parameter whose solution is easy to obtain by means of the usual methods of Calculus.

Far more serious problems appear when optimization takes into account two or more parameters: say, maximize the yield and minimize the risk. These problems would have required a more sophisticated mathematical apparatus — game theory — but, nevertheless, observe that problems with multiple criteria remain open to a considerable extent.

Far from always it is clear how to solve them and the reason here lies not only in mathematics but mainly in the "theory of choice" since it is not clear, in general, how to adjust and make commensurable criteria of distinct nature.[3]

How to correlate the beauty of the object with its price? Or the price and the quality?

It is hardly possible to construct universal means for evaluating beauty or practical value because they depend too much on the individuality of the evaluator. This is exactly why

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[1]The discussion on various types of models in economics has a long history. For separation of the "pure" theoretical law and an historical study of empirical forms of economic activity, see [**Me**]. For the present state of discussion on the type of economics, see [**Bl**].

[2]The optimization problems in economics are discussed in very broad literature, see, e.g., [**La**] solidly based on the ideas of L. Kantorovich and J. von Neumann. To a great extent this interest was also related with the general interest in the problem of the study of operations gained a great importance during WWII. The methods of the study of operations based on optimization of solutions demonstrated their effectiveness for a very broad type of problems, see, e.g., [**Saa**].

[3]See, for example, the discussion of this problem in Section 4 of Chapter 1 in the book by J. von Neumann and O. Morgenstern [**NM**].

certain trends of economic thought, in particular, the Austrian school, very skeptically considered the possibility to universalize the estimates. Such an approach certainly undermines the trust into the practical significance of the optimization problems that use the notion of "utility".

There are, however, a considerable number of optimization problems where there is no need to introduce the "utility". Such are for example, various transport problems where the subject of optimization is the time or the price of transportation.

Optimization problems provide innumerable possibilities for the skilled application of mathematical apparatus — linear, integer and dynamical programming, various versions of game theory: games with zero sum, games with non-cooperation, differential games and so on.

All these problems, however, possess a common peculiarity: these are problems with a goal. Models of solutions of such problems can be called "normative" since they are based on the evaluation of decisions.

The ontology of goal-reaching is organically in-built in these problems.

Generally speaking, the subject is utterly unnecessary for similar optimization problems. For example, there is a broad variety of variational problems in physics without any subject if, of course, one does not consider that variational principles in mechanics are testimonials of God's existence (such a possibility was subject to an active discussion immediately after the discovery of variational principles[4]).

Nevertheless, both practical economic problems on optimization and "normative models", despite the similarity of their tools, differ from the variational problems of physics: the economic theory assumes the existence of a subject with interests, a subject able to formulate goals.

In "practical economics", optimization problems are the tools that help the subject to achieve a goal rather than "nature's laws", which variational problems of mechanics are supposed to be.

Nothing similar to Hamilton's variational principle can be seen in the "normative models" for decision-making.

To what extent then can one expect the appearance of similar universal principles in the theory of economics?

This question was the center of a theoretical discourse in economics in the second half of XIX century. Such economists as C. Menger, L. Walras, W. Jevons insisted on the possibility of the existence of an abstract theory, see [**Me, Wa, Je**].

Those who defended the possibility of existence of an abstract theory in economics based on relations between "pure", "ideal" notions were actually defending theoretical economics as an independent sphere of activity, independent of successes of the practical recommendations in the realm of management.

This discussion was also connected with the existence of another danger for the newborn theoretical economics — the possibility of its engulfing by economic history, see [**Me**]. In case of such an engulfing the theoretical studies would have been reduced to the documentation of the uncountable variety of various "national economics".

An approach formulated in the works of L. Walras, W. Jevons and C. Menger and later called "neoclassical" in distinction from the "classical" formulations of A. Smith assumed the possibility to obtain precise answers to "general questions" and explanation of such

[4]Thus, one of the authors of the principle of least action, P. Maupertuis, assumed that the laws deduced from this principle are applicable not only to the world of mechanics but also to the world of animated nature, see [**Mau**].

phenomena as, for example, the surge of prices under the increase of the supply of goods, surge of banks interest under the significant accumulation of capital and so on. The founders of the neoclassical approach assumed that theoretical constructions that tie the abstract notions will enable them to go out of the realms of the "latest news" and provide both prediction and control, see [**Me**].

In particular, C. Menger saw the base of theoretical models in economics in the separation of empirical forms of economic activity — the types devoid of individual features — and in establishing logical relations between the types.

In this way, the theoretical economics was alienated both from the "economic politics", that is the technology of decision-making, and from studies on the history of economics. These studies invariably, due to the logic of the historical science, concentrated their attention on the individual peculiarities of the situations studied.

Theoretical economics became, therefore, the domain of functioning of interpretational models, i.e., the models that enable one to understand the general properties of the reality studied, the principles of its organization and functioning.

Still the considerable difference between the interpretational models of theoretical economics and the type of theoretical knowledge which in the natural sciences was used to be called "laws of Nature" was unbridgeable. The neoclassical models helped to understand certain principal peculiarities of functioning of the economic institutes, for example, the study of market equilibrium under the conditions of ideal competition but could not offer anything similar to the laws of mechanics that enable us to predict the future state of the system on the base of a small amount of general characteristics of the system and initial data.

Interpretational models of neoclassical economics are rather analogues of existence theorems in mathematics, i.e., logical means that help to discuss the existence of certain objects and their properties but tell nothing on the dynamics of these objects. In order to understand this fundamental distinction one has to make a short digression in the domain of general principles of methodology of natural sciences formed in the beginning of the Modern Age and compare these principles with methodological principles of theoretical economics.

The base of the modern science is, after Tycho Brahe, the experiment, more exactly, a methodology that assumes that there are means to verify the applicability of theoretical constructions and models, that is to compare the results obtained with the help of these models with an experiment.

Here I would like to make a very essential remark, which is usually missed not only by various laymen from the general public, but also by highly qualified experts.

The point is that any "comparison with an experiment" is hardly a comparison with natural situations encountered in Nature. Strictly speaking, by means of comparison with "natural situations" one cannot either confirm or disprove a scientific theory.

Indeed, the experiment deals not just with a mere reality. It deals with a *specially prepared* reality in which all the parameters considered to be given (and essential) in the theoretical model are under control. Actually, any experiment has no direct relation to Nature as such. This is a construction which is a corollary of theoretical abstraction (see a remarkable analysis of this problem in [**Bib**]).

For example in the "theoretical dispute" of Aristotle and Galileo on the nature of motion in physics the practice is on Aristotle's side: moving bodies, devoid of a source of energy, have a tendency to stop and assume a certain position. The idea of Galileo about the motion by inertia is a theoretical abstraction that ignores friction and resistance of the media. Nevertheless it was precisely this abstraction that Galileo realized constructing his experiments.

Actually, a thought experiment preceded the real experiment, i.e., the construction of a device that develops certain properties of the object under investigation.

Such a device creates a sort of microcosm that purifies the object and deprives it of the properties inessential for the experiment.

I intentionally stress here the role of the thought experiment as a necessary predecessor of a real experiment as a theoretical construction, which, being completely realized only in the ideal world, possesses, nevertheless, an intrinsic value and ability to prove. (M. Wertheimer suggested a very interesting analysis of the structure of Galileo's thought experiments, see [**We**].)

The extent to which the conceptual pattern of an experiment can be realized is always questionable. How can we be sure that the real device does not possess certain factors unknown or not under our control but with decisive influence on the results of the experiment? To be so sure, we should believe in the theory underlying the conceptual scheme of the experiment or more precisely in the logic behind this conceptual scheme because neglecting certain possible influencing factors during the development of the conceptual scheme is often based not on theoretical considerations but on the "natural logic" which is often implicit.

Let us make one more step. What if the experimental device spoken about in the conceptual scheme of the experiment is conceivable in principle but so complicated that it cannot be realized for technical reasons? Does it mean that we may not argue appealing to the conceptual scheme of the experiment?

Similar situations are encountered all the time.

It turned out that the conceptual scheme of an experiment can be a means of the proof even without being realized in hardware. It suffices to recall discussions of Bohr and Einstein, see [**BE**][5] on the basics of quantum mechanics, "Schrödinger's cat", "Maxwell's Demon"[6] and other examples of thought experiments. A thought experiment verifies the logic of the theory under the hypothesis of a potential realizability of however complicated technical devices whose principle is understood.

Consider, for example, "Maxwell's Demon", a thought experiment that will be needed in what follows. Working with problems of statistical physics, Maxwell suggested an idea of a technical device whose performance was intended to disprove the *second law of thermodynamics*, the law of growth of the entropy. According to the second law of thermodynamics, **heat cannot naturally pass from a cooler body to a hotter one**. Maxwell suggested to consider two volumes of gas separated by a door managed by a robot of a kind or as he called it, "Demon". We assume that the Demon is able to determine the speed of the particles of gas approaching the door and either open it or not, depending on this speed, and this way regulate the flow of the particles from one volume into another without affecting their natural motion. It seems that the Demon is able, without spending any energy,

[5]This volume comprises Bohr's papers on relativistic quantum theory and his philosophical texts regarding complementarity in physics from the years 1933 to 1958, together with two introductory essays, some unpublished manuscripts, and four relevant papers of other authors. It is supplemented by selected correspondence mainly with Heisenberg and Pauli.

The part on relativistic quantum theory documents Bohr's views on field quantization, measurability, and early particle physics that do not so much hinge on lengthy calculations and consistent schemes as on detailed reflections on the principal problems.

The part on complementarity as a "bedrock of the quantal description" presents Bohr's reaction to the Einstein-Podolsky-Rosen paper (also reprinted), that had a remarkable effect on him, quite in contrast to, e.g., Heisenberg and Pauli, who considered it as no news. Further papers deal with causality in atomic physics and general epistemological problems.

[6]For an analysis of Maxwell's thought experiment important for problems considered in our paper, see [**Bri**].

make so that the faster particles will go from a cooler volume to the hotter one, and therefore raise the temperature of the hotter volume still further, contrary to the second law of thermodynamics. This is a paradox from the thermodynamic point of view.

This paradox was repeatedly studied afterwards[7] and it became one of the cornerstones of modern information theory. L. Szilard used this example to show that the performance of such a device is impossible without exchange of information between the Demon and the gas that the Demon controls, and that this information may be defined as a negative entropy, i.e., by diminishing the entropy in the system, the Demon will invariably increase its own entropy which proves the impossibility of this type of devices.

Numerous examples of other thought experiments directed, as a rule, to elucidate fundamental questions of new theories show that a thought experiment is a powerful tool of analysis on the borderline between theory and experiment that helps the theoretical thought to penetrate the worlds inaccessible to technical control, the domains of reality where the experimental situation is not possible to be prepared and controlled artificially.[8]

Let us now return to the question formulated at the beginning of this chapter.

What is economics as science? Is it possible to experiment in economics and is it possible to construct theoretical economics along the classical pattern well known from the practice of physical studies: theory — experimental verification — theory?

At first glance nothing contradicts such an approach. Theoretical economics describes real processes and from the point of view of a "naive methodologist" a provable comparison of the theory with "practice" is possible. In other words, the "experimental verification" of a theory is possible in real life. The above arguments, however, make it manifest that the experiment, as it was understood during the last three centuries thanks to the efforts of the classics of the European scientific tradition, is hardly possible nowadays in economic investigations. In real life, there are no artificially prepared situations specifically designed for verification of theoretical ideas. And even if in certain cases (for reasons unknown, the history of Russia is exceedingly rich with such cases) "economic experiments" are being performed, the conceptual schemes of these experiments are very far from the complete description of potential influencing factors on the artificially prepared experimental situation.

We say nothing about the price of such "experiments" and how sound is the theoretical base of such "experimentation" if evaluated on the level of theoretical models in natural sciences.

The impossibility, at least at present time, of a real scientific experiment in economics does not, however, mean that the idea of an experiment as a test of theoretical models in economics failed. The rich possibilities of thought experimenting still remain. The huge help of thought experiments, for example, in the construction of quantum mechanics, leaves some hope. In such a setting of the problem we do not speak of course about verification of the quantitative predictions of the theory. But it seems quite possible to verify the theoretical models themselves and their inner logic.

Having said so, we immediately encounter a very serious question whose scale is comparable with the logical problems of quantum mechanics and which, to my mind, can be solved by analogous means. This question is **where are the limits of rationality of economic behavior?**

To what extent the economic situation formed by the choices, preferences, plans and behavior of a multitude of people can be controlled rationally? Can there be certain principal

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[7]See the paper by Szilard [**Sci**], seminal for the development of information theory.

[8]For semantics of such "twilight zone" worlds, see, e.g., [**SPU**].

constraints, logical in their nature, on the rational control similar to the second law of thermodynamics or Bohr's complementarity principle?

Nothing contradicts the assumption that thought experiments might suffice to solve such problems if the theoretical models will be formulated clearly and logically.

Conceptual models possess their own logic and, as the example of the analysis of "Maxwell's Demon" shows, the rigorous adherence to this logic may reveal the inner self-contradicting nature of the conceptual constructions, which seemingly satisfy the principles of the "natural, naive" logic.

It is precisely such analysis which we think is of primary importance in economic studies. But serious and rigorous study of the inner logic of theoretical models is only possible if these models are formulated in a sufficiently lucid language subject to logical analysis and possessing a certain inner completeness that does not let the theoretician get lost in the multitude of weak and sometimes not explicitly formulated basic assumptions. In other words, to be verifiable by a thought experiment a theoretical model should be a logical construction and not a metaphor.

As we will see in what follows, this is precisely this requirement that the current theoretical models of market economy fail to satisfy essentially.

Theoretical models of market started with the metaphor of the "invisible hand" of the market suggested by Adam Smith. Surely, the metaphor is a most powerful weapon of creative thinking. But theoretical analysis must go beyond metaphor. It should make explicit the inner logical structure of the metaphor turning the metaphor into a theoretical model and, to an extent, destructing the metaphor.

The existing economic theories of market are metaphorical to the marrow. They constantly discuss an "equilibrium", the notion metaphorically borrowed from mechanics and therefore our first problem will be the analysis of the history of the conception of this basic metaphor of economic theory. In order to turn a metaphor into a model this metaphor should have been subjected to "deconstruction", say in the same sense one deconstructs theoretical notions by M. Foucault and W. Derrida. Still, deconstruction alone is insufficient to construct a theoretical model. The deconstructed metaphor should be composed again using the "logical constructor" and only after that the logically constructed model may be fit for thought experimenting. This is precisely what we intend to do with the notion of equilibrium in market economy.

CHAPTER 2

General Equilibrium Theory and the ideas of A. Smith

In economic theory, there seems to be no notion whose importance can be compared with the notion of equilibrium.

The General Equilibrium Theory (GET) became the cornerstone of modern economic science and subjected not only theoretical investigations but even the economic politics. Strange as it may seem, but General Equilibrium Theory gained a great deal of currency and popularity in the countries with transitional economies whereas, as is manifest, the transitional economy is constantly being in a state quite distant from equilibrium, at least in its traditional reading.

Our aim here is to consider the ontological basis of GET. This will enable us to offer in the subsequent chapters a generalization of the notion of equilibrium in economics. This generalization will allow us, at least partly, to consider non-equilibrium processes with approximately the same degree of generality as the non-equilibrium statistical thermodynamics has in the framework of physics.

The creators of the General Equilibrium Theory were well aware that this theory only describes economic situations in a sufficiently small neighborhood of an equilibrium state and that the logical base for General Equilibrium Theory should be sought outside its realms, since the mechanisms that force the system to attain an equilibrium have nothing in common with the description of characteristics of the equilibrium itself.

One of the known American experts in mathematical economics wrote not so long ago, see [**Fi**]:

> "*For macro-economists ... the microeconomic theory is primarily about positions of equilibrium. The plans of agents (usually derived from the solution of individual optimization problems) are taken together, and certain variables — usually prices — are assumed to take on values that make those plans mutually consistent... In all this very little is said about the dynamics of the process that leads an equilibrium to be established... Attention is centered on the equilibria themselves... and points of non-equilibrium are only discussed by showing that the system cannot remain in such points.* "

This fundamental position is often forgotten and those who try to apply this theory in practice, assuming that "liberating" market forces and rigorous sustaining certain parameters (such as, for example, the volume of the currency or the level of the deficit of the budget) in certain limits automatically leads economy to the state of equilibrium. Such economists or, more exactly, politicians, since the one who implements an economic theory in practice becomes a politician whether one wishes that or not, should recall the warning on practical application of economic models expressed in one of the most known books in the world literature on economic theory ([**NM**]):

> "*The sound procedure is to obtain first utmost precision and mastery in a limited field, and then proceed to another somewhat wider one, and so on. This will do away with the unhealthy practice of applying so-called theories to economic or social reforms, where they are in no way useful*".

Thus it is very important to establish the limits of the basic notions used in economic theories. This applies first of all to the notion of economic equilibrium.

There is a number of key questions concerning General Equilibrium Theory which are very hard to answer.

Is General Equilibrium Theory a scientific theory or is it just a “paradigm”, an ontological base in the frame of which an economic theory should be developed? Is it possible to falsify General Equilibrium Theory? Is it possible to use it as a foundation for “normative economics”, i.e., for practical economic decisions? These questions are a subject of heated and incessant debates, see, e.g., [**Bl**], Ch.8.

The very existence of such questions during discussions testifies that the status of General Equilibrium Theory as a theory is unsettled. It is hardly possible that such questions may arise nowadays for example, concerning classical mechanics. Therefore, instead of considering General Equilibrium Theory as a base for practical solutions, we should at least try to analyze the logical foundations of General Equilibrium Theory and see what mathematical models are adequate to these foundations. Let me make several remarks that at first glance may seem rather trivial.

Any description of a real situation is a reduction. Certain parameters of the situations are being taken into account, the other ones are being ignored. For the scientific theory the problem is to find out the degree to which the inevitable reduction of the description is the subject of a conscious control. Since in the frameworks of the majority of “non-scientific” descriptions the reduction is as a rule implicit, the critics “from inside” become impossible. The description becomes the subject of faith. Precisely this type of transformation happens usually during the process of application of scientific theories. Scientific descriptions differ from “non-scientific” ones in the very fact that the reduction becomes the object of reflection. Methodological criteria clarify what “models of reality” are the base of the description. And what is very important, such a practice (i.e., reflection) needs its own “metalanguage” which is generally speaking, different from the one used for the description of reality.

To what extent are these trivial arguments used during the construction of economic models? To a large extent this is related with the understanding of the role of mathematical models in the description of reality.

When mathematical models are being used, the level of requirements to the understanding of implicit hypotheses of the model is much higher than for the usage of conceptual models. We think that, under the passage from the conceptual models of Adam Smith and the founders of the neoclassical theory to the mathematical models of economics in 1930s–50s, the understanding of the hypothesis did not improve in the majority of cases considered in the economic theory, but instead, diminished. Therefore we deem it highly important to analyze the genesis of conceptions on market equilibrium and reveal the hidden hypothesis that used to lie in the foundation of the molding representations of the classical and neoclassical economic theory.

The first thing to do is to try to understand the inner motivation and implicit suggestions lying in the foundation of the notions used by the founder of the classical economic theory, Adam Smith.

Observe, first of all, that A. Smith was not, generally speaking, an economist. At least his main interests used to lie outside the domain of economics. Smith was a moral philosopher. He was interested in the problems and principles explaining people’s behavior and correlation of the characteristics of this behavior with the common well-being. Essentially, his economic theory was precisely the theory purported to explain the character of the relations between human behavior and common well-being.[1]

[1]Before A. Smith wrote his famous book [**Sm1**], he was for a long-term plunged into studies of moral philosophy. At the age of 29, A. Smith was elected the Chair of Moral Philosophy at Glasgow University

We think that the economic representations of Smith are impossible to understand without first understanding the "model of the human being" which he put into the foundation of his economic theory (for a shrewd analysis of this aspect of A. Smith's ideas, see [**Wh**]).

For Smith, a man is, first of all, an emotional being whose behavior is determined not by reason but by feelings and emotions.

According to Smith, the "moral" expresses the communal feelings and a moral behavior is a behavior concordant with these feelings. Without difficulty Smith reveals contradictions between the individual utility and communal appraisal: he

> *remained, however, under the opinion that the first and prime reason of our approval or disapproval does not follow from the understanding what can be useful or evil to us... To the feeling of approval we constantly add our understandings on what is natural and legal which have nothing to do with the understandings on utility...*

He believed that human abilities in pure sciences are being discovered in a most wide and illustrious way especially in the so-called higher mathematics. The utility of these sciences, however, is known to precious few and, to prove it, one would require explanations which far from all can grasp. Therefore not the utility provided by these sciences is the reason of the universal respect to them. There was no discussion about this utility until one had to retort the reprimands of the people who, having no inclination to these higher sciences, tried to diminish their value accusing them of uselessness, see [**Sm2**], part IV, Ch. II.

Actually, A. Smith spent his life seeking for the principles and social institutions which could harmonize human emotions and the common prosperity. Having found nothing satisfactory in the domain of moral and political philosophies he turned to the study of economic behavior where he found, at last, the dominating emotion that governs behavior — the individual's interest and a regulative principle that matches this emotion with common prosperity — the market. Observe, however, that for A. Smith the interrelation between the individual's interest and the common prosperity was never a postulate. This was his deduction obtained from an economic analysis. Smith needed the "invisible hand" of the market not to describe the phenomenon of equilibrium. He needed the "invisible hand" to find out and explain the circumstances of a rapid economic growth. He is most interested in the questions of interrelation between well-being and growth. He has found out, for example, that the price of a labor force is highest not in the regions with the higher well-being but where the production growth is the highest.

A. Smith formulates a dynamical theory of the market in which the interests of its agents under the conditions of free competition are the driving forces that establish an equilibrium and turn themselves into this "invisible hand". He describes the influence of the "invisible hand" as follows:

and at the age of 36 he published *Theory of Moral Sentiments*. Only in 1767, being 44 years old, after he had travelled to Paris and acquainted with Turgeau, A. Smith began to study theory of economics.

> *All the people that use the land, force or capital to provide the market with goods are interested that the quantity of the goods would not exceed the actual demand whereas all the others are interested in maintaining its quantity on the level not lower this demand... If on the contrary, at some moment the amount of goods delivered to the market would be lower than the actual demand then certain constituents of its price should raise higher its natural norm. If this is a land rent, then the interests of all the other landlords will naturally prompt them to adjust more land for unearthing this product; if this is a wage or a profit then the interests of all the remaining workers and business people will soon force them to invest more labor and capital into production and delivering this item of goods to the market*, see [**Sm1**], book 1, Ch. 7.

A. Smith was not interested in the market equilibrium as such. He was interested in the conditions which establish the fastest growth of communal wealth. The problem of an "invisible hand" and the study of equilibrium were only his means to investigate the problem of harmonization of individual interests (utility) and the common prosperity (i.e., the rapid growth of the total wealth of the people). This is a principal difference between A. Smith and his neoclassical followers.

It is interesting to observe that the method Smith uses to prove his statements is essentially a thought experiment. Using an ideal logical model — a free competition and the lack of constraints on the nature of economic activity Smith proves the existence of a market equilibrium based on a logical deduction. Introduction of any constraint into his model (say restrictions on the available resources of labor or capital) will change the results.

Such a model does not, of course, allow to make quantitative predictions. The thought experiment of A. Smith can be easily formulated as a mathematical model but only on a "metaphorical level".

The equilibrium of a dynamical system is precisely a metaphor which can be applied to the description of a market "equilibrium" starting from Smith's thought experiments. A deviation of the system from the equilibrium state calls the "forces", i.e., interests that struggle to return the system into the equilibrium state.

Of course, Smith himself did not consider market equilibrium as a singular point of a system of differential equations but this metaphorical model can be easily extracted from his arguments.

Smith's ideas were developed in the first half of the XIX century by economists of the "classical school": Ricardo, J. H. Mill, and so on. But this development was not qualitative and did not considerably augment our understanding of economic processes. These theories never gave any "computational" recipes for behavior nor models of economic decision making.

The situation completely changed with appearance of works by C. Menger, L. Walras and W. Jevons who changed the intellectual landscape of research in economics.

As we mentioned above, the purpose of these studies was to construct rigorous models in whose framework by logical (C. Menger) and mathematical (W. Jevons and L. Walras) methods one could obtain significant results. To do so, the neoclassics had to essentially restrict the range of application of their analysis confining themselves to the study of market equilibrium as such. In other words, the market equilibrium became (especially in works of W. Jevons and L. Walras) not an institutional mechanism to be analyzed, but a rigorously defined situation of the market, namely, the absence of a surplus demand.

In the middle of the XX century, the development of the ideas of Jevons and Walras by A. Wald ([**W**]), K. Arrow and G. Debreu ([**D**], [**AD**]) led to creation of mathematical models of market equilibrium with developed research tools; these models digressed very far from the initial ideas of neoclassics.

To understand these highly sophisticated mathematical constructions, one has to sort out the ontological shift that took place in the passage from the classical theory to a neoclassical one. One of the central novelties here became the notion of *utility.* In order to get a possibility to construct mathematical models of the market, one had to create a universal model of behavior of its agents. Whereas the basis of such a model for the sellers was considered sufficiently obvious after works of A. Smith (as maximization of the profit realizing the private interest), it was not that easy to introduce a similar universal measure for the buyers. On the philosophical level, the notion of utility introduced and justified by I. Bentham started to play this role.

Bentham thought that human behavior in general is determined by a utility derived from decisions and actions, and the common welfare can be computed as the sum of gains and losses in totality obtained by the members of the society as a result of their decisions, CR. [**Ben**]. Bentham suggested an essential change in understanding of human behavior: for A. Smith, a private interest was only one of emotions that determine human behavior, and the role of this emotion in the general picture of behavior was a subject of the study; for Bentham, the question was solved: utility became the only driving force that determines the whole totality of behavioral acts.

This point of view was adopted by the founders of the neoclassical school and turned into an instrument of mathematical analysis of the market. With the help of the notion of utility it became possible to construct a model of rational decision making by the buyer. Thus, the hypothesis that

> *the buyer possesses clearly described goals and preferences and is capable to determine and compare utility of different goods*

became the ontological postulate of the neoclassical orthodoxy to the same extent as the hypothesis that

> *the utility the buyer obtains and the profit the seller gets exhaust the conception of common welfare obtained as a result of an economic activity.*

Sufficiently simple hypotheses on the dependence of utility on the quantity of the goods purchased (its monotonic decay) together with the principle of maximization of utility led to the "marginalist revolution" in the study of market economy namely to the understanding that the utility of the last portion of goods purchased determines the market situation.

Indeed, it is not difficult to see that the ratio of utility of this last portion to its price should be the same for all goods, otherwise it would have been possible to enlarge the common utility of goods to buy by changing the assortment of purchases.

The discovery of this fact immensely influenced the whole style of economic studies. New possibilities to study properties of the state of market equilibrium opened up. The visible ontological change discernible already in works of Bentham — placement of utility function and its properties in the center of analysis of human behavior — became fossilized. The theory of "rational choice" was formed. The analysis of decisions the buyer makes became the prime object in the study of market equilibrium. Later it was dubbed the "buyer's sovereignty", that is independence of the buyer's decisions of anything except for the utility function.

The already mentioned basic metaphor of mathematical economics emerged, namely, the problems of mathematical economy became now understood as problems on extrema of certain functions subject to given constraints (usually constraints on resources). The problem of existence and properties of the market equilibrium came to the center of research in economics.

Such a change of main goals of the study entailed the appearance of a specific mathematical apparatus: in 1950s, for the proof of existence of the point of market equilibrium, i.e., a certain vector of positive prices under a non-positive extra demand, the researchers started to use topological methods such as the fixed point theorems[2].

Of course, the conclusions of the economists of the neoclassical school, as well as the conclusions of the classics, were based on thought experiments since it is hardly possible to actually compute the utility function of the market agents. By means of a thought experiment it is possible to prove the existence of a point of market equilibrium, but it is hardly possible to determine what would this equilibrium be for given conditions on production and demand.

The most important happening in the framework of the marginalist revolution was the change in the understanding of the nature of market equilibrium.

For A. Smith, a *market equilibrium* was a stable point of a dynamical system with actually acting "forces" (interests of the market agents) that react on deviations from the equilibrium state. In other words, the metaphor of equilibrium itself turned out to be justified by comparison with the behavior of mechanical systems. In works of neoclassics and especially in mathematical interpretations of neoclassical conceptual constructions, even the relation with the etymology of the word *equilibrium* on the metaphorical level becomes lost.

It is unclear why the vector of prices which is a fixed point of a certain map and corresponds to a non-positive extra demand should be called an equilibrium point of whatever.

The model of the human in neoclassical economics becomes reduced to triviality. A behavior is called "rational" if it maximizes utility and utility is actually defined as a function which determines a "rational behavior".

This is a logical vicious circle. The market is considered to be a priori free, i.e., the conditions for the ideal competition are justified as if "taught by themselves" whereas A. Smith, for example, clearly understood that

> *a free competition is a social institute which should be maintained and the justification of conditions for free competition is a function of political power.*

Implicitly, the neoclassics started to consider *the free competition as a self-sustaining system* which is a very doubtful assumption (at least it should be seriously justified).

The General Equilibrium Theory formed on the base of neoclassics' opinions became lately not even as much scientific paradigm as the leading ideology of conservative political movements and parties in many countries despite its very doubtful performance in their economic effectiveness. One should observe that in the post-war time the highest rate of economic growth was achieved in the countries (Japan, South Korea, Taiwan and recently China and Vietnam) where the general orientation to market economy was combined with an economic policy very distinct from the ideology of neoclassic.

The faults of the economic policy inspired by the principles of the general theory of market equilibrium (as well as recent economic difficulties of the East European countries) can, nevertheless, hardly be arguments in the dispute on scientific consistency of this theory for

[2]A theorem stating that a continuous selfmap of a compact set has a fixed point is due to Brouwer. Later, Kakutani formulated a fixed point theorem for semi-continuous above maps of sets, [**Ka**]. Arrow and Debreu used this theorem to prove existence of an equilibrium in the dynamics of competition. Their idea is to construct two maps of sets: a map of the set of vectors of excess demand to the set of vectors of prices and a map of the set of vectors of prices to the set of vectors of excess demand and to apply Kakutani's theorem to the selfmap induced by the above maps on the Cartesian product of these two sets. Under certain conditions (whose fulfilment for concrete economic models is to be verified) there exists a vector of prices for which the excess demand is non-positive, cf. [**ABB**].

reasons discussed in Chapter 1. We need an inner logical analysis of the thought experiments on which General Equilibrium Theory is based.

CHAPTER 3

C. Menger, F. Hayek and D. Hume: Challenge to utilitarianism

Observe that the basic for the classical and the neoclassical economic theory idea of the market as a means to directly harmonize a private interest and common welfare was subject to a serious criticism by economists who strived to create a rigorous economic theory; first of all, by representatives of the Austrian school. Carl Menger, one of the initiators of the marginalist revolution, insisted, nevertheless, that a private interest cannot be considered as a means for automatic achievement of common welfare and the idea on an interrelation between a private interest and common welfare is not a suitable foundation for a rigorous economic theory.

According to Menger, certain economists consider the "dogma of private interest" the founding principle that pursuing a private interest on the level of an economic individuum without any influence of the political and economic measures of the government should result in the highest level of common welfare which can be achieved by the society in the given spatial and time constraints. We will not nevertheless deal here with this approach which is faulty at least in its most general form, see [**Me**], p. 83. Even if, in their economic activity, people would always and everywhere be guided exclusively by their private interests, it would be, nevertheless, impossible to assume that this feature determines economic phenomena, since from experience we know that in uncountable many cases people have faulty views on their economic interests or have no idea about the situation in economics at all. ...The presumption of such definition of the regulating force in economic phenomena, and therefore within theoretical economics, however multivalued this word might be, is not just a dogma on a private interest. It is also a dogma on "faultlessness" and "omniscience" of the people concerning the situation in economics, [**Me**], p. 84.

Menger insists that the economic theory should disregard inessential aspects in the human behavior (in particular, possible errors) giving an ideal scheme. The economic theory cannot provide with understanding of the human phenomena in all its totality and concreteness. But it may provide with understanding of one of the most important facets of human life, [**Me**], p. 87.

Menger denies the economic theory in its claim to understand real situations in which exterior factors influencing human behavior (in particular, faults, delusions and prejudices) are impossible to remove but believes that the reside after removal of these occasional factors is of considerable interest for understanding economic behavior of people.

Such a position is, of course, both vulnerable and self-contradicting because it is absolutely not obvious that, having removed essential factors that affect a real situation, the residue will be of any interest.

This is impossible to postulate. This must be proved.

In particular, one should show that eliminating such a factor as a faulty understanding of the economic situation by many market agents does not destroy the scheme of thought experiment which we used to obtain a new notion on the nature of economic life. To compare, observe that we know now that, in the physical theory of microcosm, the experimental mistakes cannot be eliminated from the theory.[1] We see, therefore, that while criticizing ontological idealization that lie in the foundation of the neoclassical theory Menger remained on the "classical" positions in quite another sense. In the sense of possibility to eliminate from the theory the so-called "inessential" factors. Nevertheless, the principal difference of his point of view from that of neoclassical orthodoxy is manifest. And therefore it is not accidental that the representatives of the neoclassical orthodoxy are rather restrained in their appraisal of Menger and the Austrian economic school as a whole.[2]

Much later the eminent representative of the Austrian school, F. Hayek, made the next step in understanding of the deeper principles of market functioning suggesting an interpretation of these principles totally different from the traditional one. He started to consider competition at the discovery procedure (in particular, circumstances and by particular people) of the facts and data on the market state which give preference to those who know these facts and can use them.

Hayek accepted the principal incompleteness of knowledge on the actual situation in economics the main circumstance of any economic activity. According to him:

It is difficult not to concede with the accusation that during the approximately 40–50 years the economists' discussions on competition based on assumptions which had the actually mirror of the reality would have made competition totally meaningless and useless. If somebody could indeed know everything that economic theory calls "data", then competition would be quite a wasteful method of adjustment to these "data"... Having this in mind it is useful to recall that each time when competition can be rationally justified it turns out that the basics for it was the lack of facts given beforehand that determine the rivals' actions.[3]

Hayek poses a principally comparative problem:

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[1]This is one of the fundamental results of quantum mechanics, see, e.g., [**JvN2**]. There exist quantities which cannot be simultaneously measured with arbitrary precision, namely such are the physical quantities whose corresponding quantum mechanical operators do not commute. The measurement problem in modern physical theory does not reduce, however, to impossibility of simultaneous precise measurements:

> "*... it goes without saying that any measurement or related process of subjective perception is related to the external physical world a new entity that does not reduce to it. Indeed, such a process leads us out of the outer world or, more correctly, leads into a non-controllable situation since in each control test the inner life of the individual is already supposed to be known.*" ([**JvN2**], Ch. VI)

[2]S. Littlechild [**Li**] characterizes the Austrian school as follows:

> "*Austrian economists are subjectivists; they emphasize the purposefulness of human actions, they are unhappy with constructions that emphasize equilibrium to the exclusion of market processes; they are deeply suspicious of attempts to apply measurement procedures to economics, they are skeptical of empirical "proofs" of economic theorem and consequently have serious reservations about the validity and importance of a good deal of the empirical work being carried on in the economics profession today.*"

Earlier, P. Samuelson ([**Sam**], p. 761) formulated his relation to certain critics of the neoclassical orthodoxy as follows:

> "*In connection with the exaggerated claims that used to be made in economics for the power of deduction and a priori reasoning by Carl Menger, by Ludwig von Mises... I tremble for the reputation of my subject. Fortunately, we have left this behind us.*".

[3]The first, it seems, to come to the idea to apply evolutionary theory to economics was A. Alchian [**Alc**]. D. North also used evolutionary approach in the study of evolutions of social institutes and first of all the inheritance law to a considerable extent, see [**NoI, NoS**].

what institutional organization in economics becomes more effective, i.e., capable to survive, and under which conditions?

Such a formulation of the question makes him to turn to the evolutionary theory for justification of deductions made. He cannot any longer consider just empirical facts, or even thought experiments, to justify his opinion:

> ... *We come to an inevitable conclusion that in cases where the competition is meaningful the validity of the theory is definitely impossible to verify empirically. We can verify it on abstract models and conjecturally in the artificially reproduced real situations where the facts on which discovery the competition is aimed are already known to the observer. But in such situations the theory has no practical value... At best we may hope to be able to establish that the communities relying on competition will eventually achieve their goals with more success than the other ones. This is the deduction which I think is remarkably confirmed by the whole history of civilization.*

This excerpt lucidly shows what determines Hayek's skepticism concerning mathematical models of market economy. Hayek's turned to evolutionary theory for justification of the advantages of market economy as compared with the methods of centralized planning let to reconsider ontological assumptions concerning the nature of society as a whole. Hayek started to consider evolution of market structures as a "third world" of a kind as compared with two other worlds — the world of "nature with its laws and the world of rational human activity" ([**H2**], Ch. 1).

Competitive structures (i.e., not only economics but also biological species) circumnavigate he thinks the limits of human rationality, thus stating the problem of creating a new type of arguments — the evolutionary epistemology (I would have said: evolutionary ontology).

Hayek is well aware that approaching the study of market economy from the position of availability of knowledge he destroys the conception on harmony of private interests and the global goals of the society. He is forced to refute the notion of economics as a system of rational "management" thus destroying the suggested by A. Smith interpretation of interest as the inner regulating force, i.e., Hayek goes much further than C. Menger who refused to accept the idea on a correspondence of a private and common interest in market conditions as a dogma (i.e., without discussion) but it seems would have nothing against accepting it as a deduction (in accordance with the general methodology of A. Smith). Hayek writes:

> "*The direct meaning of the word "management" is an organization or a social structure where somebody consciously places resources in accordance with a unique scale of goals. In the spontaneous order created by market none of this exists and it functions principally differently than the above "management". In particular, it differs in that it does not guarantee necessarily to satisfy first more important due to general opinion needs and then less important ones...*" ([**H2**], Ch. 1).

Hayek denied the notion of "management" the prime role and instead introduced the idea of "order".

This conceptual change requires a particular attention. The role of idea of "order" in statistical physics where "order" is an antonym of "chaos" is well known. In statistical physics same as in the conceptual model of economics suggested by Hayek the order establishes itself under certain circumstances since such a state becomes more probable than the chaos (for example, this is the way crystallization of the fluids under low temperatures occurs). This happens not because each molecule knows its place. The movement of molecules is only determined by local information on the position of its neighbors. The order is established because the number of possible ordered states with given energy turns out to be very high, greater than the number of chaotic states.

We can therefore try to transform Hayek's arguments based on the just mentioned metaphor of order in the statistical theory of economic activity which uses in principle the same notions as the statistical physics.

I think that Hayek himself came rather close to this idea:

"It goes without saying that it is worthwhile to try to create conditions under which the chances of any randomly selected individual for the most effective realization of his goals would have been rather high, even if it would have been impossible to predict beforehand for which particular goals these conditions will be favorable and for which unfavorable", [**H2**], Ch.1.

Therefore Hayek assumes that under the conditions when "general prosperity" would be considered simply as the sum of satisfied interests of the participants of the economic activity the market as an economic institute would ensure the attaining the "common prosperity" simply due to the high probability of satisfaction of interests of individuals. In such conditions to predict how would the "common prosperity" look like is impossible.

Such an approach to the study of market can hardly be called the study of equilibrium in the sense in which this metaphor was used by A. Smith. Here we see no "balance", no "forces" that return the system to a certain natural state. Hayek is very sensitive to this. He writes:

> "*Economists usually call the order created by competition an equilibrium. This term is not quite adequate since such an equilibrium assumes that all defects are already discovered (compare with A. Smith's thoughts that the prices inform us on "hidden" parameters such as profit or interest rate. V. S.) And the competition therefore is stopped. I prefer the notion of "order" to that of equilibrium — at least during the discussion of problems of economic politics*" ([**H2**], Ch. 12).

Hayek further makes a number of particularly interesting remarks which hint that the ideas of a fundamental relation of physical statistics with modelling of economic processes were not alien to him and his skepticism towards modelling in economics was occasioned by perhaps the fact that both conceptual constructions and mathematical methods widely used in the modern mathematical economics were in Hayek's view inadequate. It seems that Hayek was just not acquainted with the main ideas of statistical physics and therefore related even purely statistical ideas with cybernetic ideas on positive feedback, i.e., with a mechanical metaphor. (For the sake of justice observe that in N. Wiener's papers applications of the idea of the feedback to the study of behavior of complicated systems were largely inspired by his interest to the problems of statistical physics.) Studying the problem of interaction of economic agents of the market Hayek writes:

> "*This mutual adjustment of individual plans is being performed along the principle that we following the pattern of natural sciences that also turned to the study of spontaneous orders (my italics, V. S.) or self-organizing systems became known the "negative feedback"* " ([**H2**], Ch. 1).

Feeling that the market works rather spontaneously as a statistically organized system in which the order appears not because the parts have information on the whole and they, the parts of the system, make a choice in favor of the "order" but just because the order is more probable in certain conditions. Hayek still cannot completely get rid of the mechanical metaphor in the study of principles of market functioning. Actually from the above arguments Hayek deduces a principle of universal applicability of market economy refusing to distinguish the conditions and the social problems in solution of which the principles of market economy are indeed effective and the conditions and problems for which their effectiveness is doubtful. We think that such a mixture is a result of coexistence in Hayek's perceptions of two different metaphors that explain functioning of market economy — the mechanical metaphor of A. Smith and the statistical metaphor developed by Hayek himself without due logical separation. Speaking about statistical metaphor in the excerpt cited above Hayek mentions self-organization. Indeed, there were multitude attempts to prove a possibility for self-organization starting from statistical principles.[4]

Strictly speaking, a self-organization means diminishing of the entropy of the system. In the closed system this cannot happen, for self-organization we need an influx of energy from outside. In other words, if we speak about self-organization in economics, such a process should assume existence not of equilibrium but metastable states.[5]

We believe that one should be very careful applying perceptions on self-organization to the evolutionary theory (cf. Chapter 8).

Discussing the problem of self-organization Hayek says that the rules of behavior of market's agents are the result of a long cultural process. They are not natural and often are being followed contrary to the interests of the economic agents. The market as a whole exists as a system not because following the rules of behavior the market agents gain an immediate profit but because the system of rules of behavior that determines market institutes wins in the process of competition with other systems of rules of behavior. In other words, there exists a competition between the types of economic institutes and in the larger time scales that is in "macrotime" the market institutes are more effective.

This is a very radical transformation of the viewpoint on the nature of market. Placing the economic and generally speaking social institutes of the society in the evolutionary line similar to the process of evolution of species, Hayek changes the ontological level of his analysis from the analysis of separate behavioral acts to the analysis of the culturally determined systems of behavioral principles. It is not people who are being dragged into the process of competition but rather cultures and social institutes. In such an approach there is nothing left of metaphor of mechanical equilibrium. It is the dynamical systems of different types that compete. Similar systems were considered in order to describe biological mechanisms of evolution on the molecular level where the chemically reacting flows of biological quantities are the dynamical systems and the metastable states win as a result of selection, see now widely know works [**E**]; similar problems were earlier discussed from different position in [**Wdd, Th, Kas**].

We cannot consider here biological theories in any detail but we may make one principal deduction: if Hayek is right in his opinion above the nature of market competition and evolution of social institutes then the mathematical metaphors used in order to construct models describing such processes should be cardinally modified. A natural language for description of such systems is the information theory and equilibrium and non-equilibrium statistical

[4]The study of such problems started, it seems, from the book by E. Schrödinger [**Sch**]. At the end of his life, J. von Neumann studied selforganization of automata, see [**JvN**].

[5]Life as a metastable state was discussed as early as in [**W**]. In relation with inevitability of eventual equilibrium of Maxwell's demon and the environment, Wiener wrote: "Nevertheless, before the Demon becomes muddled a considerable laps of time may pass and this period may turn out to be so prolonged that we have right to say that Demon's active phase is metastable. There are no reasons to believe that metastable Demons do not exist in reality, contrariwise, it is quite probable that enzymes are such metastable Maxwell's Demons that diminish entropy if not by separation the fast particle from the slow ones but by some other method. We can very well consider the live organisms, and the Man himself, in this light."

thermodynamics. Such modifications in the language of description should naturally completely modify the theory. The models of market equilibrium should completely change both the meaning and the apparatus.

The metaphors of mechanical equilibrium, dynamical system, feedback will hardly be applicable. It is interesting to investigate if the history of economic thought contain some attempts on alternative conceptualization of economic equilibrium in the early period of the development of economic theory when the neoclassical orthodoxy was not fossilized?

Indeed, such an attempt is known, it is due to David Hume. Even before the fundamental work of A. Smith, Hume [**Hu**] wrote several essays on economic problems. In one of them, named "On Market Balance", he suggested an interesting metaphor of an economic equilibrium. Discussing the problem of relation between the amount of currency in the country and the prices Hume writes ([**Hu**], pp. 112–113):

"Let us suggest, for example, that during one night the amount of currency in Great Britain will be multiplied five-fold... Will not this raise the prices on labor and goods until neither of the neighboring countries will be able to buy anything from us while contrariwise their goods will become comparatively cheap so much so that despite possible preventive laws they will saturate our market and our money will flow out of the country until we become equal with our neighbors in relation of money and will not lose this access well which placed us in such an unfavorable position. Water, wherever it penetrates, always stands at one level. Ask physicists the cause of this phenomenon and they will answer that if water raises at one place then the raising weight of the water in this place without being in equilibrium makes the water lower its level until the equilibrium is attained."

Apparently we have here a "mechanical" equilibrium metaphor. But essentially this is the same metaphor (a liquid in joint jars), which was the initial point for the construction of thermodynamic theory. It is approximately in this way that the temperature of touching bodies becomes equal.

The meaning of Hume's metaphor is not mechanical. He draws our attention to the equalizing of the value of the essential parameter in two systems coming into an interaction. The fact that the system comes to an equilibrium is not so important. What is important is the values of the essential parameter become equal, as soon as the systems come into a contact.

Let us draw a particular attention to Hume's remark concerning possible preventive measures which nevertheless cannot forbid the attaining of an equilibrium. Hume assumes that the means of attaining equilibrium (in this case the ways of foreign goods to infiltrate the country) are so numerous that it is not worth even discussing this question: what is important is that such means will always be found. This attitude cardinally distinguishes Hume's model of equilibrium from the model of equilibrium of A. Smith who explicitly indicates the mechanism for gaining the equilibrium.

Hume's idea is close to that of F. Hayek. It is exactly the same distinction that appears between the mechanical metaphor of equilibrium and the thermodynamic metaphor in which the ways of attaining equilibrium are not known and are not important. The only important thing is that the final state is the most probable one.

Hume's ideas on the economic equilibrium did not make an influence on the economic theory comparable with the influence of A. Smith's ideas. But we see that a certain germ of a new sense, namely pregnant with possibilities theoretical metaphor contained in Hume's essay can be in principle unfolded into a theory that uses completely different language and has a different ontology, in a theory which leads in certain cases to principally different conclusions than the neoclassical orthodoxy. Hayek's approach to the market as a system in which the means of actions are impossible to calculate and predict whereas the "equilibrium" or “order” are attained as the most probable state are essentially quite concordant with Hume’s metaphorical conceptualization. Let us try to make one more step and construct an economic theory that would systematically use the language of information theory and statistical thermodynamics.

For this, we first have to “destruct” the mechanical metaphor of equilibrium.

CHAPTER 4

Metaphors of equilibrium

The theory of economics is an example of a non-trivial case in which ontological assumptions that underly mathematical models are mathematical models themselves or, to say more precisely, are mathematical metaphors. A basic mathematical metaphor for the classical model of the market is that of a *mechanical equilibrium*.

The basic idea of such a model is that small deviations of the system from the point of equilibrium produce "forces" which try to return the system to the equilibrium state.[1]

In some, very important, sense "the invisible hand" of the market in this model is equivalent to a mechanical force. The economics is considered as a dynamical system. *Time* stands as a key notion, and the mathematical structure of the economic models is represented by a system of differential equations.

In the mechanical models, the equilibrium is considered as a state at which the forces applied to the system counterbalance each other and the potential energy attains its extremum.[2] Consequently, to apply the mechanical metaphor of equilibrium in the economics, some analogs of the mechanical notions are needed. Such a conceptualization is not "harmless" at all: it implies that the system, having slightly digressed from the state of equilibrium, will return to this very state being left alone. As far as the dynamics of the market is concerned, the neo-classical economics inherited the classical approach. Here lie the roots of ideas how to revitalize the economics by means of financial stabilization, the essence of monetarist approach to vitalization of economics. According to it, it suffices to release the prices while preserving the volume of money for the system to immediately come to an equilibrium.

The practice of "shock therapies" illustrates that this is not always the case. Still, practice may lead us astray. To unearth the reasons why the economic systems refuse to come to an equilibrium as predicted, we have to deeply analyze, first of all, the "mathematical metaphor" used. The question is **are the dynamical systems adequate and sufficient metaphors for description of the equilibrium of economic systems?**

In physics, it is well-known, there are other, distinct, approaches to the conceptualization of the intuitive notion of equilibrium. Our construction is based on the thermodynamic notion of equilibrium. According to this concept, the system gets in the state of equilibrium not because it is being affected by "forces", but simply because this is the most probable state of the system, consisting of numerous parts, each of which is characterized by its independent dynamics.

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[1]Clearly, economists have in mind the stable equilibrium. A nonstable equilibrium is no less interesting, especially after Kapitsa explained how to stabilize an inverted or even slanted stick [**BP**]. The recent results on cyclic nature of (stock) markets indicate that this remark is, perhaps, deeper than one might think. *D.L.*

[2]It is worth remembering, among other things, that the subject of the modern research in mathematical economics is, as a rule, the state of equilibrium in itself. The dynamics of the system is only considered as a metaphor, pointing out the way this equilibrium state may be attained, see [**Fi**] and the works of the classics: [**W**], [**D**], [**AD**]. In general, further evolution of the theory produced such a state of affairs, in which the system's dynamics in the vicinity of the equilibrium point was completely neglected by the researchers.

This approach may refer as well to mechanical systems obeying the laws of mechanics. But, if the system is very complex, its general behavior is determined by absolutely different principles, very unlike those of mechanics.

This distinction in the mathematical description of how the system changes its state is fundamental. In terms of thermodynamic approach to equilibrium, the system, instead of evolving in time, simply changes its position in the space of macroscopic parameters, remaining on certain surface, the *surface of state*, singled out by the "equation of state".

Time is not included into a set of parameters important for the description of the system's equilibrium. The equation of state is given by (linear) dependencies between the differentials of macroscopic parameters, in other words, by a system of *Pfaff equations*.[3] By changing one or several macro-parameters of the system we simply moves the system along the surface of state.

In essence, from the mathematical point of view, the investigation of equilibrium in such a model is a problem of differential topology of the surface, described by the equation of state [**B2**].

Such a metaphor of equilibrium essentially differs from the mechanical one. Time does not occur here as an internal parameter of the system, the parameter that determines its dynamics, but as an external one.

Dependencies between the differentials of changeable macro-parameters are determined by the internal structure of the system, for example, by the energy values of subsystems.

In the beginning of the XX century, C. Carathéodory proved [**Cy**] that it is possible to logically develop the thermodynamic theory, drawing, exclusively, on the assumption that "the equation of state", i.e., the surface in space of thermodynamic variables, corresponding to a system of Pfaff equations, exists. In this case, the second law of thermodynamics is formulated as existence, in an infinitesimal neighborhood of each state of the system, of such states that cannot be reached without the change of entropy.

In other words, a simple assumption that the differentials of "generalized positions" are constrained (and this constraint is nonintegrable, in Hertz's words, *nonholonomic*), appears to be sufficient to develop the system of thermodynamic equations. Accordingly, the idea of equilibrium will look completely differently. This thermodynamic equilibrium, unlike that from a mechanical metaphor, would mean not the existence of a singular point of a solution of a system of differential equations or an extremum of a potential function, but movement along the surface of state.

In what follows I will show that there are most serious grounds to believe that the Pfaff equations (and, consequently, thermodynamic metaphor of equilibrium) are often more adequate for the description of various economic phenomena than the mechanical metaphor.

Under some additional assumptions on the thermodynamic system described by the Pfaff equations the *Le Chatelieu principle* [**LL**] is applicable. Namely, the system demonstrates a behavior obstructing influences exercised on it in the result of changes of the external macro-parameters.

The systems in economics also demonstrate such a behavior under certain conditions. For example, the increase of prices can be an incentive for the production in order to support the consumption level; attempts to impose total control over levels of production or consumption trigger the process of corruption of executive bodies, which diminish the effect of such control, and so on.

[3]A *Pfaff equation* is a particular linear equation for vector fields X; namely, any equation of the type $\alpha(X) = 0$, where α is a differential 1-form. For a modern exposition of the theory of Pfaff equations, see [**BCG**]. For another interpretation of the term, see footnote on p. 8?? and [**BCG**].

An important discovery of the past was that the phenomena of such homeostasis in physics, sometimes appearing as almost reasonable behavior, can be explained on the basis of the thermodynamic metaphor, proceeding from the very simple assumption, namely, that

for the most time the system is staying in the most probable state.

In case of economics, it may look as if the system is directed, citing A. Smith, by an "invisible hand". At the time when A. Smith was working on his book, the principles of thermodynamics were not known yet, so the "invisible hand" was interpreted in terms at hand, those of a mechanical metaphor. A. Smith's conceptual model that identifies human interests with "forces" of the market gives really serious grounds for such an interpretation.

Let us consider a thought experiment of A. Smith a bit closer. Smith assumes [**Sm1**] that an increase of commodity prices brings about an increase in at least one of the price-making components — the rent, workers' salaries or profit — giving a signal to actors, or rather forcing them, to change their behavior (to exploit more facilities, increase the offer of jobs or to expand manufacturing), and this change of behavior finally leads to the reduction of price.

There is, however, one very weak spot in the "thought experiment" of A. Smith.

(1) *The only immediately accessible information for the market actors is the price.*

Whatever changes it — up or down — the reasons for this change are not immediately revealed to the observer. It is most often not clear to the buyer at all if the changes in the price were due to change-making factors (say, raise or fall of profit) or the reason was in increase of demand? Usually, such information is the seller's most guarded secret.

In the studies of researchers from the Austrian school we find serious arguments in favor of the hypothesis that

(2) *no complete information on the hidden components of the price is available for the market actors.*

This, consequently, means that they simply cannot behave in the way, described in the A. Smith's "thought experiment". In blunt terms, this "thought experiment" was based on false assumptions. This necessitates to put under doubt the mathematical metaphor underlying the classical concepts of the market dynamics, i.e., the mechanical metaphor.

In other words, the ontology of the classical model of economics is, indeed, not indisputable.

F. Hayek [**H**] despite of his vigorous diatribes of socialist ideas of regulation of economics and adherence to market principles, negated liberal viewpoints on the role of selfishness in market economy. Hayek directly asserts that the market economy is based on the observance of moral principles, ensuring survival of the community in competition with the other communities, and that these principles not infrequently directly contradict mercantile egoistic interests.

M. Weber, in his famous work about the role of protestant ethics in genesis of capitalist economy, also developed similar views. Weber shoed that the protestant ethics (the scrupulous honesty and workaholism in the frames of the capitalism formed in the Northern Europe) are necessary conditions for richness growth.

The interests of the members of society are being fulfilled **as a result** *of abiding the moral laws in the society as a whole.*

This directly contradicts the idea of A. Smith that *the common well-being is a resulting corollary of individuals attempting to fulfill their egoistic interests.*

In neo-classical models of the economic equilibrium constructed later, such, for example, as the ones due to Arrow-Debreu-McKenzie, in order to prove the existence of equilibrium, one assumes that the subjects of economic activity maximize their utility functions. Thus, the arguments on in-built incompleteness of the information available for the market actors were ignored.

This postulate — about unavoidable incompleteness of the information — seems to be one of the basic reasons why the economists of the Austrian school rejected applicability of mathematical methods in economics. It looks as though these scientists were not so much dissatisfied with the mathematics itself, but rather with the mechanical metaphor used to construct models of equilibrium — the metaphor in which, in the end, the utility played the same role as the potential in the classical dynamical systems.

The dissatisfaction with such models of equilibrium was, however, discernible not only on the part of the opponents of mathematical economics, but also among its champions. The latter were anxious with the absence, within the framework of the mechanical metaphor, of satisfactory stability theory of the market equilibrium. Most profound disappointments were connected with impossibility to consider, in all cases, the excess demand as continuous function of the price. Controversial examples are well known, see, for example, the work by B. Arthur [**AAP**].

One can, of course, try to "improve" the theory, remaining in the confines of the mechanical metaphor and liberal dogma. We believe, however, that although the arguments of the Austrian School of economics are insufficient to completely reject possible applicability of mathematics in economics, they suffice to be the reason to change the mathematical metaphor of the equilibrium.

F. Hayek's concept of market, as the process of discovery, emphasizes the key role of *information* in the market economy (as opposed to the priority of unobservable utility functions in neo-classical models, based on the mechanical metaphor of an equilibrium). In what follows we develop a model of market equilibrium proceeding from the information theory: Brillouin showed [**Bri**] that the mathematical information theory is, in essence, identical to thermodynamics, if we identify *information* with *entropy.*

The quantity of information, obtained at the instance of interaction of the subject with the system, is measured in the information theory by the logarithm of the *relative reduction of opportunities for choice enjoyed by the subject before the information had been obtained.* It is easy to understand that such an interpretation of the information theory directly links it to the behaviorial description of the market actors, thus making entropy the most important parameter of the market interactions.

This idea is not new. At the end of 1960s, A. Wilson suggested that the entropy analysis should be used for the examination of transport flows [**WE**]. Wilson mainly used the method of the maximization of entropy to tackle transportation problems.

In his three papers published in the Russian journal *Avtomatika i Telemekhanika* in 1973[4] (nos. 5, 6, 8), L. Rozonoer formulated very general principles of "resource dynamics". Unfortunately, I was unaware of these papers while writing this book and became acquainted with them only a year after the publication of the Russian version. The basic idea of Rozonoer was to suggest a theory that could unite on most general level the economic and thermodynamic theory. On a deeper level, his aim was to formulate a theory in which the exchange of resources and the idea of equilibrium can be expressed and to extract the most general

[4]Cover-to-cover translated starting 1978 as *Automat. Remote Control.*

consequences from such a theory in the manner of the Second Law of Thermodynamics and Le Chatelieu principle. For such a purpose, Rozonoer introduced two specific functions: the "function of effect" and a "structural function", the first being an analog of entropy, the second an analog of equilibrium entropy. The definitions for these functions were suggested on the most abstract level, and the state of equilibrium was defined as a condition when the structural function reaches its maxima for subsystems of a general system, other conditions being the additive property of resources, conservation principle and the possibility of measuring resources in terms of non-negative numbers. In all his three papers, Rozonoer argued that economics with traditional concept of utility and thermodynamics with classical definition of entropy (in Carathéodory style) are two basic examples of resource dynamics. Rozonoer assumed no specification for any concrete economic or physical system. Such an approach is reasonable because Rozonoer wanted to formulate a theory with maximum abstractness and the most wide range of application.

Rozonoer succeeded in formulating a number of rather general theorems, implementing the idea of "basic exchange resource" (an analog of gold or money in economics or energy in physical thermodynamics). Rozonoer showed an analogy between pressure divided by temperature and "value of resource" divided by a "basic relation of effect", Rozonoer's term for an analog of temperature in his theory. Rozonoer obtained an impressive long list of analogies between economic theory and thermodynamics (Avtomatika i telemekhanika, 1973, no. 6, pp. 75-76).

What is interesting for our subject is to try understand why in an excellent series of papers devoted to "resource dynamics" (Rozonoer's term) Rozonoer did not suggest any methods to analyze concrete economic systems along his lines of thought. It is interesting to wonder why there was no response to his ideas from the economics community in general.

To my mind, Rozonoer considered similarities between economics and thermodynamics not as direct analogies allowing one to construct a new theory of economic equilibrium, but as a "metaphor" or, even less stringently, as an analogy, because he was not ready to adopt the idea that a "narrow", statistical definition of entropy could be used instead of the vague concept of "utility". As a result his *resource dynamics* was too "abstract", and this made it practically useless for the description of economic systems. Rozonoer incorporated into his theory only the theorems which basically were known by scholars in economics, and could be formulated within the language elaborated on the basis of General Equilibrium Theory, possibly in a more elegant if not more complicated way.

What was really important — insisting that *utility* must be considered as entropy in models formulated using a specific conceptual language — was not, unfortunately, accomplished. Using the language of *resource dynamics* it was possible to make abstract statements about economic systems, but it was difficult to derive an equation of state for a given economic system, or to analyze for such a given system the role of institutional constraints and these tasks were not performed.

The use of entropy approach is also well-known in the economic studies in relation to the estimation of uncomplete data on the basis of the ideas by E. T. Jaynes [**J**]. For details, see [**Le**], [**GJM**]. These works do not include, however, analysis of economic equilibrium. On the precedents of application of other methods from statistical physics for socio-economic studies, see, for instance, [**Dur1, Dur2**].

Nevertheless, until now, the idea to apply the notion of entropy to the study of economics was only realized for particular purposes (mainly, in relation to the transportation problems, and was not used with the aim to build up the whole theory of economic equilibrium).

The entropy alone is not sufficient to create the thermodynamic theory of economic equilibrium. To this end, we need the whole spectrum of thermodynamic variables, such as temperature, pressure, chemical potential, free energy, and so on.

The basic idea of thermodynamic approach to the analysis of economic equilibrium is as follows. If the system is described on two ontological levels — a "macroscopic" and a "microscopic" ones — and one macroscopic state is characterized by a multitude of microscopic states (their number is called the *statistical weight* of the macroscopic state), and the system will, generally, remain in the most probable state, i.e., in the state with the greatest statistical weight.

The conditions of applicability of the thermodynamic approach can be formulated in very general terms. It becomes clear thereby, that the applicability of thermodynamics goes beyond the realm of physical systems.

One can imagine The Large System that can be decomposed, or rather consists of, a huge number of Small Systems, each with its independent dynamics. We assume that the state of the Large System, and its relatively large parts, is described by a certain number of macro-parameters, which are additive, i.e., *the total values of the macro-parameters appear to be the sums of values of the same macro-parameters of parts.*

In physics, *energy* is an example of such a parameter if we neglect superficial interaction between the parts of the Large System. This is a basis for application of thermodynamic methods in physics. Let further each of the systems considered be additionally characterized by a set of micro-parameters which can have distinct values at the same value of the fixed macro-parameter. Their values are determined by the system's dynamics and, generally, can be of interest in relation with our task in one aspect only. Namely, having fixed them, an exact state of the system becomes known and one can answer, how many various micro-states correspond to one macro-state of the system.

At this stage, we can introduce the notion of statistical weight, as a number of micro-states corresponding to one macro-state, and the notion of entropy, as the measure of uncertainty of the macro-state of the system, which is a function of the number of micro-states.

If the system is such that the micro-states of the parts of the Large System are statistically independent, it is possible to compute the statistical weight of the Large System as a whole, given the statistical weights of all Small systems. For that purpose, it suffices to multiply the statistical weights of Small Parts.

If we wish the entropy be additive, we may regard it as the logarithm of the statistical weight. Since $\ln(ab) = \ln a + \ln b$, it follows that the uncertainty, or entropy, is additive.

Assuming that nothing is known about the dynamics of the system, except that it is very complex, the natural assumption for the probability value of any macro-parameter is that it is proportional to the number of the appropriate micro-states, i.e., to the statistical weight. Since the logarithm is a monotonous function, the most probable state, i.e., the state with the greatest statistical weight, is, at the same time, the state with the greatest entropy, i.e., with the greatest extent of uncertainty.

This is the core of the *second law of thermodynamics — the entropy tends to increase* — as every part of the Large System goes in the most probable state during interactions with the other parts.

If we "isolate" some part of the Large System in order to observe the distribution of the probable states, we have to consider the rest of the Large System as a *thermostat*, that is a reservoir that ensures equilibrium (in a thermodynamic sense) of the distribution of states within the subsystem isolated.

To find out the form of this distribution, it is necessary to introduce the notion of "temperature", equal to the inverse of the derivative of entropy of the Large System with respect to the macro-parameter for which there exist a conservation law.

The introduction of the parameter of equilibrium — temperature — is simply a result of the condition that *there are no flows of the conserved macro-parameter between the parts of the system.* If there are several conserved macro-parameters, then there are as many parameters of equilibrium as there are conservation laws.

Observe that we said nothing related to either physics, or physical laws and observables. All the arguments are applicable to Large Systems of any nature, subject to the above hypotheses. These arguments look rather natural, in relation to the large economic systems, if we regard the total income, the total value of products or the total value of consumption of goods as macro-parameters, whereas distribution of income and products or consumption of goods between the subjects of economic activity are viewed as micro-parameters.

In this approach, to the study of economic equilibrium, we avoid the necessity to explore the subsystem's dynamics, so the knowledge of institutional restrictions on the goods production and distribution suffices. Thus,

> *we offer not only a new approach, appropriate to describe the economic equilibrium, but, also, the instruments to investigate the impact of institutional restrictions on the state of equilibrium.*

Namely, we get an opportunity to build up a mathematical apparatus for analysis of the transaction costs theory, to suggest quantitative methods of the study of the impact of the informational asymmetry on the behavior of the market's agents and to tackle many other problems as well.

Observe that, for more than a century, the endeavors to prove thermodynamic predictions within the framework of theoretical physics by analyzing equations of motion were not successful. Born [**B1**] used to remark a radical distinction of the methods and mathematical techniques of thermodynamics from those of other branches of theoretical physics: "In the classical physics the logical processing of a branch of science is considered finalized when it is reduced to one of the chapters of the "normal mathematics". There is one astounding exception — the thermodynamics. The methods usually applied in this discipline to deduce the main postulates markedly differ from those accepted in other domains of physics."

Nobody doubted that the principles of thermodynamics work regardless of possible reductionist interpretations. The two levels of description pose the problem: how to single out certain parameters, perhaps, completely "inconspicuous" or not obvious, that govern the conditions for equilibrium, intuitively understood as the absence of significant flows between the parts of the system.

If the macro-parameters are functionally dependent, and the surface of state is differentiable, then the differentials of the macro-parameters produce are related by a system of Pfaff equations.

Thus, the idea of thermodynamic equilibrium is quite appropriate for description of economic systems. They have observable flows of money, goods and people, and, like physical systems, have two levels of description.

Consequently, the description of the economic system should be equivalent to the description of physical systems in thermodynamics, though the parameters of equilibrium — "temperature", "pressure", "chemical potential" — will certainly have quite different interpretations, the ones that mirror the peculiarities of economic systems.

Observe that thermodynamic terms had often been already used, by folk, not scientists, in relation to the economic systems in a "naive" manner, as metaphors of description: the stock exchange is "overheated", the national economic problems "boiled over" or "cooled down", stock exchange indices are associated with temperature degrees of a thermometer, and so on.

One of the aims of this work is, besides all, to show that, not rarely, there is more sense in the "naive" metaphors of such sort, than in the complex mathematical models based on mechanical metaphor of equilibrium. Proceeding from the thermodynamic metaphor, a thermodynamic theory of economics can be developed. It not only catches hold on certain realities of markets by no means less than the one built on the mechanical metaphor, but also accounts for the role of institutional restrictions for the establishment of economic equilibrium — the task unfeasible to the theories based on the mechanical metaphor.

CHAPTER 5

Entropy and temperature in models of economics

§1. Entropy and temperature

We begin our description of a thermodynamic model of economics with the simplest example. Let an economic system consist of N agents, among whom the income, constant for the system as a whole, is distributed. We assume that there are many ways to distribute income, and we are incapable to foresee all possible alternatives. This assumption fully corresponds to Hayek's concept of market in which the market is described as an arena of discoveries of new procedures and operations. For ultimate simplicity, we assume that the income is quantum, i.e., presented in integers (which is natural, as the smallest unit of currency is operational in economics).

Now, for each value of the total income E it is possible to find the quantity of modes of income distribution between the agents, as a characteristic $n(E, N)$ of this value, called the *statistical weight of the state* with income E.

At this stage we can introduce the concept of equilibrium. The idea is that two systems (under the above hypothesis) are in *equilibrium*, if the distribution function of income does not change when they enter in a contact, hence, there is no income flow between the systems. By a "contact" we understand here an "open list" of possible modes of redistribution. It is remarkable that it is possible to calculate the statistical weight regardless of uncountable variety of various institutional limitations imposed on the agents' incomes, so functions $n(E, N)$ may be different for different systems.

The given model is rather simple: at this phase of reasoning we do not really turn our face to the market. The restrictions on income may be sustained coercively, but the actual means are irrelevant for our investigation.

If two systems interact, one with the total income E_1 and the number of agents N_1, and another one, with E_2 and N_2, respectively, then the total system is characterized by the total income $E_1 + E_2$ and the number of agents equal to $N_1 + N_2$. How to find conditions necessary or sufficient for the equilibrium, i.e., for the state with no income flow between the systems?

In order to build up an appropriate theory, we have to adopt one more, extremely important, hypothesis on the nature of the systems under investigation. Namely, we assume that *all elementary states of income distribution have the same probability.*

The main ground for such an assumption is symmetry of states. As in the probability theory and in statistics, we assume equal probability of elementary events just because there are no grounds to prefer one event to another. Thus, it is of utmost importance for the theory, to include all probable states of distribution. The change of function $n(E, N)$ will, certainly, change the results obtained in this model.

In statistical physics, the literature devoted to justification the principle on equal probability of elementary states is uncountable. Still, for the majority of models used,

> *this principle remains a subject of faith, the principle that nobody was able to prove*

and we rely on it because it is justified by the remarkable effectiveness of the statistical theory and brilliant agreement of the theory with the practice.

To find the state of equilibrium, is to determine conditions under which the income is not to be redistributed between the interacting systems. For this purpose, consider a redistribution of income during an interaction. Let a certain part of income, ΔE, pass from system 1 to system 2. Then, the states of the systems change and their statistical weights become equal to $n_1(E_1 - \Delta E, N_1)$ and $n_2(E_2 + \Delta E, N_2)$, respectively.

The principle of equal probability implies that the most probable state of the integrated system is the one with the greatest statistical weight. So we should seek the maximum of function $n_{tot}(E_1, E_2, N_1, N_2)$, with the proviso that the total income $E_1 + E_2$ is a constant. If no transfer of agents from one system to another is possible, then the statistical weight of the integrated system is equal to:

$$n_{tot}(E_1, E_2, N_1, N_2) = n_1(E_1, N_1)n_2(E_2, N_2). \tag{3}$$

Since $E_1 + E_2 = \text{const}$, it follows that $\Delta E_1 = -\Delta E_2$.

Instead of seeking the maximum of n_{tot}, we can seek the maximum of $\ln n_{tot}$, because the logarithm is a monotonous function. ¿From $\ln n_{tot} = \ln n_1 + \ln n_2$ we derive the condition on the maximum. It is very simple:

$$\frac{\partial \ln n_1(E_1, N_1)}{\partial E_1} = -\frac{\partial \ln n_2(E - E_1, N_2)}{\partial E_2} \tag{4}$$

or, as $dE_1 = -dE_2$, we have

$$\frac{\partial \ln n_1(E_1, N_1)}{\partial E_1} = \frac{\partial \ln n_2(E_2, N_2)}{\partial E_2} \tag{5}$$

So, two systems are in equilibrium, if they are characterized by the same value of parameter $\frac{\partial \ln n(E,N)}{\partial E}$.

In thermodynamics, the logarithm of the statistical weightis called the *entropy* (of the system), and its derivative on energy is the inverse temperature,

$$\frac{\partial \ln n(E, N)}{\partial E} = \frac{1}{T}. \tag{6}$$

In order to reach the state of equilibrium, the interacting systems should be at the same temperature.

The italicized statement above, together with its deduction, can be found in any textbook on statistical thermodynamics [**Ki**]. Let us analyze here the adequacy of such an approach to the economic systems, at least, under the above assumptions. The economic system is in the state of equilibrium if it is rather homogeneous and there are no income flows from one of its part to another. We suppose, certainly, that the homogeneity is kept only unless there is no division into parts so tiny that significant income flows are observed.

The same postulates are available in statistical physics. Subdivision of the system into too small parts results in significant fluctuations. In physics, the question of an equilibrium of small parts of the system is solved by assuming (and this assumption is equivalent to a postulate of equal probability of elementary states), that if we observe a small part of system for "sufficiently long" time, then we will be able to adequately describe the distribution of probabilities of its state. It is one of the formulations of the so-called *ergodic hypothesis*, see [**To**].

A similar hypothesis can be made for the economic systems. We see that in our thermodynamic model, there are two extremely important characteristics — the entropy and temperature. If we do not know these parameters, we can not correctly infer the conditions for the system's state of equilibrium: as the system is in the state of equilibrium only when its subsystems have an identical temperature, and the temperature cannot be calculated without knowledge of entropy.

In physics, to measure temperature, one uses thermometers. These are special devices whose equations of state are known and calibrated; so by introducing the thermometer into a contact with a body we may find the temperature of the body by a change of the state of the thermometer. As we will see[1], (stock) markets may, to an extent, be considered as thermometers in economics.

The basic possible objection against the thermodynamic approach to economy is that the number of "particles" involved (in the given example — the number of the market agents) is much less as compared with numbers of particles in the usually considered physical systems. In the physical systems, the number of particles is comparable, as a rule, with the Avogadro number, whereas in the economic systems it is usually is $\sim 10^3 - 10^8$.

In statistics, the order of dispersion is equal to $\frac{1}{\sqrt{N}}$, where N is the number of particles in the system. Thus, in physical problems, if the statistical errors related with the fact that the number of particles is finite are completely insignificant due to the largeness of their number, in economic problems we should expect much larger errors, ~ 3 %, or less. Such errors do not look too large ones, actually, taking into account the extreme roughness of economic models.

One of the tendencies in contemporary physics is to apply thermodynamic approach to systems with a rather small number of particles $\sim 10^3 - 10^8$ (nuclear physics, cluster physics, and so on[2].), and the results appear to be quite valid not only qualitatively, but also quantitatively.

It seems that both models of interaction and the data on interactions are far from being accurate, and to strive for a better accuracy is meaningless: there are in-built limits of accuracy, like the uncertainty principle. The situation with applicability of thermodynamic approach to economics seems to be similar.

An important remark. The statistical models in physics show that the energy of the system has a peculiar quality responsible for a success of the thermodynamic theory: if the system has some parameter of inhomogeneity, then, provided there are sufficiently many particles, the system has a very sharp maximum of entropy attained in the limit as the parameter of inhomogeneity tends to zero. Thus, not only the maximally homogeneous state is most probable, but even small deviations from it are *most improbable.*

To illustrate this thesis, let us analyze the entropy of one very simple system. Suppose that the income distribution between the subjects of economic activity is arranged as follows: each subject has either zero income, or a fixed income, A assuming that the system is organized in such a way that all deviations are annihilated through special institutional redistribution mechanism.

Such an example is not so much unreal, if we recall the economic experience of some countries aspiring to implement various "levelling principles" in distribution of income, stipulating that a part of population is totally excluded from economic activities, being allowed to have a very low level of income (subsidized by a social security or by Nature). If the total number of economic subjects is N, the number of the subjects with income A is L (hence, the size of the total income is $E_{tot} = LA$), then, cleanly, number of probable states of this system would be equal to $\binom{N}{L}$.

[1] See the book [**SLR**].

[2] This idea was suggested by J. Frenkel, see [**F**]

By comparing the statistical weights of the systems with different levels of the total income, we conclude that the biggest statistical weight is the characteristic of the state with $E = LA$, where $L = \dfrac{N}{2}$ (to simplify arguments, let N be even). Introducing a parameter of inhomogeneity, $m = \left|L - \dfrac{N}{2}\right|$, we apply the Stirling formula for the factorial. We obtain the following simple approximation to the dependence of the statistical weight on the parameter of inhomogeneity:

$$n(N, m) \simeq 2^N \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{2m^2}{N}\right). \tag{7}$$

Formula (7) shows that the statistical weight (and, consequently, the entropy) has an extremely sharp maximum depending on the parameter of inhomogeneity, with width $\frac{1}{\sqrt{N}}$.

The principle of equal probability of elementary states immediately implies that the entropy of interacting systems tends to increase. The most probable state is the state with the greatest statistical weight, i.e., the state with maximum entropy. As the number of probable system's states sharply decreases with the increase of the parameter of inhomogeneity, it is hardly possible to find the system in a state for which the parameter of inhomogeneity exceeds a certain value determined by the number of particles in the system (in the example above, this value is $\frac{1}{\sqrt{N}}$).

§2. Thermostat and function of income distribution

Now we will investigate how the income is distributed among the subjects of economic activity. Within the framework of thermodynamic model, it appears that there exists a universal function of distribution, whose configuration, if number of the subjects is a constant, depends only on the temperature. This situation is well known in statistical thermodynamics, where such distribution is called as the *Boltzmann distribution*.

To examine this distribution, imagine that the economic system X is very large, and, out of it, a small part, Y, is separated. Naturally, the system X is considered as a reservoir. If we consider a certain state of the small subsystem with income E_1, the probability for the small subsystem to have income E_1 is proportional to the number of possible states of the reservoir with the range of income $E - E_1$, under the hypothesis that the total income of the system is a constant. Then the ratio of probability $P(E_1)$ for the subsystem to have income E_1, to probability to have income E_2, is equal to the corresponding probability ratio for the reservoir:

$$\frac{P(E_1)}{P(E_2)} = \frac{n(E - E_1, N)}{n(E - E_2, N)}. \tag{8}$$

Since $n = e^{S(E,N)}$, where $S(E, N)$ denotes the entropy of the system, we can express (3.1) as:

$$\frac{P(E_1)}{P(E_2)} = e^{S(E-E_1,N)-S(E-E_2,N)} = e^{\Delta S}. \tag{9}$$

If the reservoir is far larger than the subsystem in question, we can make expansion of ΔS into Taylor series in ΔE, thus obtaining in the first order $\Delta S \simeq \frac{E_2 - E_1}{T}$, where T is the temperature of the reservoir, see (6)

It follows that

$$\frac{P(E_1)}{P(E_2)} = e^{-\frac{E_1 - E_2}{T}}. \tag{10}$$

This is an approximate formula for the probability distribution for a small system.

If the number of probable states of subsystem Y with incomes E_1 and E_2 is equal to $n_Y(E_1)$ and $n_Y(E_2)$, respectively, then the distribution formula takes the form

$$\frac{W_Y(E_1)}{W_Y(E_2)} = \frac{n_Y(E_1)}{n_Y(E_2)} e^{-\frac{E_1 - E_2}{T}}. \tag{11}$$

In this formula we proceed from the probability of the state, P, to the probability of the level of income, W. Observe that no assumptions whatsoever were made, except the two: the system is homogeneous, so it is possible to subdivide it into interacting parts without producing income flows from one part to another, and the total system's income is a constant.

The arguments laid out above, are, actually, a replica of the traditional deduction of Boltzmann distribution in statistical thermodynamics.

Hereby we got something more than just “thought experiment” for the verification of the logic of reasoning. When the appropriate data are at hand, it is possible to verify this thesis with real economies, comparing, for example, the income distribution function of various sectors of economics.

The Boltzmann distribution function is realizable, with respect to the income, in any system which interacts with the reservoir kept at certain temperature. In the isolated system, on the contrary, the temperature depends on both income and entropy.

The Boltzmann distribution of income allows one to understand the relationship between the average income of the subjects inside the system, and the system's temperature. Suppose that there are no restrictions on the agent's income and all levels of income are allowed. Such a situation is commonly associated with the market economy. Then, with the help of the Boltzmann distribution function, we easily calculate the average income of the agent:

$$\bar{E} = \frac{\int_0^\infty E e^{-E/T} dE}{\int_0^\infty e^{-E/T} dE} = T. \tag{12}$$

We see that in absence of any restrictions on the agent's income, the mean income is equal to the temperature. The upper limit of integration here is equal to ∞, despite the fact that even the total income of the real system is bounded. However, since the exponent steeply decreases, this does not matter, provided the total income greatly surpasses the average income of a single agent; in the real systems this is true, of course.

As we will see, under restrictions on income the relationship between the temperature and the average income may look quite different.

In conclusion of this section, observe that, like in statistical thermodynamics, in the thermodynamic model of economics, the parameter called “statistical sum”,

$$Z_0 = \sum_E n(E, N) e^{-E/T} \tag{13}$$

is extremely useful. The sum here runs over all possible values of E. The probability of the system interacting with thermostat with temperature T can be expressed as

$$p(E) = \frac{e^{-E/T}}{Z_0}. \tag{14}$$

With a fixed number of agents, the average income in the system at temperature T is given by the formula:

$$\bar{E} = \frac{\sum\limits_{E} En(E,N)e^{-E/T}}{Z_0} = T^2 \frac{\partial \ln Z}{\partial T}. \tag{15}$$

This means that if we know how the statistical sum depends on the temperature, it is possible to obtain the value of the average income by differentiation. Again, this is fully agreeable with the standard technique of statistical thermodynamics.

Consider now one paradoxical example. It demonstrates that application of the thermodynamic approach to economic systems helps to deduce unexpected, though true, conclusions.

§3. On interaction of systems with and without restrictions on income

In this section the "spin model" with two possible values of income, 0 and A, already addressed above, will be examined in more detail. Despite its seemingly too abstract nature, this model is rather useful, as it catches some important features of systems with restrictions on income.

What the results would be once such a spin system enters into interaction with a "free" market system, the one without any restrictions on income? In the free market system, the entropy increases together with the increase of energy, and the temperature is always positive. This is not always the case for the system with restrictions on income. Actually, if the number of agents L with non-zero income surpasses the half of a total number of agents, N, the number $\binom{N}{L}$ of probable states of the system, starts to diminish, and the inverse temperature given by formula (6) becomes negative.

What is the meaning of the *negative temperature*?

This phenomenon was investigated quite well in the laser theory [**Kl**]. The negative temperature implies the existence of "inverted population", i.e., a situation, in which the levels with higher energy are more densely populated than the ones with lower energy.[3] In such a state, the system is imminently ready to release its energy at the contact with any system with positive temperature.

How can this be interpreted in terms of the income distribution described above? If two systems with positive temperature are in contact, the redistribution of income goes from the system with higher temperature to the one with the lower one. As we saw, for the free market, the temperature is equal to the average income per agent. The redistribution of income from the "richer" system to the "poorer" one takes place in accordance with our intuitive understanding.

Contrariwise, if a free market system interacts with a system with restrictions on income, then, under certain conditions, a counter-intuitive process is observed, when redistribution the income goes from the "poorer" system to the "richer" one.

Let X be a free market system, with the temperature T_X equal to the mean income per market agent. Let Y be a spin model of a market system with income bounded from above by A and with the total number of its agents N. Let $T_X > NA$ and let the number of Y's subjects with non-zero income exceed the half of their total number, so the temperature of Y is, evidently, negative.

[3]This means that the system with negative temperature is "overheated". Such a system is ready to throw out a portion of its energy at any contact with a "normal" system in equilibrium.

What is going to happen when these systems start to interact? The entropy of the joint system will increase, and the entropy of system Y **will also increase** simultaneously, thereby precipitating disorder of system Y. Therefore, Y **must** transfer a part of its income to X despite the fact that average income of the system X is already higher than the average income in Y.

As a result, system Y, with restrictions on income, will get still poorer, while system X, without any restrictions on income, will get richer and richer.

This redistribution of income will not be an effect of coercion or looting, but just a consequence of the fact that the combined system tends to acquire the most probable state. As for coercion, it may play a certain role, but not in the form of X raping Y, but coercion inside system Y, aimed at obstructing any rise of income above certain level.

Applying the above most simple model to real economic situations, we can draw two important conclusions:

1) the policy of restriction of income is dangerous if the contact with a free market surroundings is unavoidable;

2) if such a policy was already embarked on, and the economy in which it was implemented is isolated from the free market, then the effect of the "market reforms" will depend on the sequence of two main steps, the release of incomes and the establishment of contact with the free market surroundings.

For the country with a "non-market" economy, it would be necessary first to release the income from restrictions and then to "open" the economy to the market only **after the equilibrium had been attained**. Otherwise, the resources of the country will be "sucked out" outward before the equilibrium is established.

Of course, our arguments are based on a very simple idealized model. Still, our reasoning gives an explanation of different results of market reforms observed in the Eastern Europe, on the one hand, and in China and Vietnam, on the other.

In the Eastern Europe, the reforms were conducted by means of a "shock therapy": the national economies were "opened up" for outside activities without preliminary creation of market institutions inside the country.

The results were catastrophic (flight of capital and collapse of production).

In China and Vietnam the reverse order was implemented: first, the income was released from restrictions inside the country and then the market was gradually opened up for outsiders, in accordance, more or less, with the process of creation of a domestic market. Such a policy resulted in an astoundingly rapid economic growth.

Consider another case when a system with a restriction on income interacts with a free market system.

Let the "spin system", a community of N workers, with fixed individual income A, have L vacancies. Obviously, for A fixed, the system's temperature is negative for $L > \frac{N}{2}$.

This means that

> **in the state of equilibrium, the free market would prefer that half of the workers should not get salary**,

i.e., should be unemployed, because this is a far more probable state than the other ones with a lower rate of employment. So, this model includes unemployment as an inherent characteristic of the system in the state of equilibrium.

For the neo-classical school of economics, the following famous paradox stood as a stumbling block:

why the market equilibrium is not realized when the free labor force market is in operation[a]*?*

[a]The problem of unemployment is of utmost importance for the discourse on economic equilibrium, as it represents by itself a counter-example for neo-classical theory of equilibrium, stating that no excess demand is possible. This served as one of greatest incentives for the emergence of institutional approach to the economy. See [**Wi**], [**Af2**].

In the above model this paradox is dismissed.

A negative temperature may also emerge in the systems with restrictions on the salary's *from above*. Indeed, to increase the total value of wages in the situation of a salary-restricted economy with the value of salary bounded from above by A, *means* that as the total value of wages E equals NA, the entropy vanishes. This implies that, at least for some interval $E_0 < E < NA$, the temperature, $T = \frac{1}{\frac{\partial S}{\partial E}}$, is negative, i.e., at any contact with a free market the value of total wages will fall, at least to E_0.

If, moreover, there is a *minimal wage value*, B, the number of gainfully employed will be bounded from above:

$$L < \frac{E_0}{B}. \tag{16}$$

This restriction will keep salary from dropping down, and necessitate a certain rate of unemployment, which will be $\geq \alpha = \frac{N-L}{N}$. In reality, the level of wages is always limited "from below" by the level of biological survival (at least, in towns, where Nature's resources are unattainable). Therefore, if salary is bounded, the opportunity for unemployment is created.

The analysis given above shows that **unemployment emerges in the state of equilibrium** at the positive temperature if the wages are bounded simultaneously from above and below.

The above can be considered as a proof of Keynesian arguments on the reasons of unemployment. In real situations, the administration aspires to put a limit on the salary "from above", while the trade unions (are supposed to) do it "from below". This combination **must**, according to our model, result in the emergence of unemployment.

§4. Migration potential

In the previous sections we showed some applications of the thermodynamic approach to economics and interpreted in terms of economics several thermodynamic parameters.

One more very important parameter is known in statistical physics under the name the *chemical potential*. Here it will be referred to as *migration potential*. It describes the state of equilibrium in the systems when the number of agents is not fixed.

Take two such systems, assuming that the total number of their agents is a constant, $N_1 + N_2 = N$, but the agents can migrate from one system to another. Consider the problem: *under what conditions these two systems will get into an equilibrium?*

Having applied the standard technique, it is possible to find out when the entropy of two interacting systems, characterized by restrictions on the total amount of income and the total number of agents, will be maximal. The total entropy is the sum of entropies of the two systems. As a result, we get two conditions (as a corollary of $dN_1 = -dN_2$ and $dE_1 = -dE_2$ implied by $dN = 0$ and $dE = 0$):

$$\frac{\partial S_1(N_1, E_1)}{\partial N_1}\Big|_{E_1} = \frac{\partial S_2(N_2, E_2)}{\partial N_2}\Big|_{E_2}; \tag{17}$$

$$\frac{\partial S_1(N_1, E_1)}{\partial E_1}\Big|_{N_1} = \frac{\partial S_2(N_2, E_2)}{\partial E_2}\Big|_{N_2}. \tag{18}$$

The factor $\mu = -T\frac{\partial S(N,E)}{\partial N}\Big|_E$ will be called the *migration potential.* When the temperatures are equal, the two systems in diffusion contact, i.e., when migration of agents from one system to another is allowed, are in equilibrium (i.e., in the most probable state), if their migration potentials are equal.

As δN_1 agents from the second system migrate to the first one, the change of entropy will be as follows

$$\delta S = \delta S_1 + \delta S_2 = \frac{\partial S_1}{\partial N_1}\Big|_{\delta N_1} + \frac{\partial S_2}{\partial N_2}\Big|_{\delta N_2} = \left(\frac{\mu_2}{T} - \frac{\mu_1}{T}\right)\delta N_1 \tag{19}$$

(since $\delta N_2 = -\delta N_1$ and $\delta N_1 > 0$).

This makes clear what happens with the systems, if their migration potentials are not equal. If $\mu_2 < \mu_1$, the change of entropy with the increase of the number of agents in the first system is positive, hence, it is a change of state towards the maximal probability.

In other words, the agent flow proceeds from the system with a larger migration potential to the system with a smaller migration potential.

Now, we can find the relative probability of various states for the system interacting with a thermostat and exchanging with it either income, or agents, or both. In the same way as before, we obtain

$$\frac{P_Y(E_1, N_1)}{P_Y(E_2, N_2)} = \exp\left(S(E_1 - E_2, N - N_1) - S(E - E_2, N - N_2)\right), \tag{20}$$

where (E_1, N_1) and (E_2, N_2) are different states of subsystem Y, interacting with the thermostat, while E and N are the total income and the total number of agents, respectively, in the total system (sub-system Y plus the thermostat).

On expansion into Taylor series and simplifications, we obtain:

$$\frac{P_Y(E_1, N_1)}{P_Y(E_2, N_2)} = \frac{\exp\left(\dfrac{N_1\mu - E_1}{T}\right)}{\exp\left(\dfrac{N_2\mu - E_2}{T}\right)} \tag{21}$$

This means that the probability for subsystem Y to exchange income and agents with thermostat is proportional to the Gibbs's factor

$$\exp\left(\frac{N\mu - E}{T}\right). \tag{22}$$

As it was mentioned earlier, the thermostat is just the remaining part of the large system, out of which the subsystem Y was isolated. As other thermodynamic parameters, we obtained the Gibbs's factor in a way usual for statistical physics.

A question arises: *are the hypotheses on the total system's income and the number of agents too binding and non-realistic?*

It looks that hypotheses of this kind are always needed in the study of the properties of ideal systems. We observe very similar problems in statistical physics as well. There, certainly, no absolutely closed system can be found, still, nevertheless, the theory "works".

In real economic systems, the number of agents is, of course, much smaller than that in physical systems, but, on the other hand, the expected accuracy of prediction is not so high, either.

The Gibbs's factor paves the way for introducing a very useful parameter, called in statistical thermodynamics the *large statistical sum*:

$$Z = \sum_{N=0}^{\infty} \sum_{k} \exp\left(\frac{N\mu - E_k(N)}{T}\right). \tag{23}$$

The large statistical sum allows one to compute, effortlessly, some important parameters of the system. For instance, the mean number of the system's agents can be found by differentiation with respect to Z.

Set $\lambda = e^{\mu/T}$. Then the large statistical sum takes the form:

$$Z = \sum_{N} \sum_{k} \lambda^N e^{-\frac{E_k}{T}}. \tag{24}$$

Now it is easy to deduce (see [**LL**]) that the mean number of agents is equal to

$$\bar{N} = \lambda \frac{\partial}{\partial \lambda} \ln Z. \tag{25}$$

This relation is very important. It provides with a way to determine λ in the investigated systems by equating the number of agents in the system to $\langle N \rangle$.

Here is an example from the economics of migration, aimed to demonstrate the usage of the migration potential. Consider two systems: A, with N_A vacancies, and B, with N_B vacancies, with the total of n agents capable to freely migrate between the two systems. Let the per capita income in A be E_A, that in B be E_B. Besides, we assume that both systems are immersed in a much greater system, a thermostat with temperature T.

How do probabilities of filling vacancies in systems A and B depend on the parameters of the model? The equilibrium, considered here as the most probable state of the joint system, will be determined by the system's temperature and migration potential. Using the large statistical sum, one easily obtains the function of distribution of agents for A and B, see [**Ki**].

Let one vacancy in system A be isolated out as a subsystem. Assuming that this subsystem is in equilibrium with the remaining part of the system, it is possible to determine the large statistical sum of the subsystem. As this subsystem can only be in one of the two possible states, with income 0 (the empty state) and with income E_A, the large statistical sum is equal to:

$$Z_A = 1 + \lambda e^{-\frac{E_A}{T}}, \text{ where } \lambda = e^{\mu/T}. \tag{26}$$

Similarly, for system B:

$$Z_B = 1 + \lambda e^{-\frac{E_B}{T}} \tag{27}$$

As systems A and B are supposed to be in equilibrium, and, therefore, their migration potentials are equal and λ is the same for both systems. Hence, the probability of one vacancy in system A to be occupied is equal to:

$$\varphi_A = \frac{\lambda e^{-\frac{E_A}{T}}}{1 + \lambda e^{-\frac{E_A}{T}}}. \tag{28}$$

Similarly, for system B:

$$\varphi_B = \frac{\lambda e^{-\frac{E_B}{T}}}{1 + \lambda e^{-\frac{E_B}{T}}}. \tag{29}$$

So we have:

$$\frac{\varphi_A}{\varphi_B} = \frac{\lambda + e^{\frac{E_B}{T}}}{\lambda + e^{-\frac{E_A}{T}}}. \tag{30}$$

Since the total number of the system's agents is equal to n, we have

$$n = N_A\varphi_A + N_B\varphi_B. \tag{31}$$

We determine the migration potential from this equation; we substitute it into the expressions for φ_A and φ_B thus finding the probabilities for agents to belong to A and B, respectively, in the equilibrium state. It is clear that as the temperature of the thermostat changes, the migration potential also changes, because the relative probability for the joint system $A+B$ to have total income E depends on the Boltzmann's factor $e^{-\frac{E_A}{T}}$.

Now, consider the case when the numbers of vacancies in the systems in question are equal and both of them are almost totally occupied, i.e., both φ_A and φ_B are close to 1. This means that we can expand φ_A and φ_B with respect to $\lambda^{-1}e^{-\frac{E_A}{T}}$ and $\lambda^{-1}e^{-\frac{E_B}{T}}$. We will confine ourselves to the first order terms. Simple calculations result in:

$$\frac{\varphi_A}{\varphi_B} = \frac{x\left(1-\frac{\Delta}{N}\right)+1}{x+\left(1-\frac{\Delta}{N}\right)}, \tag{32}$$

where $N = N_A = N_B$ is number of vacancies, $\Delta = 2N - n$ is total number of vacancies in the joint system, and $x = e^{\frac{E_A-E_B}{T}}$.

Clearly, if $E_A - E_B$ is far greater than the temperature, then the ratio of probabilities becomes independent of the income, and tends to $1 - \frac{\Delta}{N}$. If $E_A - E_B$ is small, we have

$$e^{\frac{E_A-E_B}{T}} \sim 1 + \frac{E_A - E_B}{T} \text{ and } \frac{\varphi_A}{\varphi_B} \sim 1 - \frac{(E_A - E_B)\Delta}{2TN}. \tag{33}$$

Thus, not the values of E_A and E_B are essential, but the ratio $\frac{E_A-E_B}{T}$, the parameter determined by the thermostat, that is by the environment.

Assuming that the thermostat is a free market system, and, therefore, the temperature in it is equal to the mean income, it becomes manifest that to find the relative probability of employment in two systems, the ratio of the difference of wages to the mean wage in the environment is essential.

So far we had considered very simple model problems of employment, in order to demonstrate a possible way of solving this kind of problems in general. Clearly, one can consider more realistic assumptions, e.g, the investigate migration processes in interacting systems characterized by distinct restrictions on income.

Such models are of particular importance in the study of regional economies.

CHAPTER 6

The thermodynamics of prices

§1. A setting of the problem

Let us try now to extend the thermodynamic approach to the study of markets and prices. For this purpose, consider a simplified model situation. It is necessary, nevertheless, to simplify with caution, so as not to lose the most essential situational characteristics.

Our first model is intended to analyze how the size of the flow of goods and money influences the market prices.

In the above sections we showed that if the money flow is constant, it is possible to introduce the concept of equilibrium in the income distribution. To implement this, we have to know the entropy of the system, hence its temperature. Using temperature we are able to find out the conditions under which the system stays in equilibrium with the environment.

Now, we add to this model a flow of goods. It is still possible to calculate the new entropy and introduce an additional parameter of equilibrium. In what follows we will discuss how this additional parameter can be interpreted in terms of the price of the goods. Moreover, it is also possible to deduce the *equation of the market state*, i.e., to find dependence between the flow of goods, the price, the number of buyers and the temperature.

First, consider an intuitive concept of the market and the market equilibrium, and then try to formalize it.

The market is characterized by the presence of goods which are sold, and money spent at purchases. Let $V(t)$ be the amount of "units of goods" sold per unit of time during which the buyers spend $E(t)$ of units of money. We say that the market is *stationary* if $V(t)$ and $E(t)$ are constants independent of time and all goods are bought, i.e., there is no accumulation of goods in the hands of the sellers. While the market functions, the deals are made, i.e., agreements on exchange of some of the goods for some money.

Bargaining regulations are typical for the market. Some kinds of deals may be ruled out, for instance, the ones bidding price too high or too low. What is important for us is that the regulations on deals do not vary with time.

Observe that even if the price regulations are not officially fixed, some rules exist anyway, e.g., the ones that ensure contracts' fulfillment. Thus, the market is a social institution due to existence of rules of dealing. No doubt, the rules imply that a control apparatus should be present, its purpose to enforce the rules. It has to be entitled with certain enforcement powers, in order to punish violators and guarantee reliability of contracts. In this sense, the market is certainly not the arena for totally spontaneous activity of its agents, but an organized social institution.

Consider a model with several co-existing markets able to interact: exchange goods and money resources. The intuitive concept of equilibrium of these markets is that the situation in each of these markets remains, in certain essential aspects, the same even after the interaction. Clearly, this cannot take place for any values of the amounts of goods and money V_1, E_1 and V_2, E_2, in each system, respectively. Besides, the interaction can vary, i.e., include an exchange only of money resources, only of goods, or both.

Consider the simplest case, when the values of the flows of money and goods are discrete, as is the case in real life:

$$E_n = nE_0, \quad V_m = mV_0. \tag{34}$$

In addition, let the values of n and m be sufficiently large, to make it possible to treat small changes of flows as insignificant and differentiate. Let N be the number of buyers in our model. (Generally speaking, N should be the number of bargains, but it is more convenient to consider N as the number of buyers.) The market is assumed to be stationary in the above sense.

The set of probable states of the market is the set of all contracts, allowed by rules, i.e., the ways of distribution of the goods between buyers. Let us examine the market with a single seller assuming that the seller is not capable to influence the flow of arriving goods.

For this model, introduce the entropy in the same way as earlier, namely, as the logarithm of the number of probable states. In this case, the entropy depends on both the total money expenditure, E, and the total amount of purchased goods, V. Various states of the market can be regarded as legalized by the rules of distribution of the total supply of money and goods among N buyers. Observe that now we study the "market of the buyers", so, we do not have to include in the entropy the distribution of goods among sellers.

If, however, we do take them into consideration, we have to incorporate in the model the distribution of the money among the sellers. This is a far more complex task.

To determine the price, it does not matter where the buyer purchased the goods. What is important, is how many goods he or she has received and how much he or she has paid.

So, for the analysis of the "buyer's market", the entropy, due to the presence of many sellers at the market, is not considered.

To avoid all these complications, we could have counted the number of contracts, instead of the numbers of sellers and buyers. Such an assumption, however, requires to introduce the migration potential corresponding to possible changes of the number of contracts. This case will be considered later.

By analogy with the pressure in statistical thermodynamics, the notion of *marginal price* P is introduced as follows:

$$P = T\frac{\partial S}{\partial V}. \tag{35}$$

A bit later we will justify the introduction of this parameter. The marginal price is a characteristic that points out whether or not the two interacting systems are in equilibrium.

First, consider the simplest case, referred to as the "free market". In this model there are no restrictions on goods and money distribution among the buyers. This means that the buyer can pay any price — either infinitesimally low or infinitely high.

The free market is rather easy to study, because the total number of probable market states is the product of the number of possible distributions of goods between the market agents times the number of possible distributions of money between the same agents. Thus, the statistical weights of the system are obtained by multiplying the statistical weights determined by the flows of goods (V) and money (E). Consequently, the entropy of the system (the logarithm of the statistical weight) consists of the two components: one, is determined by the flow of goods, the other one, by the flow of money:

$$S(E, V) = S(E) + S(V). \tag{36}$$

Let the money and goods flows be quantized, as in (34). In what follows we will derive the equation of state for such a system.

First, compute the number of probable distributions of goods and money flows among N agents of the market. The statistical weight $g(E_n, N)$ of the flow E_n , distributed among N agents is equal to the number of non-negative integer solutions of the equation

$$n = x_1 + \cdots + x_N. \tag{37}$$

The calculational technique for such equations is well-known. Namely, add 1 to each x_i. Then the quantity to be determined is the number of positive integer solutions of the equation

$$n + N = y_1 + ... + y_N. \tag{38}$$

Now, let us subdivide the integer segment of length $n+N$ into N integer subsegments. Thus, the statistical weight is equal to

$$g(E_n, N) = \binom{N-1+n}{N-1} = \frac{(N-1+n)!}{n!(N-1)!}. \tag{39}$$

For sufficiently large n and N, the Stirling formula gives

$$g(E_n, N) \approx \frac{1}{\sqrt{2\pi}} \frac{(N+n-1)^{N+n-\frac{1}{2}}}{N-\frac{1}{2}^{(}N-1)n+\frac{1}{2}^{n}}. \tag{40}$$

Hence,

$$\begin{aligned} &S(E_n) = \ln g(E_n, N) \approx \\ &(N+n-1)\ln(N+n-\tfrac{1}{2}) - (N-1)\ln(N-\tfrac{1}{2})n\ln(n+\tfrac{1}{2}) - \tfrac{1}{2}\ln 2\pi. \end{aligned} \tag{41}$$

The temperature T is calculated in terms of $\frac{\partial S}{\partial E}$:

$$\frac{1}{T} = \frac{1}{\varepsilon}\frac{\partial S(E_n)}{\partial n} \approx \frac{1}{\varepsilon}\ln\frac{N+n-\frac{1}{2}}{n+\frac{1}{2}}. \tag{42}$$

For $n \gg N$, which is a rather natural condition for the free market, the expression for the inverse temperature is:

$$\frac{1}{T} = \frac{\partial S(E_n)}{\partial n} = \frac{1}{\varepsilon}\frac{N}{n} = \frac{N}{E}. \tag{43}$$

This means that the temperature $T = \frac{E}{N}$ is equal to the mean value of income per capita.

To compute $\frac{\partial S}{\partial V}$ is a completely analogous matter (since the entropy S can be described as the sum of two terms, one of which only depends on E, the other one, on V, see (36)):

$$\frac{P}{T} = \frac{\partial S(V_m)}{\partial V_m} = \frac{1}{W}\frac{N}{m} = \frac{N}{V}. \tag{44}$$

Thus, a relation between the marginal price, the goods flow and the "temperature" of the free market is of the form

$$P = T\frac{N}{V}. \tag{45}$$

As $TN = E$, it is easy to show that for the free market *the marginal price is equal to a median price*. This, actually, allows to define the *price* as $T\frac{\partial S}{\partial V}$.

In presence of restrictions on price, the marginal price may deviate from the median one.

Manifestly, the equation of the free market (45) is totally analogous to the equation of the ideal gas.

In our arguments, **the marginal price has the same thermodynamic origin as the temperature or the chemical potential**, i.e., the theory is completely derived from thermodynamic principles. The marginal price is the parameter of equilibrium, identical in meaning to the temperature. It indicates whether or not the interacting systems are in the equilibrium.

Indeed, the same pattern of reasoning as at the introduction of the temperature, can be repeated and we similarly prove that two interacting systems for which the exchange of the goods is possible are capable to be in the state of equilibrium only when their marginal prices for the goods are equal.

Indeed, let the statistical weights of the interacting systems depend on the flows of money and goods, E and V, while the total values of both parameters are constants (speaking about a flow, we mean the expenditure capacity per unit time):

$$E_1 + E_2 = E, \qquad V_1 + V_2 = V, \tag{46}$$

then the statistical weight of the combined system is

$$g_1(E_1, V_1) \cdot g_2(E - E_1, V - V_1). \tag{47}$$

The probability of the state becomes maximal when the function (47) attains its maximum. Introducing entropy as $\ln g$, the conditions for the entropy extremum become

$$\left(\frac{\partial S_1}{\partial E_1} dE_1 + \frac{\partial S_1}{\partial V_1} dV_1\right) + \left(\frac{\partial S_2}{\partial E_2} dE_2 + \frac{\partial S_2}{\partial V_2} dV_2\right) = 0. \tag{48}$$

Taking into account that

$$dE_1 = -dE_2, \qquad dV_1 = -dV_2, \tag{49}$$

we express the conditions of equilibrium as follows:

$$\left.\frac{\partial S_1}{\partial E_1}\right|_{V_1} = \left.\frac{\partial S_2}{\partial E_2}\right|_{V_2}, \qquad \left.\frac{\partial S_1}{\partial V_1}\right|_{E_1} = \left.\frac{\partial S_2}{\partial V_2}\right|_{E_2} \tag{50}$$

The first expression, as we already discussed, means the coincidence of "temperatures". The second one means coincidence of values of the marginal prices.

Thus, the coincidence of values of the marginal prices is a necessary condition for the interacting markets to be in the state of equilibrium.

Now, proceed to the case that the market is not "free"; let some restrictions be imposed on its operation.

Let us investigate, first, what is going to happen if something similar to the "allocation system" is adopted, i.e., the possibilities to buy goods are forcefully restricted. Recall also our assumption that the market is *stationary*, which means that the flows of goods and money are completely distributed among the market participants.

Suppose that none of the market agents is allowed to get more goods than a certain amount, K. In all the other respects the distribution remains free, and — this is of utter importance — *does not depend on the money distribution*, i.e., there are no explicit restrictions on the prices. This means, that the statistical weight of the state with V goods, and E money is still the product of statistical weights:

$$g_N(E, V) = g_N(E) g_N(V), \tag{51}$$

while the entropy includes two components, one of which depends on E, and the other one on V:

$$S_N(E, V) = S_N(E) + S_N(V). \tag{52}$$

To determine the marginal price, we should known $\left.\frac{\partial S}{\partial V}\right|_{E,N}$, i.e., we should compute $S_N(V)$. This appears to be a difficult combinatorial problem: *find the number of non-negative integer* *solutions of the equation*:

$$m = x_1 + x_2 + \cdots + x_N, \text{ where } x_i \leq K \text{ for each } i. \tag{53}$$

The solution of this problem is distinguished by a special feature, which renders meaningful conclusions.

It is clear that if $m > KN$ this problem has no solutions at all. The logarithm of the number of solutions is the system's entropy $S_N^K(V_m)$. This means that when m (i.e., the flow of goods) approaches KN from below, then at some point the entropy begins to decrease. But this, in turn, means that $\frac{\partial S(V_m)}{\partial V}$ becomes negative. Taking into account that the distribution of money flow is free, which implies positive temperature, the result would be that, under restrictions, the system's marginal price becomes negative when the volume of the flow of goods becomes sufficiently large.

This means that during interaction with other systems with positive marginal prices, our "allocation system" with imposed restriction on consumption of the goods will begin to eject goods at dumping prices: any more-than-zero price of the exported goods will contribute to destabilization of the equilibrium.

Of course, it is possible to say that when the flow of goods is close to the value of KN, there is no sense to restrict the distribution. The fact, however, is that the marginal price can become negative *long before* the flow of goods approaches the critical value, KN.

For instance, for $K = 1$, this will happen for $m > \frac{N}{2}$.

Since the introduction of restrictions such as in our example is a political decision, the question should be addressed: **how the persons responsible to make such decision could learn that the marginal price has already became negative**?

Indeed, the price of market transactions is positive, and the median price is also positive. In order to know how far the system that interacts with other systems is from the state of the equilibrium, it is necessary to calculate certain, not directly observable, parameters — the temperature and the derivative of the entropy with respect to the flow of goods.

As we have just shown, this problem is rather tough, even for a very simple model.

We see, nevertheless, that the restrictions imposed on the market can create, in the range of the system's states, a zone of "latent instability".

Let us consider now whether the zones of latent instability would emerge if the price is bounded from below. This model is very important indeed because real market systems are hardly free: it is not realistic to sell the goods, on the large scale, at the price below the production cost.

If in our model the price of the bargain is bounded from below, then the entropy of the system cannot be described any more as the sum of two components, each depending on only one variable, one being the entropy of the flow of goods, the other one the entropy of the money flow.

The mathematical problem of computing the entropy is to determine $\ln g_N(E, V_m, \lambda)$, where λ is a parameter bounding the prices from below, and $g_N = \sum g_N(y_1, \ldots, y_n)$, where $g_N(y_1, \ldots, y_n)$ is the number of positive integer solutions of the equation

$$m = x_1(y_1) + x_2(y_2) + \cdots + x_n(y_n), \tag{54}$$

In (54) y_n is an arbitrary partition of the number n into N non-negative integers, and

$$x_i(y_i) = 0 \text{ for } y_i = 0 \text{ and } x_i(y_i) < \frac{y_i}{\lambda} \text{ for } y_i > 0. \tag{55}$$

(Here x is the allowed number of goods in the deal while y is the corresponding amount of money.)

This is a still more difficult calculational problem. But, as in the previous cases, some qualitative dependencies can be discovered relatively easy. If we increase the goods flow but the money flow does not increase simultaneously, the problem has no solutions. Indeed, any possible price remains below the allowed level once $V_{k,m}$ is great enough for limitations on x_i, where the y_i are $\leq n$ and n is such that $ne = E$), i.e., the entropy $S_N^{\lambda}(E, V)$ vanishes.

Like in the previously given model, the marginal price becomes negative earlier than that because the distribution of the money flow is free, the temperature is positive, but the derivative $\frac{\partial S_N^{\lambda}(E,V)}{\partial V}$ at values of V smaller than but close to V_K becomes negative.

So, we observe the same effect as under the terms of the restrictions on goods, namely, the latent zone of instability with a negative marginal price, and the same consequences — an outward dumping of goods by the system striving to reach an equilibrium.

Apparently, this explains phenomena of mass destruction of the goods for the sake of maintaining the level of prices during the crises of overproduction. It is not the "malicious will" of owners of the commodities, but simply the shortest way to the most probable state of the system: getting rid of the goods the system assumes a statistical equilibrium, i.e., a most probable state.

As a byproduct of our arguments, we see that **introducing restrictions in the system we provoke occurrences of zones of latent instability**. The borders of these zones are extremely scarcely discernible even for the simplest models.

Such parameters, as the derivatives of the entropy, e.g., the temperature and the marginal price, are crucial for the description of the behavior of systems with restrictions after they became engaged in interactions with similar systems or with free-market systems.

The marginal price becomes a very important parameter, a major parameter of the market equilibrium. It coincides with the median price only for the free market systems. In the systems with restrictions, it may be negative. If this happens, the system is unstable.

Thus, restrictions on economic activity can, by no means, be held as "harmless". First of all, **no restriction is harmless because the range of its influence is unclear, the range within which its influence renders the system unstable**.

Observe that the so-called "economic" reasons for various restrictions on deals (e.g., the minimal price determined by the cost of production) turn out, at deeper scrutiny, purely political reasons: the cost of production often can not be lowered "thanks" to a monopoly, i.e., a political control of the market. In other words, the restrictions imposed on the freedom of market activity, are capable to generate uncertainties in the system, instead of, as politicians use to think and preach during the election campaigns, making it more predictable.

The above study of the market with restrictions on prices, is related to the problem which lately aroused considerable interest in theoretical economics. In 1970, G. Akerloff published an article [**Af1**] that soon enjoyed much popularity. The article dealt with the markets with "asymmetric" information, i.e., markets, where the seller and buyer have different opportunities to estimate the quality of the item on sale.

Akerloff showed, further developing his contention in later publications [**Af2**], that, for asymmetric markets, **the presence of low quality goods and dishonest sellers can result not only in sweeping away of high quality goods from the market, but may also result in a collapse of the market as such**.

Briefly, Akerloff's idea was as follows. Assume that the article of goods (say, a used car) may appear to be of low quality with certain probability, q (and, accordingly, of high quality with probability $1 - q$). Assume further that the buyer "knows" to an extent these *a priori* probability because q may reflect the ordinary index of production rejected by the factory.

But the seller in question knows about the article he or she is selling far better.

The buyer just follows the general opinions concerning the item. By this reason, the seller of a *good* second-hand car is seldom able to get the real price for it: the buyer wants to insure himself or herself against the dishonest seller.

The seller of the *bad* car, contrariwise, enjoys all the chances to get for it more money than it is really worth. As a consequence, good cars are "washed away" from the market, and the bad ones dominate. Using the standard technique of the utility functions, Akerloff has shown that under certain conditions the equilibrium may be unattainable, e.g., when the graph representing dependence of supply on prices does not overlap with the graph representing dependence of demand on prices. Akerloff interprets the problem of asymmetric markets linking it to the *Grasham law* which states that the "bad" money oust the "good" money out of circulation: people tuck away the "good" money. Akerloff conjectures that such an approach gives an opportunity to estimate the losses of the market from crooked dealings.

Akerloff concludes: the price of dishonest behavior amounts not just in the losses of the buyer, but also in the undermining consequences for honest businesses [**Af2**].

The model with restrictions on prices from below discussed above, shed, in our view, some new light on the problem of asymmetric markets. We believe that Akerloff has proved not so much the possibility of collapse of the asymmetric market, but rather the impossibility to apply the standard equilibrium techniques used in neo-classical analysis. Indeed, even in an asymmetric case, the flow of deals will last anyway, but for the study of the market state, the definition of equilibrium, based on the notions of statistical thermodynamics, looks to be more appropriate.

Indeed, consider two markets in interaction, one, with restrictions on prices from below, and another one, without such restrictions (say, a market of used cars). Consider a model with an "asymmetric information", we see that a market with restrictions on price from below is a market of "good" cars, while the market without restrictions is a market with defective cars (Akerloff uses a slang term: "lemons") in circulation. As we have shown above, at a certain value of parameters of the flow of goods and "temperature" of the market (namely, at low temperature), the marginal price in the market with restrictions (here: in the market of "good" cars) becomes negative. This means that this market is collapsing. But **the actual collapse will only happen at a rather low temperature**. At higher temperature the market of high quality cars is quite viable. Thus, it is possible to make some amendments to Akerloff's statements on the market of "lemons", and on the opportunities to operate honest business in the developing countries, where "standards of honesty" are low.

The market of "good" cars will not rarely be on the brink of collapsing but will not actually and totally collapse. According to our thermodynamic approach, the crucial factor lies not in the standards of honesty, but in the "low temperature" of the market. If the temperature is not low any more, the "honest market" (the one with restrictions on prices from below) is quite capable to coexist with the market of "lemons". The marginal price formally corresponds to the pressure and the only thing needed to render the market "alive" is that the "pressure" in the market of "good" cars were not below a certain minimal value.

Next, consider in more detail the thermodynamic model of the market with prices bounded from below. For this purpose, we apply the technique of large statistical sum. But, first of all, let us make some remarks.

If we forget sellers and buyers, and only consider the amount of contracts, N, then the temperature of the market, according to our previous consideration, can be defined as:

$$T = \frac{E}{N}, \tag{56}$$

where E is the amount of money.

Introduce the concept of a "potential contract" which means the possibility of striking a deal, with the goods flow $V_l = lV_0$, and money flow $E_k = kE_0$, where V_0 and E_0 are the least values of the respective flows. Then some further simplification would be expedient. Introducing discrete time labelled by moments T_k, we assume that the goods and money flows only occur at discrete time moments. Such a model of market resembles the market of von Mises [**Mis**]. It allows us to treat every deal as a single-time event.

Bearing in mind the statistical aspect of the model, it looks quite natural to consider the market as the set of simultaneously coexisting systems covering all admissible deals, and to carry out averaging over the ensemble instead of averaging over time.

The potential contract $C(E_k, V_l)$ with parameters V_l, E_k may be "filled in" with a real content, but also may remain "empty". The notion of potential contract corresponds to the notion of an orbital in quantum mechanics. We are able now to refer to the set of potential contracts, $C = (E_k, V_l)$ subject to some restriction, for instance, the contracts with $\frac{E_k}{V_l} < Q$ may be ruled out (which, actually, mirrors the restriction on prices from below, Q being the least allowed price).

Now, apply the standard technique of large statistical sum. If $N \ll nm$, where n, m stand for the total amount of goods and money in the market, respectively, the market discussed is a "classical one": the "population density" of every contract is very low ($\sim \frac{N}{nm}$), so, we can neglect the probability that one "potential contract" may be "filled in" with two real contracts with equal values of V_l and E_k. This means that, in the large statistical sum, we can neglect the terms of degree ≥ 2 with respect to λ. Hence, the probability for the potential contract to be "filled in" is given by the formula

$$W(k,l) = \lambda e^{-\frac{E_n}{T}} = \lambda W_0(k,l). \tag{57}$$

If the number of actual contracts is equal to N, it is possible to find λ because N can be found by summing over all probability values $W(k,l)$ for all potential contracts:

$$N = \lambda \sum_{l,k} W_0(k,l). \tag{58}$$

Therefore

$$\lambda = \frac{N}{\sum\limits_{l,k} W_0(k,l)}. \tag{59}$$

Since $\lambda = e^{\frac{\mu}{T}}$, where μ is the migration potential, we can express μ as follows:

$$\frac{\mu}{T} = \ln \frac{N}{\sum\limits_{l,k} W_0(k,l)}. \tag{60}$$

So, the entropy of the system can be derived from the formula

$$\frac{\partial S}{\partial N}\Big|_{E,V} = -\frac{\mu}{T}. \tag{61}$$

Substituting this value for $\frac{\mu}{T}$, we find that

$$S = -\int_0^N \frac{\mu}{T} dN = \int_0^N \ln N dN + N \ln \sum_{k,l} W(k,l), \tag{62}$$

or

$$S = -N \ln N + N + N \ln \sum_{k,l} W(k,l). \tag{63}$$

The sum $\varphi(T,V) := \sum W(k,l)$ depends, generally, on T and V.

Now, differentiating the entropy with respect to V we derive the equation of state (35)

$$\frac{P}{T} = \frac{\partial S}{\partial V}\Big|_{E,N}. \tag{64}$$

It requires to calculate $\varphi(T,V)$.

First, consider the already discussed case, the one without restrictions on prices:

$$\varphi(T,V) = \sum_{k=1}^{\infty} C(k) \cdot e^{-\frac{kE_0}{T}}, \tag{65}$$

where $C(k)$ is the number of contracts with E_k. Since a certain definite number $m = \frac{V}{W}$ of potential contracts which can be “filled” exists for each value of E_k, it follows that $C(k) = m$. In order to simplify the calculations, we assume that n is very large ($E = nE_0$), and the upper limit of summation is equal to ∞. We thus obtain

$$\varphi(T,V) = \frac{V_0}{V} \frac{e^{-\frac{E_0}{T}}}{1 - e^{-\frac{E_0}{T}}} = \frac{V_0}{V} \frac{1}{e^{-\frac{E_0}{T}} - 1}. \tag{66}$$

Hence, the already known result:

$$\frac{P}{T} = \frac{\partial S}{\partial V}\Big|_{E,N} = N \frac{\partial}{\partial V} \ln \varphi(V,T) = \frac{N}{V}. \tag{67}$$

Now, let us calculate $\varphi(T,V)$ for the case of prices bounded from below. The number of potential contracts $C_k(E_k,Q)$ now depends on E_k and the minimal price, Q:

$$C_k(E_k,Q) = \sum_{n=1}^{\frac{kE_0}{QW}} 1 = \frac{kE_0}{QV_0}. \tag{68}$$

In this case $\varphi(T,V)$ splits into the two components.

One of them is obtained by summation up to $k = \frac{mQW}{E_0}$, hence, $C_k(E_k,Q) = \frac{kE_0}{QV_0}$. For these values of k, the number of potential contracts depends on k. For very large k such dependence vanishes since we assume that the amount of money used in any potential contract suffices to ensure any of $m = \frac{V}{W}$ possible values of goods purchased:

$$\begin{aligned} \varphi(T,V) = & \sum_{k=1}^{\frac{QV}{E_0}} C_k(E_k,Q) e^{-\frac{kE_0}{T}} + \sum_{k=\frac{QV}{E_0}}^{\infty} m e^{-\frac{kE_0}{T}} = \\ & \sum_{k=1}^{\frac{QV}{E_0}} \frac{kE_0}{QV_0} e^{-\frac{kE_0}{T}} + \sum_{k=\frac{QV}{E_0}}^{\infty} \frac{V}{V_0} e^{-\frac{kE_0}{T}}. \end{aligned} \tag{69}$$

After simplifications we have

$$\varphi(T,V) = \varphi_1(T,V) \left(\frac{E_0}{QV_0} - \frac{E_0}{QV_0} e^{\frac{-VQ}{E_0}} \right), \tag{70}$$

where

$$\varphi_1(T) = \sum_{k=1}^{\infty} k e^{\frac{kE_0}{T}} = \frac{e^{\frac{E_0}{T}}}{\left(e^{\frac{E_0}{T}} - 1\right)^2}. \tag{71}$$

Now, substitute $\varphi(T,V)$ in the expression for the entropy, so the equation of state takes the form:

$$\frac{P}{T} = \frac{\partial S}{\partial V}\Big|_{E,N} = N\frac{\partial}{\partial V}\ln\varphi(T,V) \tag{72}$$

or

$$\frac{P}{T} = \frac{NQ}{E_0}\frac{1}{e^{\frac{QV}{E_0}} - 1}. \tag{73}$$

The equation of state (73) is very interesting. When $Q \ll \frac{T}{V}$, equation (73) turns into the familiar equation of the ideal gas:

$$\frac{P}{T} = \frac{N}{V}. \tag{74}$$

But, as it turns out, even small restriction on price from below completely transforms the equation of state.

In the range of larger V, the marginal price P (i.e., the equilibrium price) becomes smaller than the minimal price of the contract, which means that the market collapses. Such a situation is equivalent to the negative marginal price for the free market. It is quite clear that if the least price is implemented, the large volumes of goods cannot be sold in equilibrium provided the temperature is bounded. This does not mean that the absence of deals. This means that the equilibrium cannot be attained.

If $T = \frac{E}{N}$, the minimal price is equal to Q, and $V = \frac{E}{Q} = \frac{TN}{Q}$, then a part of goods equivalent to $V - \frac{TN}{Q}$ is impossible to realize.

A hasty observation may lead to the premature conclusion that the critical mass of the goods when the price is bounded from below will lie in a neighborhood of

$$V_{crit} = \frac{TN}{Q}. \tag{75}$$

But, as is clear from the above analysis, this is not the case: in reality the critical mass is considerably smaller. Namely, assuming that

$$P = \frac{TNQ}{E_0}\frac{1}{e^{\frac{QV_{crit}}{T}} - 1} > Q \tag{76}$$

or

$$\frac{TN}{E_0} > e^{\frac{QV_{crit}}{T}} - 1, \tag{77}$$

we obtain

$$V_{crit} = \frac{E_0}{Q}\ln\left(\frac{TN}{E_0} - 1\right). \tag{78}$$

As a result, instead of $V_{crit} = \frac{TN}{Q} = \frac{nE_0}{Q}$, deduced by the "naive" argument, V_{crit} is close to $\ln V$, i.e., is of several orders of magnitude smaller. Therefore, the market collapses much earlier than it seems at the first glance.

Coming back to the market with "lemons", this means that the ousting of good cars by the bad ones will happen much earlier than the naive *a priori* assumptions allow for.

Clearly, the asymmetric information narrows the market's "zone of stability" to a very small size. Thus, our analysis justifies Hayek's conjecture that the market exists thanks to the culture of "honest business", and survives in the process of competition of business cultures.

The investigation above also points to another condition for existence of markets with high quality goods. The market operating with high quality goods has to be somehow separated from the market of low quality goods. Otherwise, with the influx of high quality commodities, the market becomes unstable.

The model outlined above shows that due to a limited supply of high quality goods, their market can coexist and interact with the market of "bad" goods, but only as the "market of elite", which means that outside its aegis the market is stunted by the flaws in the "business code of behavior". An alternative is the certification of goods.

We see that, in a sense, our theory exorcises "human component": the equilibrium theory in economics becomes identical to that of statistical physics.

It is also clear that hypothesis on equal probability of market states — the basis of any statistical theory — may be justified only if, following Hayek, **we consider the human activity a principal factor subject to incomplete information on what's on at the market**. Otherwise the hypothesis on equal probability of market states becomes absurd. It is as a result of human ability to discover new opportunities together with all possible errors, the hypothesis of equal probability becomes natural.

Human component is essential in one more aspect. The market requires abiding certain rules of contracts. In our theory we assume this as a given reality. This, however, is extremely important. Even if contracts are not restricted by "external" rules, **the mere necessity to abide the contracts makes the market into a social institute**. The behavior of this institute, considered as a machine, in particular, its thermodynamic properties are of huge interest, but, first of all, such a machine — such an institute — must be created and being created they should be maintained.

§2. Two markets with two items of goods

Consider now the problem of the market equilibrium for the two markets with two items of goods, capable to replace each other. The market with replaceable goods poses one of the most known problems of the mathematical economics. The study of such markets prompted the marginalist revolution.

In terms of the neo-classical theory, to work out this task, it is necessary to define the utility functions of the goods, the function depending on the volume of consignments. When the derivative of the utility function with respect to the volume of consignment decreases as this volume grows, the techniques of the classical analysis show that the summary utility function reaches its maximum at such a volume of the consignment of goods that the increase of utility by one unit of the expenditure is equal for all nomenclature of goods.

Intuitively, this is a plausible statement. If, at the expenditure of one unit of funds per one item of goods, it were possible to increase the total utility of the purchased goods by replacing one item by another, this replacement have been implemented, until the utility had not dropped due to the increase of the goods' volume.

We see that the dependence of utility on volume is indeed very important in terms of maintenance of sustaining the system's stability: otherwise all funds could have been invested in the most useful goods only.

Let us now study the problem of replaceable goods in the framework of thermodynamic approach.

Clearly, it makes sense to speak about replacement only if there is a certain equivalence between goods. If there is no equivalence, we can not proceed. So, let such an equivalence exist in the model, i.e., n units of item 1 are equivalent (never mind, in what sense precisely) to m units of item 2. The equivalence relation can be used to find the maximum of entropy, in exactly the same manner, as it was done before for only one item of goods.

It is important to observe that, in our hypothesis, the equivalence relation does not depend on the flow of goods. If such dependence takes place, the theory of statistical equilibrium is, all the same, possible to deduce, but it is a bit more involved.

The most probable or, what is the same here, equilibrium, state of the system is attained as the entropy of the system attains its maximum. To find the maximum of entropy, we use the equivalence relation between two items of goods, in completely the same fashion as earlier on these pages.

Thus, consider a market with N agents, with a flow of money E and flows of items of two goods, V_A and V_B. To find the entropy of system, we have to count the number of probable deals, i.e., the number of ways to attribute to each agent four numbers $(E_{A_i}, E_{B_i}, V_{A_i}, V_{B_i})$ that show how much money was spent by an agent per unit time on purchasing the goods A and B, and how many goods had been bought over the same period, respectively. Adding up over all agents we obtain the values of macroscopic parameters of the system.

Observe that in order to estimate the system's state, it does not suffice to simply fix the amount of money spent by the ith agent because some microscopic states may be ruled out, owing to, say, a restriction of the prices from below.

In the general case, the dependence of the entropy on macroscopic parameters cannot be represented as the sum of components each depending on one or two parameters. For the free market, however, this is so.

Consider the system with entropy $S(E_A, E_B, V_A, V_B)$. Replacing E_A by E_B and using the fact that $E_A + E_B = E$ we see that

$$\delta S = \frac{\partial S}{\partial E_1}\delta E_1 + \frac{\partial S}{\partial E_2}\delta E_2 = \delta E_1 \left(\frac{\partial S}{\partial E_1} - \frac{\partial S}{\partial E_2} \right). \tag{79}$$

In the state of equilibrium, the derivatives $\frac{\partial S}{\partial E_1}$ and $\frac{\partial S}{\partial E_2}$ are equal and V_A and V_B are fixed. If there is an equivalence between the items A and B, we may "unite" these items and consider the equilibrium problem. Let $V_A = V_A^0 n_A$, $V_B = V_B^0 n_B$ where n_A and n_B are the amounts of goods, purchased per unit time.

Having introduced V_A and V_B we imply that the replacement relation is known, i.e., we have a common unit of measurement. So, again, we have:

$$\begin{gathered} V_A + V_B = V, \qquad \delta V_A = \delta V_B, \\ \delta S = \tfrac{\partial S}{\partial A}\delta A + \tfrac{\partial S}{\partial B1}\delta B = \delta A \left(\tfrac{\partial S}{\partial A} - \tfrac{\partial S}{\partial B1} \right) \end{gathered} \tag{80}$$

and the condition for the equilibrium state can be expressed as:

$$\frac{\partial S}{\partial V_A} = \frac{\partial S}{\partial V_B}. \tag{81}$$

Observe that we made no assumptions on utility, but only on possibility to replace the goods by other goods.

Now consider the free market where we know how the entropy depends on macro-parameters. As we outlined above, the equilibrium conditions with respect to the money flow are of the form:

$$\frac{\partial S}{\partial E_A} = \frac{N}{E_A}, \qquad \frac{\partial S}{\partial E_B} = \frac{N}{E_B}. \tag{82}$$

As $E_A + E_B = E$, the equality of $\frac{\partial S}{\partial E_A}$ and $\frac{\partial S}{\partial E_B}$ result in $E_A = E_B = \frac{E}{2}$. Similar calculational techniques for $\frac{\partial S}{\partial V_A}$ and $\frac{\partial S}{\partial E_B}$ produce $\frac{\partial S}{\partial V_A} = \frac{N}{V_A}$ and $\frac{\partial S}{\partial V_B} = \frac{N}{V_B}$ and, with the constraint that $V_A + V_B = V$, $V_A = V_B = \frac{V}{2}$. Returning to the condition of substitution, expressed by $V_A = V_A^0 n_A$, $V_B = V_B^0 n_B$, it means:

(83) $$\frac{n_A}{n_B} = \frac{V_B^0}{V_A^0},$$

i.e., the volume of goods, purchased in the equilibrium state of the free market, is inversely proportional to its "replacement capacity".

Recall our earlier assumption that the equilibrium of the money flow is attained. The conditions for the equilibrium of the goods flow takes the form

(84) $$T\frac{\partial S}{\partial V_A}\Big|_{E_A,E_B} = T\frac{\partial S}{\partial V_B}\Big|_{E_A,E_B}.$$

If the money flow is constant, TdS can be interpreted as the expenditure on the purchase of the last, "marginal" portion of the goods. So, we have obtained the well-known marginalists's formulation:

> *In the state of equilibrium, the amount of money spent on the last portion of goods per unit of replacement capacity is the same for all items of goods represented at the market.*

§3. The market at constant temperature

Above we have considered models with economic systems with a fixed flow of money. In the case of actual markets, it is of course very difficult to determine how the actual flow of money looks like. Moreover, the flow of money is not a parameter that affects the equilibrium of the system, i.e., even if we know what are the flows of money in two distinctly organized systems, we cannot say if the systems will come to an equilibrium after they come into a contact.

Therefore it is much more interesting to consider economic models at a constant temperature since the temperature is a parameter of equilibrium.

To pass to new variables, we can apply a mathematical technique called *Legendre transformation*. The geometric meaning of the Legendre transformation is that, for the new variables in a system given by a Pfaff equation (i.e., a linear constraint on the differentials of the initial variables that single out a surface), one uses the coordinates on the tangent plane for this surface, see [**Cou**].

This technique enables one to obtain a large number of new relations. Under the passage to new independent variables the extensive quantity whose conservation was used to derive the parameter of equilibrium is replaced by a new one in which this parameter enters linearly. It turns out that such a reparameterization of basic quantities is very useful. In statistical thermodynamics, the quantities obtained after Legendre transformation are usually called *potentials*. They possess a number of remarkable properties.

Let us look how to work with the Legendre transformation sending independent variables (E, V) into independent variables (T, V). If we start from the main relation for our theory

(85) $$\boxed{dE = TdS + \mu dN - PdV},$$

then, assuming that the temperature T and the number of particles N are constants, we immediately derive by dividing (85) by dV that

$$P = -\left.\frac{\partial E}{\partial V}\right|_{T,N} + T\left.\frac{\partial S}{\partial V}\right|_{T,N}. \tag{86}$$

If we introduce the function $F = E - TS$ — the Legendre transform, then

$$P = -\left.\frac{\partial F}{\partial V}\right|_{T,N} \tag{87}$$

Therefore given F, differentiating yields the value of the price at the constant temperature and constant number of market agents.

This (87) is a very essential relation. Let our system interact with a thermostat at temperature T. Now we are not bounded by a particular value of the flow of money: the system exchanges its money with the thermostat and it suffices to assume that the total system plus thermostat satisfy the law of money conservation.

It is not difficult to see that if the flow of goods changes from V_1 to V_2, then at constant temperature the corresponding variation of the flow of money will be equal to

$$\int_{V_1}^{V_2} P dV = -\int_{V_1}^{V_2} \left.\frac{\partial F}{\partial V}\right|_{T,N} dV = F(V_1) - F(V_2). \tag{88}$$

Under the increase of the flow of goods the flow of money spent in order to make such an increase possible becomes equal to the increment of the function F.

In statistical thermodynamics, F is called the *free energy*. We will call F the *free flow of money*. It is easy to see that $F = -\ln Z$ where Z is the statistical sum of the system.

Indeed,

$$F = E - TS = \sum_i E_i W_i - T\sum_i W_i \ln W_i, \tag{89}$$

where $W_i = \dfrac{e^{E_i/T}}{Z}$ is the probability of the system to be in the state i. Substituting this W_i in the formula for F and computing the difference we get

$$F = -\ln Z. \tag{90}$$

Observe that F is a negative quantity. The relation (88) shows that the free flow of money are exactly the highest expenditures possible in the system to increase the flow of goods.

Let us show now that a free flow of money is extremal at the most possible configuration and at a constant temperature and constant flow of goods. Indeed, in the system which interacts with a thermostat, the total flow of money is preserved:

$$dE_C + dE_T = 0, \tag{91}$$

where dE_C is the infinitesimal increment of the flow of money in the system and dE_T is the infinitesimal increment in the flow of money of the thermostat.

The total entropy of the system $S_C + S_T$ is maximal, and therefore $dS_C + dS_T = 0$ and the temperature T of the system is determined by the thermostat and

$$dE_T = TdS_T. \tag{92}$$

Therefore

$$dE_T = -dE_C = TdS_T = -TdS_C \tag{93}$$

or

$$dE_C - TdS_C = 0, \quad dF_C = 0 \tag{94}$$

In other words, F has an extremum at the point of equilibrium between the system and the thermostat. This extremum is the minimum which follows from the maximality of the entropy. For any change of the situation which leads the system out of the equilibrium, the total sum of increments of the entropies of the system and the thermostat is

$$\Delta S_C + \Delta S_T \leq 0. \tag{95}$$

But the energy can only sneak into the system due to the diminishing of the thermostat's entropy:

$$\Delta E_C = -T\Delta S_T \tag{96}$$

and since $\Delta S_T \leq -\Delta S_C$, it follows that

$$\Delta F_C = \Delta F_C - T\Delta S_C \geq 0. \tag{97}$$

This means that

the system is in equilibriumat the minimum of F at constant V and T, see [**Ki**].

This is a very important property. It means that we can find the equilibrium point by looking for the minimum of a function which can be computed if we know the statistical sum. To find the statistical sum, it suffices to know the statistical weights of the states with distinct admissible values of the money flow and the temperature. All other parameters of the system (the flow of money and the price) can be found from the simple formulas:

$$E = -T^2 \left.\frac{\partial}{\partial T}\frac{F}{T}\right|_{V,N}, \tag{98}$$

$$P = -\left.\frac{\partial F}{\partial V}\right|_{T,N}. \tag{99}$$

It is easy to see that the Legendre transformations can be applied differently passing to other independent variables, for example, T and P. In this case in statistical thermodynamics one uses the *thermodynamic potential*

$$\Phi := E - TS + PV. \tag{100}$$

We will retain this name in our case as well. It is not difficult to show that the thermodynamic potential attains a minimum in an equilibrium at constant values of T and P. For the thermodynamic potential, we have the following relations:

$$d\Phi = -SdT + VdP, \quad S = -\left.\frac{\partial \Phi}{\partial T}\right|_P, \quad V = \left.\frac{\partial \Phi}{\partial P}\right|_T. \tag{101}$$

These relations imply (thanks to possibility to interchange the order of differentiation) an important relation:

$$-\left.\frac{\partial S}{\partial P}\right|_T = \frac{\partial}{\partial P}\left.\frac{\partial \Phi}{\partial T}\right|_P = \frac{\partial}{\partial T}\left.\frac{\partial \Phi}{\partial P}\right|_T \tag{102}$$

or

$$-\left.\frac{\partial S}{\partial P}\right|_T = \left.\frac{\partial V}{\partial T}\right|_P. \tag{103}$$

Now we can construct a thermometer to define the absolute temperature of an arbitrary system.

One should, of course, observe that to apply such a thermometer in economics will be not as easy as in physics. In physics, a thermometer is an arbitrarily graded physical system which comes into a heat contact with the system to be measured and by the change in the state of thermometer which is supposed to be in the heat equilibrium with the body to be measured the "conditional temperature" is defined.

Further, if we know the equation of the state of the thermometer, the conditional temperature expressed, say, in the volume of mercury, as one does in the medical thermometers, one can derive the absolute temperature.

This procedure, however, can be performed even without beforehand knowledge of the equation of the state of the thermometer by performing a number of measurements of thermodynamic quantities.

Let $T = T(T_{arb})$, where T_{arb} is an arbitrarily graded scale of a thermometer. Relations (103) imply

$$\left.\frac{\partial E}{\partial P}\right|_T = T \left.\frac{\partial S}{\partial P}\right|_T = -T \left.\frac{\partial V}{\partial T}\right|_P . \tag{104}$$

We can express $\left.\frac{\partial V}{\partial T}\right|_P$ in the "empirical scale" in terms of T_{arb}:

$$\left.\frac{\partial V}{\partial T}\right|_P = \left.\frac{\partial V}{\partial T_{arb}}\right|_P \frac{\partial T_{arb}}{\partial T} \tag{105}$$

This implies that

$$\left.\frac{\partial E}{\partial P}\right|_{T_{arb}} = -T \frac{\partial V}{\partial T_{arb}} \frac{\partial T_{arb}}{\partial T} \tag{106}$$

or

$$\frac{\partial T_{arb}}{\partial T} = -\frac{\left.\frac{\partial V}{\partial T_{arb}}\right|_P}{\left.\frac{\partial E}{\partial P}\right|_{T_{arb}}} \tag{107}$$

The right-hand side of this relation only involves the functions that can be measured on the conditional scale ([**LL**]). If we can measure them we can, therefore, determine the law of dependence of the conditional scale on the relative temperature.

For real economic systems, this would mean the necessity to have a certain model market which one could append to an arbitrary market and measure the variation of the derivative of the flow of goods with respect to the conditional temperature at the fixed price and the derivative of the flow of money with respect to the price at the fixed conditional changes.

To collect such statistical data for a large number of artificially created conditions is hardly possible in reality. Observe, however, that such a device is possible as a thought experiment. In other words, a recovery of the scale of the absolute temperature from a collection of statistical data is possible, in principle.

Indeed, the fact that there is an analogue of the gas thermometer in economics — free market — is a favorable circumstance. As we saw above, we can calculate the absolute temperature of the free market in terms of the mean value of the flow of money per market agent. Therefore we have thermostats with different temperatures in the form of free markets of considerable volume and, at favorable circumstances, we might be able to use them as instruments to study non-free markets.

Creation of such favorable circumstances, however, is rather expensive and requires a great number of large-scale economic experiments whose price is difficult even to imagine. They can, however, be replaced, to an extent, by the study of historical cases where such experiments were performed for some reasons but it would be hardly possible to collect a sufficient number of cases for one particular non-free market.

It seems that the only possible way to study the properties of non-free markets is mathematical modelling or computer simulation, the results of which can be compared with the results of analysis of specially selected historical cases.

CHAPTER 7

Thermodynamic inequalities and Le Chatelieu's principle

§1. Thermodynamic inequalities

In the framework of the theory developed in this book, we get an interesting possibility to derive inequalities that relate different variables in an economic system in precisely the same way as similar inequalities are obtained in thermodynamics, namely thanks to the technique of the change of variables.

The inequalities obtained are, as we suggest, precisely the "economic laws" Carl Menger wrote about. These inequalities are essentially corollaries of assumptions that under any spontaneous changes the system only increases its entropy, i.e., tries to pass to the most probable state. The technique used in the proof of inequalities is based on Jacobians (108). Recall that the Jacobian of two functions of two variables (we will not need more complicated cases) is the determinant of partial derivatives

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}. \tag{108}$$

Obviously, under composition of U and V the Jacobian changes its sign

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(v,u)}{\partial(x,y)}. \tag{109}$$

Moreover,

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = \left.\frac{\partial u}{\partial x}\right|_y. \tag{110}$$

The multiplicativity of Jacobians reflects the chain rule:

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(q,p)}\frac{\partial(q,p)}{\partial(x,y)}. \tag{111}$$

By using the properties of Jacobians we may easily derive various relations between derivatives of the parameters of the system (109). Actually, the basic inequality $\delta S \leq 0$ in the equilibrium state can be expressed by a multitude of ways that show what restrictions imposes this inequality onto relations between various variables in the system and their derivatives. Consider an important example.

The thermodynamic potential Φ given as $E - T_0 S + P_0 V$, where T_0 and P_0 are fixed, attains its minimum at the equilibrium point. Therefore

$$\delta\Phi = \delta E - T_0\delta S + P_0\delta V > 0 \tag{112}$$

under deviations from equilibrium. Let us decompose the flow of money, as a function of entropy S and the flow of goods V, into the variations $\delta\Phi$ and δV up to the second order of magnitude:

$$\delta E = \frac{\partial E}{\partial S}\delta S + \frac{\partial E}{\partial V}\delta V + \frac{1}{2}\left(\frac{\partial^2 E}{\partial S^2}(\delta S)^2 + 2\frac{\partial^2 E}{\partial S \partial V}\delta S\delta V + \frac{\partial^2 E}{\partial V^2}(\delta V)^2\right). \tag{113}$$

Further, since at the equilibrium point, where

$$\frac{\partial E}{\partial S} = T \quad \text{and} \quad \frac{\partial E}{\partial V} = -P, \tag{114}$$

we have $T = T_0$ and $P = P_0$, it follows that substituting δE into $\delta\Phi$ and cancelling terms linear in δS and δV we get

$$\frac{\partial^2 E}{\partial S^2}\delta S^2 + 2\frac{\partial^2 E}{\partial S \partial V}\delta S \delta V + \frac{\partial^2 E}{\partial V^2}\delta V^2 > 0. \tag{115}$$

This inequality holds provided the following two conditions are fulfilled:

$$\frac{\partial^2 E}{\partial S^2} > 0, \qquad \frac{\partial^2 E}{\partial S^2}\frac{\partial^2 E}{\partial V^2} - \left(\frac{\partial^2 E}{\partial S \partial V}\right)^2 > 0. \tag{116}$$

The first of them means that, under the constant flow of goods, to increase the flow of money, we have to increase the temperature if it is positive or decrease the temperature if it is negative. Indeed, by the definition of the temperature,

$$\frac{\partial^2 E}{\partial S^2} = \left.\frac{\partial T}{\partial S}\right|_V = \frac{T}{T \cdot \left.\frac{\partial S}{\partial T}\right|_V}. \tag{117}$$

In statistical thermodynamics, the expression in the denominator is called the *heat capacity under constant volume*. In our case this is the quantity by which we have to augment the flow of money in the system in order to heat it by a unit of temperature. We will call it *the heat capacity of the market under constant flow of goods* C_V. We thus derive that

$$\frac{T}{C_V} > 0. \tag{118}$$

This is the first of thermodynamic inequalities we intend to obtain.

The second one can be expressed in terms of Jacobians as

$$\frac{\partial\left(\left.\frac{\partial E}{\partial S}\right|_V, \left.\frac{\partial E}{\partial V}\right|_S\right)}{\partial(S, V)} > 0. \tag{119}$$

or, which is the same, as

$$\frac{\partial(T, P)}{\partial(S, V)} < 0, \tag{120}$$

where T is the temperature and P is the price. Let us replace variables S, V by T, P. We get

$$\frac{\partial(T, P)}{\partial(S, V)} = \frac{\frac{\partial(T, P)}{\partial(T, V)}}{\frac{\partial(S, V)}{\partial(T, V)}} = \frac{\left.\frac{\partial P}{\partial V}\right|_T}{\left.\frac{\partial S}{\partial T}\right|_V} = \frac{T}{C_V}\left.\frac{\partial P}{\partial V}\right|_T < 0. \tag{121}$$

Having recalled the preceding inequality (118) we deduce that

$$\left.\frac{\partial P}{\partial V}\right|_T < 0, \tag{122}$$

In other words,

> *under the increase of the flow of goods at constant temperature the prices plunge down.*

Observe that we have proved a quite general and far from obvious statement. Note also how this statement was derived. We have started from the fact that $\delta\Phi > 0$, and having expanded δE into a series with respect to δS and δV, obtained a new inequality. But this inequality means that $\delta\Phi$ has a strict minimum at the equilibrium point. The relation obtained is nothing but the condition on the convexity of the domain determined by the condition $\Phi > \Phi_0$ in a neighborhood of the equilibrium point. Under the passage to new variables the transformations of Jacobians fix the same fact. The thermodynamic inequalities obtained is just the same statement expressed in other terms.

This is an extremely general method for obtaining thermodynamic relations. Usually this circumstance is not being stressed in the courses of thermodynamic but in the base of various thermodynamic relations lie simple statements on convexity or concavity of certain functions primarily the entropy and also the free energy and thermodynamic potential.

The fact that for a twice differentiable function to have a strict maximum the second differential should be negative is sufficient to make statements on convexity of this function in a neighborhood of the maximum. We have seen what inequalities are needed to express this fact in terms of the derivatives. We can express these inequalities in various coordinate systems; the property of the function to be convex in a neighborhood of the maximum is coordinate invariant. This is exactly the fact that lies in the method of obtaining these inequalities. This is the place to draw a parallel with the role of convex admissible domains in the standard problems of mathematical economics.

In these problems the convexity is important to establish such properties as the uniqueness of the maximum of the utility function or the existence of a fixed point of the maps. We see that there exists something in common between the standard approach of mathematical economics and the thermodynamic approach developed in this book: the differential and topological invariants are characters of importance. From a very abstract point of view any "equilibrium theory" under all the distinctions between mechanical and thermodynamic approaches mentioned above should be a theory of critical points of maps, i.e., a branch of differential topology [**M**]. Regrettably, as we have already mentioned, this fact did not draw due attention.

Let us now define the expression for heat capacity of the market under constant price as the change in the flow of money needed to raise the temperature by one unit under the constant price in the form

$$C_P = T\frac{\partial S}{\partial T}\bigg|_P . \tag{123}$$

Let us now try to compare C_V and C_P — the heat capacities of the market under constant price and constant value:

$$C_V = T\frac{\partial(S,V)}{\partial(T,V)} = T\frac{\dfrac{\partial(S,V)}{\partial(T,P)}}{\dfrac{\partial(T,V)}{\partial(T,P)}} = \frac{T\left(\dfrac{\partial S}{\partial T}\bigg|_P \dfrac{\partial V}{\partial P}\bigg|_T - \dfrac{\partial S}{\partial T}\bigg|_T \dfrac{\partial V}{\partial P}\bigg|_P\right)}{\dfrac{\partial V}{\partial P}\bigg|_T} = T\,\frac{\partial S}{\partial T}\bigg|_P - T\frac{\dfrac{\partial S}{\partial T}\bigg|_T \dfrac{\partial V}{\partial P}\bigg|_P}{\dfrac{\partial V}{\partial P}\bigg|_T}. \tag{124}$$

If we substitute into this formula the earlier obtained relation $\frac{\partial S}{\partial P}\Big|_T = -\frac{\partial V}{\partial T}\Big|_P$ and recall that $C_P = T\frac{\partial S}{\partial T}\Big|_P$, we get

$$C_P = C_V - T\frac{\left(\frac{\partial V}{\partial T}\right)_P^2}{\frac{\partial V}{\partial P}\Big|_T}. \tag{125}$$

We have just established that $\frac{\partial P}{\partial V}\Big|_T < 0$ which means that, at positive temperature, we have

$$C_P > C_V. \tag{126}$$

In other words, in order to heat the market by a unit of temperature under the constant flow of goods we need lesser flow of money than in order to perform the same under the constant price, i.e.,

under the increase of the flow of money the temperature of the market under the constant flow of goods grows faster than under the constant price.

Translating this statement into the language of economic politics, we may say that in the conditions of restriction onto import the inflation in the system grows faster than under the conditions of free market if the world prices are constant. (Under the conditions of free market the price will be determined by the prices of the world market.)

Of course as we have repeatedly underlined, statements of this type are only valuable for the models. Under a model we understand here ideally typically abstraction in the sense of Weber, abstraction from certain though essential perhaps but immaterial for the *given model factors*. Nevertheless, it was only with this type of models that theoretical economics dealt with, starting with Walras, Javons, Menger through Arrow, Samuelson and Debreu.

§2. The Le Chatelieu principle

The *Le Chatelieu principle* reflects the following remarkable fact:

systems in their strife to the most probable state are capable to exert a certain resistance to the modification of external conditions. Under such a modification they produce "forces" that resist the modifications.

In order to see this, let us consider a system consisting of a market submerged into a universum. Suppose there are two parameters, α and x that parameterize the market and S is the entropy of the system as a whole. Let further the parameter α characterize the inner equilibrium of the market, i.e., if $\frac{\partial S}{\partial \alpha} = 0$ this means that the market is already in the most probable state but not necessarily in the equilibrium with the universum. If, moreover, $\frac{\partial S}{\partial x} = 0$ this means that the equilibrium exists also between the universum and the market. Under disequilibrium conditions the quantities $A = -\frac{\partial S}{\partial \alpha}$ and $X = -\frac{\partial S}{\partial x}$ will be nonzero. Since in equilibrium the entropy should be maximum this imposes certain restrictions not only on A and X but also on their derivatives with respect to α and x in the precisely the same way as in the preceding section where we have obtained thermodynamic inequalities. Under equilibrium we should have

$$
\begin{gathered}
A = 0, \quad X = 0, \\
\left.\frac{\partial A}{\partial \alpha}\right|_{\alpha} > 0, \quad \left.\frac{\partial X}{\partial x}\right|_{\alpha} > 0, \\
\left.\frac{\partial X}{\partial x}\right|_{\alpha} \left.\frac{\partial A}{\partial \alpha}\right|_{x} - \left(\frac{\partial X}{\partial \alpha}\right)_{x}^{2} > 0.
\end{gathered} \tag{127}
$$

Let now as a result of a certain external influence the equilibrium of a market with the universum is violated. This means that X does not vanish any more and the parameter x of the equilibrium of the market with the universum will change. Accordingly, X will change by ΔX_H. This will lead to the fact that the inner parameter α that determines the equilibrium of the market as such will also change. In turn, the change of the parameter x will lead to a new change of the value of X as compared with the initial deviation so the resulting deviation at $A = 0$ (i.e., under the restitution of the market equilibrium) will be equal to ΔX_K.

The question is where will the processes that restore the market equilibrium move the quantity ΔX? It turns out that these processes lead to the diminishing of the initial deviation, i.e., $|\Delta X_H| > |\Delta X_K|$. Indeed,

$$
\begin{gathered}
\Delta X_H = \left.\frac{\partial X}{\partial x}\right|_{\alpha} \Delta x, \\
\left.\Delta X_K\right|_{A=0} = \left.\frac{\partial X}{\partial x}\right|_{A=0} \Delta x.
\end{gathered} \tag{128}
$$

The inequality desired is obtained by the method similar to the one described in the preceding section:

$$
\left.\frac{\partial X}{\partial x}\right|_{A=0} = \frac{\partial (X, A)}{\partial (x, A)} = \frac{\dfrac{\partial (X, A)}{\partial (x, \alpha)}}{\dfrac{\partial (x, A)}{\partial (x, \alpha)}} = \left.\frac{\partial X}{\partial x}\right|_{\alpha} - \frac{\left.\dfrac{\partial X}{\partial \alpha}\right|_{x}^{2}}{\left.\dfrac{\partial A}{\partial \alpha}\right|_{x}}. \tag{129}
$$

Since thanks to equilibrium $\left.\frac{\partial A}{\partial \alpha}\right|_{x} > 0$, we deduce the result desired. Taking into account that

$$
\left.\frac{\partial X}{\partial x}\right|_{\alpha} \left.\frac{\partial A}{\partial \alpha}\right|_{x} - \left(\left.\frac{\partial X}{\partial \alpha}\right|_{x}\right)^{2} > 0, \tag{130}
$$

we have

$$
0 < \left.\frac{\partial X}{\partial x}\right|_{A=0} < \left.\frac{\partial X}{\partial x}\right|_{\alpha} \tag{131}
$$

or

$$
|\Delta X_K| < |\Delta X_H|, \tag{132}
$$

i.e., the processes that restore equilibrium inside the market do indeed partly compensate the influence leading to the destruction of the equilibrium with the universum. We see that systems of thermodynamic type possess certain homeostatic properties that try to resist the disturbance of the equilibrium state. This is the Le Chatelieu principle.

Let us explain the obtained result more graphically. Fig. 1 depicts the situation considered.

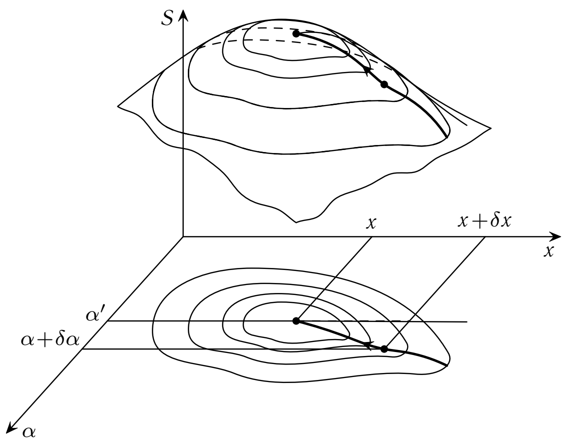

FIGURE 1

After changing x by δx the system loses equilibrium. Nevertheless, the equilibrium begins to be reestablished along the parameter α which in order to maximize the entropy moves into the position α' in which $\left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=\alpha'} = 0$. Obviously, during this we have

$$|\delta S(x + \delta x, \alpha')| > |\delta S(x + \delta x, \alpha' + \delta\alpha')| \,. \tag{133}$$

It is clear that the steepness of the surface $S(x, \alpha)$ will be lesser after a certain increase of the entropy under the process of relaxation of the system along the parameter α.

We see again that the character of the singularity of the function S namely the presence of a maximum, and therefore the convexity of $S(x, \alpha)$ determines essentially the validity of the inequalities constituting the essence of the Le Chatelieu principle.

The Le Chatelieu principle allows one also to derive thermodynamic inequalities. Thus if for a parameter x we take the market's entropy and for a parameter α the flow of goods V, then the Le Chatelieu principle will imply the already known inequalities

$$C_P > C_V > 0. \tag{134}$$

If for x we take the flow of goods to the market and for α the entropy of the market then the Le Chatelieu principle implies

$$\left.\frac{\partial P}{\partial V}\right|_S < \left.\frac{\partial P}{\partial V}\right|_T < 0 \tag{135}$$

In other words, the change of the price under the change of the flow of goods by a unit under the constant entropy is negative and its absolute value is smaller than the change of the price under the change of the flow of goods by one under the constant temperature.

Observe that our deduction of thermodynamic inequalities for economic models *does not differ at all* on the deductions of thermodynamic inequalities in statistical physics.

These inequalities do not depend on the nature of the systems. They are just another form of expression of a fundamental inequality of thermodynamics directly related with the law of entropy growth, namely that in equilibrium we have $\delta S \leq 0$ plus the assumption on differentiability of thermodynamic functions.

In other words, both the Le Chatelieu principle and the thermodynamic inequalities are corollaries on our hypothesis that, in equilibrium, the system attains the most probable state and this most probable state embodies the maximum of a twice differentiable function.

Essentially, thermodynamics is not a physical theory. **Thermodynamics is a theory of how our knowledge on the possible states of elementary systems armed with conjectures on a priori probabilities of these states determines the most probable (equilibrium) states of the more complicated compound systems**.

CHAPTER 8

Market fluctuations

§1. Mean values of fluctuations

Markets fluctuate. This is well known. If we observe how market parameters vary with time for example, consider prices or the amount of goods sold we observe certain fluctuations of these parameters about their equilibrium states. The reasons for these fluctuations may be various but for us it is necessary to differentiate two important types of fluctuations.

The first type of fluctuations is the result of the fact that markets can rarely if ever be considered isolated. As a rule, markets are parts of larger markets and even if the system as a whole, i.e., not only the directly observed part but other connected with it are in equilibrium certain random deviations are possible which will be the larger the smaller system is being considered.

The second type of fluctuations are fluctuations due to speculations and we will study them at the end of this chapter.

We have already seen that the values of the mean quadratic deviation are inversely proportional to the square root of the number of the economic agents of the market. In what follows we will show what kind of theory can be used for calculating mean quadratic fluctuations. This theory is again identical to the fluctuation theory known from the statistical physics. But in our case — being applied to economics — the fluctuation theory is especially viable because market fluctuations is a quite accessible procedure and we will see in what follows the mean value of market fluctuations become related with thermodynamic parameters of the markets. This means that the measurements of the mean values of fluctuations can be used to determine thermodynamic parameters of the markets in particular to define the temperature.

It is hardly needed to explain how important this may be for the construction of a theory. We obtain at last in our hands the measurement tool that can replace special experimental conditions.

Nevertheless, everything said about the artificial reality of the experiment remains valid. In addition to purely probabilistic factors related with the peculiarities of the market structure the fluctuations of the quantities to be measured such as prices and volumes of the goods will be affected by other non-market factors such as social, political, demographic and so on.

Our thermodynamic model of the market ignores these extra facts whereas in reality we cannot get rid of them at best reduce their influence to a minimum by selecting specific moments for the measurements or excluding certain data. As before we face the same dilemma. In order to actually verify a theory one needs to create an artificial reality or, at least, select particular cases which will be close to an artificial reality so that the influence of the factors not included in the model will be reduced to a minimum.

In this sense to measure fluctuations in the given systems at hand is much more economic way to study economic systems than special constructing of experimental situations which among other things can hardly be possible because of an incredibly high price of such experiments. Thus, let us see how the study of market fluctuations can replace such special experiments. In this section we will assume that market fluctuations appear by themselves because of the qualities of the market as such, i.e., we will consider the enforced changes of macroparameters of the market and land speculations non-existent. In the next section we will consider the reaction of the system on the enforced changes. Moreover, what is very essential, we will only consider fluctuations in a neighborhood of the equilibrium point. We will see further that there exist domains of disequilibrium of the markets where a random fluctuation may lead to macroscopic and sometimes however great deviations from the equilibrium — a catastrophe.

It is natural to assume in complete agreement with the man postulate of the statistical theory that the probability of the system to be in a state described by a macro-parameter X is proportional to the number of microstates corresponding to this value of the macroparameters. In other words, the probability $W(X)$ of the system to be in the macro-state X is proportional to the statistical weight $g(X)$ of this state. Recall that these basic principles of the statistical theory are never proved. We define the elementary microstates to be equally probable. This easily implies that the probability of the system to be in the macro-state X is proportional[1] to the exponent of the entropy of this macro-state just due to the definition of the entropy as the logarithm of the statistical weight:

$$W(x) = g(x) = e^{\ln g(z)} = e^{S(x)}. \tag{136}$$

We see therefore that if we are only interested in the relation between macro and micro parameters of the system, there is no difference between physical and economic systems.

If we consider a system in an equilibrium state X_0 we can expand the entropy $S(X_0+\Delta X)$ in the series with respect to X. Since, by definition of the equilibrium state, the entropy is maximum at we have

$$\left.\frac{\partial S}{\partial X}\right|_{X_0} = 0, \qquad \left.\frac{\partial^2 S}{\partial X^2}\right|_{X_0} < 0, \tag{137}$$

Again the properties of the singular point of a map enable us to construct a theory. We see that the probability $W(x)$ of the system to possess the value X of the macro-parameter that differs by ΔX from the equilibrium state X_0 is proportional to

$$W(X_0 + \Delta X) = e^{S_0 - \frac{1}{2}\frac{\partial^2 S}{\partial X^2}\Delta X^2}. \tag{138}$$

Since the exponent decreases very rapidly as the argument grows, the role of $W(X_0+\Delta X)$ for large values of ΔX is insufficient actually and we can with good accuracy obtain the normalized constant for the probability distribution integrating W along ΔX from $-\infty$ to ∞.

Thus we obtain

$$\int_{-\infty}^{\infty} dW(X_0 + \Delta X) = A \int_{-\infty}^{\infty} e^{-\frac{\Delta X^2}{2}} \left.\frac{\partial^2 S}{\partial X^2}\right|_{X_0} d\Delta X = 1 \tag{139}$$

or

$$W(X_0 + \Delta X) = \sqrt{\frac{1}{2\pi} \left.\frac{\partial^2 S}{\partial X^2}\right|_{X_0}}\; e^{-\frac{\Delta X^2}{2}\frac{\partial^2 S}{\partial X^2}\Delta X^2 \; X_0}. \tag{140}$$

It is not difficult to deduce from here the mean square of the fluctuation:

$$\overline{\Delta X^2} = \sqrt{\frac{1}{2\pi} \left.\frac{\partial^2 S}{\partial X^2}\right|_{X_0}} \int_{-\infty}^{\infty} (\Delta X)^2 e^{-\frac{\partial^2 S}{\partial X^2}\Delta X^2} d\Delta X = \frac{1}{\left.\frac{\partial^2 S}{\partial X^2}\right|_{X_0}}. \tag{141}$$

[1]In (136), we set the proportionality coefficient equal to 1.

Therefore, for the probability of systems deviation from the equilibrium state, we obtain the Gauss distribution:

$$W(X) = \frac{1}{\sqrt{2\pi\overline{\Delta X^2}}} e^{-\frac{\Delta X^2}{2\overline{\Delta X^2}}}. \tag{142}$$

Since ΔX is small and the probability steeply drops as X grows, we can simply find the mean square of any function $f(X)$ by expanding it into the Taylor series and confining to the first term:

$$\overline{\Delta f^2} = \left(\frac{\partial f}{\partial X}\bigg|_{X=X_0} \right)^2 \overline{\Delta X^2}. \tag{143}$$

To compute the mean of the product of the fluctuations of thermodynamic quantities, observe that the mean of the fluctuation vanishes thanks to the symmetry of the distribution function relative to the point $\Delta X = 0$. The mean of the product of the fluctuations of independent values $\overline{\Delta\alpha\Delta b}$ also vanishes since for the independent values we have $\overline{\Delta\alpha\Delta b} = \overline{\Delta\alpha} \cdot \overline{\Delta b} = 0$.

Now consider the mean of the product of fluctuations, which are not independent. We will need approximately the same technique of dealing with thermodynamic quantities that we have already used to derive thermodynamic inequalities. If a fluctuation occurs in a portion of the market, which is in equilibrium, this means that we can assume the temperature of the universum and the price constants in the first approximation (for a fluctuation). This in turn means that the deviations of the flow of money from the equilibrium will be given by thermodynamic potential in accordance with arguments given in §6.3[2]

$$\Delta\Phi = \Delta E - T_0\Delta S + P_0\Delta V, \tag{144}$$

where T_0 — is the equilibrium temperature and P_0 — is the *equilibrium price.*

Indeed, the entropy of the market is a function on the money flow. If we perform a modification of this flow in a part of the system then the entropy of the system as a whole will change:

$$\Delta S = \frac{\partial S}{\partial E} \cdot \Delta E|_{P_0,T_0} = T_0\Delta\Phi. \tag{145}$$

The changes of Φ can be found by expanding E into the series with respect to δS and δV. Observe here that the distribution function depends on the total change of the entropy of the system under the fluctuation whereas δS and δV — are the changes of entropy and the goods flow only for the separated part of the system.

In the same way as above we have

$$\begin{aligned} \Delta\Phi &= \Delta E - T\delta S + P\delta V \\ &= \frac{1}{2}\left(\frac{\partial^2 E}{\partial S^2}\delta S^2 + 2\frac{\partial^2 E}{\partial S \partial V}\delta S\delta V + \frac{\partial^2 E}{\partial V^2}\delta V^2 \right) \\ &= \frac{1}{2}\left(\delta S\delta \left(\frac{\partial E}{\partial V}\bigg|_S \right) \right) = \frac{1}{2}(\delta S\delta T - \delta V\delta P) \end{aligned} \tag{146}$$

(where the first order terms cancel).

Here we see the mathematical meaning of the variation of the thermodynamic potential $\Delta\Phi$. It shows how much the money flow deviates from the tangent plane to the surface of state $E = E(V, T)$.

[2]Here we follow [**LL**].

We can now express this change of $\Delta\Phi$ in various coordinate systems. Having selected for example, the variables $\delta V, \delta T$ we can express δS and δP in terms of $\delta V, \delta T$. After simplifications we obtain

$$\delta P \delta V - \delta T \delta S = -\frac{C_V}{T}\delta T^2 + \left.\frac{\partial P}{\partial V}\right|_T \delta V^2. \tag{147}$$

The probability of fluctuation under the deviation of the systems from equilibrium is accordingly proportional to the product of two factors depending on δV and δT

$$W(\delta T, \delta V) \approx e^{-\frac{1}{2T}\ -\frac{C_V}{T}\delta T^2 + \left.\frac{\partial P}{\partial V}\right|_T \delta V^2}\ , \tag{148}$$

i.e., $\overline{\delta V \delta T} = 0$.

It is not difficult to compute the mean quadratic values of the fluctuations by comparing $W(\delta T, \delta V)$ with the Gauss distribution formula[3]:

$$\begin{aligned} \overline{(\delta T)^2} &= \frac{T^2}{C_V}, \\ \overline{(\delta V^2)} &= -T\left.\frac{\partial V}{\partial P}\right|_T. \end{aligned} \tag{149}$$

This gives for the mean value the following value:

$$\overline{\delta P \delta V} = \overline{\left(\left.\frac{\partial P}{\partial V}\right|_T \delta V - \left.\frac{\partial P}{\partial T}\right|_V \delta T\right)\delta T} = \left.\frac{\partial P}{\partial V}\right|_T \overline{\delta V^2} = -T. \tag{150}$$

This relation enables us to measure the temperature of the market by computing the mean of the product of the fluctuation of the market by the fluctuation of the flow of goods. Since the averaging over the ensemble can be replaced by averaging over time, we obtain the following formula for the empirical computation of the market's temperature:

$$T = -\frac{1}{2T}\lim\int_0^T (V(t) - \bar{V})(P(t) - \bar{P})dt. \tag{151}$$

Both the dependence of the price on time and the dependence of the flow of goods on time are accessible, in principle, data, say, for the stock market. Therefore if we assume that no external factors not determined by the structure of the market as such influence the prices and the flows of goods then we have a means to measure thermodynamic parameters of markets.

§2. Fluctuations in time

Let us see how one can consider the dependence of fluctuations on time in the system led out of the equilibrium state. Observe here that we may only consider the not too large deviations from equilibrium states but, on the other hand, not too small ones.[4]

If the initial deviation of the equilibrium state is very tiny, the dynamics of the fluctuations will not differ from the chaotic spontaneous fluctuations. If, on the other hand, the initial deviation is very large, one has to take into account the non-linear effects on the dependence on the speed of the deviation of the quantity under the study on the value of the initial deviation.

[3]See, e.g., [**Tr**].

[4]Voluminous literature is devoted to the study of time series corresponding to the market prices, see, e.g., [**Mi**]. We will not discuss here various methods of analysis and prediction of prices since they are based on universal mathematical properties of time series and various probabilistic hypotheses whereas in our approach the market is considered as a system with a specific property — equation of state — which takes into account the structure of dependence of the entropy of macroparameters of the market.

Therefore we will confine ourselves to a linear case that is we will assume that in the dependence of the speed with which the quantity returns to the equilibrium state on the deviation we can ignore all the in the Taylor series expansion except the first one.

Speaking about practical applications of such an approach for prediction of behavior of time series, say of the price on the share and stock markets this means that reasonable predictions can be only made for short periods of time when the prices are still capable to return to the equilibrium state but the time spans are still larger than the value of dispersion (see Fig. 2 on which overbraced are relaxation periods; the time segments $[t_{i_n}, t_{i_k}]$ (subscripts n and k stand for the first letters of Russian words "beginning" and "end") are the periods during which a prediction can be significant). One can determine the value of dispersion of ΔX by means of the arguments from the preceding section.

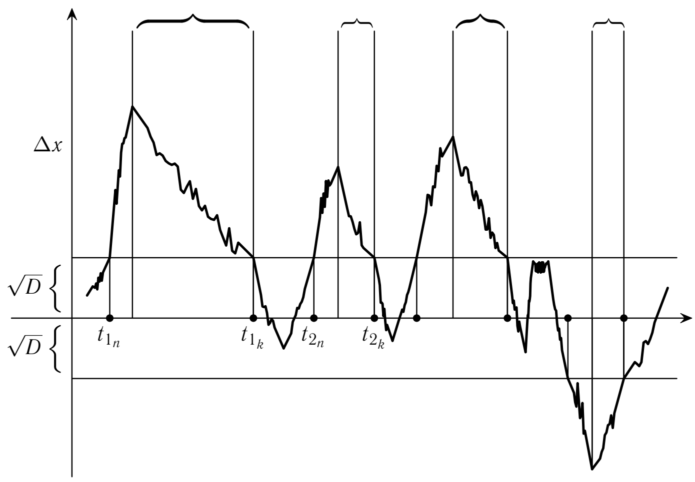

FIGURE 2

Observe that such a prediction is possible far from all markets. As we will see in what follows in order for the purely thermodynamic approach to work it is necessary that the "shadow of future" does not affect too much the behavior of the market agents and the existence of a certain symmetry between the sellers and the buyers.

In other words, the thermodynamic approach will hardly be effective for stock exchange, where the fulfillment of both of the above requirements is hard to imagine but it can certainly be applicable for the commodity markets.

In order to construct the theory of time fluctuations we have to introduce an important value called autocorrelation function. It is defined as the average over the ensemble of the product of the values of the quantities studied separated by a fixed time interval t_0:

$$\varphi(t_0) = \langle \Delta X(t) \Delta X(t + t_0) \rangle. \tag{152}$$

This quantity enables to determine the spectral density of the fluctuations, that is the probability that the "frequency" of the fluctuation belongs to a certain interval. Here speaking about "frequency" of fluctuations we use a metaphorical language because in actual fact we have in mind the existence of processes with a certain characteristic relaxation time t_0.

The frequency is inversely proportional to the relaxation time:

$$\omega = \frac{1}{t_0}. \tag{153}$$

In a more formal presentation, the arguments on a relation of the frequency of fluctuations and the relaxation time are as follows. Consider the Fourier transform of $\Delta X(t)$:

$$\Delta X_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Delta X(t) e^{i\omega t} dt. \tag{154}$$

Then $\Delta X(t)$ can be considered as the inverse Fourier transform of ΔX_ω:

$$\Delta X = \int_{-\infty}^{\infty} \Delta X_\omega e^{-i\omega t} d\omega. \tag{155}$$

This expression for $\Delta X(t)$ can be substituted into the definition of the autocorrelation function for $\Delta X(t)$:

$$\varphi(t_0) = \left\langle \iint_{-\infty}^{\infty} \Delta X_\omega \Delta X_{\omega'} e^{-i(\omega+\omega')t} e^{-i\omega t_0} d\omega d\omega' \right\rangle. \tag{156}$$

Now observe that, in accordance with the main principles of statistical thermodynamics, the averaging over the ensemble can be replaced by averaging over time.

$$\varphi(t_0) = \iint_{-\infty}^{\infty} \Delta X_\omega \Delta X_{\omega'} \left(\frac{1}{T} \int_{-T}^{T} e^{-i(\omega+\omega')t} dt \right) e^{-i\omega t_0} d\omega d\omega'. \tag{157}$$

In the limit as $T \to \infty$ the integral in parentheses becomes an expression for the delta function $\delta(\omega + \omega')$. Therefore for the autocorrelation function $\varphi(t_0)$ we get the expression

$$\varphi(t_0) = \int_{-\infty}^{\infty} \Delta X_\omega^2 e^{-i\omega t_0} d\omega. \tag{158}$$

or, by performing the Fourier transformation, we obtain an expression for the spectral density

$$X_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t_0) e^{i\omega t_0} dt_0. \tag{159}$$

This expression is known as the Wiener-Hinchin theorem[5]

Now we can better understand how the frequencies of the fluctuations and the relaxation time are related. Let the speed with which the variable ΔX returns to the equilibrium position (i.e., to 0) only depends on the values of this variable itself

$$\frac{d\Delta X(t)}{dt} = f(\Delta X). \tag{160}$$

Expanding $f(X)$ into the Taylor series and taking into account that $f(0) = 0$ (i.e., in equilibrium the rate of change is equal to 0) and selecting all the terms of the expansion except the linear one we obtain

$$\frac{d\Delta X(t)}{dt} = -\lambda \Delta X, \tag{161}$$

where $\lambda > 0$, i.e., $\Delta X(t) = \Delta X(0)e^{-\lambda t}$.

[5]The above arguments do not follow from this theorem, they only illustrate the idea. For the deduction, see [**W1**]. For the role of this theorem in the study of physical time series, see [**W2**].

Substituting this expression for $\Delta X(t)$ into the formula for the autocorrelation function we get

(162) $$\varphi(t_0) = \langle \Delta X^2 \rangle e^{-\lambda t}.$$

We have to recall here that we

a) neglect the higher terms in the expansion of $\Delta\dot{X}(t)$ with respect to ΔX

b) consider the values of ΔX greater than the typical involuntary fluctuations, i.e., $|\Delta X| < \sqrt{D}$, where D is the dispersion of the thermodynamic fluctuations of ΔX.

Now we can determine λ if we know the autocorrelation function:

(163) $$\int\limits_{-\infty}^{\infty} \varphi(|t_0|)dt_0 = \langle \Delta X^2 \rangle 2 \int\limits_{0}^{\infty} e^{-\lambda t_0} dt_0 = \langle X^2 \rangle \frac{2}{\lambda}.$$

This means that if we have a time series $\Delta\dot{X}(t)$ we can having computed the autocorrelation function

(164) $$\varphi(t_0) = \lim_{T\to\infty} \frac{1}{T} \int\limits_{0}^{T} \Delta X(t) \Delta X(t+t_0) dt.$$

and integrating it with respect to time obtain the weight with which the variable ΔX returns to equilibrium, i.e., obtain the constant

(165) $$\lambda = \frac{2\langle X_2 \rangle}{\int_0^\infty \varphi(t_0)dt_0} = \frac{2\pi\langle X^2 \rangle}{X_\omega X^2(0)}.$$

where $X^2_\omega(0)$ is the spectral density at zero frequency in accordance with the Wiener-Hinchin theorem.

Observe that for the exponential autocorrelation function the spectral density is as follows

(166) $$X^2_\omega = \frac{\lambda}{\pi(\omega^2 + \lambda^2)} \langle X^2 \rangle.$$

For frequencies smaller than λ, the spectral density is approximately equal to $\frac{1}{\pi\lambda}\langle X^2 \rangle$, i.e., for the frequencies smaller than the inverse relaxation time of the system, we see that the probabilities for ΔX to have such frequency component are approximately equal.

What is the "physical meaning" of the above analysis of thermodynamic fluctuations? Such an analysis is of value when the system considered has two distinct relaxation times.

One — very small — to determine an incomplete equilibrium, i.e., such an equilibrium when it is meaningful to introduce thermodynamic variables for parts of the system. The other one — the large one — corresponds to the total equilibrium.

Speaking about markets this means that it is meaningful to subdivide the market into parts such that each part can be described by a thermodynamic model. Speaking practically, this means that for $\Delta X(t)$ we can take, for example, the difference of prices at different markets since the relaxation time of this parameter is greater than the relaxation time on a particular market. It would be interesting, of course, to consider the processes of globalization of the economics from this point of view.

Observe that to apply the above theory of thermodynamic fluctuations to the study of price relaxations on markets should be implemented with utmost caution for the reasons that we consider in the next section.

§3. "The shadow of future" and the collective behavior at market

Now we have to consider a very important question directly related with the study of market fluctuations. How do fluctuations affect the perceptions of the market agents on the future?

Generally speaking, the market agents have different information on the situation. In accordance with discussion in Chapter 3, it is difficult to expect any coordinated behavior of the market agents anywhere, in particular, in a neighborhood of the equilibrium. Indeed, the very lack of such a coordinated behavior characterizes the "extended order" F. Hayek wrote about. The one who discovers new possibilities, new types of behavior gets an advantage and the multitude of such possibilities is unlimited. Moreover it is unknown.

In the case when a certain stereotype of behavior starts to dominate one should not expect the growth of "order". Contrariwise, one should expect its destruction. This is precisely what happens when a certain idea becomes common for a considerable majority of market agents: for example to invest into a particular type of activity or company. In this case, the shares quickly become overvalued and this sooner or later (usually relatively soon) becomes clear thus influencing a new wave of spontaneously coordinated behavior, this time to withdraw money from the corresponding activity.

Such situations lead not just to market fluctuations, but to considerable oscillations and sometimes to a total transition of the market. Examples of this type are quite numerous. It suffices to recall economic catastrophes in Mexico and South East Asia during the 1990s.

A decisive role in such spontaneously coordinated behavior of the market agents is played by the "shadow of future" that is, perceptions on a possible development of the situation. The result turns out unexpected for the participants because their collective behavior leads precisely to the very result that each of them tries to avoid.

A similar effect is well known in so-called non-cooperative games and is best studied with an example of the game called "prisoner's dilemma". The innumerable literature is devoted to this topic and here is not the place to discuss this problem, still observe that it is precisely in non-cooperative games (though in a somewhat different sense) the radically important role of the "shadow of the future" in the molding of spontaneous patterns of collective behavior have been singled out[6].

In this case the following problem becomes of interest: how are the small fluctuations of the system related to patterns of spontaneous behavior that totally change the market situation? In other words: what is the role of spontaneously formed collective behavior in the problem of stability of market economy?

If small fluctuations of market parameters help to form spontaneous collective behavior that destroys the market equilibrium then the market becomes evolutionary unstable despite of the fact that in "neoclassical" sense such a market should possess an equilibrium.

Lately similar questions are in the center of attention of researchers that try to leave "neoclassical" orthodoxy and extend the frameworks of economic studies in particular in connection with the study of the influence of technical innovations to economics[7]. Here we will confine ourselves to the simple model of "speculative behavior" which shows under what conditions the market fluctuations can be considered as thermodynamic ones.

The study of the process of molding of the stock market price is of particular interest both for creating forecasting models and from purely theoretical point of view since this price is a good example illustrating how a directed activity of a multitude of people based on individual forecasts and decision making leads to a formation of a certain collective variable.

[6]See [**Ax**] and discussion in [**SW**].

[7]See [**AAP**].

The problem of forecasting stock market prices requires a very detailed study of the concrete situation and discovery of a number of factors not only of economic but also of political character. An attempt of construction in mathematical model taking into account all these factors is doomed to failure. Nevertheless observe that many of external factors and also a number of economic factors (for example the level of actual demand of an item of goods) may remain constant during a sufficiently long time though the prices are subject to constant fluctuations. The reason for these fluctuations is a speculative activity. Analysis of dependence of stock market prices on time shows us that for sufficiently short lapses of time the nature of fluctuations often possesses certain common peculiarities. This hints to study speculative oscillations provided the "long-ranged" factors are constant. This makes it possible to model the modification of prices making use of the difference in the "time scale" for price fluctuations caused by speculative activities and oscillations resulting by "long-ranged" factors.

Let us abstract from the real conditions of stop market functioning making several simplifications.

Define the "mean price" $\bar{X}(t)$ as the ratio of the mean amount of a money $\bar{P}(t)$ spent for the purchase of the goods per unit of time to the mean value of goods $\bar{Q}(t)$ sold at the same time:

$$\bar{X}(T) = \frac{\bar{P}(t)}{\bar{Q}(t)}. \tag{167}$$

Recall that in accordance with Chapter on the free market this ratio coincides with the marginal price that determines the market equilibrium, where $\bar{P}(t)$ and $\bar{Q}(t)$ are slowly changing quantities whose value is determined by the productive powers and other slowly changing factors.

Speculative activities lead to the change of both $P(t)$ and $Q(t)$ and this in turn leads to the change of price. The "instant" price $X(t)$ depends on $\Delta P(t)$ and $\Delta Q(t)$:

$$X(t) = \frac{\bar{P}(t) + \Delta P(t)}{\bar{Q}(t) + \Delta Q(t)}. \tag{168}$$

For short lapses of time we may assume that $\bar{Q}$ and $\bar{P}$ are time-independent (this is possible due to the difference of time scales of the changes between the speculative and long-ranged factors.

The idea of the "shadow of future" discussed above suggests that the models of molding the market price should radically differ in structure from traditional mechanical models of equilibrium. The mechanical models of equilibrium describe a future state on the base of our knowledge of the past. Contrariwise, the market price is essentially formed as a result of interaction of goal-minded systems (in other words as a result of correlation of models of the future by people taking decisions to purchase a certain amount of goods). The market agents act on the base of predictions they have and therefore the market price depends on the predictions that the market agents that take decisions stake to. These predictions may depend not only on the price value in the past and present but also on their evaluation of the direction of development of long-ranged factors, on political situation and various other factors.

It is precisely the fact that the market price is molded as a result of forecasts which leads to a certain unpredictability of the market prices. Indeed, in order to predict the price one has to predict the forecasts of each separate market agent.

If we confine ourselves to a simple assumption that certain extra amount of goods and money appearing on the market depends on a possible profit we can express the market by means of the following equation

$$X(t) = \frac{\bar{P} + \sum\limits_l f_l(X(t), \widetilde{X}_l(t+T))}{\bar{Q} + \sum\limits_k \varphi_k(X(t), \widetilde{X}(t+T))}, \tag{169}$$

where l is the index that characterizes the buyers k is the index characterizing the sellers, $\widetilde{X}(t+T)$ is the predicted price for the period T and φ_k and f_l are the functions that characterize the relation of an additional flow of goods and money on the instant and predicted price.

In this form the equation is too general and not fit for investigations but it can serve as a starting point for further simplifications leading to more tangible equations. Our main problem will be investigation of the conditions for which we observe an equilibrium type of price fluctuations — oscillations about a certain mean value which slowly varies perhaps together with the volume of goods Q and the volume of money P.

Let us simplify as follows:

1) Assume that all the buyers use the same forecast and all the sellers use the same forecast (though these forecasts are not necessarily identical);

2) The increase of the offer is proportional to a possible (predicted) profit per unit of goods;

3) The increase of demand is also proportional to a possible profit.

Under these assumptions equation (169) takes the form

$$X(t) = \frac{\bar{P} + \alpha(\widetilde{X}_P(t+T) - X(t))}{\bar{Q} + \beta(\widetilde{X}_Q(t+\theta) - X(t))}, \tag{170}$$

where $\widetilde{X}_P(t+T)$ is the buyer's prediction who use the basis prediction time T and $\widetilde{X}_Q(t+\theta)$ is the seller's prediction who use the basis prediction time θ.

We should expect that T and θ may be rather different, i.e., the market is, generally speaking, asymmetric[8].

Let us simplify further. It is natural to assume that the prediction is determined by the expansion of the price $X(t)$ in the Taylor series with respect to time and terms higher second order are neglected. It is difficult to conceive the influence of the derivatives of the price greater than the second one on human perception: the eye usually catches the first and second derivatives from the form of the curve.[9] Thus the equation (170) takes the form

$$X(t) = \frac{\bar{P} + \alpha\left(T\dot{X} + \frac{T^2}{2}\ddot{X}\right)}{\bar{Q} + \beta\left(\theta\dot{X} + \frac{\theta^2}{2}\ddot{X}\right)}. \tag{171}$$

We have the following alternatives:

a) We may confine ourselves to the first derivatives thus obtaining the equation

$$X(t) = \frac{\bar{P} + \alpha T\dot{X}}{\bar{Q} + \beta\theta\dot{X}}; \tag{172}$$

[8]Cf. Akerloff's hypothesis on market's asymmetry, [**Af1**].

[9]Some, more perceptive, observers penetrate into even finer details: speaking about "the rate of change of inflation" President Nixon implicitly took into account the third derivative, whereas President Gorbachev once mentioned in Pravda that "the tendency of declining of growth rate of our economy has developed lately" which demonstrated his awareness of effects of the fifth, if not sixths, derivative. *D.L.*

b) We may study the more complicated equation (171)

Case (a) is rather simple. Resolving (172) for $\dot{X}$ we obtain

$$\dot{X} = \frac{\bar{P} - \bar{Q}X}{BX - A}, \tag{173}$$

where $B = \beta\theta$ and $A = \alpha T$.

Certainly one can integrate this equation but we will study it in a simpler way. The change of variables $Z = \dfrac{B}{A}X$ leads us to

$$F\dot{Z} = \frac{\Phi - Z}{Z - 1}, \quad \text{where} \quad F = \frac{A}{Q}, \quad \Phi = \frac{\bar{P}B}{\bar{Q}A}. \tag{174}$$

If $F > 0$ (Fig. 3a), that is $A > 0$ and $\Phi > 1$, we have a stable equilibrium at the point $Z = \Phi \left(X = \dfrac{\bar{P}}{\bar{Q}} \right)$.

If $F > 0$ and $\Phi < 1$ (Fig. 3b), then the equilibrium at $Z = \Phi \left(X = \dfrac{\bar{P}}{\bar{Q}} \right)$ is *unstable*. Under a small increase of the price it steeply grows up to the value corresponding to $Z = 1 \left(X = \dfrac{A}{B} \right)$ and under small diminishing falls to zero.

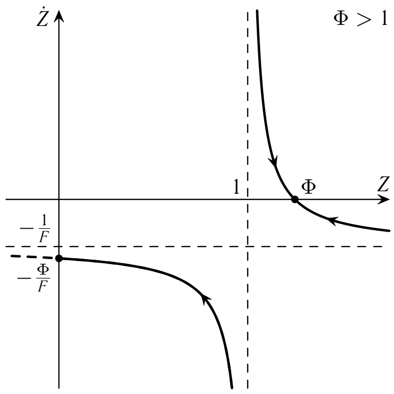

FIGURE 3a

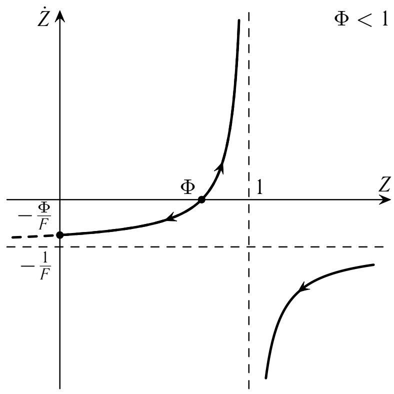

FIGURE 3b

The case $\Phi = 1$ is extremely interesting. In this case there is no equilibrium at all since $\dot{Z} = -\dfrac{1}{F} \left(\dot{X} = \dfrac{Q}{B} \right)$.

For $B < 0$ (Fig. 3c), the price continuously grows whereas for $B > 0$ (Fig. 3d) it falls to zero.

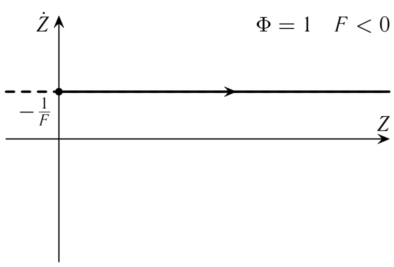

FIGURE 3c

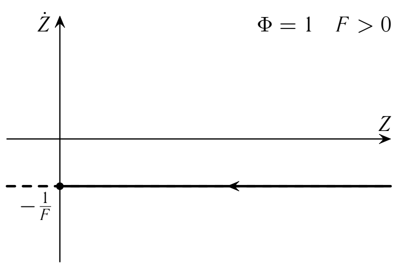

FIGURE 3d

(Obviously only the half-plane $X > 0$ is practically meaningful.) In this situation everything depends on the nature of the sellers' forecast, i.e., do they believe that the raising the price will stimulate the production or one should hide the goods and wait till its price grows up.

Clearly, if the supplies are restricted or if there is a possibility to shrink the production, the case $B < 0$ is realized. In other words, in the case of a monopoly of the seller and objective restrictions on supplies, the price grows unboundedly.

Apparently, **this model adequately describes the phenomenon of sudden plummeting of prices during social unrests and wars and also the inflation in the case of restricted production** (in particular, the growth of prices of the black market in 1970s–1980s in the countries with "real socialism").

Let us now consider another type of prognosis, which takes into account the second derivative of price. Resolving equation (171) for the second derivative we get

$$\ddot{X} = \frac{\bar{P} + \alpha T\dot{X} - X(Q + \beta\theta\dot{X})}{\beta\frac{\theta^2}{2}X - \alpha\frac{T^2}{2}}. \tag{175}$$

We may study this equation by standard means, see, e.g., [**Tr**]. Introduce a new variable $\dot{X} = Y$ and in the system of equations obtained eliminate time by dividing $\dot{X}$ by $\dot{Y}$. We get

$$\frac{dY}{dX} = \frac{\bar{P} + \alpha TY - X(\bar{Q} + \beta\theta Y)}{Y\left(\beta\frac{\theta^2}{2}X - \alpha\frac{T^2}{2}\right)}. \tag{176}$$

We can simplify this equation by setting

$$k_1 = \alpha T, \quad k_2 = \alpha\frac{T^2}{2}, \quad k_3 = \beta\theta, \quad k_4 = \beta\frac{\theta^2}{2}. \tag{177}$$

We get

$$\frac{dY}{dX} = \frac{\bar{P} + \bar{Q}X}{Y(k_4X - k_2)} + \frac{k_1 - k_3}{k_4x - k_2}. \tag{178}$$

Here we see that if $k_1 = k_3$ then the system is an equilibrium at the point $Y = 0$ that is $X = \dfrac{\bar{P}}{\bar{Q}}$ is the expected equilibrium point.

However, if $k_1 \neq k_3$ then $X = \dfrac{\bar{P}}{\bar{Q}}$ is not an equilibrium point.

This is an astonishing result that shows that under asymmetric conditions taking into account the second derivative we eliminate equilibrium. Asymmetric markets behave totally unexpectedly.

So far our considerations resulted in rather unexpected corollaries: The "shadow of future" for the linear forecast restricts the domain of a stable equilibrium but even for the asymmetric forecasts of sellers and buyers it does not totally eliminate the equilibrium.

Adding the second derivative into the forecast (which amounts, actually, to professionalisation of the forecast) completely eliminates equilibrium for the asymmetric markets. In other words, the market becomes globally unstable.

This deduction is extremely important for our further analysis. It means that there appears a possibility to manipulate the market using restricted resources. In other words, one can turn the market in a "heat machine" of sorts by creating a symmetry in the ways the market agents perceive the future.

CHAPTER 9

“Heat machines” in economics

§1. Speculation and “heat machines”

An interesting question: can one extract money from the market making use of its thermodynamic properties?

It is well known that in physical systems one can extract energy using the difference of temperatures of two systems. The corresponding device is called a heat machine. In the simplest form the performance of a heat machine is as follows. The working body is heated by means of a source of high temperature T_1 and forced to perform a work and then it is cooled in a media with temperature T_2 after which the cycle is repeated. The performance of such a machine becomes possible thanks to the dependence of the volume of the body on temperature: expanding the body is capable to perform work.

In most elegant formulation this works as follows: on the level surface of the system, say $P = P(S, T)$ one can draw curves of isotherms $P = P(S, T_0)$ for different T_0 and curves of adiabats $P = P(S_0, T)$ for different S_0. On the coordinate net obtained we separate any curvilinear rectangle bounded by two isotherms (for T_1 and T_2) and two adiabats (for S_1 and S_2). Moving the system along the boundary of this curvilinear rectangle we will perform the Carnot cycle.[1] Depending on the direction of the movement a work will be either expanded or gained and its value is equal to

$$A = \int T dS = T_1(S_2 - S_1) - T_2(S_2 - S_1). \tag{179}$$

This relation enables to determine the efficiency of the heat machine, i.e., the part of the heat that can be turned into work.

Since the expanded heat can be represented as $Q_3 = T_1(S_2 - S_1)$ this portion is equal to $\frac{T_1 - T_2}{T_1}$ where T_1 is the temperature of the heater and T_2 is the temperature of the cooler. We ask: is it possible to use the principle of the heat machine, i.e., the idea of turning heat into the mechanical energy in economics?

In a sense, the answer is trivial: yes, of course, this is possible and is performed since long ago. The simplest example of such an operation is a purchase of goods where it is plentiful at low price and selling where it is insufficient at high price. If the temperatures of the markets are equal the profit obtained is equal to $V(P_2 - P_1)$. This expression corresponds to the transformation of the difference of pressures obtained through heating into the mechanical energy.

If we perform sufficiently many such operations then the amount of goods at market 1 will go down and its price will rise, whereas the amount of goods at market 2 will grow and its price will go down. We see that we cannot make a cycle. The market business results in levelling the price though a temporary difference between the prices can be used to gain a profit.

[1]For a detailed discussion of the Carnot cycle and its role in thermodynamics, see, e.g., [**So**].

Now suppose that we are able to regulate the temperature of the market. As we saw in Chapter 6, the price depends not only on the volume of the goods delivered but also on the temperature. Now we have a possibility to realize a simple scheme: cool the market — buy goods, heat the market — sell goods. The profit will again be determined by the formula $V(P_2 - P_1)$ where V is the volume of goods bought at temperature T_1 and sold at temperature T_2. But since $P_1 = \dfrac{T_1 N}{V}$ and $P_2 = \dfrac{T_2 N}{V}$ the profit resulting in such operations will be equal to

$$D = \frac{1}{2}\left(\frac{T_1 N}{V} - \frac{T_2 N}{V + \Delta V}\right)\Delta V = \frac{N \Delta V}{V}\left(T_1 - T_2\left(1 - \frac{\Delta V}{V}\right)\right). \tag{180}$$

Selling an additional portion of goods will lower the price and the time during which the profit is gained constitutes only half of the cycles period (assuming that switching the temperatures is instantaneous).

In principle, such a scheme can be realized if we have two markets with different temperatures and market monopolies. From the expression for the profit we see that in order to gain profit we should satisfy the requirement

$$\frac{T_1 - T_2}{T_2} > \frac{\Delta V}{V}, \tag{181}$$

i.e., there is a natural restriction on the amount of the extra goods ΔV to the amount of goods on the market V. The scale of speculations is also restricted by the difference of markets' temperatures.

We see that the nature of cyclic speculations on markets is the same as that of Carnot cycle. The question is how to ensure sufficiently large difference of temperatures. In the case where spatially separated markets interact we can make use of the natural difference in temperatures or the difference in the levels of delivery of goods on the market which may arise for multitude of reasons: thus for example, the plants producing spices do not grow in Europe which in the Middle Ages amounted to a natural difference in prices at respective markets. To regularly gain profit one needs a market monopoly, otherwise the prices would have been essentially equalized. It was also stupid to import too many spices to Europe.

The case of two weakly connected markets is trivial.

What is not trivial, we think, is consideration of speculative operations as a heat machine. What is also non-trivial is the possibility to manipulate the market's temperature. Consider the share market where the price of the share is determined by the demand and the demand is mainly determined by the firm's perspectives. The information on the firm's perspectives is a very delicate matter and not very reliable one. Therefore the majority of market agents are governed by the general trend, i.e., they try to forecast the price on the base of its change.

This is precisely the situation considered in sec. 8.3. In this case in certain circumstances (for example, for asymmetry of forecasts of the seller and the buyer) the behavior of the price may be unstable due to sudden changes in the flow of money or goods delivery. But it is precisely the amount of money flow that determines the temperature under the constant number of market's agents. Under certain conditions this parameter becomes subject to influences caused by fluctuations. This in turn means that being able to govern fluctuations that is, using relatively small resources to sell or buy shares we can steeply change the market's temperature. In this case it is easy to realize the cyclic heat machine able to extract money from the share market. The cycle looks as follows. One sells a relatively small amount of shares of a company, small but sufficient to cause a collective reaction, i.e., instability. After a considerable fall of the price one buys shares causing new instability leading to the price of shares and the situation returns to square one but the "hares" collect the difference between the total price of shares sold (at high price) and bought (at low price).

The cycle can be performed in the opposite direction playing "for raising". These differences in the direction of how one circumvents the cycle precisely correspond to the difference between the heating machine and the fridge in the Carnot cycle.

In the first case the player gains extra money, in the second one extra shares. The effectiveness of the operation depends on how much one can change the temperature of the market: to lower it ("Bull game") or raise ("Bear game").

This effectiveness is very dependent of course how close to instability the market is. Generally, to perform such operations in order to "disbalance" the market and change the temperature one needs considerable resources. A good example of such operations are speculations on international markets of currency. Here it is very interesting to compare the points of view of politicians of countries whose currency is plagued by such operations and the actors of the international currency markets.

In 1997, at a conference of the International Monetary Fund in Hong-Kong the Prime Minister of Malaysia, M. Mahatir, called for necessity to ban or at least restrict speculations on the international monetary market and accused the actors of this market in particular, the known American businessman G. Soros in intentional damaging Malaysian currency (as a result of currency speculations the Malaysian ringgit dropped 20% just before that). Mahatir's claim caused immediately further drop of ringgit by 5%.

Replying to Mahatir's accusation G. Soros claimed that speculations do not affect healthy currencies and only weak or overvalued currencies can suffer such attacks[2]. It is interesting to analyze such debates in the light of the above model. The analysis in Section 8.3 implies that stability domains of the market can considerably vary the sizes or totally disappear depending on the nature of forecast used by market agents. For factors forming the forecast one can take not only the study of the price changes during a preceding period but also rumors, statements of politicians, influential financiers and so on.

Under these conditions there is generally no objective stability of the market. Its stability depends on the nature of prognoses of the majority of its agents. But these subjective prognoses are also a part of reality and sometimes they are very flexible but can be extremely conservative. Therefore the result of speculations depends essentially on the following factors: on the degree the prognoses are conservative and on their nature (what exactly is subject to be forecast) and finally on the amount of goods used to perform a speculation (this is in close connection with the prognoses: financiers believe that there is no sense to try to disbalance a stable market).

Obviously, the larger resources an agent or a group of market agents starting to play possesses, the higher probability of their success. However, to precisely predict the result is difficult since in order to predict it one should know very well the nature of prognosis of the other participants in the market play and potential resources of the one who is attacked.

But for the case of speculations on currency markets these factors are usually very well known. G. Soros manifestly knows what he speaks about saying that without being sure in success there is no sense to start the operation. There is however, an asymmetry in potential losses of the sides. Having started an unsuccessful speculation the financier risks but a little: having not obtained the effect desired — the steep change of the market's temperature — one can return to square one without essential losses. But for the country whose currency is being attacked the stake is the amount of national wealth and possible its evaluation

[2]On the dialog between the Prime Minister of Malaysia Mahatir and G. Soros, see Far Eastern Economic Review, 1997, September 25, October 2, **160** (39), p. 40.

(through the cross-currency exchange rate) becomes subject to the nature of prognosis of the dealers of the currency market, i.e., on subjective factors that governments can influence it seems much less than large-scale financiers.

To predict a possible behavior of the government is not an easy task either. We know from history long periods when many governments refused to make their currencies freely convertible. Therefore too great successes of financiers in the usage of heat machines in international finances may result in sharp reaction of interested governments.

The above model of the market shows a possibility to use "heat machines". But one should understand that financial heat machines are external devices in relation to the market and not its parts. This actually is one of the principal deductions we obtained as a result of application of the notion of heat machine to economics.

The market can well exist without financial heat machines: they are not necessary components of the market.

They are intellectual superstructures over the market that enable one to extract profit not from production of goods or selling them but from cyclic operations, i.e., away to extract money from the market agents who are already sufficiently poor, that is, do not have possibilities sufficient to influence the market situation.

Unfortunately this is precisely the position occupied by the population of various countries, the population having on their hands paper money issued by the government.

Certainly the role of financial "heat machines" in globalization of economic processes is huge. Thanks to them and also to the world market we establish a certain analogue of a global economic equilibrium. To create a certain "code of behavior" in this domain of economic activity may become necessary at least to avoid the growth of economic populism which threatens the freedom of market relations.

§2. The government and the economics

Let us see what are the possibilities of the government concerning means to extract resources from the economic system in our thermodynamic approach to economics.

For the country it is vitally important to have a reliable pattern of taxation. A natural way to extract resources is to tax transactions and immovable property. In the case of immovable property and also transactions controlled by the government (for example, when the goods cross the border of the country) there are no problems. The problem appears when the collection of resources is based on taxation on non-controllable transactions. Such a scheme turns out to be extremely ineffective.

The most notable example is the income tax. In order to collect this tax we should have a possibility to control at least in principle transactions of tens of millions of people. So here the principle of "Maxwell's Demon" starts to work: it is impossible to control the transactions (i.e., get the corresponding information) without spending some resources.

The firm must possess a documentation on transactions, whereas to force private persons to lead such documentation is practically impossible and to verify all the transactions is rather difficult since the number of potential sources of income is practically infinite.

The verification of the income also costs money, its own price of transaction. In order for this verification to be effective its probability should be rather high. This means that to perform these verifications full-scale (for tens of millions of tax payers) one should spend huge amount of money with a low probability to find tax evasions. Transactions not declared are usually never registered. In the countries with a considerable portion of "black" or "gray" economy such a method of tax collection is extremely inefficient and amounts to huge losses for the government because both "black" and "gray" transactions by definition cannot become taxable: to declare above them will mean at least the loss of a source of income for the tax payer in future.

Since the price of verifications depends on their number it is obvious that the effectiveness of taxation is inversely proportional to the number of subject of taxation and this is the main argument in favor of refusal of income tax and replacing it by the value added tax or turnover tax and the tax[3] on immovables. Obviously, this being implemented, the effectiveness of taxation will be much higher.

Why then despite the obvious drawbacks of this institution the majority of countries still collect the income tax?

Arguments on social justice can hardly be a sufficient justification. It is not difficult to correct the taxes by means of a value added tax collecting it from the luxury items at highest rate.

Provided the argument on social justice can, to an extent, be applicable for the explanation of the practice of income tax with the countries with democratic government (say, in the light of economic illiteracy of the majority of populace) it is difficult to find reasons for the effectiveness of this argument in authoritarian governments.

Among other things, the collection of income tax is doubtful from the liberal law system's point of view since it actually abandons the presumption of innocence principle during accusations in tax evasion and contradicts the principle fixed in many democratic constitutions according to which the man does not have to witness against himself or herself.

This is why in the USA (say) the tax payer has a right to refuse to fill in the tax declaration (and in this case the tax is being computed by taxation organs on the base of their own estimation of the subject's expenditures).

The inefficiency of the income tax striping off the government the incomes from the "black" and "gray" sectors of economics and a possibility to replace it by the value-added tax became subject of serious discussions lately in the USA in relation with the growth of these sectors of economics (especially drugs selling).

The above analysis justifies another interpretation of continuation of the practice of income tax collecting: it can be considered as a means of a political and economic control over subjects. It is precisely the totalitarian nature of this institution which is it seems the main argument in favor of its existence.

It does not give much money to the government but it establishes who is rich and who is poor — the information of important political flavor. It is not by accident that the income tax was discovered in Venice by the "democratic oligarchy" notorious by its sophisticated means of control over citizens' behavior.

From the point of view of our thermodynamic model of economics income tax collecting is equivalent to absolutely unthinkable and as the study of the paradox of "Maxwell's Demon" show simply non-realizable way to separate system's energy. Indeed, just imagine a construction of a robot which takes from each of the particles of gas a percentage of its kinetic energy and in turn depends on energy.

This metaphor (which as we have seen above with the examples of the study of economic processes by means of statistical thermodynamics is much more than just a metaphor) makes inefficiency of income tax absolutely manifest. Observe, however, that various governments retaining this economically inefficient and costly institute for political goals use it in order

[3]Income tax was introduced in merchant societies (e.g., Venice and the Netherlands), so small in size that rich people knew each other and could perceive the size of each other's wealth. As the society grows, a Demon enters into play.

to extract money from the populace much more effective economic methods we discussed in the preceding section: manipulation with the monetary market, valuable papers and shares.

Since the central banks of the countries are the largest reservoirs of the resources, it is not difficult for them to realize various heat machines which enable them (of course in the limits bounded by the possibilities of total destruction of the economics) extract considerable resources from the economic system which represents interrelated markets at different temperatures. Since some of these markets, in particular share markets, are very easy to control, that is to change their temperature almost does not require any financial input, it follows that to control them, only mass media or just controlled "leakages of information" from the decisive organs are usually sufficient.

The possibilities of governments to extract distributed resources from economic systems are practically unlimited up to a total destruction of economics.

This is one of the main arguments which stimulates the civil society usually through the means of parliamentary control to rigidly curb the activity of their governments. Empirically this is testified by the practice of creation of central banks supposedly independent on governmental decisions and also (in the cases when due to the peculiarities of political culture to ensure independence of the central bank is difficult) the creation of currency boards that tie the national currency to one or several stable foreign currencies in order to exclude possibilities of political manipulations in economics performed by central banks under the pressure of the respective governments.

CHAPTER 10

The limits of rationality: the thermodynamic approach and evolutionary theory

§1. Rationality and uncertainty in economics

In the preceding chapters we have shown how to construct a mathematical theory of economic equilibrium without the notion of utility. In this approach we have to answer: what is rationality in economics?

If the prices are formed not as a result of maximization of utility but due to the fact that certain states of the system turn out to be far more probable than the other states (have greater entropy) what is the role of human decisions in economics? To what extent these decisions can rely on a rational analysis at all? What can one know about the economic situation?

In order to answer these questions we have to introduce a very important distinction concerning the types of knowledge on the situation and the types of economic decisions. The first ones are subdivided into "micro-knowledge" and "macro-knowledge" and the second ones into "micro-decisions" and "macro-decisions". Without making this distinction it is impossible to discuss the problem in general since the character and possibilities of application of "micro-knowledge" and "macro-knowledge" are totally different and the difference in uncertainties the subject of economic activity encounters with in the domain of "micro-decisions" and "macro-decisions" are cardinal.

The owner of a shop or manager of a small firm in their search for acceptable deals is confronted with uncertainty of prices.

The prices are different in different places and since there are many possible sellers the uncertainty of the situation is related first of all with the fact that not the whole information on the market is available. Somewhere perhaps nearby there is a big lots seller capable to sell at a price lower than the ones known to the shop's owner but to find this seller in the chaos of market is sometimes very difficult.

The "micro-knowledge" is first of all the knowledge about such perhaps rare cases, the knowledge where and when one should turn in order to obtain the goods at a low price and to sell it at high price. Such knowledge is based on the connections and is a result of participation in an informal informational networks. However perfect the formal system of market information would be the informal contacts and operative knowledge will always give an advantage at least because the information cannot be instantly included into the formal commonly available net and even it is, one has to be dexterous in extracting it. There is always a certain time lag between the moment of availability of a possible deal and the general spread of this information whereas the resources for making a deal may be exhausted faster than the official information becomes available. The macro-knowledge is an understanding of the global situation on the market, the knowledge of general tendencies in the developing of prices, availability of resources, possible consequences of the decisions that influence the market situation as a whole. The character of uncertainty for the "macro-knowledge" and "macro-decisions" is totally different than on the "micro" level. In many ways the economic "macro-knowledge" is determined by the ideas on the character of the economic equilibrium, in other words, the equilibrium metaphors used in the analysis of the economic situation.

The character of rationality differs accordingly with the differences in the character of knowledge on the micro and macro levels. Lately, in the theoretical economics, the study of the rationality problem became one of the central topics, see [**HH**].

The insufficiency of the model of the rational choice to explain economic phenomena becomes more and more obvious especially in connection with the growing interest to the study of economic institutions, in particular, property rights. In his time, R. Coase [**Co**] pointed out that the economic equilibrium depends on the prices of transactions whereas the prices of transactions are directly related with the property rights, see [**Wi, NoI**]. Therefore the work of Coase initiated in economics an interest to the study of social institutions. It is not difficult to find out that institutionalized processes of decision-making are regulated not by a rational choice between alternatives with the help of utility functions but certain routine rules to a large extent traditional.

The discovery of this fact required a serious modification on the point of view of the nature of human activity in economics. If not all human activity is determined by utility functions then how to construct equilibrium models? To answer this question in the framework of the neoclassical approach is very difficult, if possible at all.

Thus in economic theory in addition to the "instrumental rationality" related with utility one more type of rationality appears (the "procedural" in terminology of Hargreaves–Heap). H. Simon intensively studied the procedural rationality both in his works on theoretical economics [**Si**] and in connection with his activity related with artificial intelligence. H. Simon connected the appearance of procedural rationality with the restriction of computational possibilities of humans (theory of "bounded rationality"). In other words, being unable to compare all the possible alternatives H. Simon relies on the procedures formed from experience and they become conditional agreements that form the structure of social institutes.

Chess play is a good metaphor for an explanation of this situation: one cannot evaluate and compare all the moves allowed and one is forced to use the standard schemes, debuts, difficult combinations, general positional principles, and so on. It seems that the problem is more serious here than the near restricted ability for calculations. We think we have to admit that life is not a play with fixed rules. The list of alternatives is open: the alternatives of actions can be created by our mind. But in this case there should appear at least one more type of rationality, which we would like to call ontological rationality. There should exist certain rules that regulate the inclusion of alternatives into the list and in principle regulating generating alternatives, see [**Sco, SB1**].

This can be only performed if we rationalize the fact that the world in the perception of a subject possesses a certain ontology. In other words, there exist certain ways to perceive what is real and what is essential and should be included into the consideration. There should also exist mechanisms for generating alternatives. Here obviously metaphors and examples are of huge importance, see [**Sco**].

Apparently Hargreaves–Heap had something close to this conception introducing the notion of "expressive rationality". He wrote that the expressive rationality is defined by the universal human interest in understanding the world in which we live. It is thanks to our purposefulness we have to give sense to the world: the world must be rationally described if we want to act in it. We have in mind the necessity of a cosmology which answers the question on the meaning of this, on the foundations for that, on the interrelations of an individual and the society and so on, see [**HH**].

We believe that the conception of ontological rationality better grasps the problem concentrating attention on the most principal question in the theory of social sciences: where and how the human takes the alternatives from which he or she chooses?

The assumption that the alternatives are given externally introduced in the theory of rational choice does not correspond to reality and rudely bounds the sphere of human freedom.

We believe that from the positions of cognitive analysis of the decision-making process one can justify the existence of three types of rationality. The first phase of this process — the formulation of alternatives — is an activity which possesses its own logic and does not reduce to cataloging of alternatives already known or enforced by the environment. It is connected with ontological rationality.

The second phase of the process — a procedural development of alternatives (something similar to the plans at general headquarters compiled in case of possible conflicts) corresponds to procedural rationality.

The third phase — the evaluation of constructed and procedurally formed alternatives with the help of a given[1] hierarchy of values — corresponds to the theory of rational choice (instrumental rationality).

Of course this is a very simplified scheme. In reality the phases coexist, a cyclic return to a previous phase is possible and so on. But such a scheme gives at least a framework for a much deeper analysis of human activity in general and economic activity in particular than the theory of rational choice which has eliminated the two most important phases of the process of decision-making — the ontological and procedural ones.

The presence of such a scheme makes it possible to explain how the thermodynamic approach works in economics. The most essential is the fact that various subjects not only possess a different choice of alternatives (and some of these alternatives are better and some are worse from the point of view of optimization criterion) but tend to discover new alternatives.

Under conditions of institutional restrictions imposed by historical tradition, political pressure of interested groups, and so on, only the process of competition at large ensures the equilibrium in the sense we introduced, that is the equilibrium as the lack of flows between the parts of the system.

The presence of institutional restrictions in economics stunned the economists of the neoclassical school for a long time. It is interesting to note that R. Coase in his well known paper [**Co**] poses a seemingly strange question: **why do firms exist?**

This question is sudden only for the one who perceives the world through the neoclassical ontology. Indeed, why a part of transactions became integrated inside of corporations instead of being performed through market relations?

The answer to such questions required a reconsideration of the methodology of economic studies. Together with the problem of fundamental uncertainty the interest to which was initiated by works by Knight [**Kn**] and Keynes [**Ke**] and also the accent on the importance of the study of *human actions* under the uncertainty conditions, the action made by all the representatives of the Austrian school after von Mises [**Mis**] the institutional analysis forced theoreticians to pass from the study of equilibrium to the study of evolution of economic institutes. The main idea of this passage is to try to discover on the level of evolutionary process something we cannot discover in the activity of an individual market agent (realization of utility's maximality) because of the uncertainty and institutional restrictions.

[1]For the social sources that "give" or predestine the hierarchy of values, see, e.g., E. Berne, *Games People Play : The basic handbook of transactional analysis.* Ballantine Books, 1996, 216 pp.

§2. Rationality and evolution

As a result of mathematization of the neoclassical economics the "invisible hand" practically disappeared from the economic theory being reduced to the existence theorem for a vector of prices. The powerful metaphor of self-regulating of the market by means of individual interests which convinced A. Smith that in economics, unlike political sciences, a principle of spontaneous equilibrium acts provided an ontological justification of the economic theory for more than a century. A gradual dissolving of this metaphor in abstract mathematical constructions could not but worry theoreticians. F. Hayek returned, as we saw above, to the idea of "invisible hand" [**HRS**] suggesting to consider the evolutionary process as such an "invisible hand".

In 1950, A. Alchian published a paper [**Alc**] in which on the model level he showed how the evolution can replace human rationality. Alchian's initial position was that market uncertainty devalues instrumental rationality. He suggested to consider the process of natural selection as a means that determines which type of business activity survives and which does not. As a filter of evolution, Alchian suggested to consider the value of profit as a result of economic activity provided the successful patterns of business activity are copied by other participants of the economic process which guarantees proliferation of the corresponding pattern.

Alchian's work continues to provoke animated discussions until now [**Lo**].

Certain positions of his work were criticized: in particular, the possibility to reproduce the pattern [**Wnt**] the assumption on sufficiency of the positive profit as a filter of evolution [**Da**] but general very high evaluation of his work is based on the belief that Alchian indeed managed to return to economics the strong version of the "invisible hand".

In this way again now on the level of modern science we obtain a mechanism that establishes a rational order by a means not depending on the degree of rationality of particular participants of the process. This order is not a result of somebody's plan and is achieved by decentralized activity. Alchian's works demonstrated great possibilities of the information theory in economic analysis. One of his followers recently wrote:

> "*My own understanding of economics was under heavy influence of Alchian's ideas. I would have formulated the base of his teaching in the phrase: "Everything in economics is information theoretical...*"." [**V**]

Alchian's ideas heavily influenced the development of institutional economic theory in works of O. Williamson, D. North and others [**Wi**].

Indeed, despite an obvious attractiveness of the idea to take into account in the economic theory of the prices of transactions, property rights, and so on, it was totally unclear how to perform this in the framework of the neoclassical model of the rational choice. The difference of the institutionalized behavior from the behavior in the frameworks of the models of the rational choice is in the practical difficulty to ascribe any utility value to institutional procedures based on conventions. Even if one performs this inside the intra-institutional behavior it is still totally unclear how to relate such a utility with the effectiveness of the institution as a whole.

The idea to consider evolutionary process as a global understanding of sort which guarantees rationality by selecting and eliminating the rules of behavior unable to compete gave a theoretical foundation for the institutional analysis comparable with its force of conviction with the neoclassical equilibrium theory but essentially surpasses it in applicability to study real economic institutions. The institutional scope in theoretical economics created a considerable revival of the interest to the ideas of the "Austrian school" practically completely forgotten in 1950–60s[2]

Nevertheless, the evolutionary approach did not quite solve the problems that the economic analysis of market processes faced when took into account such factors as the restriction of information, restrictions in behavior related with the cultural tradition, taking into account of the prices and transactions, and so on. The heart of the problems consists in the utmost labor consuming feature of the modelling of evolutionary processes and feasible to perform only for very simple systems. Moreover, for solutions of such problems there is no developed analytical tools. Mainly evolutionary models are being demonstrated by computer simulations. This does not preclude one to make important theoretical deductions but the poverty of the analytical apparatus clearly manifests the weakness of the approach.

The evolutionary analysis deals with micro-knowledge and micro-decisions but cannot say practically anything on macro-knowledge and macro-decisions[3]. Meanwhile it is impossible to deny the role of macro-decisions. It goes without saying that a certain spontaneous order is being developed in the process of evolutionary selection. But what to do in the cases when one has to "correct" the activity of the "invisible hand" which depends itself on certain macro-parameters, such as for example financial and taxation law, the practice of licensing of certain types of economic activity, and so on?

The evolutionary theory can hardly help in this case[4]. The uncertainty problem and insufficiency of information on macro-level exists nevertheless and is represented perhaps at a greater scale than on micro-level. Our book is devoted to the development one more, thermodynamic, way to introduce the "invisible hand" to economics.

In Chapter 4 we have discussed the relation between the mechanical and thermodynamic versions of the "invisible hand". What is the relation between the "invisible hand" of the evolution and the "invisible hand" of thermodynamics?

§3. Evolution and thermodynamics

In order to apply the evolutionary approach to the study of economic processes, one has first of all to have a clear structure of the evolutionary theory. This theory contains two extremely important but weakly related aspects. The first aspect concerns the birth of innovations in the system and the second one the mechanism that selects innovations.

In the theory of biological evolution, the first aspect caused a huge amount of disputes. The distinction between the principal versions of the evolutionary theory — Darwinism and Lamarkism — is related with the different understanding of this aspect of the theory. In Darwin's theory innovations are totally random whereas in Lamark's theory they are the result of education. At present we have no direct experimental testimonies in favor of inheritance of features obtained. But this does not mean at all that Darwin's theory has no difficulties.

The main difficulty of Darwin's theory is how to explain appearance of a complicated construction as a result of random mutations: such a construction consists of elements which can be only created by separate independent mutations but it gives a preference in the selection only when all the necessary elements are already present and assembled into a functioning mechanism such as a hand or an eye.

[2]See, e.g., [**NoI**]. Observe that a revival of interest to the Austrian school of economics is related with the Nobel prize in economics awarded to F. Hayek.

[3]For a discussion of potential of the evolutionary theory, see [**Ho**].

[4]Cf. the discussion of potentialities of evolutionary theory in [**Ho**].

We do not have to discuss these problems here since thankfully the economic theory does not have to speak about a birth of a new structure (this is a result of human activity) but the selection problem stands in full height. Essentially the selection problem is a purely thermodynamic problem. One has to determine what is the selection of the "organisms" or the "nutritional niches" in order to understand for example how the restriction of the number of "nutritional niches" affects the distribution of the "organisms" or the appearance of new "organisms" with different properties modifies the general distribution over the "nutritional niches".

It is not difficult to see that in the process of selection evolutionary stable states are being created and we can identify them with the equilibrium states of thermodynamic systems. Since in the evolutionary approach to the economic theory there is no need to introduce notions of generations, inheritance and genes [**Ho**] the evolutionary theory applied to economics becomes much simpler than in biology.

Here we only speak about the mechanism that filters institutional innovations. As A. Alchian observed, this is a pure problem of information theory [**Alc**]. But due to the practical identity of the information theory and thermodynamics the problem of innovation filtrations becomes a problem of search for an equilibrium distribution function for economic "organisms" over certain "micro-parameters" determined by the structure of the evolutionary filter. Consider one more example how this idea works. Suppose we have N firms each of which being characterized by a certain annual profit ε. Define the structure of the filter as follows: the profit should be positive. This is sufficient in order to construct the distribution of firms according to their incomes under the equilibrium — that is, at temperature T and the migrational potential $\mu(T)$.

The meaning of the approach consists in a way to define the distribution function of the firms according to their incomes having given the spectrum of possible values of the income and assuming that each of these values can be "populated" by any number of firms and also taking into account that the system is in equilibrium.

The equilibrium is understood here in the sense that if we separate the system into parts according to the parameters not related with the study of income distribution (for example, considering geographical location and assuming that this parameter and the income are independent) then the distribution function is preserved under such partition of the system. In this case the preservation of the number of "organisms" or the income is inessential. We only need an empirical assuredness in the existence of the invariant distribution function for the corresponding equilibrium parameters in our case - the temperature and migrational potential. We can consider this system as a number of systems that appear and disappear during an infinite number of equal time intervals. If we observe the stability of the distribution function of the "organisms" over the parameters of the filter this is an equilibrium.

To analyze such a system the technique of the large statistical sum can be applied. In this case separating one of the states of the income and computing for it the large statistical sum we obtain

$$Z = \sum_{n=0}^{\infty} \lambda^n e^{-n\frac{\varepsilon}{T}} = \frac{1}{1 - \lambda e^{-\frac{\varepsilon}{T}}}. \tag{182}$$

We consider the spectrum of possible states consisting of positive equidistant quantities $\varepsilon_n = n\varepsilon_0$. We have selected the discrete version of this spectrum to simplify the solution of the model problem.

For the probability function of a particular state of the income being filled we get

$$\langle n(\varepsilon)\rangle = \frac{\sum\limits_{n=0}^{\infty} n x^n}{\sum\limits_{n=0}^{\infty} x^n}. \tag{183}$$

where $x = \lambda e^{-\frac{\varepsilon}{T}}$.

Computing this expression we obtain for (183) identical to the expression for the Bose-Einstein distribution function in the statistical physics

$$\langle n(\varepsilon)\rangle = \frac{1}{e^{\frac{\varepsilon-\mu}{T}} - 1}. \tag{184}$$

Recall that T is an equilibrium parameter defined from the equation

$$\frac{\partial S(E)}{\partial \varepsilon} = \frac{1}{T}. \tag{185}$$

We can compute the number of ways to fill in the income level of the systems in large placing N firms in order with the fixed level of income and therefore can compute the temperature and the migrational potential μ determine it as earlier from the condition

$$\sum_{\varepsilon} \langle n(\varepsilon, \mu)\rangle = N. \tag{186}$$

We have obtained a very interesting distribution function. It is well known that in the systems with Bose-Einstein statistics an effect of “Bose-condensation” is observed. Namely, at low temperatures the particles condensate on the base level of the energy of the system. It is not difficult to see that in our case the same will happen.

We obtain the following expression for the occupation of the zero-th level of income:

$$n(0, T) = \frac{1}{e^{-\frac{\mu}{T}} - 1} = N_0. \tag{187}$$

For $T = 0, \quad N(0, T) = N$ we can obtain the value of migrational potential μ on T

$$\mu(T) = -T \ln\left(1 + \frac{1}{N}\right). \tag{188}$$

or

$$\lambda = 1 - \frac{1}{N} \quad \left(\lambda = e^{\frac{\mu}{T}}\right). \tag{189}$$

Having known the occupation functions of the levels we can compute how many particles (firms) will occupy the non-zero levels of income:

$$N_1 = \sum_{n=1}^{\infty} \frac{1}{e^{\frac{(\varepsilon_n - \mu(T))}{T}} - 1}. \tag{190}$$

Now we can obtain the value of the temperature of “Bose condensation” starting from the fact that at this temperature the number of particles on the base level of the system is equal to the number of particles on the excited levels:

$$N_0(T_0, \mu) = N_1(T_0, \mu) = \frac{N}{2}. \tag{191}$$

We have two equations to determine two parameters: T_0 and μ_0 where μ_0 is defined in terms of the occupation of the base level

$$\mu_0 = -T \ln\left(1 + \frac{2}{N}\right). \tag{192}$$

and T_0 is found from the equation

$$\frac{2}{N} = \sum_{n=1}^{\infty} \frac{1}{e^{\frac{nT\varepsilon_0}{T_0} + \ln\left(1+\frac{2}{N}\right)} - 1}. \tag{193}$$

To estimate this expression we can replace the sum by the integral assuming ε_0 very small and in principle compute $T_0 = \varphi(N, \varepsilon_0)$.

Further having calculated the mean income E of the system we may obtain the dependence of the temperature on the value of the income for given N_0. This procedure means that the equilibrium condition can be expressed in terms of parameters T, μ equally well as in terms of parameters E, N. We will not do this here since the model is too rough.

Here only one deduction of the model is essential for us, namely at a finite non-zero temperature the lowest level of income will be occupied by a "microscopic" portion of the firms, i.e., in the asymptotical limit for N large we have

$$\frac{N_0}{N} \approx O(1). \tag{194}$$

Now suppose that conditions slightly changed for example one has to pay an additional tax for each transaction. This means that the income of each firm will diminish by a fixed value. We obtain the well-known effect of a "crash", an essential portion of existing firms cease to exist. Observe again that the main peculiarity of the considered model is the Bose-Einstein occupation function of the values of the spectrum of possible income.

We see that under certain conditions the system becomes unstable with respect to small modifications of external parameters though it is in equilibrium. Under a small increase of the price of transactions the "macroscopic" number of firms dies out.

This is just one example of how to use statistical thermodynamics for analysis of the performance of a filter of the evolutionary process.

Here we just wanted to show that the survival in the evolutionary theory can be described in the frames of the thermodynamic model and thus establish a relation between the evolutionary and thermodynamic approaches to the description of economic processes.

CHAPTER 11

Conclusion

In this book we tried to show that a new metaphorical frame is possible in the theoretical economics the frame which preserves in the most strong form the idea of spontaneous order or "invisible hand" and at the same time provides with richer analytical possibilities than neoclassical theory of equilibrium or evolutionary modelling.

The thermodynamic approach shows that macro-parameters in the economic system become related by the equation of state, i.e., lie on a surface whose form is determined by a Pfaff equation. In order to able to say how the macro-parameters of the system will change under variation of one of them one has to know this equation.

Generally speaking, a success in attempts to move economics in a certain direction by changing the macro-parameters will be only achieved if we know the equation of the state. The system with a sufficiently exotic equation of state (and apparently the economic system with non-stabilized markets) may as a result of seemingly rational attempts to modify its state arrive not where a "rational" politician pushes it but to quite other state.

We believe that this corollary is of huge importance for macro-economics. Uncertainties one has to deal with in the process of macro-decisions are much more complicated than uncertainties of the same system on a micro-level as follows from the thermodynamic approach to the economics. The rationality of macro-decisions becomes "tied" in much stronger sense than the one H. Simon had in mind. And the problem is not only in the restricted ability of humans to calculate, but in our ignorance of the constraints imposed on the system.

Once again, the deduction one can make from the above analysis: without knowing the equations of the state of the system we cannot speak about rational macro-decisions.

In the absence of the knowledge of the equation of state one can only speak about empirical decisions in macro-economics. At present the equations of the state of economic systems not only are unknown but such vital parameters that determine the state of economic systems as temperature and migrational potential are not being taken into account.

We think that economic macro-theory is now on approximately the same level the study of heat processes was during the period of reign of the phlogiston theory, that is before the laws of thermodynamics were discovered.

The economics is being globalized and this globalization provides new unexpected possibilities for construction of gigantic heat machines in economics which enable those who controls international financial flows to work with national economic systems in approximately the same way as what worked with the steam engine. Recall that the steam engine was invented long before the thermodynamic relations became subject of scientific analysis.

In the process of macro-economic decisions two dangers work. One — the main one — is the wide application by politicians by the standard procedures that do not take into account the "equation of the state" of national economies and what is remarkable, the politicians pay no responsibility for the damages caused by such actions, whereas there is no evolutionary effective mechanism for selection of macro-decisions on the national level. Institutional mechanisms of democracy are too weak and work satisfactory as a rule only in places where politicians have worked out a certain procedural practice of working with economics anyway.

On the other hand, the procedures of using “heat machines” in the sphere of international finances are subject to ruthless rules of evolutionary selection based on the profitability of the economic activity and therefore are effective but totally void of any control from those whose well-being actually depends on the procedural manipulations in the globalized world economics.

I hope that the approach presented in this book will help at least to a small extent diminish the uncertainty and increase the degree of rationality of macro-economic decisions based not on the ideological dogmas but as a result of establishing an interrelation of micro- and macro-analysis in theoretical economics.

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APPENDIX 1

Physics as a tool in sociology

§1. Social changes and tacit knowledge

It is well known that under certain circumstances sudden transformations that change the most important parameters of social structures can take place in human societies. Sudden revolutionary changes may touch the relation of the members of the society to the religion, or economic, or political system. Until now there are no models except, perhaps, the catastrophe theory able to describe such processes however rigorously.

Despite of the fact that an uncountable amount of literature is devoted to revolutions of all types (social and intellectual) the nature of the jump-like modification of the mass conscience cannot be grasped by purely functional models. One can, of course, guess that further exploitation might ensue in a revolt, and that the widening of corruption in the church hierarchy might lead to success of a new religious trend. Such happenings, however, do not always occur and to determine the precise moment of time or a concrete situations when such jump-like changes of mass patterns of behavior is beyond the limits of the possibilities of any theoretical analysis.

The most serious obstruction for such mass changes is the conservative nature of the social patterns of behavior. The institutional structure of the society is targeted precisely to maintaining such conservativeness, see [**1**]. At the same time, the factor that obstructs any institutional changes is the fact that the knowledge determining the functioning of social institutes is a two-level one.

In addition to formal rules abided by the members of the society — the rules that formulate and restrict the human behavior — there is a huge volume of non-formalized knowledge, and it is the latter that makes the functioning of the visible social forms actually possible. Since these informal rules (tacit knowledge) are rooted very deeply, they are seldom an object of reflection, and therefore are subject to conscience changes with more difficulty than rules understood, see [**2**], [**3**].

§2. The paradox of changes

The above described situation leads to the paradox of changes: rational (meaning: understood) changes of formal social institutes do not touch the existing deep tacit knowledge that constitute their basis. But, on the other hand, without changing the formal rules how can the tower of tacit knowledge be created, the tower that should be a pillar of these changes?

This paradox of the hen and the egg is the stumbling point of any however developed evolutionary theory (the theory of biological evolution is, perhaps, the most graphic example).

The moment two ontological changes become objects of the study of an evolution theory (for example, in biology: the phenotype and genotype) the problem of relation between them and the interrelation between changes that take place on these two levels become subjects of heated discussions. Here, I think, lie the roots of incessant discussion between neo-Darwinists and neo-Lamarcists in the theory of biological evolution, see [**4**].

Negation by neo-Darwinism of the principal possibilities of directed influence of the environment on the genotype leads to logical difficulties in the theory, whereas at the same time there are no examples of such an influence.

In the theory of social evolution, there are, fortunately, no such logical difficulties that make the development of the evolutionary theory in biology so painful: the directed influence of the environment on the tacit knowledge is quite obvious. However, the presence of interaction in both directions between the two ontological levels of social reality does not, nevertheless, eliminate the paradox of changes.

Heated debates on the possibility to use the theory of rational choice in order to explain revolutionary changes, such as the one at the seminar organized by Swedish Collegium of Future Studies in Social Sciences in 1995, is one of many examples. If the selection of rules of behavior is subject to a rational choice, then how to describe the mass support of revolutionary changes that lead to equally large-scale worsening of social conditions?

Ethnic conflicts similar to recent conflict in Bosnia are subject to a rational explanation even less. One can, of course, try to interpret such phenomena on the base of national spirit in the spirit of theoreticians of nationalism. But from the methodological point of view such theories encounter serious counterarguments, see, e.g., [**5**].

It seems vitally necessary to have a theory that unifies phenomena of social evolution and avoids both the logical difficulties of the theory of rational choice and the additionally non-observable entities.

§3. The Interaction Space

We assume that the reader got acquainted himself or herself with ideas of application of thermodynamics to economic systems in the main text. The only difference is that transaction costs in social networks can be considered in the same way as a flow of money to the market.

In physical systems of many particles, the interaction takes place in a space. The interaction depends on the distance and all reasonable mathematical models take this into account. All interactions occur somewhere. The space in physics is the place that rooms the changes. Some difficulties in creation of mathematical models of social systems are related first of all with the fact that, obviously, social interactions occur not in a physical space.

Of course, the actors are placed in a physical space, but does their interaction depend on the distance between them?

Manifestly, for very large distances, the possibility to interact depends on the level of communication technology. But it is precisely this dependence that robs the physical space of social sciences of the meaning it has in physics. If the possibility to interact depends on technology, then the physical space ceases to be universal. Ten kilometers in mountains is more than 100 km in the steppe or along the sea with the sufficient means of transportation provided.

The structure of social space in this example depends on the means of transportation. Therefore the natural desire of the researcher to consider the physical space as the one that hosts social interactions is inapplicable. If the physical space does not fit to host social interactions, what can we offer as the space of social interactions?

The answer to this question is one of the main objectives of this Appendix. And with all its simplicity the answer seems to be rather unexpected. If the object of a social study is the social network, each relation of which is characterized by a transaction cost, why shouldn't we consider a graph of a social network that is the set of vertices–agents joined by edges–interactions?

The problem is immediately solved. Thus, a formally defined graph is a typological structure insensitive to the actual distance between the vertices. One can express such graphs differently but as a mathematical object any two of its presentations are indistinguishable, see Fig. A1.

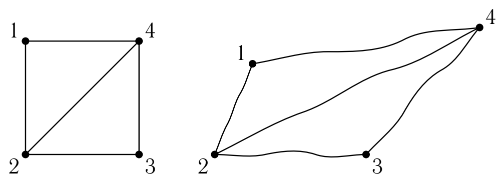

FIGURE A1

Therefore, for a model of a social system, we can consider the totality of agents with certain patterns of behavior and connected into a social network represented by a graph.

To the edges of this graph we ascribe transaction costs depending on the patterns of behavior at the vertices of the graph. Nothing prevents us from expanding the ontology of the model and considering that the costs depend not only on the patterns of behavior but also on the patterns of tacit knowledge. Such an approach to the definition of the space of interaction admits a very important generalization.

Suppose we know the construction of the stable relations in the social network. Does it mean that the social agents do not interact outside these stable channels?

Obviously not. Therefore we must have a possibility to introduce transaction costs between the agents not directly connected in the social network. How is this done?

The answer is again very simple. Having the graph of stable relations we may introduce the distance between the vertices taking for this distance the least number of edges that form a path that connects these two vertices. It is not difficult to see that thus defined integer distance satisfies the axioms of the metric space. Therefore, having a graph that represents a social network, we can introduce an interaction that depends on the (integer) distance in a totally classical way.

If we consider the common for all social agents patterns of inner states as analogues of the spin of physical particles, then for the space determined by the graph of social network we obtain a formal description equally identical to the description of a physical system by means of the Hamiltonian of the interaction. If we want to study the properties of such a system at equilibrium, it suffices to compute the statistical sum as a function on the temperature:

$$Q = \sum_{\sigma} \left\langle e^{-H/T} \right\rangle , \tag{195}$$

where $H = \sum_{ij} V(\sigma_i \sigma_j)$ and were σ run over all possible states of "mental patterns".

Considering the expression for the heat capacity of such a system, i.e., the ratio of the change in transaction cost to the change in temperature in terms of the derivatives of the statistical sum, we may find a point of phase transition in a social system in exactly the same way as one does this for physical systems. Now we have a means to obtain mathematically absolutely non-trivial corollaries describing behavior of the social systems under the influence of significant external social and economic forces, the structure of the social networks, and the level of economic prosperity.

Finally, we have obtained a possibility to regularly construct and investigate by mathematical methods highly non-trivial models of social systems, predict behavior of such models in various situations and compare these predictions with experimental historical data, and search for known and previously unknown effects.

§4. Phase Transitions in Social Systems

Social and political studies under such formulation of the problem begin to correspond to natural sciences at the level of probability. We have, of course, taken into account general considerations with respect to the nature of social models. Since we are interested in the thermodynamic limit, i.e., the behavior of the system when its size (the number of agents-nodes) grows without bound, then it is clear that such a thermodynamic limit exists only for some types of social systems.

To obtain a thermodynamic limit of the model, we should be able to expand the graph without changing its macrostructure. There exist several simple, but important from the point of view of description of social networks, graphs of such type. One such is, for example, the graph of the n-dimensional lattice (Fig. A2 a), the graph of a regularly branching tree (Fig. A2 b), the complete graph of all interactions (Fig. A2 c). Such graphs can be enlarged without changing their type.

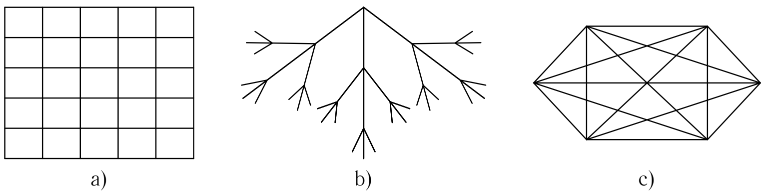

Figure A2

Let us make one more important remark. The paradox of this approach that eliminates the physical space from the models of social networks and replaces it by the space of a graph of social interaction is very interesting from the point of view of tractability of statistical models.

The point is that a number of exactly solvable statistical models (in particular, the spin statistics of the *Boethe tree*) do not have any corollary in physics.

To realize such models, one needs not a three-dimensional but an infinite-dimensional physical space.

Conversely, in the study of statistical behavior of social networks, these exactly solvable models are very meaningful. It suffices to observe that the *Boethe tree* is just the graph describing the structure of a regular social hierarchy (Fig. A2b).

The question of the existence of a phase transition in a social hierarchy — i.e., its collapse or essential inner transformation — is one of the principal problems in the study of social changes. And, within the framework of the above described model, this problem has an exact analytical solution.

In what follows we consider several similar results and their interpretation. Now consider several simple network models.

First, consider a social network in which each agent has a fixed and equal for all agents number of neighbors in the network and in which there are only two patterns of behavior (agent's states). We may further assume that only nearest neighboring agents interact. Assume also that the transaction costs for all pairs of interacting neighbors are equal and only depend on the combination of two possible patterns of the agents' behavior:

$$V(\sigma_i, \sigma_j) = J\sigma_i\sigma_j, \tag{196}$$

where σ_i and σ_j can be equal to either 1 or -1. The system obtained is known in physics as the Ising model.

Observe that the Ising model can describe two essentially different situations:

(1) an analogue of the spin system or
(2) the case of replacement.

In the first case, each of the agents can change the pattern of behavior.

In the second case, there are two types of agents incapable to change their own patterns of behavior but capable to replace each other in the nodes of graph. The second case, where the interaction between the agents of two types (A and B) is given by three values of transactional expenditures V_{AA}, V_{BB} and V_{AB}, is reduced to the Ising model but the statistical sum in the Hamiltonian of the interaction accrues by the term corresponding to an external field.

Certain important conclusions concerning the social network represented by the replacement solution can be made immediately as a result of its formal reduction to the spin model. It turns out that the quantity

$$2J = \frac{1}{2}(V_{AA} + V_{BB}) - V_{AB} \tag{197}$$

is a very essential parameter. If this parameter is negative, then the agents of types A and B will tend to randomly replace each other in the network whereas, if it is positive, then under the decrease of the total costs (the sum of all transaction costs) the network will fragment into clusters consisting of agents of type A and agents of type B.

In a real social system (in which types A and B are ethnic or social groups), ethnic or social cleansing correspond to this process.

Now consider a social network in which each agent has $2n$ neighbors. The most easy way to realize such a network is to form of an n-dimensional lattice. Simple thermodynamic arguments allow one to determine whether in this case the one-phase state (when both patterns of behavior are the same) is stable or not.

First, consider the case of a one-dimensional network. If, in such a system, we allow domains with different patterns of behavior, then we allow the changes of the level of transaction costs. The appearance of one borderline causes the total change of the transaction costs by the doubled constant of interaction between two neighbors.

At the same time the borderline can occur at N distinct places, where N is the length of the network. Therefore the change of the free energy can be estimated as

$$\Delta F = \Delta E - T\Delta S = 2J - T\ln N, \tag{198}$$

so ΔF becomes negative for sufficiently long net. The negative value of the change of the free energy in the process of creation of the disordered state means that this ordered state is thermodynamic unstable (this argument is due to Peierls, see [**6**]).

Somewhat more sophisticated arguments prove that, in the network represented by a two-dimensional lattice, the domains with orientation opposite to that of dominant one are unstable at low temperatures.

The same applies also for lattices of higher dimensions.

In other words, the dissident clusters in the scenario described by the Ising model of social networks in dimensions > 1 are only stable at temperatures above a critical one, and therefore in this case a phase transition is possible.

Though very rough, this model catches one very essential aspect of the situation. The diminishing of the temperature in the network causes fragmentation of the model. Therefore the tolerance of the system to dissidents is a corollary of a sufficiently high level of well-being.

It is very well known empirically that democratic regimes tend to exist stably under the level of per capita income exceeding three–four thousand dollars. It is interesting to try to relate this empirical fact with phase transitions in network models.

With the help of Peierls arguments we can also evaluate the stability of clusters in regular transitions (i.e., in nets corresponding to Boethe lattices). The results obtained here are rather astonishing: The branches of the hierarchical tree are unstable with respect to reorientation. But this is not the case for the inner domains of the hierarchical tree in which, for sufficiently low temperatures, the "dissident" clusters cannot occur. Once again the social projection of these corollaries is not without interest. It is very well known that under lower temperature conditions (that is when the total costs diminish) rigid hierarchies tend to appear. This rigidity of hierarchies melts with the heightening of the temperature that is with the growth of income. The history of collapses of autocratic regimes illustrates that the dissident groups are usually formed on the middle levels of autocratic hierarchies (the branches fall off) and the social changes by means of coups occur much more often than people-driven revolutions.

It goes without saying that the Ising model of transactional interactions is too rough, if not rude, to describe social networks. It is not difficult, however, to improve it by considering the mean field approximation ([**7**]). Under such an approach the interaction of each agent with its nearest neighbors is also influenced by a mean field of interactions with the more distant agents; let this interaction be self-consistent.

The obtained approximate description of the system is known as the Curie-Weiss model, which also has a phase transition for a sufficiently low temperature. In other words, in terms of the social theory the result consists in the fact that for the low total income there is a tendency towards a uniform behavior.

§5. Certain Conclusions

In this note we have no possibility to analyze in detail a larger number of statistical models of social networks. To do so, though not truly difficult, would require a much larger amount of work.

We have no possibility to discuss interesting mathematical details of the models of phase transitions in social nets, either. The level of mathematical apparatus necessary for a reasonable discussion of such models far exceeds the level of complexity of the mathematical apparatus usually applied in social studies.

To understand the inner quantitative peculiarities of such models, it is natural to use modern representations of the spectral analysis of operators in functional spaces, Feynman integrals, and certain basic notions of the modern quantum field theory and statistical mechanics, such as renormalization group and scaling. [1]

Such a situation naturally poses several serious questions pertaining to the realm of the theory of science.

First of all, the very fact that thermodynamic approaches and, speaking more broadly, the methods of statistical mechanics have not been, so far, widely used in the social theory is astonishing. As in disordered physical systems, in social systems, the distributions of a certain conserved parameter among the elements of the system is of huge importance, which guarantees the success of thermodynamic approach to the study of such systems.

Furthermore, in social systems, the parameters of order can be naturally introduced which allows one to apply physical models of phase transitions for the description of social systems. The utility of application of statistical models of social systems, in spite of their roughness, can hardly cause any doubt.

[1]The nature of the mathematical machinery useful in the study of statistical theory of social networks can be appreciated, for example, with the help of the books [**8, 9**].

The perspective of development of such methods is, nevertheless, considerably impaired by the social structure of science and higher education.

It seems to me that the general level of mathematical education at the universities preparing researchers in the domain of social sciences does not correspond to the level necessary for the active work with models of such sophistication.

On the other hand, the interest to work in the domain of social studies among experts with sufficient mathematical training — theoretical physicists and mathematicians — is restrained precisely due to the widespread understanding among theoretical physicists and mathematicians of dullness of social sciences, that is impossibility for the representatives of the natural and precise sciences to apply their intellectual potential in its totality.

One of the main tasks of this note was to demonstrate how faulty these prejudices are. We believe that at the moment the social sciences represent one of the most promising domains for application of the enormous stock of methods and models worked out in mathematics, theoretical and mathematical physics, especially lately.

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APPENDIX 2

Economic dynamics as geometry

If the macro-parameters of an economic system are functionally dependent, and the surface of state is differentiable, then the differentials of the macro-parameter are related by a system of Pfaff equations.

In fact, we may consider a general system described by parameters $x, \dots, z$. If we change the state of this system, small changes may be considered as differentials $dx, \dots, dz$. Generally speaking, if the system is constrained by internal links, the differentials of parameters are not independent. They are related by linear equations of the form

$$A(x, \dots, z)dx + B(x, \dots, z)dy + \dots = 0. \tag{199}$$

We can find a number of such Pffaf equations that single out our system.

Such a general case is too complicated for analysis. Let us restrict ourselves with a simpler case with one Pfaff equation and a number of functional dependencies between parameters of our system:

$$f(x, \dots, z) = 0, \quad g(x, \dots, z) = 0, \dots. \tag{200}$$

We can change the system of coordinates and minimize the number of independent parameters of the Pfaff equation. The smallest such number is called the *class* of the Pfaff equation. The class of one equation is always odd ([**R**]).

We have just obtained a system completely identical to the generalized Hamiltonian system studied by Dirac [**Di1**] (see also Faddeev [**F**], Pavlov [**P**]). Formally, all functional constraints are equivalent and each one may be considered as a Hamiltonian function.

In the analysis of generalized Hamiltonian system, time plays an outstanding role. In our case, we do not need to distinguish the role of one parameter. We may choose one constraint $f(x, \dots, z) = 0$ equation on the surface defined by the constraints (199), see [**R**], Chap. IX. It is possible to relate a vector field to each field of characteristic directions. We may act by this vector field on any function $g(x, \dots, z)$. The result is called the *Jacobi*, or *Legendre*, or *contact, bracket* $\{f, g\}_{K.b.}$ (see [**R**], Chap. IX).

The difference between the Jacobi and Poisson brackets consist of the fact that the Poisson brackets are defined on an even-dimensional space with a bivector whose Schouten bracket ([**Gr**]) with itself vanishes (if the bivector is nondegenerate, it is equivalent to a nondegenerate closed differential 2-form that determines a symplectic space). The Jacobi brackets are defined not for differential form but for a Pfaff equation in the space with odd dimension.

To find a k-dimensional integral surface on the surface singled out by the equations of functional constraints is only possible if all Jacobi brackets between the constraints vanish on this surface [**R**], Chap. IX. If some of these brackets do not vanish, we must include the result of bracketing into the set of constraints and calculate Jacobi brackets between all the elements of the new set. After several iterations we may come to one and only one of the following results:

(a) The number of constraints exceeds $k + 1$, so no integral surface of the Pfaff equation exists.

(b) The process terminates and we get a surface on which we can find an integral surface of our Pfaff equation. In absence of singularities this surface simply consists of k-dimensional integral surfaces of the Pfaff equation.

Despite the simplicity of the argument the consequences could be very important for the economic systems. **We need a check of coherence of constraints imposed on the system**. This means that, in the case of economic system, control actions cannot be arbitrary. Any additional constraint imposed on the system must be checked on compatibility with other constraints, it should not destroy the very existence of a solution of the system of equations.

Consider a hypothetical example. Imagine that, in a certain country, the currency board is established. This means that an additional link is created between the country's economy and the world economy. Could we be sure that such an additional link would not destroy the equilibrium of the country's economy?

One more remark. It is known that, for some equations of generalized Hamiltonian systems, there exist certain hidden, "non-physical" variables [**D2**]. This means that, under certain conditions, there exist functions of parameters of the system that could be changed without visible effect. A contemporary physical theory (gauge theory) attributes a great importance to the role of such cases. It is rather interesting to investigate what kind of effects could be connected with that type of equations. We may conclude here with one very general and challenging statement: The integral surface of a system of Pfaff equations has some geometrical features. It is possible to define, for example, different connections on this manifold, using the natural frame bundle, or to define characteristic classes. All such geometrical objects may be interpreted as dynamic objects, if dynamics is introduced properly.

We may change the direction of inference and interpret equilibrium economical dynamics as geometry.

Of course, this is only general definition of a very complex program of future studies, but I am convinced that the realization of such a program could change significantly our understanding of economic processes.

Editor's remarks. The idea of locality. Basically, when we describe our world by mathematical equations, we may use two distinct types of theories: local and non-local ones. Non-local theories are described by integral or integro-differential equations.

Locality of a theory means that any change of the system's parameters of a point is defined by its infinitesimally small area i.e., the long-distant interaction is absent. In such a case, the mathematical model of the system is represented by differential equations (perhaps, of arbitrary order, non-linear, etc.).

Any system of differential equations may be reformulated as a system of Pfaff equations [**BCG**]. Some of these systems determine an integrable distribution, others determine non-integrable distributions; Hertz called such distributions *nonholonomic*. A branch of mathematics I am working in (representation theory) provides with tools to analyze — at least, in principle — on a qualitative level solvability of a given system of differential equations and stability of its solutions, if any exist.

This book shows how to represent economic theories as nonholonomic systems.

How to solve differential equations. I could never understand why engineers (or any other customer who needs to solve differential equations in earnest, in real life) never use criteria for formal integrability of differential equations Goldschmidt suggested in late 1960's (for an exposition, see [**BCG**]). As a byproduct, these criteria yield an approximate solution. It is just inconceivable that these criteria are only good for nothing but theoretical discussions and no type of differential equations is "convenient" enough for these criteria to be implemented in real computations.

The following facts constituted, perhaps, the stumbling block:

(a) psychological threshold: these criteria are formulated in terms of Spencer *cohomology* — a new and scary term, not clear if worth learning;

(b) even now no efficient code for computing cohomology exists (although the problem obviously involves lots of repeated computations and sparse matrices, none of these features was ever exploited); at the time Goldschmidt suggested his criteria the situation was much worse;

(c) the criteria embraced only "one half" of all equations.

To explain (c), recall that a theorem of S. Lie ([**KLV**]) states that all differential equations are of the two types: for one, all its symmetries are induced by point transformations, for the other one, all its symmetries are induced by contact transformations. Goldschmidt's criteria fit only differential equations whose symmetries are induced by point transformations, they did not embrace differential equations whose symmetries are induced by contact transformations. Now, recall that the contact structure is the simplest of *nonholonomic* structures.

The first nonholonomic examples were from mechanics: a body rolling with friction over other body. Among various images that spring to mind, a simplest is that of a bike, or just a ball, rolling on an asphalt road. At the point of tangency of the wheel or the ball with asphalt the velocity is zero. This is a *linear* constraint.

More generally, a manifold (no dynamics) is said to be *nonholonomic* if endowed with a nonintegrable distribution (a subbundle of the tangent bundle). A famous theorem of Frobenius gives criteria of local integrability: the sections of the subbundle should form a Lie algebra. One often encounters *non-linear* constraints: switching the cruise control of your car ON you single out in the phase space of your car a distribution of spheres over the configuration space.

On my advice this book is appended with a paper by Vershik with first rigorous and lucid mathematical formulations of nonholonomic geometry and indications to various similar structures in several unexpected, at the time Vershik's paper was written, areas (like optimal control or macro-economics, where nonlinear constraints are also natural). Vershik summarizes about 100 years of studies of nonholonomic geometry (Hertz, Carathéodory, Vrănceanu, Wagner, Schouten, Faddeev, Griffiths, Godbillon; to his list we should add that Internet returns hundreds of thousands entries for "nonholonomic", and its synonyms (anholonomic, "sub-Riemannian", "Finsler", "cat's problem" and "autoparallel"; there seems to be more, actually, nonholonomic dynamical systems than holonomic ones; finally "supergravity" is also a nonholonomic structure, albeit on *super*manifolds).

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APPENDIX 3

Mathematics of nonholonomicity (A. M. Vershik)

§1. Dynamics with constraints

The purpose of this article[1] is to give a detailed and, to a large extent, self-contained account of results, to raise a number of questions on dynamics with constraints on the tangent bundle of a smooth manifold. The classical problems of this kind are the problems of nonholonomic mechanics, nonclassical ones — the problems of optimal control and economic dynamics. The investigation of these topics from the point of view of global analysis was started fairly recently (see [**VF1, VF2, G**]). Such a treatment needs a detailed study of geometry of the tangent bundle, connections and other notions necessary for general Lagrangian mechanics and, particularly, for the theory of nonholonomic problems. The present article continues the investigations of geometry of the tangent bundle and dynamics on it. For standard facts from the geometry of manifolds and the Riemannian geometry, see [**KN, BC, SS**].

1.1. Lagrangian dynamics. The Lagrangian formalism is based on a procedure that allows one to invariantly construct a special (see (201) vector field given an arbitrary C^2-smooth function on the tangent bundle (this function is called the *Lagrangian*). Formally, this construction uses only two canonical objects present in the tangent bundle of any manifold: the *principal tensor* and the *fundamental vertical field*. These structures are invariantly defined in [**VF1, VF2, G**].

The Lagrangian mechanics studies the structure of trajectories of special vector fields — their integrability, stability, integrals, and so on — as dependent on the Lagrangian.

Such entities and principles of classical mechanics as forces, virtual displacements, variational principles can be completely described in terms of geometry of the tangent bundle. In addition to that, there is a number of geometric notions insufficiently used so far but, probably, important for mechanics, e.g., the notion of connection.

Recall here the following fundamental result due to Levi-Civita (later elaborated by Synge et al.) which connects mechanics with geometry:

> *any mechanical system with quadratic Lagrangian moves by inertia along the geodesics of the corresponding Riemannian manifold.*

[1] This is an edited (by me, *D.L.*), as far as English is concerned, version of the namesake article published in Yu. Borisovich, Yu. Gliklikh (eds.), *Global Analysis — Studies and Applications. I.*, Lecture Notes in Mathematics 1108, 1984, 278–301. I tried to return this difficult to read version of English to match the original Russian transcript of Vershik's inspiring lecture at Voronezh Vinter sschool: Vershik, A. M. Classical and nonclassical dynamics with constraints. (Russian) In: Yu. G. Borisovich and Yu. E. Gliklikh (eds.) *Geometry and topology in global nonlinear problems, 23–48, Novoe Global. Anal.*, Voronezh. Gos. Univ., Voronezh, 1984. The article expounds the joint paper of A. Vershik and L. Faddeev (Lagrangian metrics in invariant form. In: Problems of Theoretical Physics. vol. 2, Leningrad, 1975 = Selecta Math. Sov. 1:4 (1981), 339–350) and gives a mathematical description of several aspects of nonholonomic manifolds, i.e., manifolds with a nonintegrable distribution. A similarity of the constrained dynamics for the linear constraints with fields of cones in the optimal control is observed. I also tried to update references and eliminate typos. *D.L.*

Levi-Civita also defined the Riemannian connection. In [**VF1, VF2**] (see §2) this theorem was generalized to the non-Riemannian connections that appear in the nonholonomic case.

The translation of Lagrangian mechanics into the language of geometry was initiated in the works [**VF1, VF2, G**], but it is not entirely completed. For an invariant formulation of *d'Alembert's principle* — the most general local principle of mechanics also valid for dynamics with constraints, see [**VF2**].

Observe that the Hamiltonian mechanics in an invariant form (symplectic dynamics) gained much more attention than the Lagrangian one. This is quite understandable, because the symplectic structure is a universal object of analysis and geometry (see [**A**]). Yet, from the point of view of mechanics, examples of symplectic dynamics are scantier, such dynamics have no equivalent for certain mechanical notions (force, constraint, and several more).

Apart from that, the presence and necessity of two parallel formalisms (the Lagrangian and Hamiltonian ones) in the quantum theory indicates that they might be unable to completely replace each other on the classical level, either. In this article we primarily use the Lagrangian formalism, passing to symplectic geometry only for reduction and examples (see §4).

1.2. The dynamics with constraints. At least three branches of science — nonholonomic mechanics, optimal control and dynamics of economics — lead to necessity of considering the following generalization of Lagrangian dynamics:

> *given a submanifold (e.g., a subbundle) or a distribution (a field of subspaces) in the tangent bundle TQ of a manifold Q, construct a dynamics so as the trajectories could not leave the given submanifold, or, equivalently, the vector field would belong to the given distribution.*

In some problems (e.g., of optimal control) the value of vector field at each point might belong to a submanifold with boundary or even with corners.

These problems were almost nowhere considered from the point of view of global analysis and coordinate free differential geometry, save the study of nonholonomic dynamics in [**VF2**]. The initial aim of the article [**VF2**] was to comprehend nonholonomic mechanics in an invariant form and to revise Lagrangian mechanics in conformity with this comprehension.

The fields of cones or polytops in the tangent bundle (see [**A, V**]) are important in optimal control and economic dynamics.

There exist two different constructions of dynamics with constraints, each of construction leading to reasonable mathematical problems. A choice between the two possibilities lies outside mathematics.

Let a field of submanifolds in the tangent spaces be given either as the set of zeros or as the set of non-positivity of a collection of functions and, additionally, let there be given a Lagrangian or, more generally, a functional. Then one can:

1) consider the conditional variational problem of minimizing a certain functional (action, time, and the like) provided the trajectories belong to the given submanifold, and, with the help of an appropriate version of the Lagrange method, derive the Euler-Lagrange equation, which in other words can be expressed, as we will see, as a vector field,

2) consider the projection of the vector field that corresponds by 1) to the Euler-Lagrange equation of the unconditional problem (on the whole tangent bundle) at every point of the given submanifold onto the tangent space to the submanifold at this point, and again obtain a vector field.

Both these vector fields tangent to the submanifold of constraints, are *special* (201), and therefore determine a dynamics with constraints. But, generally speaking, these fields do not coincide. In the first case, the constraints are "in-built" in the Lagrangian, in the second case, only the reactions of constraints take effect.

It is the second construction this should be used for description of the movement of mechanical systems. In a number of special cases (holonomic constraints, Chaplygin's case — a special case of linear constraints, and so on), both constructions lead to the same vector field. (Among a considerable number of works on nonholonomic mechanics, we mention here [**Ch, D, NF, G**], where one can find further references. I know no work on this topic that uses an invariant approach except [**VF2**], [**Gl**].)

Faddeev and me shown in [**VF2**] that the second construction corresponds to the general d'Alembert's principle, and, in this situation, **no variational principle corresponds to it**.[2]

Of course, the principle is a postulate verified by practice, and by no means a theorem.

An implicit abusing of the two approaches favored confusions in the foundations of nonholonomic mechanics until recently. Starting with the first works of classics, these confusions were present, in some form, in almost all mathematical textbooks on Calculus of Variations.

For example, in mathematics, the conditional Lagrange problems with non-integrable conditions on derivatives are called *nonholonomic*. This usage might give the reader an idea that the nonholonomic mechanical problems are conditional variational problems with non-integrable constraints. As we observed above, this is not true. (Of all the textbooks I know, the difference between the "problems with nonholonomic constraints" and "problems of nonholonomic mechanics" is distinguished in [**Sm**] only.)

The first construction is employed in optimal control and other applications of dynamics with constraints. However, it is also possible to use the second construction.

Many technical difficulties appearing in the treatment of dynamics with constraints are due to the absence or insufficient usage of the adequate geometric apparatus. We mean here, first of all, the coordinate-free theory of distributions and connections. With these notions instead of vague "quasi-coordinates" and sophisticated procedures of exclusion, the account becomes simple and the whole dynamics turns into the source of new lucid mathematical problems whose solution will promote new effective applications.

1.3. Contents and main results. In this article, we give a detailed account of the theory of the problems with constraints on the tangent bundle. In §2, we briefly, with certain innovations and simplifications, recall the results from [**VF1, VF2**], i.e., we give an invariant derivation of the Euler-Lagrange equations for an arbitrary Lagrangian and an invariant study of equations with constraints.

In §3, we analyze the systems with a quadratic Lagrangian (the Newton equations), and the systems with linear constraints. For this purpose, we define a new geometric object — the *reduced connection* on a subbundle of the tangent bundle of a Riemannian manifold. To my mind, this object corresponds to the old coordinate notion of "nonholonomic manifold" which appeared in a number of geometric works written in the 1930s–40s (V. V. Wagner, Schouten, Vrănceanu). The geodesic flow for this connection is the main object to study.

The most interesting is the case of an *absolutely nonholonomic constraint*: the corresponding phase space is then endowed with a new (non-Riemannian) metric, and the *Hopf-Rinow theorem* (3.4.2) in the usual formulation is false for this metric.

[2]Boldface is mine: this important observation might explain the in-built (and, perhaps, impenatrable) obstacles in quantizing supergravity since the Minkovski superspace is nonholonomic. *D.L.*

Further in §3 we consider the general problem with linear constraints and quadratic Lagrangian, and give a complete proof of the following theorem:

> *The gliding of a system with quadratic Lagrangian and linear constraints occurs along the geodesics of the reduced connection in the subbundle that the constraints single out.*

In a less precise form this theorem is given in [**VF1, VF2**].

§4 is devoted to the main example — the group theory (symmetry study) of the problems with constraints. We formulate the problems of rolling without sliding as group-theoretical ones and consider gliding. The main statement of this section is a reduction theorem analogous to that of symplectic dynamics. We reduce the problem to dynamics on the dual of a Lie algebra. It is interesting to find out the cases of its integrability.

In §5, we briefly consider non-classical problems for the fields of indicatrices. The peculiarity of these problems is that the constraints are given by inequalities; in each tangent space, they single out submanifolds with boundary or with corners. In problems of optimal control (stated in terms of differential inclusions) and of dynamics in economics, there appear analogous fields of convex subsets in the tangent spaces — analogs of distributions (linear subspaces) in the classical case, see [**V, VCh, Ger**].

Here it is necessary to use the extremal posing of the problem, because the constraints work in an active form (through control) and not through reactions.

These problems are also very close to the problem of infinite dimensional convex programming. In the simplest case, this phenomenon was noted in [**V, T-K**].

We also formulate a number of open problems.

§2. Geometry of the tangent bundle and Lagrangian mechanics

2.1. Background. Let Q be a smooth (always C^∞) connected manifold without boundary, TQ its tangent bundle, $\pi\colon TQ \longrightarrow Q$ the canonical projection, let $T^2Q = T(TQ)$ and $d\pi\colon T^2Q \longrightarrow TQ$.

The tangent spaces at the points $q \in Q$ and $(q,v) \in TQ$ are denoted by T_q and $T_{q,v}$, respectively. A vector field X on TQ is called *special* if

$$d\pi(X_{q,v}) = v. \tag{201}$$

The term "special vector field" is a synonym of the terms "virtual displacement" in Mechanics and "second order differential equation" in Calculus.

The vertical tangent vectors, i.e., the vectors tangent to the fiber $T_q \subset TQ$ over q of the vector bundle TQ, form the subspace $\tilde{T}_{q,v} \subset T_{q,v}$. Evidently, $\tilde{T}_{q,v}$ can be identified with $\tilde{T}_q$ and, by the same token, with T_q. Thus, we have the following canonical monomorphism

$$\gamma_{q,v}\colon T_q \longrightarrow T_{q,v} \tag{202}$$

and a $(1,1)$-tensor

$$\tau_{q,v} = \gamma_{q,v} d\pi_{q,v} \tag{203}$$

on TQ.

The tensor field $\tau = \{\tau_{q,v} \mid (q,v) \in TQ\}$ is called the *principal tensor field* on TQ. This field, as a map of vector fields, annihilates all the vertical vector fields (because they are annihilated by $d\pi$) and only them. The range of τ is the subbundle consisting of vertical vector fields.

The coordinate form of τ is as follows:

$$\tau\left(a\frac{\partial}{\partial q}+b\frac{\partial}{\partial v}\right)=a\frac{\partial}{\partial v}. \tag{204}$$

The dual tensor field τ^* acts on forms:

$$\tau^*(adq+bdv)=bdq. \tag{205}$$

The range and the kernel of τ^* coincides with the bundle of horizontal 1-forms on TQ (i.e., the forms that annihilate the vertical fields).

The vertical field on TQ with coordinates

$$\Phi_{q,v}=\gamma_{q,v}v=\sum v_i\frac{\partial}{\partial v_i} \tag{206}$$

will be called the *fundamental* one.

As is easy to see,

(207) a field X is special if and only if $\tau X=\Phi$.

Indeed, in local coordinates, $X_{q,v}=\sum v_i\frac{\partial}{\partial q_i}+...$ and $(\tau X)_{q,v}\sum v_i\frac{\partial}{\partial q_i}=\Phi$.

With these two notions — τ and Φ — we formulate all of Lagrangian mechanics in invariant way.[3]

Let L be a Lagrangian, i.e., a linear on fibers smooth function on TQ (i.e., a 1-form on Q). Then $\tau^*(dL)=\frac{\partial L}{\partial v_i}dq_i$ is a horizontal 1-form called the *momenta field of the Lagrangian*.

In mechanics, the horizontal 1-forms describe forces, and the integral of such a form along an integral curve of a special field is the work performed by the force along this curve.

The *Lagrangian 2-form* is $\Omega_L=d(\tau^*(dL))$ (cf. [**G**], Ch. 11, §1); in coordinates, we have:

$$\Omega_L=\frac{\partial^2 L}{\partial v_i\partial v_j}dq_idv_j+\frac{\partial^2 L}{\partial v_i\partial q_j}dq_idq_j. \tag{208}$$

The image of this 2-form under the Legendre transformation (provided the Hessian $\frac{\partial^2 L}{\partial v_i\partial v_j}$ is non-degenerate) is the canonical 2-form $\sum dp_idq_i$ on the cotangent bundle. The *energy* (or the *Hamiltonian*) is

$$H_L=dL(\Phi)-L, \tag{209}$$

and the *Lagrangian force on the virtual displacement* (special field) X is

$$\Omega_L(X,\cdot)-dH_L. \tag{210}$$

This is a horizontal 1-form, its value on the vector field Y can be interpreted as the work of the Lagrangian force along X on Y.

In our terms, *d'Alembert's principle* (the principle of virtual displacements) is formulated as follows:

> *On the vector field that determines the real trajectories of motion, the Lagrangian force is equal to the exterior force ω. In particular, if the exterior force vanishes, then so does the Lagrangian force*:
>
> $$\Omega_L(X,\cdot)-dH_L=\omega. \tag{211}$$

[3]L. D. Faddeev and me defined the tensor τ in 1968. A report on invariant construction was made for the Leningrad Mathematical Society in May 1970 (see Uspekhi Mat. Nauk, 1973, v. 28, no. 4, p. 230). Later I learned about the book [**G**], published in French in 1969 and translated into Russian in 1973, where τ, called a "vertical endomorphism", was considered and where the invariant form of the Euler equations (without constraints) was given. The work [**VF2**] has been accomplished in 1971 and was sent to press in early 1973.

In coordinates, the *Euler-Lagrange equations* are:

(212) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \quad \text{where } i = 1, \dots, \dim Q.$$

If the 2-form Ω_L is non-degenerate, then one can easily find the vector field X_L itself (i.e., solve the equations with respect to second derivatives) in a Hamiltonian form:

(213) $$X_L = \Pi_L(dH_L),$$

where Π_L is the bivector given by the expression

(214) $$\Omega_L(\Pi_L(\omega), Y) = \omega(Y).$$

The fact that the field X_L is special follows from local formulas ([**VF2**]).

The form Ω_L is non-degenerate if and only if $\det(\frac{\partial^2 L}{\partial v_i \partial v_j}) \neq 0$.

Eq. (213) is the equation of motion written in the Hamiltonian form. The matrix $\Gamma_L = (\frac{\partial^2 L}{\partial v_i \partial v_j})$ determines a quadratic form on the fiber and is connected with Π_L by the relation[4]

(215) $$\Gamma_L^{-1} = \Pi_L \tau^* = -\tau \Pi_L.$$

The case where L is a positive definite quadratic form on the fibers of TQ, i.e., L is a Riemannian metric on Q, is of special interest. In this case formula (211) can be reduced to the form of the Newton equations by means of the Riemannian connection on Q (see §4).

Observe that the Lagrangian can be considered as a closed (rather than exact) 1-form on TQ. All the arguments are valid for this case, but the Hamiltonian is defined only locally. This theory is being actively studied in recent years [**Nov**].

2.2. Constraints. Consider a dynamical system on a smooth manifold Q. A *constraint* is a distribution $S \subset TQ$, and a *dynamical system on Q with constraint S* is a dynamical system for which the velocities at every point $q \in Q$ belong to S. Accordingly, the most general notion of constraint in mechanics (and in the theory of the second order equations in general) is defined as a subbundle in T^2Q.

It is often more convenient to consider the codistributions, i.e., the subbundles of T^*Q. Then the above distribution S can be regarded as the annihilator of a codistribution.

A *constraint* on the phase space TQ is a codistribution θ on TQ. A *dynamical system concordant with the constraint* θ is a special vector field X on TQ such that $\theta(X) = 0$ at every point. In coordinates:

(216) $$\theta = Span(a_{ik}dq_k + b_{ik}dv_k \mid i = 1, \dots, m).$$

We do not require here the codistribution to be integrable, but in the majority of "usual" examples they are: θ is given as the linear span of differentials of a set of functions on TQ, these functions (or rather their zeros or the level sets) determine a constraint in the usual sense of this word. Such a constraint can be called a *functional* one. In this case all vector fields and forms are considered only on the level sets of these functions rather than on the whole TQ without particular reservations.

[4]Cf. [**VF2**], where "horizontal forms" on p. 132, 7-th line from below, should be replaced by "quotient space modulo horizontal forms" and where τ^* means what τ means here.

A classical example of this kind: linear constraints — functions on TQ linear in velocities:

$$\begin{aligned}\theta = Span\left(\varphi_i(q,v) = \sum_k a_{ik}v_k \mid i = 1, ..., m\right) = \\ Span\left(\sum_k a_{ik}dv_k + \sum_{k,j} \tfrac{\partial a_{ik}}{\partial q_j} dq_j \mid i = 1, ..., m\right).\end{aligned} \tag{217}$$

The corresponding subset of TQ is the subbundle formed by the codistribution

$$Span\left(\sum_k a_{ik}dv_k \mid i = 1, ..., m\right). \tag{218}$$

The principal tensor field permits one to define the notion of *constraint reactions*. By that we mean the horizontal codistribution $\tau^*\theta$, where θ is the constraint.

Any 1-form belonging to $\tau^*\theta$ is called a *constraint reaction force*. A constraint is said to be *admissible* if $\dim \tau^*\theta = \dim\theta$, i.e., if the codistribution θ has no horizontal covectors at any point (recall that the kernel of τ^* is the space of horizontal 1-forms).

A constraint is said to be an *ideal* one if it annihilates the fundamental vector field Φ.

2.2.1. STATEMENT. 1) *If a constraint is admissible, then there exist a special vector fields that satisfies this constraint.*

2) *If a constraint is an ideal one, then the* 1*-forms that represent constraint reactions vanish on cycles lifted to TQ from Q (i.e., "do no work").*

PROOF. 1) Since the special fields X are exactly those for which $\tau X = \Phi$, we have to establish solvability of the linear system

$$\tau X = \Phi, \quad \theta(X) = 0. \tag{219}$$

Let

$$\begin{aligned}X &= \sum_i v_i \tfrac{\partial}{\partial q_i} + \sum f_i \tfrac{\partial}{\partial v_i}; \\ \theta &= Span\left(\sum_i (\theta^1_{ki}dq_i + \theta^2_{ki}dv_i) \mid 1 \le k \le m\right); \\ \tau^*\theta &= Span\left(\sum_i \theta^2_{ki}dq_i \mid 1 \le k \le m\right).\end{aligned} \tag{220}$$

Since $\dim \tau^*\theta = \dim\theta$, it follows that $\mathrm{rk}(\theta^1, \theta^2) = \mathrm{rk}\theta^2$; but

$$\theta(X) = \sum_i (\theta^1_{ki}v_i + \theta^2_{ki}f_i) = 0; \tag{221}$$

hence the system

$$\sum_i \theta^2_{ki}f_i = \sum_i \theta^1_{ki}v_i \quad (k = 1, ..., m) \tag{222}$$

is solvable (for f). It is easy to see that, in this case, the number of linearly independent special vector fields is $\ge \dim Q - \dim\theta$.

2) Let ξ be a cycle in Q (i.e., $\xi\colon S^1 \longrightarrow Q$, where S^1 is the circle, is a continuous map), let $\tilde\xi$ be its lift to TQ. Then

$$\int_{\tilde\xi} \tau^*\theta = \int_{S^1} \langle \tau^*\theta, \dot\xi\rangle = \int_{S^1} \langle \theta, \tau\dot\xi\rangle = \int_{S^1} \langle\theta, \Phi\rangle = 0. \tag{223}$$

□

Let a distribution θ be given as a system of Pfaff equations for X (for some functions φ_i on TQ):

$$\theta = Span(X \in \mathrm{Vect}(Q) \mid d\varphi_i(X) = 0 \ \text{ where } \ i \in I). \tag{224}$$

2.2.2. STATEMENT. *If the functions φ_i are homogeneous in v of homogeneity degree one, then the constraint θ is an ideal one.*

PROOF. It follows from a theorem of Euler that, on the zero set of the collection of functions φ_i, where $i \in I$, we have

$$\begin{gathered} \sum_j v_j \tfrac{\partial \varphi_i}{\partial v_j} = \lambda \varphi_i, \\ \text{and hence } \sum_j v_j \tfrac{\partial \varphi_i}{\partial v_j} = 0 \text{ on the set of zeros of the } \varphi_i, \end{gathered} \tag{225}$$

i.e., $\langle \varphi_i, \Phi \rangle = 0$. □

Now let us proceed to derivation of the equations of constrained dynamics (cf. [**VF2**]). Let $\alpha = \{\alpha_i\}_{i \in I}$ be a set of 1-forms that define a distribution on Q, let L be the Lagrangian, let H_L and Ω_L be the corresponding Hamiltonian and the Lagrangian 2-forms (see above). As before, we proceed from the d'Alembert principle:

$$\Omega_L(X, \cdot) - dH_L = \omega, \tag{226}$$

where ω is the constraint reaction force that causes the vector field X to be found be compatible with the constraints

$$\alpha_i(X) = 0 \ \text{ for } i \in I. \qquad \square \tag{227}$$

2.2.3. THEOREM. *If the Lagrangian L is non-degenerate, and the Hessian $\left(\frac{\partial^2}{\partial v_i \partial v_j}\right)$ is positive definite at all points (q, v), then, for every admissible constraint α, there exists a special vector field X concordant with constraints (227) and satisfying d'Alembert's principle (211), where ω is a certain constrained reaction force.*

PROOF. Let us solve the system composed of (226), (227) for X and ω. Since Ω_L is non-degenerate, there exist an antisymmetric 2-field Π_L of maps from the space of 1-forms to that of vector fields defined from

$$\Omega_L(\Pi_L(\rho), Y) = \rho(Y) \ \text{ for any 1-form } \rho. \tag{228}$$

Then we have

$$X = \Pi_L(dH_L + \omega), \tag{229}$$

and by (227) we have (for $i \in I$)

$$\langle \alpha_i, \Pi_L(dH_L + \omega) \rangle = 0 \tag{230}$$

or

$$\langle \alpha_i, \Pi_L(dH_L) \rangle = -\langle \alpha_i, \Pi_L(\omega) \rangle \ \text{ and } \ X = X_L + \Pi_L(\omega), \tag{231}$$

where $X_L = \Pi_L(dH_L)$.

By the definition of constraint reaction we seek ω in the form $\tau^* \rho$, where $\rho \in \alpha$, i.e,

$$\langle \alpha_i, X_L \rangle = -\langle \alpha_i, \Pi_L(\tau^* \rho) \rangle. \tag{232}$$

Since the constraint α is admissible, τ^* preserves its dimension. As the Hessian, and hence Π_L, is positive definite, the restriction of Π_L onto $\tau^* \alpha$ is also positive definite, and therefore non-degenerate. The determinant of the system of equations (232) for ρ is nonzero at every point:

$$\rho = \sum_{j=1}^{m} \lambda_j \alpha_j, \quad \sum_{j=1}^{m} \lambda_j \langle \alpha_j(\Pi_L(\tau), \alpha_i \rangle = -\alpha_i(X_L). \tag{233}$$

Here the λ_j are the desired coefficients of expansion — the Lagrange multipliers. □

REMARKS. 1) The vector field desired is the sum of a special vector field for the constraint-free problem and an additional vector field $\Pi_L(\omega)$, the latter field being vertical, since ω is a horizontal 1-form. Thus, one can obtain the field for the constrained problem from the field of the constraint-free problem by means of a projection. This projection consists in adding a certain vertical field (the horizontal component being unchanged) whereupon the field becomes tangent to the constraint. This projection depends on the Lagrangian (see §3).

2) If the Hessian, even a non-degenerate one, were not positive definite, its restriction on a certain codistribution would, possibly, be not of the maximal rank, i.e., the determinant of system (233) could vanish for a certain constraint. Similarly, if the constraint were not admissible, the rank of the system could drop for a certain Lagrangian. In this sense, both conditions of the Theorem are necessary for the existence of a solution.

3) One can allow violation of these conditions on submanifolds of TQ of lower dimensional (and this is inevitable for certain problems). In this case the vector field desired can have singular points.

4) A coordinate expression of equations (233) is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j \alpha_j^i, \quad i = 1, \ldots, m. \tag{234}$$

Here one finds the λ_j from the conditions of concordance of the solution to the constraints:

$$\sum_{j=1}^{m} \alpha_j^i dq_i + \tilde{\alpha}_j^i d\dot{q}_i = 0, \quad \text{where } \ j = 1, ..., m, \tag{235}$$

where $\alpha_j = (\alpha_j^1, ..., \alpha_j^n, \tilde{\alpha}_j^1, ..., \tilde{\alpha}_j^n)$ are the coordinates of the constraining forms in variables $q_1, ..., q_n,\ v_1, ..., v_n$ and the $d\dot{q}_i$ are taken from (234). In the case of functional constraints (exact forms α), equations (235) turn into functional relations

$$\varphi_j(q, \dot{q}) = 0, \quad \text{where } \ j = 1, ..., m, \tag{236}$$

i.e., into common constrained equations. In particular, for linear homogeneous functional constraints, they turn into the conditions

$$\sum_{i=1}^{n} a_j^i(q)\dot{q}_i = 0, \quad \text{where } \ j = 1, ..., m. \tag{237}$$

Once again observe that (234) and (235) are not, generally speaking, the Euler-Lagrange equations for any conditional variational problem: the Lagrange multipliers enter the right-hand side of the Euler equation, rather than the Lagrangian. Thus, universally adopted motion equations of nonholonomic Lagrangian mechanics are derived from the general d'Alembert principle by means of invariant structures in the tangent bundle.

§3. Riemannian metric and reduced connection

3.1. Various classical formulations for Riemannian manifolds. If $2L = g$ is a Riemannian metric on Q, i.e., a quadratic symmetric positive definite form on T_q for every q, and, by the same token, a quadratic on fibers function on TQ, then the vector field of the dynamical system with the Lagrangian L admits many descriptions. In textbooks, one usually proves the equivalence of these descriptions implicitly. Let us recall the most important formulations, several simplifications having been fixed previously. Let

$$L = H_L = \frac{1}{2}g. \tag{238}$$

Then T^*Q can be identified with TQ by means of the metric[5], and Ω_L is the canonical 2-form on TQ. The manifold TQ is also a Riemannian one with the metric G defined on T^2Q from the decomposition $T_{q,v} = T_{q,v}^{\mathrm{vert}} \oplus T_{q,v}^{\mathrm{hor}}$ (as Euclidean spaces), where $T_{q,v}^{\mathrm{vert}}$ and $T_{q,v}^{\mathrm{hor}}$ are the vertical and horizontal subspaces of $T_{q,v}$, respectively (they exist due to existence of the Riemannian connection). Each of these subspaces is canonically isomorphic to T_q (thanks to the isomorphisms $d\pi\colon T_{q,v}^{\mathrm{hor}} \simeq T_q$; and $T_{q,v}^{\mathrm{vert}} = \tilde{T}_{q,v}$, see §2), hence, each of them is endowed with the form g_q. The explicit block form of the tensors τ and τ^* (with respect to the decomposition $T = T^{\mathrm{vert}} \oplus T^{\mathrm{hor}}$) is very simple:

$$\tau = \begin{pmatrix} 0 & 1_a \\ 0 & 0 \end{pmatrix}, \quad \tau^* = \begin{pmatrix} 0 & 0 \\ 1_b & 0 \end{pmatrix}, \quad \text{where } a = \dim T^{\mathrm{vert}} = b = \dim T^{\mathrm{hor}}. \tag{239}$$

The horizontal forms (fields) are connected with vertical forms (fields) by a fixed isometry, because every vector from T_q has the horizontal lifting in $T_{q,v}$, and $T_{q,v}^{\mathrm{vert}} \simeq T_q$.

Finally, the matrix form of Π_L is $\begin{pmatrix} 0 & 1_a \\ -1_a & 0 \end{pmatrix}$. All these facts follow from elementary geometry of Riemannian spaces.

The vector field X_L corresponding to the motion with the Lagrangian $L = \frac{1}{2}g$ can be described by any of the following principles:

A) X_L is a Hamiltonian field in TQ, i.e., $X_L = \Pi_L(dg)$, i.e., $\Omega_L(X_L, \cdot) = dg(\cdot)$.

The field X_L is horizontal with respect to the Riemannian connection because it determines the geodesic flow. Hence, its integral curves permit another description:

B) Integral curves of the field X_L are the curves $x(\cdot)$ in Q satisfying the Newton equation $\nabla_{\dot{x}}\dot{x} = 0$ (in coordinates, $\ddot{x}^k = \Gamma_{ij}^k \dot{x}^i \dot{x}^j$) lifted to TQ.

C) The field X_L is the field corresponding according to (213) to the Euler equation for the variational problem of minimum length (the *principle of least action*).

The equivalence of the formulations of principles A and B is an important fact usually proved by comparing formulas or by establishing an equivalence with principle C. However, A and B are local principles (d'Alembert's and Gauss's, respectively), and their nature is distinct from that of principle C.

Ordered as generality deminishes, these principles are: A, C, B[6].

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[5]One can also identify TQ with T^*Q for arbitrary non-degenerate Lagrangians, too (the Legendre transformation), but it is important that in our case this identification is linear on fibers, hence, it preserves linearity of constraints.

[6]The principle B seems to have more generality than formulated here. I think that the objects like connection and covariant derivative exist for a more general class of Lagrangians than Riemannian metrics.

3.2. Linear constraints and the reduced connection. Now, let us proceed to the specialization of equations from §2 for the case of the Riemannian metric and linear constraints. In mechanics, one usually considers the linear constraints. This accounts for linearity of constraints in the majority of nonholonomic applied problems (rolling, etc.). Moreover, the possibility of realizing non-linear constraints in mechanical problems has been an open problem for a long time, cf. [**NF**].

Let Q be a Riemannian manifold with metric g, let β be a codistribution on Q, and let $\beta^\perp$ be the distribution annihilating β. From here on we make no distinction between distributions and codistributions since the metric allows us to identify 1-forms with vector fields. However, we determine every constraint by its annihilator, hence, we denote the distribution of admissible vectors as $\beta^\perp$. The set of pairs

$$T^\beta Q = \{(q, v) \mid \beta_q(v) = 0\} \tag{240}$$

is a subbundle of TQ. The subbundle $T^\beta Q$ determines a constraint and is described by the system of equations

$$\varphi_i(q, v) := \langle a_i(q), v\rangle = 0, \quad i = 1, ..., m, \tag{241}$$

where $a_i(q)$ are 1-forms forming a basis of the space of sections of β at q. The forms that correspond to our constraint are determined by the set of differentials of the functions φ_i. We do not need these forms thanks to the following statement (true for arbitrary manifolds).

3.2.1. Statement. *Reactions of the linear constraints defined by a codistribution β on Q form the codistribution $(d\pi)^*\beta$ on TQ, see sec. 2.2. The linear constraints are admissible and ideal ones.*

Proof. For $i = 1, ..., m$, we have

$$\tau^*(d\varphi_i) = \tau^*\{\partial_q a_i dq + a_i dv\} = \{a_i dq\} = (d\pi)^* a_i. \tag{242}$$

The ideal nature of constraints follows from homogeneity in v; admissibility is evident. □

Now, define connections in $T^\beta Q$. By the definition, any connection in $T^\beta Q$ is an adjoint connection in the principal fiber bundle $B^\beta Q$ of partial frames, where the fiber over $q \in Q$ is the space of all frames in the subspace $\beta_q^\perp$, and the structure group is $GL(\text{rk }\beta^\perp)$. Since $T^\beta Q$ is a vector bundle, the connection can be determined by a covariant derivative for the fields compatible with $\beta^\perp$.

First, let us prove the following general lemma (without using the metric).

3.2.2. Lemma. *Let Q be an arbitrary manifold with a linear connection determined by a covariant derivative ∇. Given a subbundle $T^\beta Q$ in TQ, and, for every fiber R_q (a subspace of T_q), a projection $F_q\colon T_q \longrightarrow R_q$ smoothly depending on q. Then*

$$\widetilde{\nabla}_X(Y) := F\nabla_X(Y) \quad \text{for any } X, Y \in \Gamma(T^\beta Q)$$

determines a connection in the subbundle.

Proof. Let us verify that the properties of connection are satisfied for $\widetilde{\nabla}$ (see [**KN**]). Of the four properties, only the following one is nontrivial:

$$\widetilde{\nabla}_X(\lambda Y) = (X\lambda)Y + \lambda\widetilde{\nabla}_X(Y) \text{ for any function } \lambda. \tag{243}$$

This property holds since

$$\begin{aligned} &F\nabla_X(\lambda Y) = X(\lambda Y)) + F\lambda(\nabla_X(Y) = \\ &\lambda\widetilde{\nabla}_X(Y) + X\lambda \cdot FY \end{aligned} \tag{244}$$

and $FY = Y$ for any $Y \in T^\beta X$. Hence, $\widetilde{\widetilde{\nabla}}$ is a connection in $T^\beta X$. □

Let Q be a Riemannian manifold and β, $\beta^\perp$ distributions on Q. At every point q, define the orthogonal projection $F_q\colon T_q \longrightarrow \beta_q^\perp$ by means of the Riemannian metric on Q, and the connection $\widetilde{\widetilde{\nabla}} := \nabla^\beta$ by Lemma 3.2.2. The connection $\widetilde{\widetilde{\nabla}}$ (on $T^\beta Q$) is said to be a *reduced* one.

3.2.3. THEOREM. *In local coordinates, the reduced connection $\widetilde{\widetilde{\nabla}}$ in the subbundle $T^\beta Q$ is expressed as:*

$$\nabla^\beta_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k, \quad i, j, k = 1, \dots, m, \tag{245}$$

where the Γ^k_{ij} are the Christoffel symbols of the Riemannian connection but X_i, X_j, X_k are the linearly independent (coordinate) fields from $\beta^\perp$ only.

PROOF. Since F is the orthogonal projection onto $\beta^\perp$, eq. (245) follows from the expression for the covariant derivative. □

3.2.4. REMARKS. 1) If the distribution $\beta^\perp$ is an involutive one, then ∇^β is the induced Riemannian connection on the fibers of the bundle determined by ∇^β.

2) The reduced connection can be extended to a (non-Riemannian) connection in TQ, but **there hardly exists any** *canonical* **extension**.[7]

3) The expression

$$\nabla^\beta_X(Y) - \nabla^\beta_Y(X) - [X, Y] = F([X, Y]) - [X, Y] \tag{246}$$

does not vanish if $\beta^\perp$ is not involutive. On the other hand, the symbol Γ^k_{ij} is symmetric. This means that one can not define torsion by formula (246). **It well may be true that the torsion for ∇^β can not be defined at all**.[8]

4) The curvature form is rather sophisticated[9] and is studied insufficiently. A most interesting problem is to find the holonomy groups for ∇^β.

3.3. Inertial motion with quadratic Lagrangian and linear constraints. Now, apply the results of §2 and subsec. 3.1, 3.2 to the initial problem.

3.3.1. THEOREM. *Let Q be the position space, $L = \frac{1}{2}g$, where g is a Riemannian metric on Q. Let $T^\beta Q = \{(q, v) \mid \beta_q(v) = 0\}$ be the subbundle corresponding to a distribution β. Then the equations of motion for the dynamical system with the Lagrangian L and linear constraints β are equations of the geodesics for the reduced connection ∇^β.*

PROOF. By Theorem 2.2.3 the field desired exists in $T^\beta Q$ because the constraint is admissible; it is given by the formula $X_L = X + \Pi_L(\omega)$, where X is the vector field for the system without constraints (geodesic field), and ω is a 1-form from the codistribution of the constraint reactions. This means that ω is a covector (hence, vector) field on Q lying in β. Thus, $\Pi_L(\omega)$ is a vertical field. It is chosen so that the field X_L be concordant with the constraint, i.e., the vertical projection of X_L lies in the image of $\beta^\perp$.

On the other hand, the connection ∇^β can be represented as

$$\nabla^\beta = F\nabla = \nabla + S, \tag{247}$$

[7]Boldface is mine. Here, Vershik was too pessimistic, as one can see in [**L**]. *D.L.*

[8]Boldface is mine. Fortunately (this is an important place), Vershik was too pessimistic here also, as one can see in [**L**]. *D.L.*

[9]It is not if defined as in [**L**]. *D.L.*

where S is a $(1,2)$-tensor (the difference between any two covariant differentiations is a $(1,2)$-tensor), i.e., $S = F^{\perp}\nabla$, where $F^{\perp}$ is the projection on β. Hence, any horizontal vector of the connection ∇^{β} differs from a horizontal vector of the connection ∇ in a vertical vector lying in β and is concordant with the constraint. The decomposition into vertical and horizontal components is unique, and hence the special horizontal field of the connection ∇^{β} at the points $(q,v) \in T^{\beta}Q$ and the field X coincide. □

3.3.2. Corollary. *The equations of motion for our problem can be represented as*

$$\nabla^{\beta}_{\dot{X}_i}\dot{X}_j = 0 \tag{248}$$

or

$$\ddot{X}^k = \sum_k \Gamma^k_{ij}\dot{X}^i\dot{X}^j, \quad i,j,k = 1,...,m = \dim\beta^{\perp}, \tag{249}$$

where the coordinates are chosen so that $\dot{X}^1,\dots,\dot{X}^m$ form a basis in the fiber of the distribution $\beta^{\perp}$. If there are exterior forces, potential, etc., then one makes a change in (248) *in the usual way:*

$$\nabla^{\beta}_{\dot{X}_i}\dot{X}_j = \omega_1, \tag{250}$$

where ω_1 is the vector field of exterior forces (the gradient in the potential case).

If the distribution $\beta^{\perp}$ is involutive, then (248) is the equation of geodesics in the fiber of the bundle, and if the distribution is geodesic (i.e., its fiber is completely geodesic, see [**KN**]), then the solution of (248) coincide with the solution of the constraint free problem.

3.3.3. Remark. As we have said already, for the dynamical problem, one usually uses the field X_L (in our terms: the field of geodesics) only. Connections were not generally used even for constraint-free problems. As one can see from above, their roles are rather essential.

3.4. Non-holonomicity and the Hopf-Rinow theorem. Of the most interest is the case when the involutive hall of the distribution $\beta^{\perp}$ generates the whole bundle. By the Frobenius theorem this happens when the brackets of the vector fields from $\beta^{\perp}$ generate the whole Lie algebra of vector fields on Q.

3.4.1. Problem. *Describe the distributions of dimension m in general position and involutivity of these distributions. (One can suppose that, for $m > 1$, the distribution generates the whole T_q for generic points $q \in Q$, and the dimension of the involutive hull diminishes at singular points.)*

3.4.2. Statement. (The Hopf-Rinow theorem, cf. [**KN**], Ch. 3, §6, and [**JJ**]) *The connection $\beta^{\perp}$ on a complete Riemannian manifold is complete.*

3.4.3. Corollary. *If the distribution $\beta^{\perp}$ is nonholonomic, then, for any two points $q_1, q_2 \in Q$, there exists a continuous curve connecting these points and consisting of pieces of the geodesics of the connection ∇^{β} . In other words, the geodesic flow is transitive.*

At the same time not every two points can be immediately connected by a ∇^{β}-geodesic.[10] The following interesting problem arises.

On a Riemannian manifold Q with a nonholonomic distribution $\beta^{\perp}$, define the following new metric:

$$d_{\beta}(q_1,q_2) = \inf\{l(\tau) \mid \tau(0) = q_1,\ \tau(1) = q_2,\ \dot{\tau} \in \beta^{\perp}\}, \tag{251}$$

[10]For the most lucid example, see [**Poi**]. *D.L.*

where τ is a smooth curve, $l(\cdot)$ the length with respect to the initial Riemannian metric. This infimum is not attained on the geodesics of the connection ∇^β, and hence the classical *Hopf-Rinow theorem* is not valid in this case.

3.4.4. PROBLEM. *Describe intrinsically the d_β-type metrics on Riemannian manifolds. (Compare with the "space of geodesics" defined by Busemann* [**B**].*)*

An example of this kind was considered by Gershkovich [**Ger**].

§4. Group mechanics with constraints

4.1. Problem of rolling and its group model. The most popular example of a nonholonomic problem is the following one: two bodies (e.g., a ball and a plane) move (from force of inertia or in a field) so that the linear velocities of both bodies at the point of contact coincide (no sliding). The reader interested in the traditional technique of derivation of the constraint equations is referred to special literature. Here we describe (apparently for the first time) the group model of this problem in a reasonably general formulation. The group example is a central one here, as in the case of ordinary dynamics.

Let G_1 and G_2 be two arbitrary connected Lie groups determining position of each body in a moving coordinate system, G_1^o and G_2^o their stationary subgroups (provided the point of contact is fixed). The coincidence of velocities at the point of contact is expressed as an isomorphism $s\colon \mathfrak{g}_1/\mathfrak{g}_1^o \simeq \mathfrak{g}_2/\mathfrak{g}_2^o$, where $\mathfrak{g}_i$ is the Lie algebra of G_i. Now consider $Q = G_1 \times G_2$ as the position space and, in $(TQ)_e = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, single out the subspace $\mathfrak{M}$ spanned by

$$(a_1, a_2) \text{ such that } s \circ t_1(a_1) = t_2(a_2), \tag{252}$$

where $t_i\colon \mathfrak{g}_i \longrightarrow \mathfrak{g}_i^o$ are the canonical projections. Clearly, $\mathfrak{M}$ is a linear subspace of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. By transferring this subspace to all points of $G = G_1 \times G_2$ by means of left translations we obtain a distribution (linear elements with coinciding velocities at the points of contact). This is the principal model.

Now one can consider a Lagrangian (left-invariant for inertial motions) and apply the methods developed in §3 to form the equations of motion. For a more general scheme, one does not need to specialize the group as the direct product of two (or more) groups. The most natural form of our scheme is the following one.

Let G be an arbitrary connected Lie group, $\mathfrak{M}$ a subspace in the Lie algebra $\mathfrak{g}$ (usually, a complement to a subalgebra). The distribution $\beta^\perp$ is the distribution of the left translates of $\mathfrak{M}$ in TG. Such a model will be called a *group* one.

4.1.1. EXAMPLE. 1) $G = SO(3)$, $\mathfrak{g} = \mathfrak{o}(3)$. This is the rotation of a solid body with a fixed point and a zero linear velocity at it.

2) $G = SO(3) \times \mathbb{R}^2$, $\mathfrak{g} = \mathfrak{o}(3) \oplus \mathbb{R}^2$, $\mathfrak{M} = \{(a, h) \mid \pi(a) = h\} \subset \mathfrak{g}$, where $\pi\colon \mathfrak{o}(3) \longrightarrow \mathfrak{o}(3)/\mathfrak{o}(2) \simeq \mathbb{R}^2$. This is a description of a rolling ball on a rough plane.

4.2. Reduction. For the case of a left-invariant Lagrangian, we can reduce the proposed group model to a system on the Lie algebra. Observe that the problems with constraints are not symplectic, i.e., the fields that appear do not preserve, generally, any 2-form, hence, the question about integrals and reduction for these systems should be considered separately. Reduction of the order of these systems is less than that of symplectic systems. The following statement refers to a general system with constraints.

4.2.1. STATEMENT. (See [**VF2**].) *If the constraint α is an ideal one, then the energy H_L is a motion integral for system with the Lagrangian L.*

PROOF.

$$\begin{aligned} &\tfrac{d}{dt}H_L = X(dH_L) = \Omega_L(\Pi_L(dH_L), X) = \\ &\Omega_L(\Pi_L(dH_L), \Pi_L(dH_L)) + \Omega_L(\Pi_L(dH_L), \Pi_L(\omega)) = \\ &\omega(\Pi_L(dH_L) = \omega(X_L) = \tau^*\rho(X_L) = \\ &\rho\tau(X_L) = \rho(\Phi) = 0. \qquad \square \end{aligned} \tag{253}$$

4.2.2. THEOREM. *Let the position space be a Lie group G, and the (linear) constraints and the (quadratic) Lagrangian of the system are left-invariant. Then the phase flow of the system is the fiber bundle whose base is the set of flows whose vector fields are the projections of the Euler fields (without constraints) on the subspace of constraints in the Lie coalgebra, and the fiber is the set of conditionally periodic flows on the group.*

PROOF. We identify TG and T^*G by means of the Lagrangian (or, for semi-simple groups, by means of the Killing form). The field X that describes the motion of the system commutes with the left-invariant fields A on G. Indeed, by hypothesis we have

$$[X, A] = [X_L, A] + [X, \Pi_L(\omega)] = 0. \tag{254}$$

Hence, the partition into orbits of the natural (Hamiltonian) action of the group G in T^*G is invariant under the flow of the field X. Thus, the base of the fiber bundle is $T^\beta G/G \simeq \mathfrak{M} \subset \mathfrak{g}^*$, where $\mathfrak{M}$ is the subspace determined by the constraints and the fiber is the group G. The flow on the fiber is determined by the motion on orbits, i.e., on the group, and commutes with the left translations, and hence it reduces to the flow on left cosets with respect to a maximal torus, the latter flow being conditionally periodic on each of these cosets. The trajectories are determined by their initial vectors at a certain point, e.g., at the unity element of the group.

Proceed now to the base. Recall that for the constraint-free systems one can define the moment map $dH\colon T^*G \longrightarrow \mathfrak{g}$, see, e.g., [**A**]. On $\mathfrak{g}^*$, we have the Euler equation:

$$\dot a \ = \{dH(a), a\}, \tag{255}$$

where $a \in \mathfrak{g}^*$ and $\{\cdot,\cdot\}$ is the Poisson bracket. For the constraint-free system, the Euler equation determines the motion on the base.

By Theorem 4.2.2, in our case the vector field defined on the subspace of constraints $\mathfrak{M}$ is the orthogonal projection of the field corresponding to the Euler equation on $\mathfrak{M} \subset \mathfrak{g} = \mathfrak{g}^*$. Thus, the equation on $\mathfrak{M}$ takes the form

$$\dot a \ = P\{dH(a), a\}, \tag{256}$$

where P is the orthogonal (with respect to the Lagrangian) projection of $\mathfrak{g}$ onto $\mathfrak{M}$. $\square$

4.2.3. REMARK. 1) Actually, we asserted that the projection of the initial field onto constraints commutes with the group factorization. But, unlike the generalized Noether theorem, here one can not assert that the elements from the center of the enveloping algebra are integrals of motion.[11] Hence, the system obtained on $\mathfrak{M}$ does not preserve, generally, the orbits of the coadjoint action.

2) If the Lagrangian is not quadratic, the reduction is also possible, but in this case the Legendre transformation on the fiber of TG is non-linear, the quotient space T^*G/G has no linear structure, and the reduced system is rather sophisticated.

4.2.4. Example. Consider the problem of rolling by inertia of two n-dimensional solid bodies. In this case, we have (in notations of sec. 4.1):

$$\begin{aligned} &G_1 = G_2 = SO(n), \quad G_1^o = G_2^o = SO(n-1), \quad G = G_1 \times G_2; \\ &\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{o}(n) \oplus \mathfrak{o}(n), \quad \mathfrak{g}^o := \mathfrak{g}_1^o = \mathfrak{g}_2^o = \mathfrak{o}(n-1), \\ &\mathfrak{M} = \operatorname{diag} \mathfrak{g}^o \oplus (\mathfrak{g}_1^o \oplus \mathfrak{g}_2^o), \\ &\text{where } \operatorname{diag} \mathfrak{g}^o = \{(a,a) \mid a \in \mathfrak{g}^o\} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2. \end{aligned} \tag{257}$$

Hence,

$$\mathfrak{M} = \{(a+b, a+c) \mid a,b,c \in \mathfrak{o}(n-1)\}. \tag{258}$$

Let P be the projection onto $\mathfrak{M}$. (In our case, the orthogonal projection P does not depend on the inertia tensors.) Then, for $i,j = 1,...,n-1$, we have

$$P(X_{ij}^1, X_{ij}^2) = \begin{pmatrix} X_{ij}^1 + X_{ij}^2 & X_{ij}^1 \\ -X_{ij}^1 & 0 \end{pmatrix} + \begin{pmatrix} X_{ij}^1 + X_{ij}^2 & X_{ij}^2 \\ -X_{ij}^2 & 0 \end{pmatrix}. \tag{259}$$

Let L_1, L_2 be the inertia tensors of the bodies. Then the equations of motion in $\mathfrak{M}$ have the form

$$\begin{aligned} \dot{a}^{(1)} &= P((a^{(1)})^2 I_1 - I_1 (a^{(1)})^2), \\ \dot{a}^{(2)} &= P((a^{(2)})^2 I_2 - I_2 (a^{(2)})^2), \end{aligned} \tag{260}$$

for any $(a^{(1)}, a^{(1)}) \in \beta$. The projection P connects both equations.

I do not know to qualitatively describe the motion for this system.

In the same manner one can take another stationary subgroup (e.g., $SO(n-k)$ that corresponds to the Stifel manifold; however, I do not know whether the constraint-free problem is integrable or not).

§5. Non-classical problems with constraints

The following problems arise in optimal control and economics dynamics.

5.1. Problem. *Let Q be a manifold, TQ its tangent bundle, and we are given a field B of submanifolds $B(q)$ (with boundary or corners) in fibers T_q of TQ. We have to minimize a certain functional of the boundary value problem, the tangent vectors to admissible curves lying in the given submanifold:*

$$\inf\{F(q(\cdot)) \mid \dot{q} \in B(q) \subset T_q;\ q(a) = \bar{q},\ q(b) = \bar{\bar{q}}\}. \tag{261}$$

[11]The reduction of Hamiltonian systems under the regular actions of Lie groups (generalizations of the Noether theorem) was considered and rediscovered by many researchers. For Lie groups, I defined it as early as 1968 (the reports were made at seminars in Leningrad and Moscow State Universities and at a conference in Tsakhkadzor in 1969). The well-known 2-form discovered by A. A. Kirillov in 1962 ([**A**]) naturally appears from the canonical form in T^*G in the process of reduction. A list of literature on this topic (by no means complete) is given in the book [**A**] and in the surveys [**P**], the latter being devoted to the symmetries of more complicated nature.

The problem in this formulation is said to be problem in contingencies (differential inclusions).

One passes to this formulation from the traditional one in the following way: $B(q)$ is the set of right-hand sides of differential equations when controls run over the admissible region.

Usually, $B(q)$ are convex solids, in particular, polyhedral sets (cones or polytops). In connection with this it is important to consider the theory of fields of such sets as a generalization of the theory of distributions (involutivity, singularities, etc.). For another motivation (connected with other problems), see [**VCh, Ger**].

5.2. Problem. *Consider the following principle and apply it to these problems.*

> *One can construct the field corresponding to the Euler equation for the constraint-free problem and then project this field onto the constrain manifold as we did it above.*

In the interior points, the field does not change under this projection, but jumps (switches) can appear on the boundary. Constraints of this kind are not ideal in general.

5.3. Problem. *Find out problems of optimal control for which the field desired is obtained by the above principle.*

Mostly, the field of sets $B(\cdot) \subset TQ$ is defined by means of inequalities. For example, let $B(\cdot)$ be the field of polyhedral sets $B(q) = \{v \mid A(q)v \leq b(q)\}$, where $b(\cdot)$ is a map from Q to $\mathbb{R}^m$ and $A(\cdot) \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. In this case, we have the following general problem:

$$\inf\{\Phi(q(\cdot)) \mid A(q)\dot{q} \leq b(q),\ q \in Q\}. \tag{262}$$

Thus, this is a field of problems of linear (if Φ is linear) or convex programming in the tangent bundle TQ. Similar problems were considered in the linear theory of optimal control and also (for $Q = \mathbb{R}^1$ or $Q = [a, b] \subset \mathbb{R}^1$) in the theory of continuous economic models.

5.4. Problem. *For the above problems, relate Pontriagin's optimality principle with the duality theorem of convex programming. Conjecturally, these two statements are equivalent, as one can easily verify for the case when $A(\cdot), b(\cdot)$ are constant on $Q = \mathbb{R}^n$.*

5.5. Problem. *A problem close to the classical ones from §3 and §4 consists in finding a minimum in the following situation. Let Q be a Riemannian (or Finsler) manifold, $B(\cdot)$ a field of convex sets (polytops, spheres, etc.) in TQ. It is convenient to suppose that $B(q)$ is central-symmetric for all q. One should find the shortest (in the sense of metric) curves that are admissible for the restrictions $q \in B(q)$.*

The obtained new metric (non-Riemannian) seems to be similar to the metric from §3. Manifolds with these metrics possess, perhaps, unusual properties. (I did not see this formulation of the problem in the literature, but, possibly, similar questions were stated and solved. There is a considerable number of works that indirectly refer to the question discussed. We confine ourselves to the reference to the monograph [**T-K**].)

Study metrics on smooth manifolds appearing in these problems, see sec. 3.4, cf. [**Ger**].

5.6. Problem. *In non-classical dynamics (as well as in arbitrary dynamics), the most interesting example of a problem with constraints is the group one. Let a Lie group act transitively on a manifold M, and let the field of constraints be invariant under this action. If the functional is invariant too, then one can, conjecturally, reduce the problem (in the same manner as in §4) to the Lie algebra if M is a group.*

The difficulty appearing here is that the reduction permits one to study only the Cauchy problem rather than the boundary value problem (same as in §4). This circumstance is not substantial for classical mechanics without constraints, because usually (e.g., for Riemannian complete manifolds) the set of solutions of the Cauchy problem includes solutions of all boundary value problems (the Hopf-Rinow theorem). Here this is not the case and one should search for a solution considering not only the result of reduction but the whole fiber bundle.

5.7. Problem. *Describe the reduction for group problems of optimal control and the structure of the corresponding fiber bundle.*

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