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STRUCTURES OF $G(2)$ TYPE AND NONINTEGRABLE DISTRIBUTIONS IN CHARACTERISTIC p

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Abstract. Lately we observe: (1) an upsurge of interest (in particular, triggered by a paper by Atiyah and Witten) to manifolds with $G(2)$-type structure; (2) classifications are obtained of simple (finite dimensional and $\mathbb{Z}$-graded vectorial) Lie superalgebras over fields of complex and real numbers and of simple finite dimensional Lie algebras over algebraically closed fields of characteristic $p > 3$; (3) importance of nonintegrable distributions in (1) – (2).

We add to interrelation of (1)–(3) an explicit description of several exceptional simple Lie algebras for $p = 5$ (Melikyan algebras), for $p = 3$ (Brown, Ermolaev, Frank, and Skryabin algebras) as subalgebras of Lie algebras of vector fields preserving certain nonintegrable distributions analogous to (or identical with) those preserved by $G(2)$, $O(7)$, $Sp(4)$ and $Sp(10)$. The description is performed in terms of Cartan-Tanaka-Shchepochkina prolongs — a main tool in constructing simple Lie superalgebras of vector fields with polynomial coefficients — and is similar to descriptions of these superalgebras. We give presentations of some algebras. Our results illustrate usefulness of Shchepochkina's algorithm and **SuperLie** package: one family of simple Lie algebras found in the process might be new.

In memory of Felix Aleksandrovich Berezin

§1. Introduction

The exceptional (non-serial) simple finite dimensional Lie algebras, although of considerable interest lately on account of various applications ([AW, B, CJ, FG]) are far less understood than, say, $\mathfrak{sl}(n)$, cf. [A, B]. Who, experts including, can nowadays lucidly explain what is $\mathfrak{g}(2)$[1] or $\mathfrak{f}(4)$ or $\mathfrak{e}(6) - \mathfrak{e}(8)$?! Definitions in terms of octonions, although beautiful ([BE]), do not really help to understand these algebras. Descriptions in terms of defining relations (as in [GL1]) are satisfactory for computers, not humans. Together with [Sh2], this paper gives some applications of Shchepochkina's general algorithm [Shch] describing Lie algebras and Lie superalgebras in terms of nonintegrable distributions they preserve. Berezin who taught all three of us, liked to read classics and advised his students to. We return to Cartan's first, now practically forgotten, description of Lie algebras, not necessarily simple or exceptional ones, in terms of nonintegrable distributions they preserve; we intend to apply it to ever wider range and begin with $\mathfrak{o}(7)$, $\mathfrak{sp}(4)$ and $\mathfrak{sp}(10)$.

Turning to simple Lie algebras over algebraically closed fields $\mathbb{K}$ of characteristic $p > 0$ we encounter more and more *seemingly* "strange" examples. In [S], Strade listed all simple finite dimensional Lie algebras over $\mathbb{K}$ for $p > 3$ and selected examples for $p = 3$. We mainly use notations from [S] except for Skryabin algebras Y with appropriate adjectives: we think that Skryabin's own notation Y with its inherent implicit question mark is more appropriate. For various cases of classification of simple finite dimensional Lie algebras for $p = 3$ due to Kuznetzov, see [Ku]; for further references, see [Sk], [Sh14], [Y], [LSh]. Deformations and the case of $p = 2$ are mentioned in passing, they will be studied elsewhere.

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[1]We denote the exceptional Lie algebras in the same way as the serial ones, like $\mathfrak{sl}(n)$; we thus avoid confusing $\mathfrak{g}(2)$ with the second component $\mathfrak{g}_2$ of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$.

Melikyan algebras, still described as something somewhat mysterious and usually only for $p = 5$, are, as we will see, no more mysterious than $\mathfrak{g}(2)$ for which [Shch] recalls Cartan's lucid description. We observe that, for $p = 2, 3$, Melikyan algebras are the conventional special vectorial (divergence free) Lie algebras.

For $p = 3$, the two Brown algebras, and their deformations, were until now given by means of Cartan matrices A with only implicit defining relations (3), see below. As we pass to the Skryabin algebras, the mist thickens so much that one example ($\mathrm{BY}(\underline{N})^{(1)}$) was only partly described, cf. [Sk, S]; the Lie superalgebras $\mathrm{SBY}(\underline{N})^{(1)}$ we single out might be even new (previously unobserved) simple Lie algebras.

A. Kostrikin and Shafarevich used flags in description of simple Lie algebras in characteristic $p > 0$. In these descriptions, (nonintegrable) distributions appear twice: as associated with flags and with Lie algebras of depth > 1. Kostrikin felt the importance of the Cartan prolong and its generalization to algebras of depth > 1 but his voice, even amplified by the authority of an ICM talk, was not heard, except by Elsting, Ermolaev and Kuznetsov who buried their results in a little-known journal Izvestiya Vusov. Examples Kostrikin and his students unearthed (for example, Melikyan and Ermolaev algebras, followed by Skryabin algebras), as well as Kuznetsov's interpretations — practically identical to ours — of several of the known algebras, are all obtained as such (generalized) Cartan prolongs. Still, no general definition of generalized prolongs — a most vital tool — was ever formulated for $p > 0$ (except a tentative one in [FSh]; [Shch] positively answers questions of [FSh]); a similarity between these examples and Shchepochkina's constructions of simple Lie superalgebras, as well as "nonstandard" regradings, was mentioned only in [KL] and never before or after. This fact and the lack of lucid algorithm for constructing generalized prolongs was a reason why the examples we consider (and a lot remains) still had to be elucidated and interpreted.

Remarkably, an interpretation we have in mind — **the description of (the exceptional simple) Lie algebras as preserving a nonintegrable distribution and, perhaps, something else** — WAS repeatedly published; first, by Cartan, cf. [C]. At least, for $p = 0$. But this aspect of [Y], as well as of [C], passed unnoticed. In [Y], Yamaguchi lucidly described Tanaka's construction of generalized prolongs and considered, among other interesting things, two of the three possible $\mathbb{Z}$-gradings of $\mathfrak{g}(2)$ related with "selected"(see eq. (14)) simple roots and interpreted $\mathfrak{g}(2)$ as preserving the nonintegrable distributions associated with these $\mathbb{Z}$-gradings. The initial Cartan's interpretation of $\mathfrak{g}(2)$ used one of these distributions without indication why this particular distribution was selected. Later Cartan considered another distribution which characterizes Hilbert's equation $f' = (g'')^2$, see [Y]. Larsson [La2] considered the remaining, third grading of $\mathfrak{g}(2)$ and several (selected randomly, it seems) gradings of depth ≤ 2 of $\mathfrak{f}(4)$ and $\mathfrak{e}(6)$–$\mathfrak{e}(8)$. These and similar results for other algebras looked as *ad hoc* examples. Shchepochkina's algorithm ([Shch]) describes Lie algebras and superalgebras $\mathfrak{g}$ of vector fields as **generalized** Cartan prolongs and **partial** prolongs ([LSh]) in terms of nonintegrable distributions $\mathfrak{g}$ preserves and is applicable to fields of prime characteristic. Such an interpretation (except partial prolongs) was known to the classics (Lie, É. Cartan) but an *explicit* description of the Lie (super) algebras in terms of nonintegrable distributions they preserve was only obtained (as far as we know) for *some* of the "selected"gradings of some algebras. We believe it is time for a thorough study of all possible distributions related with simple Lie (super)algebras, and start with [GL3].

With Shchepochkina's algorithm we immediately see that various examples previously somewhat mysterious, e.g., Frank algebras, are just *partial prolongs* — analogs of the projective embedding $\mathfrak{sl}(n+1) \subset \mathfrak{vect}(n)$. Likewise, $\mathfrak{g}(2)$ is a partial prolong if $p = 5$ or 2, whereas the Melikyan algebras are complete prolongs; $\mathfrak{g}(2)$ is the complete prolong only if $p \neq 5, 2$.

Our results are obtained with aid of **SuperLie** package, see [Gr, GL2], and are applicable to many other Lie (super)algebras. An arXiv version [GL4] of this paper contains lists of defining relations for the positive nilpotent parts of the Frank algebras and some of the Skryabin algebras.

§2. Background

2.1. Notations. We usually follow Bourbaki's convention: if G is a Lie group, then its Lie algebra is designated $\mathfrak{g}$, except in characteristic $p>0$ where the modern tradition does not favor Gothic font. The ground field $\mathbb{K}$ is assumed to be algebraically closed; its characteristic is denoted by p. The elements of $\mathbb{Z}/n$ are denoted by $\bar{a}$, where $a\in\mathbb{Z}$.

For the list of simple Lie superalgebras and background on Linear Algebra in Superspaces, see [LSh]. All this super knowledge is not a must to *understand* this paper, but comparison of super and $p>0$ cases is instructive. Recall that there are two major types of Lie superalgebras (symmetric and "skew"):

(SY) For symmetric algebras, related with a Cartan subalgebra is a root decomposition such that (sdim is *superdimension*)

$$\mathrm{sdim}\mathfrak{g}_\alpha=\mathrm{sdim}\mathfrak{g}_{-\alpha}\ \text{ for any root }\ \alpha; \tag{1}$$

(SK) For skew algebras, related with a Cartan subalgebra is a root decomposition such that (1) fails. (Usually, skew algebras can be realized as **vectorial** Lie superalgebras — subalgebras of the Lie superalgebra of vector fields $\mathfrak{vect}(n|m)=\mathfrak{der}\mathbb{K}[x,\theta]$, where $x=(x_1,\dots,x_n)$ are even indeterminates and $\theta=(\theta_1,\dots,\theta_m)$ are odd indeterminates. Of course, symmetric algebras can also be realized as subalgebras of $\mathfrak{vect}(n|m)$, but this is beside the point.)

On vectorial superalgebras, there are two types of trace. The divergences (depending on a fixed volume element) belong to one of them, various linear functionals that vanish on the brackets (traces) belong to the other type. Accordingly, the *special* (*divergence free*) subalgebra of a vectorial algebra $\mathfrak{g}$ is denoted by $\mathfrak{sg}$, e.g., $\mathfrak{vect}(n|m)$ and $\mathfrak{svect}(n|m)$ abbreviated for $p>0$ to $W(n|m)$ and $S(n|m)$; the superscript $^{(1)}$ (never $'$) singles out the derived algebra — the traceless ideal.

2.2. Integer bases in Lie superalgebras. Let $A=(a_{ij})$ be an $n\times n$ matrix. A *Lie superalgebra* $\mathfrak{g}=\mathfrak{g}(A)$ *with Cartan matrix* $A=(a_{ij})$, is given by its *Chevalley generators*, i.e., elements $X_i^\pm$ of degree ±1 and $H_i=[X_i^+,X_i^-]$ (of degree 0) that satisfy the relations (hereafter in similar occasions either all superscripts $\pm$ are $+$ or all are $-$)

$$[X_i^+,X_j^-]=\delta_{ij}H_i,\quad [H_i,H_j]=0,\quad [H_i,X_j^\pm]=\pm a_{ij}X_j^\pm, \tag{2}$$

and additional relations $R_i=0$ whose left sides are implicitly described, for a general Cartan matrix, as

(3) "the R_i that generate the maximal ideal I such that $I\cap Span(H_i\mid 1\le i\le n)=0$. "

For simple (finite dimensional) Lie algebras over $\mathbb{C}$, instead of implicit description (3) we have the following explicit description (Serre relations): Normalize A so that $a_{ii}=2$ for all i; then the off-diagonal elements of A are non-positive integers and

$$(\mathrm{ad}X_i^\pm)^{1-a_{ij}}(X_j^\pm)=0. \tag{4}$$

A way to normalize A may affect reduction modulo p: Letting some diagonal elements of the **integer** matrix A be equal to 1 we make the Cartan matrices of $\mathfrak{o}(2n+1)$ and Lie superalgebra $\mathfrak{osp}(1|2n)$ (for definition of Cartan matrices of Lie superalgebras, see [GL1]) indistinguishable (this accounts for their "remarkable likeness" [RS], [Ser]):

$$\begin{pmatrix}\ddots&\ddots&\ddots&\vdots\\ \dots&2&-1&0\\ \dots&-1&2&-2\\ \dots&0&-2&2\end{pmatrix}\ \text{ or }\ \begin{pmatrix}\ddots&\ddots&\ddots&\vdots\\ \dots&2&-1&0\\ \dots&-1&2&-2\\ \dots&0&-1&1\end{pmatrix} \tag{5}$$

For Lie superalgebras of the form $\mathfrak{g}=\mathfrak{g}(A)$, there exist bases with respect to which all structure constants are integer. Up to the above indicated two ways (5) to normalize A, there is only one such (*Chevalley*) basis, cf. [Er].

When $p=3$ and 2, it may happen that $a_{ii}=0$ (if $p=0$ and $p>3$, then $a_{ii}=0$ implies $\dim\mathfrak{g}(A)=\infty$). It is natural to study this case in terms of vectorial Lie algebras.

For vectorial Lie superalgebras, integer bases are associated with $\mathbb{Z}$-forms of $\mathbb{C}[x]$ — a supercommutative superalgebra in a (ordered for convenience) indeterminates $x = (x_1, ..., x_a)$ of which the first m indeterminates are even and the rest n ones are odd ($m+n = a$). For a multi-index $\underline{r} = (r_1, \dots, r_a)$, we set

$$u_i^{r_i} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{\underline{r}} := \prod_{1\leq i\leq a} u_i^{r_i}.$$

The idea is to formally replace fractions with $r_i!$ in denominators by inseparable symbols $u_i^{r_i}$ which are well-defined over fields of prime characteristic. Clearly,

$$u^{\underline{r}} \cdot u^{\underline{s}} = \binom{\underline{r}+\underline{s}}{\underline{r}} u^{\underline{r}+\underline{s}}, \quad \text{where} \quad \binom{\underline{r}+\underline{s}}{\underline{r}} := \prod_{1\leq i\leq a} \binom{r_i+s_i}{r_i}. \tag{6}$$

For a set of positive integers $\underline{N} = (N_1, ..., N_m)$, denote

$$\mathcal{O}(m; \underline{N}) := \mathbb{K}[u; \underline{N}] := Span_{\mathbb{K}}(u^{\underline{r}} \mid r_i < p^{N_i} \text{ for } i \leq m \text{ and } r_i = 0 \text{ or } 1 \text{ for } i > m). \tag{7}$$

As is clear from (6), $\mathbb{K}[u; \underline{N}]$ is a subalgebra of $\mathbb{K}[u]$. The algebra $\mathbb{K}[u]$ and its subalgebras $\mathbb{K}[u; \underline{N}]$ are called the *algebras of divided powers*; they are analogs of the algebra of functions.

The simple vectorial Lie algebras over $\mathbb{C}$ have only one parameter: the number of indeterminates. If $\mathrm{Char}\mathbb{K} = p > 0$, the vectorial Lie algebras acquire one more parameter: $\underline{N}$. For Lie superalgebras, $\underline{N}$ only concerns the even indeterminates. Let

$$\mathfrak{vect}(m; \underline{N}|n) \text{ a.k.a } W(m; \underline{N}|n) := \mathfrak{der}\mathbb{K}[u; \underline{N}]$$

be the general vectorial Lie algebra.

For Lie superalgebras $\mathfrak{g} = \mathfrak{g}(A)$, if $a_{ii} = 0$, then x_i^+ and x_i^- generate an analog of the Heisenberg Lie algebra: if $x_i^{\pm}$ are even, we denote this Lie algebra $\mathfrak{hei}(2; p; \underline{N})$, where $N \in \mathbb{N}$. Its natural representation is realized in the Fock space of functions $\mathcal{O}(1; \underline{N})$; it is indecomposable for $\underline{N} > 1$ and irreducible for $\underline{N} = 1$.

If $x_i^{\pm}$ are odd, they generate $\mathfrak{sl}(1|1; \mathbb{K})$; all its nontrivial irreducible representations are of dimension $1|1$.

2.3. On modules over vectorial Lie algebras. For simple complex vectorial Lie algebras with polynomial or formal coefficients considered with a natural x-adic topology (as algebras of continuous derivatives of $\mathbb{C}[[x]]$), Rudakov described the irreducible continuous representations. Up to dualization (passage to induced modules), all such representations either depend on k-jets of the vector fields for $k > 1$ and are coinduced, or $k = 1$ and then the irreducible representations are realized in the spaces of tensor fields and also are coinduced, except for the spaces Ω^i of differential i-forms which have submodules $Z^i := \{\omega \in \Omega^i \mid d\omega = 0\}$. The spaces Z^i are irreducible since (Poincaré's lemma) the spaces Ω^i constitute an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m \longrightarrow 0. \tag{8}$$

For details, as well as a review in super setting, see [GLS]. At the moment, there is no complete description of irreducible representations of simple finite dimensional vectorial Lie algebras for $p > 0$, but in what follows we will only need the following version of Rudakov's result easy to obtain by Rudakov's method:

Theorem. *Since Rudakov's description of d in (8) is given in terms of integers, reducing modulo p is possible but this is not all: additional new invariant operators appear over $\mathbb{K}$; it is natural to interpret them as integrals depending on $\underline{N}$:*

$$\begin{gathered} 0 \longrightarrow \mathbb{K} \longrightarrow \Omega^0(\underline{N}) \xrightarrow{d} \Omega^1(\underline{N}) \xrightarrow{d} \Omega^2(\underline{N}) \xrightarrow{d} \dots \\ \xrightarrow{d} \Omega^m(\underline{N}) \xrightarrow{\int_{\underline{N}}} \mathbb{K} \longrightarrow 0 \end{gathered} \tag{9}$$

by setting

$$\int_{\underline{N}} f du_{\underline{N}} = \text{the coefficient of } u^{\tau(\underline{N})} \text{ in the expansion of } f(u), \tag{10}$$

where

$$\tau(\underline{N}) = (p^{N_1} - 1, \dots, p^{N_m} - 1). \tag{11}$$

Remark. This description is not, perhaps, new, but we never saw it formulated, cf. [Ho].

Set $Z^i(\underline{N}) := \{\omega \in \Omega^i(\underline{N}) \mid d\omega = 0\}$ and $B^i(\underline{N}) := \{d\omega \mid \in \Omega^{i-1}(\underline{N})\}$. As is easy to see by induction on m, the sequence (9) is not exact: the space $H^i(\underline{N}) := Z^i(\underline{N})/B^i(\underline{N})$ is spanned by the elements

$$u_{i_1}^{\tau(\underline{N})_{i_1}} \dots u_{i_k}^{\tau(\underline{N})_{i_k}} du_{i_1} \dots du_{i_k}. \tag{12}$$

For any $W(m;\underline{N}|n)$-$\mathcal{O}(m;\underline{N}|n)$-bimodule M with the $W(m;\underline{N})$-action ρ, we denote by $M_{A\,\mathrm{div}}$ a copy of M with the affine $W(m;\underline{N}|n)$-action given by

$$\begin{array}{l} \rho_{A\,\mathrm{div}}(D)(\mu) = \rho(D)(\mu) + A\,\mathrm{div}(D)(\mu) \\ \text{for any } D \in W(m;\underline{N}|n),\ \mu \in M \text{ and } A \in \mathbb{K}. \end{array} \tag{13}$$

After Strade, we denote the space $Vol(m;\underline{N})$ of volume forms by $\mathcal{O}(m;\underline{N})_{\mathrm{div}}$; denote the subspace of forms with integral 0 by

$$\mathcal{O}'(m;\underline{N})_{\mathrm{div}} = Span(x^a vol(u) \mid a_i < \tau(\underline{N})_i),$$

where $vol(u)$ is the volume element in coordinates u. The spaces B^i are irreducible.

2.4. $\mathbb{Z}$-gradings. Recall that every $\mathbb{Z}$-grading of a given vectorial algebra is determined by setting $\deg u_i = r_i \in \mathbb{Z}$; every $\mathbb{Z}$-grading of a given Lie superalgebra $\mathfrak{g}(A)$ is determined by setting $\deg X_i^\pm = \pm r_i \in \mathbb{Z}$. For the Lie algebras of the form $\mathfrak{g}(A)$, we set

$$\deg X_i^\pm = \pm\delta_{i,i_j} \quad \text{for any } i_j \text{ from a selected set } \{i_1, \dots, i_k\} \tag{14}$$

and say that we have "**selected**"certain k Chevalley generators (or respective nodes of the Dynkin graph). Yamaguchi's theorem cited below shows that, in the study of Cartan prolongs defined in sec. 2.5, the first gradings to consider are the ones with $1 \leq k \leq 2$ "**selected**"Chevalley generators. In this paper we consider $k = 1$.

For vectorial algebras, filtrations are more natural than gradings; the very term "vectorial" means, actually, that the algebra is endowed with a particular (*Weisfeiler*) filtration, see [LSh]. Unlike Lie algebras, the vectorial Lie **super**algebras can sometimes be regraded into each other; various realizations as vectorial algebras are described by means of one more parameter — regrading $\underline{r}$ — with a "*standard grading*" as a point of reference:

$$\begin{array}{l} \mathfrak{vect}(m;\underline{N}|n;\underline{r}) \ \text{ a.k.a } W(m;\underline{N}|n;\underline{r}) \ := \mathfrak{der}\mathbb{K}[u;\underline{N}], \text{ where} \\ \deg u_i = r_i \ \text{ is a grading of } \mathcal{O}(m;\underline{N}|n). \end{array} \tag{15}$$

For $W(m;\underline{N}|n)$, the standard grading is $\underline{r} = (1, \dots, 1)$. For the contact algebras $\mathfrak{k}(2n+1,\underline{N})$ that preserve the Pfaff equation $\alpha(X) = 0$ for $X \in \mathfrak{vect}(2n+1)$, where

$$\alpha = dt - \sum_{i \leq n} (p_i dq_i - q_i dp_i), \tag{16}$$

the standard grading is $\deg t = 2$ and $\deg p_i = \deg q_i = 1$ for any i.

2.5. Cartan prolongs. Let $\mathfrak{g}_0$ be a Lie algebra, $\mathfrak{g}_{-1}$ a $\mathfrak{g}_0$-module. Let us define the $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \oplus_{i\geq -1}\mathfrak{g}_i$ called the *complete Cartan prolong* (the result of the Cartan *prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. Geometrically the Cartan prolong is the maximal Lie algebra of symmetries of the G-structure (here: $\mathfrak{g}_0 = \mathrm{Lie}(G)$) on $\mathfrak{g}_{-1}$. The components $\mathfrak{g}_i$ for $i > 0$ are defined recursively.

First, recall that, for any (finite dimensional) vector space V, we have

$$\mathrm{Hom}(V, \mathrm{Hom}(V, \dots, \mathrm{Hom}(V, V)\dots)) \simeq L^i(V, V, \dots, V; V),$$

where L^i is the space of i-linear maps and we have $(i+1)$-many V's on both sides. Now, we recursively define, for any $i > 0$ and $v_1, \dots, v_{i+1} \in \mathfrak{g}_{-1}$:

$$\mathfrak{g}_i = \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) \mid X(v_1)(v_2, v_3, ..., v_{i+1}) = X(v_2)(v_1, v_3, ..., v_{i+1})\}.$$

Let the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ be faithful. Then, clearly,

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* := \oplus\mathfrak{g}_i \subset \mathfrak{vect}(m) = \mathfrak{der}\ \mathbb{C}[[x_1, \dots, x_m]], \quad \text{where } m = \dim\ \mathfrak{g}_{-1}.$$

Moreover, setting $\deg x_i = 1$ for all i, we see that

$$\mathfrak{g}_i = \{X \in \mathfrak{vect}(m) = \mathfrak{der}\ \mathbb{C}[[x_1, \dots, x_m]] \mid \deg X = i,\ [X, \partial] \in \mathfrak{g}_{i-1} \text{ for any } \partial \in \mathfrak{g}_{-1}\}.$$

Now it is subject to an easy verification that the Cartan prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ forms a subalgebra of $\mathfrak{vect}(n)$. (It is also easy to see that $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is a Lie algebra even if $\mathfrak{g}_{-1}$ is not a faithful $\mathfrak{g}_0$-module.)

2.6. Nonholonomic manifolds. Tanaka-Shchepochkina prolongs. Let M^n be an n-dimensional manifold with a nonintegrable distribution $\mathcal{D}$. Let

$$\mathcal{D} = \mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \mathcal{D}_{-3} \dots \subset \mathcal{D}_{-d}$$

be the sequence of strict inclusions, where the fiber of $\mathcal{D}_{-i}$ at a point $x \in M$ is

$$\mathcal{D}_{-i+1}(x) + [\mathcal{D}_{-1}, \mathcal{D}_{-i+1}](x)$$

(here $[\mathcal{D}_{-1}, \mathcal{D}_{-i-1}] = \mathit{Span}\,([X, Y] \mid X \in \Gamma(\mathcal{D}_{-1}), Y \in \Gamma(\mathcal{D}_{-i-1})))$ and d is the least number such that

$$\mathcal{D}_{-d}(x) + [\mathcal{D}_{-1}, \mathcal{D}_{-d}](x) = \mathcal{D}_{-d}(x).$$

In case $\mathcal{D}_{-d} = TM$ the distribution is called *completely nonholonomic*. The number $d = d(M)$ is called the *nonholonomicity degree*. A manifold M with a distribution $\mathcal{D}$ on it will be referred to as *nonholonomic* one if $d(M) \neq 1$. Let

$$n_i(x) = \dim \mathcal{D}_{-i}(x); \qquad n_0(x) = 0; \qquad n_d(x) = n - n_{d-1}. \tag{17}$$

The distribution $\mathcal{D}$ is said to be *regular* if all the dimensions n_i are constants on M. We will only consider regular, completely nonholonomic distributions, and, moreover, satisfying certain transitivity condition (19) introduced below.

To the tangent bundle over a nonholonomic manifold $(M, \mathcal{D})$ we assign a bundle of $\mathbb{Z}$-graded nilpotent Lie algebras as follows. Fix a point $pt \in M$. The usual adic filtration by powers of the maximal ideal $\mathfrak{m} := \mathfrak{m}_{pt}$ consisting of functions that vanish at pt should be modified because distinct coordinates may have distinct "degrees". The distribution $\mathcal{D}$ induces the following filtration in $\mathfrak{m}$:

$$\begin{aligned}\mathfrak{m}_k = \quad & \{f \in \mathfrak{m} \mid X_1^{a_1} \dots X_n^{a_n}(f) = 0 \text{ for any } X_1, \dots, X_{n_1} \in \Gamma(\mathcal{D}_{-1}),\\ & X_{n_1+1}, \dots, X_{n_2} \in \Gamma(\mathcal{D}_{-2}), \dots, X_{n_{d-1}+1}, \dots, X_n \in \Gamma(\mathcal{D}_{-d}) \text{ such that}\\ & \sum_{1\leq i\leq d} i \sum_{n_{i-1}<j\leq n_i} a_j \leq k\},\end{aligned} \tag{18}$$

where $\Gamma(\mathcal{D}_{-j})$ is the space of germs at pt of sections of the bundle $\mathcal{D}_{-j}$. Now, to a filtration

$$\mathcal{D} = \mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \mathcal{D}_{-3} \dots \subset \mathcal{D}_{-d} = TM,$$

we assign the associated graded bundle

$$\mathrm{gr}(TM) = \oplus \mathrm{gr}\mathcal{D}_{-i}, \quad \text{where } \mathrm{gr}\mathcal{D}_{-i} = \mathcal{D}_{-i}/\mathcal{D}_{-i+1}$$

and the bracket of sections of $\mathrm{gr}(TM)$ is, by definition, the one induced by bracketing vector fields, the sections of TM. We assume a "transitivity condition": The Lie algebras

$$\mathrm{gr}(TM)|_{pt} \tag{19}$$

induced at each point $pt \in M$ are isomorphic.

The grading of the coordinates (18) determines a nonstandard grading of $\mathfrak{vect}(n)$ (recall (17)):

$$\begin{aligned} &\deg x_1 = \ldots = \deg x_{n_1} = 1,\\ &\deg x_{n_1+1} = \ldots = \deg x_{n_2} = 2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\deg x_{n-n_{d-1}+1} = \ldots = \deg x_n = d. \end{aligned} \tag{20}$$

Denote by $\mathfrak{v} = \underset{i\geq -d}{\oplus} \mathfrak{v}_i$ the algebra $\mathfrak{vect}(n)$ with the grading (20). One can show that the "complete prolong" of $\mathfrak{g}_-$ to be defined shortly, i.e., $(\mathfrak{g}_-)_* := (\mathfrak{g}_-, \tilde{\mathfrak{g}}_0)_* \subset \mathfrak{v}$, where $\tilde{\mathfrak{g}}_0 := \mathfrak{der}_0\mathfrak{g}_-$, preserves $\mathcal{D}$.

For nonholonomic manifolds, an analog of the group G from the term "G-structure", or rather of its Lie algebra, $\mathfrak{g} = \mathrm{Lie}(G)$, is the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a subalgebra of the $\mathbb{Z}$-grading preserving Lie algebra of derivations of $\mathfrak{g}_-$, i.e., $\mathfrak{g}_0 \subset \mathfrak{der}_0\,\mathfrak{g}_-$. If $\mathfrak{g}_0$ is not explicitly indicated, we assume that $\mathfrak{g}_0 = \mathfrak{der}_0\,\mathfrak{g}_-$, i.e., is the largest possible.

Given a pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ as above, define its *Tanaka-Shchepochkina prolong* to be the maximal subalgebra $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \underset{k\geq -d}{\oplus} \mathfrak{g}_k$ of $\mathfrak{v}$ with given non-positive part $(\mathfrak{g}_-, \mathfrak{g}_0)$. For an explicit construction of the components, see [Sh14], [Y], [Shch].

2.7. Partial prolongs and projective structures. Let $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ be a depth d Lie algebra; $\mathfrak{h}_1 \subset \mathfrak{g}_1$ be a $\mathfrak{g}_0$-submodule such that $[\mathfrak{g}_{-1}, \mathfrak{h}_1] = \mathfrak{g}_0$. If such $\mathfrak{h}_1$ exists (usually, $[\mathfrak{g}_{-1}, \mathfrak{h}_1] \subset \mathfrak{g}_0$), define the ith partial prolong of $(\underset{i\leq 0}{\oplus} \mathfrak{g}_i, \mathfrak{h}_1)$ for $i \geq 2$ to be

$$\mathfrak{h}_i = \{D \in \mathfrak{g}_i \mid [D, \mathfrak{g}_{-1}] \in \mathfrak{h}_{i-1}\}. \tag{21}$$

Set $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$ and call $\mathfrak{h}_* = \underset{i\geq -d}{\oplus} \mathfrak{h}_i$ the Shchepochkina *partial prolong* of $(\underset{i\leq 0}{\oplus} \mathfrak{g}_i, \mathfrak{h}_1)$, see [Sh14]. (Of course, the partial prolong can also be defined if $\mathfrak{h}_0$ is contained in $\mathfrak{g}_0$.)

Example. The $SL(n+1)$-action on the projective space P^n gives the embedding $\mathfrak{sl}(n+1) \subset \mathfrak{vect}(n)$; here $\mathfrak{sl}(n+1)$ is a partial prolong of $\mathfrak{vect}(n)_{i\leq 0} \oplus \mathfrak{h}_1$ for some $\mathfrak{h}_1$.

Yamaguchi's theorem. Let $\mathfrak{s} = \underset{i\geq -d}{\oplus} \mathfrak{s}_i$ be a simple finite dimensional Lie algebra. Let $(\mathfrak{s}_-)_* = (\mathfrak{s}_-, \mathfrak{g}_0)_*$ be the Cartan prolong with the maximal possible $\mathfrak{g}_0 = \mathfrak{der}_0(\mathfrak{s}_-)$.

Theorem ([Y]). *Over $\mathbb{C}$, equality $(\mathfrak{s}_-)_* = \mathfrak{s}$ holds almost always. The exceptions (cases where $\mathfrak{s} = \underset{i\geq -d}{\oplus} \mathfrak{s}_i$ is a partial prolong in $(\mathfrak{s}_-)_* = (\mathfrak{s}_-, \mathfrak{g}_0)_*$) are*

1) *$\mathfrak{s}$ with the grading of depth $d = 1$ (in which case $(\mathfrak{s}_-)_* = \mathfrak{vect}(\mathfrak{s}_-^*)$);*

2) *$\mathfrak{s}$ with the grading of depth $d = 2$ and $\dim \mathfrak{s}_{-2} = 1$, i.e., with the "contact" grading, in which case $(\mathfrak{s}_-)_* = \mathfrak{k}(\mathfrak{s}_-^*)$ (these cases correspond to "selection" of the nodes on the Dynkin graph connected with the node for the maximal root on the extended graph);*

3) *$\mathfrak{s}$ is either $\mathfrak{sl}(n+1)$ or $\mathfrak{sp}(2n)$ with the grading determined by "selecting" the first and the ith of simple coroots, where $1 < i < n$ for $\mathfrak{sl}(n+1)$ and $i = n$ for $\mathfrak{sp}(2n)$. (Observe that $d = 2$ with $\dim \mathfrak{s}_{-2} > 1$ for $\mathfrak{sl}(n+1)$ and $d = 3$ for $\mathfrak{sp}(2n)$.)*

Moreover,

$$(\mathfrak{s}_-, \mathfrak{s}_0)_* = \mathfrak{s} \tag{22}$$

holds almost always, even in these exceptional cases $1) - 3)$. The cases where (22) fails (the ones where a projective action is possible) are $\mathfrak{sl}(n+1)$ or $\mathfrak{sp}(2n)$ with the grading determined by "selecting" only one (the first) simple coroot.

§3. Examples from the literature

In this section, we list several Lie algebras more or less as described in [S]; in the next section we give their interpretations in terms of (partial) prolongs: no version of Yamaguchi's theorem is yet available for $p > 0$. For a general algorithm that describes the nonholonomic distributions these Lie algebras preserve, see [Shch]. We mainly consider the simplest $\underline{N} = (1 \dots 1)$ and eventually skip it whenever possible.

Melikyan algebras for $p = 5$. For any prime p, on the space $\mathfrak{g}_{-1} := \mathcal{O}(1; \underline{N})/\text{const}$ of "functions (in one indeterminate u) modulo constants", the skew-symmetric bilinear form (see (10))

$$(f, g) = \int_{\underline{N}} f g' du_{\underline{N}},$$

is nondegenerate. Hence $W(1, \underline{N})$ is embedded into $\mathfrak{sp}(p^{\underline{N}} - 1)$. So we can consider the prolong

$$\mathfrak{g}_* = \mathop{\oplus}\limits_{i \geq -2} \mathfrak{g}_i := (K(p^{\underline{N}})_-, \mathfrak{c}W(1, \underline{N}))_* \subset K(p^{\underline{N}}),$$

where $\mathfrak{cg} = \mathfrak{g} \oplus \mathbb{K}\, z$ is the trivial central extension of $\mathfrak{g}$. This construction resembles Shchepochkina's construction of some of exceptional simple vectorial Lie superalgebras [Sh14]. Whatever this prolong $\mathfrak{g}_*$ is for $\underline{N} > 1$ or $p > 5$, either $\mathfrak{g}_1$ is zero or the complete prolong is a known simple Lie algebra.

Melikyan observed that, for $p = 5$ and $\underline{N} = 1$, the prolong $(K(5)_-, \mathfrak{c}W(1, \underline{1}))_* \subset K(5)$ is a new simple Lie algebra, Me. Melikyan's only available publication lacked details: he did not write for which 5-tuples $\underline{N}$ it is possible to generalize the construction to $K(5, \underline{N})$ and the ground for Melikyan's claim that $\underline{N}$ can only have 2 parameters was unclear. The following are the vital for constructing the complete prolong terms of Me, as elements of $K(5)$, given both in terms of the indeterminates $t; p_1, p_2, q_1, q_2$ (see 16) and in terms of the u^r corresponding to $W(1, \underline{1})$; let z be the grading operator in $K(5)$ corresponding to the generating function t:

(23)

	$\mathfrak{g}_0 \simeq W(1) \oplus \mathbb{K}\, z$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$
spanned by	$-q_2^2 + p_1p_2 \leftrightarrow u^4 \frac{d}{du}$, $-q_1p_1 + 2q_2p_2 \leftrightarrow u^3 \frac{d}{du}$, $-q_1q_2 - 2p_2^2 \leftrightarrow u^2 \frac{d}{du}$, $-q_1p_2 \leftrightarrow u \frac{d}{du}$, $-q_1^2 \leftrightarrow \frac{d}{du}$; t	$p_1 \leftrightarrow u^4$, $p_2 \leftrightarrow u^3$, $q_1 \leftrightarrow u^2$, $q_2 \leftrightarrow u$	1

Kuznetsov [Ku1] found another description of Me($\underline{N}$). From Yamaguchi's theorem cited above we know that, over $\mathbb{C}$, the Tanaka-Shchepochkina prolong of $(\mathfrak{g}(2)_-, \mathfrak{g}(2)_0)$ (in any $\mathbb{Z}$-grading of $\mathfrak{g}(2)$) is isomorphic to $\mathfrak{g}(2)$. There are two $\mathbb{Z}$-gradings of $\mathfrak{g}(2)$ with **one** "selected"generator: one of depth 2 and one of depth 3. Kuznetsov observed that, for $p = 5$, the non-positive parts of $\mathfrak{g}(2)$ in the grading of depth 3 are isomorphic to the respective non-positive parts of the Melikyan algebras in one of their $\mathbb{Z}$-gradings. Let $U[k]$ be the $\mathfrak{gl}(V)$-module which is U as $\mathfrak{sl}(V)$-module, and let a fixed central element $z \in \mathfrak{gl}(V)$ act on $U[k]$ as k id. Then

(24)

$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-3}$
$\mathfrak{gl}(2) \simeq \mathfrak{gl}(V)$	$V = V[-1]$	$E^2(V)$	$V[-3]$

So it is natural to conjecture that, for $p = 5$, Melikyan algebras Me($\underline{N}$) are *complete*[2] Tanaka-Shchepochkina prolongs $(\mathfrak{g}(2)_-, \mathfrak{g}(2)_0)_*$ of total symmetries preserving a nonholonomic structure whereas $\mathfrak{g}(2)$ is a *projective* type subalgebra in Me($\underline{N}$). In this realization, it remains unclear what are the admissible values of $\underline{N}$.

Kuznetsov [Ku1] gives yet another realization. As spaces, and $\mathbb{Z}/3$-graded Lie algebras, we have:

$$\text{Me}(\underline{N}) := \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}} \simeq W(2; \underline{N}) \oplus \tilde{W}(2; \underline{N})_{2\text{div}} \oplus \mathcal{O}(2; \underline{N})_{-2\text{div}}\ , \tag{25}$$

where $\tilde{W}(2; \underline{N})$ is a copy of $W(2; \underline{N})$ endowed, together with each element, with a tilde to distinguish from (the elements of) $W(2; \underline{N})$. Let v be a short for $vol(u)$; observe that we have the following

[2]Me($\underline{N}$) could be smaller than what it actually is, the complete prolong.

identifications for $m = 2$, where $du_1du_2 = v$:

$$\begin{array}{ll} \text{for any } p, & du_iv^{-1} = \mathrm{sign}(ij)\partial_j \ \text{ for any permutation } (ij) \text{ of } (12) \\ \text{for } p = 5, & v^{-4} = v, \quad du_1du_2v^2 = v^3 = v^{-2}. \end{array} \tag{26}$$

The $\mathfrak{g}_{\bar 0}$-action on the $\mathfrak{g}_{\bar i}$ is natural; the multiplication in $\mathrm{Me}(\underline{N})$ is given by the following formulas (in line 2 we use $du_iv = \mathrm{sign}(ij)\partial_jv^2$, see (26)):

$$\begin{array}{lcl} [f_1v^{-2}, f_2v^{-2}] & = & 2\,(f_1d(f_2) - f_2d(f_1))\,v^4 = \\ 2\,(f_1\partial_1(f_2) - f_2\partial_1(f_1))\,du_1v + cycle & = & 2\,(f_2\partial_2(f_1) - f_1\partial_2(f_2))\,\tilde\partial_1v^2 + cycle(12); \\ [fv^{-2}, \tilde Dv^2] := -[\tilde Dv^2, fv^{-2}] & = & fD; \\ [\sum f_i\tilde\partial_iv^2, \sum g_j\tilde\partial_jv^2] & = & [\sum \mathrm{sign}(ik)f_idu_kv, \sum \mathrm{sign}(jl)g_jdu_lv] = \\ (f_1g_2 - f_2g_1)\,du_1du_2v^2 & = & (f_1g_2 - f_2g_1)\,v^{-2}. \end{array} \tag{27}$$

The standard $\mathbb{Z}$-grading is given by setting ([S]):

$$\deg u^{\underline{r}}\partial_i = 3|\underline{r}| - 3, \quad \deg u^{\underline{r}}v^{-2} = 3|\underline{r}| - 2, \quad \deg u^{\underline{r}}\tilde\partial_iv^2 = 3|\underline{r}| - 1. \tag{28}$$

This realization allows one to easily compute the dimensions of $\mathrm{Me}(\underline{N})$ and its homogeneous components, shows that $\underline{N}$ depends on at least 2 parameters but does not preclude more. The upper bound on the number of independent parameters of $\underline{N}$ comes from the classification.

Melikyan algebras for $p = 3$. Shchepochkina's realization [Shch] of the non-positive part of $\mathrm{Me}(\underline{N})$, identical to that of $\mathfrak{g}(2)$ in a $\mathbb{Z}$-grading (58), only involves ±1 as coefficients in $\mathfrak{g}_-$ and $\pm1, \pm2$ in $\mathfrak{g}_0$ and so invites to study the prolongs $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ for $p = 3$, and $(\mathfrak{g}_-)_*$ for $p = 2$; this is being done.

Another approach is to interpret decomposition (25): In $W(3; \underline{N, 1})$, consider a nonstandard $\mathbb{Z}$-grading:

$$\deg u_1 = \deg u_2 = 0; \quad \deg u_3 = 1\ . \tag{29}$$

Let $u = (u_1, u_2)$, $v = u_3$; $\partial_i = \partial_{u_i}$; $\partial = \partial_v$. Then $W(3; \underline{N, 1})$ can be represented as a direct sum of the following spaces and (non-canonically) $W(2; \underline{N})$-modules (here $\langle T\rangle$ denotes the space spanned by the elements of a set T):

$$\begin{array}{lll} W(2; \underline{N}) \simeq \langle f_i(u)\partial_i\rangle; & W(2; \underline{N}) \simeq \langle vf_i(u)\partial_i\rangle; & W(2; \underline{N}) \simeq \langle v^2f_i(u)\partial_i\rangle; \\ \mathcal{O}(2; \underline{N}) \simeq \langle f(u)\partial\rangle; & \mathcal{O}(2; \underline{N}) \simeq \langle f(u)v\partial\rangle; & \mathcal{O}(2; \underline{N}) \simeq \langle f(u)v^2\partial\rangle\ . \end{array} \tag{30}$$

If we recall that $2 \equiv -1 \mod 3$, we see that the corresponding decomposition of $S(3; \underline{N, 1})$ is of the form (25):

$$\begin{array}{l} W(2; \underline{N}) \simeq \langle \sum f_i(u)\partial_i - (\sum \partial_i(f_i(u)))\,v\partial\rangle; \\ W(2; \underline{N})_{\mathrm{div}} \simeq \langle vf_i(u)\partial_i - (\sum \partial_i(f_i(u)))\,v^2\partial\rangle; \\ \mathcal{O}(2; \underline{N})_{-\mathrm{div}} \simeq \langle f(u)\partial\rangle\ . \end{array} \tag{31}$$

Thus, the Melikyan algebras for $p = 3$ are $S(3; \underline{N, 1})$. Having observed this we recalled that Shen [Sh] had noticed that, for $p = 2$, $\mathfrak{g}(2) \simeq S(3; \underline{1, 1, 1})$. It is natural then, for $p = 2$ and 3, to consider $S(3; \underline{N, 1})$ as complete prolongs of $\mathfrak{g}(2)$-type.

Melikyan algebras for $p = 2$. It is also natural to consider the prolongs of non-positive parts of $\mathfrak{g}(2)$ in its various $\mathbb{Z}$-gradings for $p = 2$ ($p = 3$ does not fit for obvious reasons) and Brown [Br] did just it: As spaces, and $\mathbb{Z}/3$-graded Lie algebras, let

$$L(\underline{N}) := \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1} \oplus \mathfrak{g}_{\bar 2} \simeq W(2; \underline{N}) \oplus \mathcal{O}(2; \underline{N})_{\mathrm{div}} \oplus \mathcal{O}(2; \underline{N})\ . \tag{32}$$

The $\mathfrak{g}_{\bar 0}$-action on the $\mathfrak{g}_{\bar i}$ is natural; the multiplication in $L(\underline{N})$ is given by the following formulas:

$$
\begin{array}{rcl}
[fv, g] & = & fH_g;\\
[f, g]: & = & H_f(g)v,
\end{array}
\tag{33}
$$

where $f \mapsto H_f$ is the assignment of the Hamiltonian field to the Hamiltonian function $f \in \mathcal{O}(2; \underline{N})$:

$$H_f = \frac{\partial f}{\partial u_1}\partial_2 + \frac{\partial f}{\partial u_2}\partial_1.$$

Define a $\mathbb{Z}$-grading of $L(\underline{N})$ by setting

$$\deg u^{\underline{r}}\partial_i = 3|\underline{r}| - 3, \quad \deg u^{\underline{r}}v = 3|\underline{r}| - 2, \quad \deg u^{\underline{r}} = 3|\underline{r}| - 4. \tag{34}$$

Now, set $\mathrm{Me}(\underline{N}) = L(\underline{N})/L(\underline{N})_{-4}$. This algebra is not simple, because $\mathcal{O}(2; \underline{N})_{\mathrm{div}}$ has a submodule of codimension 1; but $\mathrm{Me}(\underline{N})^{(1)}$ is simple.

As is easy to see, the non-positive parts of $\mathfrak{g}(2)$ and $\mathrm{Me}(\underline{N})$ are isomorphic; it remains to find out if the complete Tanaka-Schepochkina prolong of this part is $\mathrm{Me}(\underline{N})$.

$\underline{p = 3}$**: Brown algebras** Consider the $r \times r$ Cartan matrices for $r = 3, 2$ (of course, -1 is the same as 2 modulo 3, but -1 is more conventional):

$$\begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1\\ -1 & 0 \end{pmatrix}. \tag{35}$$

Although relations (3) are implicit, it is known ([S]) that the dimensions of algebras $Br(r)$ given by the matrices (35) are equal to 29 and 10, respectively (assuming the usual rules (2), (3) of constructing $\mathfrak{g}(A)$ from A).

Kostrikin [Ko] described a 3-parameter family containing $Br(2)$ and acknowledged that Rudakov was the first to observe that, if $p = 3$, then for **any** irreducible $\mathfrak{sl}(2)$-module V, the Cartan prolong $\mathfrak{g} := \oplus\mathfrak{g}_i$, where $\mathfrak{g}_{-1} = V$ and $\mathfrak{g}_0 = \mathfrak{sl}(2)$ or $\mathfrak{gl}(2)$ whose center acts on $\mathfrak{g}$ as the grading operator, is a simple Lie algebra. Nobody, it seems, published so far exact descriptions of this Cartan prolong (36), (37) nor were particular cases studied (see (62)).

There are not that many irreducible $\mathfrak{sl}(2)$-modules; all such modules are listed in [RS]: for $p = 3$, there is just one module of dimension 2 (the identity one; it yields $\mathfrak{h}(2; \underline{N})$ and $\mathfrak{k}(3; \underline{N})$) and a 3-parameter family $\mathbb{T}(a, b, c)$ of 3-dimensional modules given by the following matrices, where $a \neq bc$:

$$\tilde X^- = \begin{pmatrix} 0 & 0 & c\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix} \quad \tilde H = \begin{pmatrix} a - bc & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -a + bc \end{pmatrix} \quad \tilde X^+ = \begin{pmatrix} 0 & a & 0\\ 0 & 0 & a\\ b & 0 & 0 \end{pmatrix}$$

$$\text{normalized as} \quad \tilde X^+ = \begin{cases} \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ \frac{b}{a} & 0 & 0 \end{pmatrix} & \text{if } a \neq 0 \text{ and then } H = \begin{pmatrix} 1 - bc & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1 + bc \end{pmatrix}\\ \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix} & \text{if } b \neq 0 \text{ and then } H = \begin{pmatrix} -c & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & c \end{pmatrix} \end{cases}$$

So, $Br(2; a, b, c)$ depends, actually, on two parameters. In realization by vector fields we have

$$
\begin{array}{ll}
\tilde X^- = cu_1\partial_3 + u_2\partial_1 + u_3\partial_2, & \tilde H = (a - bc)\,(u_1\partial_1 - u_3\partial_3)\,,\\
\tilde X^+ = au_1\partial_2 + au_2\partial_3 + bu_3\partial_1, & \tilde E = u_1\partial_1 + u_2\partial_2 + u_3\partial_3.
\end{array}
\tag{36}
$$

Indeed,

$$[\tilde X^+, \tilde X^-] = \tilde H, \qquad [\tilde H, \tilde X^\pm] = \pm(a - bc)\tilde X^\pm.$$

If $a\neq bc$, the change

$$\tilde{X}^{\pm}\mapsto X^{\pm}:=\sqrt{a-bc}\tilde{X}^{\pm};\qquad \tilde{H}\mapsto H:=(a-bc)\tilde{H}$$

leads to the standard commutation relations, so we drop the tilde. Set also $E=\sum u_i\partial_i$.

Set $\mathfrak{g}_{-1}=\mathbb{T}(a,b,c)=Span(\partial_1,\partial_2,\partial_3)$, where weight$(u_1)=-$weight$(u_3)=w:=a-bc$, weight$(u_2)=0$. Let us compute the Cartan prolong assuming that $\mathfrak{g}_0$ contains $\mathfrak{sl}(2)$.

For $\underline{N}=(111)$, $\mathfrak{g}_2=0$; the space $\mathfrak{g}_1$ is easy to get by hands; it is spanned by (for $a\neq 0$):

$$\begin{array}{|lc|c|}\hline \partial_3^*:= & (au_2^2+bcu_1u_3)\partial_1+a(cu_1^2+u_2u_3)\partial_2+(acu_1u_2-(a+bc)u_3^2)\partial_3 & -w\\ \partial_2^*:= & (u_1u_2+bu_3^2)\partial_1+(u_2^2+u_1u_3)\partial_2+(u_2u_3+cu_1^2)\partial_3 & 0\\ \partial_1^*:= & -((a+bc)u_1^2-bu_2u_3)\partial_1+(au_1u_2+bu_3^2)\partial_2+(bcu_1u_3+au_2^2)\partial_3 & w\\ \hline\end{array}\tag{37}$$

The commutators are

(38)

	∂_1^*	∂_2^*	∂_3^*
∂_1	$H+aE$	X^-	cX^+
∂_2	$-X^+$	E	aX^-
∂_3	bX^-	X^+	$aE-H$

Since $[\partial_2^*,\partial_2]=E$, it follows that $\mathfrak{g}_0$ must be equal to $\mathfrak{gl}(2)$ and can not equal to $\mathfrak{sl}(2)$. Now, set $Br(2;a,b,c):=\mathfrak{g}$.

Occasional isomorphisms If $a\neq 0$, then

$$Br(2;1,b,c)\simeq Br(2;1,c,b),\quad Br(2;1,0,0)\simeq Br(2).\tag{39}$$

Particular cases. Since $Br(2;1,0,0)=Br(2)$ has the same non-positive part as $\mathfrak{k}(3)$, it follows that $Br(2)$ is a partial Cartan prolongation. All Kostrikin's examples $L(\varepsilon)$ also have the same non-positive parts as $\mathfrak{k}(3)$, so $L(\varepsilon)$ is a deformation $Br(2,a)$, where $\varepsilon=\frac{a}{2-a}$, of $Br(2)$ and only one member of the parametric family is Cartan prolongation. Each $L(\varepsilon)$ can be embedded into $K(3;\underline{1})$: Set ([S]):

$$\begin{aligned}&X_1^-=q^2,\qquad X_2^-=p,\\ &X_1^+=-p^2,\quad X_2^+=\delta(apq^2-qt),\ \text{where } \delta=\begin{cases}1 & \text{if } a=2,\\ \dfrac{1}{a+1} & \text{if } a\neq 2.\end{cases}\end{aligned}\tag{40}$$

Then

$$h_1=pq,\quad h_2=\begin{cases}pq-t & \text{if } a=2,\\ \dfrac{a-1}{a+1}pq-\dfrac{1}{a+1}t & \text{if } a\neq 2.\end{cases}$$

The Cartan matrix is

$$\begin{pmatrix}2&-1\\ \alpha&0\end{pmatrix},\ \text{where } \alpha=\begin{cases}-1 & \text{if } a=2,\\ \dfrac{a-1}{a+1} & \text{if } a\in\mathbb{K}\setminus\{1,2\}.\end{cases}$$

Kostrikin observed [Ko] that $Br(2,a)$ can be deformed into an algebra $L(\varepsilon,\alpha,\beta)$ which may be identified with $Br(2;a,b,c)$ for some a,b,c.

Remark. There is no "contact" analog of $Br(2;a,b,c)$ because there is no nondegenerate skew-symmetric bilinear form on $\mathfrak{g}_{-1}=\mathbb{T}(a,b,c)$.

$\underline{p=3}$**: Skryabin algebras.** To describe them, observe that, as far as $W(3,\underline{N})$-action is concerned, we have the following identifications: $v^2 = v^{-1}$ and (cf. (26))

$$du_i du_j v^{-1} = \mathrm{sign}(ijk)\partial_k \text{ for any permutation } (ijk) \text{ of } (123). \tag{41}$$

Observe that, as $W(3,\underline{N})$-modules, $S(3,\underline{N}) \not\simeq Z^2(3,\underline{N})$, more precisely, $S(3,\underline{N})$ is not a $W(3,\underline{N})$-module: As a simple Lie algebra, $W(3,\underline{N})$ has no submodules in the adjoint representation. And, contrary to what is stated in [Sk, S], neither $S(3,\underline{N})$ nor $Z^2(3,\underline{N})$ are $\mathcal{O}(3;\underline{N})$-modules.

The deep Skryabin algebra. As spaces, and $\mathbb{Z}/4$-graded Lie algebras, we have, see [Sk]:

$$\begin{array}{l}\mathrm{DY}(\underline{N}) := \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1} \oplus \mathfrak{g}_{\bar 2} \oplus \mathfrak{g}_{\bar 3} \simeq \\ W(3;\underline{N}) \oplus \mathcal{O}(3;\underline{N})_{-\mathrm{div}} \oplus \Omega^1(3;\underline{N})_{\mathrm{div}} \oplus Z^2(3;\underline{N}) \ , \\ \mathrm{DY}(\underline{N})^{(1)} \simeq W(3;\underline{N}) \oplus \mathcal{O}(3;\underline{N})_{-\mathrm{div}} \oplus \Omega^1(3;\underline{N})_{\mathrm{div}} \oplus B^2(3;\underline{N}) \ .\end{array} \tag{42}$$

In particular (hereafter $|\underline{N}| = \sum N_i$),

$$\dim \mathrm{DY}(\underline{N}) = 3^{|\underline{N}|+2} + 1, \quad \dim \mathrm{DY}(\underline{N})^{(1)} = 3^{|\underline{N}|+2} - 2. \tag{43}$$

The multiplication in $\mathrm{DY}(\underline{N})$ is given by the $W(3,\underline{N})$-invariant bilinear differential operators acting in the spaces of tensor fields entering (42). Over $\mathbb{C}$, all such operators are described [Gr0]; to describe even unary operators is an open problem for $p > 0$. For $p = 3$, the following formulas for multiplication reveal presence of new invariant operators. Here, brackets in lines 3 and 5 are anti-symmetrized by definition, cf. (27), $f, g \in \mathcal{O}$, $\omega^i \in \Omega^i$:

$$\begin{array}{lcl} [fv^{-1}, gv^{-1}] & = & (gdf - fdg)v; \\ [fv^{-1}, \omega^1 v] & = & -d(f\omega^1); \\ [fv^{-1}, \omega^2] & = & f\omega^2 v^{-1} \in \mathfrak{g}_{\bar 0}; \\ [\omega^1_1 v, \omega^1_2 v] & = & \omega^1_1\omega^1_2 v^{-1} \in \mathfrak{g}_{\bar 0}; \\ [\omega^1 v, \omega^2] & = & \omega^1\omega^2 v \in \mathcal{O}_{2\mathrm{div}} \simeq \mathcal{O}_{-\mathrm{div}}; \\ [\sum f_i(u) du_j du_k, \sum g_i(u) du_j du_k] & = & (f_3 g_2 - f_2 g_3) du_1 v + \mathit{cycle}(123). \end{array} \tag{44}$$

Set ([Sk]):

$$\begin{array}{ll} \deg u^{\underline{r}}\partial_i = 4|\underline{r}| - 4, & \deg u^{\underline{r}} du_i v = 4|\underline{r}| - 2, \\ \deg u^{\underline{r}} v^{-1} = 4|\underline{r}| - 3, & \deg u^{\underline{r}} du_i du_j = 4|\underline{r}| - 1. \end{array} \tag{45}$$

For $\mathfrak{gl}(3) = \mathfrak{gl}(V)$, where $V := V[-1]$, we have $E^2(V) \simeq (V[-1])^* \simeq V^*[-2]$, and $E^3(V) \simeq \mathbb{1}[-3]$, where $\mathbb{1}$ is the trivial $\mathfrak{sl}(3)$-module. Then $\mathrm{DY}(\underline{N})$ can be defined as the Tanaka-Shchepochkina prolong with the following non-positive part (here $i, j = 1, 2, 3$):

(46)

	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-3}$	$\mathfrak{g}_{-4}$
$Span_{\mathbb{K}}$	$x_j\partial_i$	$du_i du_j$	$du_i v$	v^{-1}	∂_i
	$\mathfrak{gl}(3) \simeq \mathfrak{gl}(V)$	$V = V[-1]$	$E^2(V)$	$E^3(V)$	$V[-4]$

The big Skryabin algebra. $\mathrm{BY}(\underline{N})$ was only described so far ([Sk, S]) as having the following non-positive part:

(47)

$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-3}$
$\mathfrak{gl}(3) := \mathfrak{gl}(V)$	V	$E^2(V)$	$E^3(V)$

The middle Skryabin algebra. As spaces, and $\mathbb{Z}/2$-graded Lie algebras, we have

$$\mathrm{MY}(\underline{N}) \simeq W(3;\underline{N}) \oplus \Omega^1(3;\underline{N})_{\mathrm{div}}. \tag{48}$$

with multiplication given by (44). Set ([S]):

$$\begin{aligned} \deg u^{\underline{r}}\partial_i &= 2|\underline{r}| - 2,\\ \deg u^{\underline{r}} du_i v &= 2|\underline{r}| - 1. \end{aligned} \tag{49}$$

Then $\mathrm{MY}(\underline{N})$ can be defined as the generalized prolong with the following non-positive part (here $i, j = 1, 2, 3$):

(50)

	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$
$Span_{\mathbb{K}}$	$u_j\partial_i$	$du_i v$	∂_i
	$\mathfrak{gl}(3) \simeq \mathfrak{gl}(V)$	V	$E^2(V)$

and therefore the non-positive part $\mathrm{MY}(\underline{N})_{\leq 0}$ coincides with $\mathfrak{o}(7)_{\leq 0}$ in the grading with the last "selected" root.

The five small Skryabin algebras. The dimensions of these algebras $\mathfrak{g}$ are distinct, as well as the structures they preserve, so they should be considered as separate entities. Here we only consider one of these algebras — $\mathrm{Y}_{(1)}(\underline{N})$ — defined as the generalized prolong with the following non-positive part:

(51)

	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$
$Span_{\mathbb{K}}$	$\mathfrak{sl}(3) \simeq \mathfrak{sl}(V)$	$V = Span(du_i)_{i\leq 3}$	$E^2(V) = Span(\partial_i)_{i\leq 3}$

Clearly, $Y_{(1)}(\underline{N}) \simeq \mathrm{SMY}(\underline{N})$ and, as spaces,

$$\begin{aligned} &\mathrm{SMY}(\underline{N}) \simeq S(3;\underline{N}) \oplus Z^1(3;\underline{N}), \qquad \text{and } \mathrm{SMY}^{(1)}(\underline{N}) \simeq S(3;\underline{N})^{(1)} \oplus B^1(3;\underline{N}),\\ &\dim \mathrm{SMY}^{(1)}(\underline{N}) = 3^{|\underline{N}|+1} - 3. \end{aligned} \tag{52}$$

The other small Skryabin algebras are filtered deforms to be considered elsewhere.

$\underline{p = 3}$**: Frank algebras.** $\mathrm{Fr}(n)$ has the same non-positive part as $\mathfrak{g} := K(3; (1, 1, n))$, where $\underline{N} = (1, 1, n)$ corresponds to the ordered set (q, p, t); and hence same non-positive part as $\mathfrak{sp}(4)_{\leq 0}$ in the grading with the first "selected" root. As $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ has two lowest weight vectors: qt, and pq^2. Strade gives all homogeneous components of $\mathrm{Fr}(n)$, in particular,

$$\mathrm{Fr}(n)_1 = Span(p^2q - pt,\ pq^2 + qt) \subset \mathfrak{g}_1.$$

$\underline{p = 3}$**: Ermolaev algebras.** As spaces, and $\mathbb{Z}/2$-graded Lie algebras, we have:

$$\begin{aligned} &\mathrm{Er}(\underline{N}) := W(2;\underline{N}) \oplus \mathcal{O}(2;\underline{N})_{\mathrm{div}}, \qquad \text{and } \mathrm{Er}(\underline{N})^{(1)} = W(2;\underline{N}) \oplus \mathcal{O}'(2;\underline{N})_{\mathrm{div}}\\ &\dim \mathrm{Er}(\underline{N})^{(1)} = 3^{|\underline{N}|+1} - 1. \end{aligned} \tag{53}$$

Recall that $v^2 = v^{-1}$ and observe that (cf. (41))

$$du_i v^{-1} = \mathrm{sign}(ij)\partial_j \ \text{ for any permutation } (ij) \text{ of } (12). \tag{54}$$

For any $f\ v, g\ v \in \mathcal{O}(2;\underline{N})_{\mathrm{div}}$, set

$$[fv, gv] = (fdg - gdf)v^{-1} = (f\partial_2(g) - g\partial_2(f))\partial_1 + (g\partial_1(f) - f\partial_1(g))\partial_2; \tag{55}$$

define the other products canonically. Define the $\mathbb{Z}$-grading of $\mathrm{Er}(\underline{N})$ by setting in the standard $\mathbb{Z}$-grading of $\mathcal{O}(2;\underline{N})$:

$$\mathrm{Er}(\underline{N})_i := W(2;\underline{N})_i \oplus (\mathcal{O}(2;\underline{N})_{\mathrm{div}})_{i+1} \quad \text{for any } i \geq -1. \tag{56}$$

$\mathrm{Er}(\underline{N})$ is defined as the Cartan prolong of the following non-positive part:

$$\begin{array}{lcl}\mathrm{Er}_{-1} & = & Span_{\mathbb{K}}(\partial_1, \partial_2;\ vol);\\ \mathrm{Er}_0 & = & Span_{\mathbb{K}}(u_i\partial_j;\ u_k vol \mid i,j,k = 1,2).\end{array} \tag{57}$$

Questions we address for all the above algebras. 1) What are the structures the algebras preserve?

2) What are the complete and partial Tanaka-Shchepochkina prolongs of the non-positive and negative parts of $\mathfrak{g}$ corresponding to the $\mathbb{Z}$-gradings obtained by setting $\deg X_i^\pm = \pm\delta_{i,i_0}$ for one or several selected indices i_0?

3) What are the defining relations explicitly?

4) What are the natural generators of $Br(2; a, b, c)$, and hence natural relations? Conjecturally, the answer is similar to [GL5].

5) Why can't we consider obvious analogs of $Br(r)$ for $r > 3$? Presumably, they are of infinite dimension; which of these $\mathbb{Z}$-graded algebras are of polynomial growth?

The above problems are resolved in what follows, at least, partly. The following questions are open:

6) In what follows we impose no restrictions on $\underline{N}$; the construction of prolongation imposes them automatically. The result for $\mathrm{BY}(\underline{N})$ surprised us; it is clear now that, for $p = 3$ and 2 and all vectorial algebras, to find the number of parameters $\underline{N}$ depends on is an open problem. Constructing the prolongs of $\underset{-d\leq i\leq 0}{\oplus}\mathfrak{g}_i$ when $d > 1$, we observed that $\underline{N}_j > 1$ only if $\deg u_j > 1$. What is the reason for this?

7) What are the analogs of Weisfeiler gradings for infinite dimensional simple vectorial Lie algebras in the limit as variable coordinates of $\underline{N}$ tend to ∞? (The corresponding preserved structures are most interesting.)

§4. Interpretations

Melikyan algebras. The following realization of $\mathfrak{g}(2)_- \oplus \mathfrak{g}(2)_0$ by vector fields is obtained by Shchepochkina's algorithm:

(58)

$\mathfrak{g}_0$	$X_- = u_2\partial_1 + u_1u_2^2\partial_4 + u_2^3\partial_5 + u_5\partial_4$, $X_+ = u_1\partial_2 + 2u_1^3\partial_4 + u_4\partial_5$, $h_1 = u_1\partial_1 + u_3\partial_3 + 2u_4\partial_4 + u_5\partial_5$, $h_2 = u_2\partial_2 + u_3\partial_3 + u_4\partial_4 + 2u_5\partial_5$
$\mathfrak{g}_{-1}$	$\partial_1 - u_2\partial_3 - u_1u_2\partial_4 - u_2^2\partial_5;\quad \partial_2$
$\mathfrak{g}_{-2}$	$\partial_3 + u_1\partial_4 + u_2\partial_5$
$\mathfrak{g}_{-3}$	$\partial_4;\quad \partial_5$

Set $\deg u_1 = (1, 0)$, $\deg u_2 = (0, 1)$. This determines the other degrees ($\deg u_3 = (1, 1)$, $\deg u_4 = (2, 1)$, $\deg u_5 = (1, 2)$). Unlike $p = 0$ case, the complete prolong of $\mathfrak{g}(2)_- \oplus \mathfrak{g}(2)_0$ strictly contains $\mathfrak{g}(2)$ (the underlined components) and has (for the simplest $\underline{N}$) the following irreducible components as $\mathfrak{g}_0$-modules given by their highest weights:

deg	dim	highest weights of components	deg	dim	highest weights of components
-3	2	$\underline{(-1,-2)}$	23	2	$(12,11)$
-2	1	$\underline{(-1,-1)}$	22	1	$(11,11)$
-1	2	$\underline{(0,-1)}$	21	2	$(11,10)$
0	4	$\underline{(1,-1)}, \underline{(0,0)}$	20	4	$(11,9),(10,10)$
1	2	$\underline{(1,0)}$	19	2	$(10,9)$
2	4	$(2,0), \underline{(1,1)}$	18	4	$(10,8),\ (9,9)$
3	6	$(3,0),\ \underline{(2,1)}$	17	6	$(10,7),(9,8)$
4	3	$(3,1)$	16	3	$(9,7)$
5	6	$(4,1),\ (3,2)$	15	6	$(9,6),(8,3)$
6	8	$(5,1)$,$(4,2)$	14	8	$(9,5),\ (8,7)$
7	4	$(5,2)$	13	4	$(8,5)$
8	8	$(6,2),\ (5,3)$	12	8	$(8,4),(7,5)$
9	10	$(7,2),\ (6,3)$	11	10	$(8,3),\ (7,4)$
10	5	$(7,3)$			

Computer experiments show that without restrictions on $\underline{N}$ the complete prolong only depends on two parameters, as theory [S] predicts, explicitly: $\underline{N} = (1,1,1,N_4,N_5)$.

Brown algebras. Let $x_i^{\pm}$ be the preimage of the generators $X_i^{\pm}$ relative (3). By abuse of notations we will often write $x_i^{\pm}$ instead of $X_i^{\pm}$; let x_i be either all x_i^+ or all x_i^-.

$\underline{Br(2)}$: Basis (of $Br(2)_{\pm}$):

$$x_1,\ x_2,\ [x_1,x_2],\ [x_2,[x_2,x_1]].$$

Here, x_2^+ and x_2^- generate $\mathfrak{hei}(2;3;\underline{1})$ on which h_1 acts as an outer derivation. The Fock space representation $\mathcal{O}(1;\underline{1})$ of $\mathfrak{hei}(2;3;\underline{1}) \Subset \mathbb{K}h_1$ (hereafter $\mathfrak{a} \Subset \mathfrak{i}$ is a semidirect sum of algebras, where $\mathfrak{i}$ is an ideal) is irreducible of dimension 3.

Therefore, the non-positive terms of the simplest $\mathbb{Z}$-gradings ($\deg x_{i_0}^{\pm} = \pm 1$) are:

(59)

i_0	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$
1	$\mathfrak{hei}(2;3;\underline{1}) \Subset \mathbb{K}h_1$	$\mathcal{O}(1;\underline{1})$	$-$
2	$\mathfrak{gl}(2) \simeq \mathfrak{gl}(V)$	V	$E^2(V)$

The first grading tempts us to investigate if there is a nontrivial Cartan prolong of the pair $\mathfrak{g}_0 = \mathfrak{hei}(2;3;\underline{N}) \Subset \mathbb{K}h_1$ and $\mathfrak{g}_{-1} = \mathcal{O}(1;\underline{N})$. But for the prolong to be simple, irreducibility is needed, while $\mathcal{O}(1;\underline{N})$ is irreducible $\mathfrak{hei}(2;3;\underline{N})$-module only for $N = 1$.

Observe that in the second grading, the non-positive terms are the same as the non-positive terms of $K(3)$, and hence same as those of the Frank algebras and same as those of $\mathfrak{sp}(4)$ with one (first) "selected"simple root. Moreover, even the defining relations between the positive (negative) generators are the same as the Serre relations of $\mathfrak{sp}(4)$ although the Cartan matrix of $Br(2)$, to say nothing of $Br(2,a)$, is different:

Theorem. *For any $a \in \mathbb{K} \setminus \{1,2\}$, the defining relations between the positive (negative) generators of $Br(2;a)$ are*

$$\operatorname{ad}^2 x_1(x_2) = 0; \qquad \operatorname{ad}^3 x_2(x_1) = 0. \tag{60}$$

So the defining relations for the Chevalley generators of $Br(2;a)$ are of the same type as Serre relations, but recovered from the Cartan matrix according to different (as compared with the $p=0$ case), and so far unknown, rules, cf. [GL1]. (Although the general rules are not known, the answer for $Br(2;a)$ and $Br(3)$ is now obtained: relations (60), (65).)

$Br(2;a,b,c)$**: Particular cases**.

Theorem. *Let a,b,c be such that the $\mathfrak{sl}(2)$-module $\mathbb{T}(a,b,c)$ is irreducible (i.e., $a\neq bc$). Then $\mathfrak{g}_2\neq 0$ only for $a=0$.*

Proof. Direct computations with aid of **SuperLie**. □

Clearly, if $a=0$, we can divide X_+ by b, setting $b=1$. As one can verify directly, $Br(2;0,1,c)_{\leq 0}$ has the following kth Cartan prolong for $\underline{N}=(11n)$ and $1\leq k\leq 3^n-2$, where $w:=$ weight of $u_3=$ $-$weight of u_1 (note that $\mathfrak{g}_{3^n-1}=0$):

(61)

elements of $\mathfrak{g}_k$	their weights
$u_3^{k+1}\partial_1$	$(k+2)w$
$\left(u_2u_3^k-cu_1^2u_3^{k-1}\right)\partial_1+u_3^{k+1}\partial_2+cu_1u_3^k\partial_3$	$(k+1)w=(k-2)w$
$-u_1u_3^k\partial_1+u_3^{k+1}\partial_3$	kw

In particular, $\mathfrak{g}_1$ is spanned (compare with (37)) by

(62)

$\partial_1^*:=$	$(u_2u_3-cu_1^2)\partial_1+u_3^2\partial_2+cu_1u_3\partial_3$	$-w$
$\partial_2^*:=$	$u_3^2\partial_1$	0
$\partial_3^*:=$	$-u_1u_3\partial_1+u_3^2\partial_3$	w

The commutators are

(63)

	∂_1^*	∂_2^*	∂_3^*
∂_1	H	0	X^+
∂_2	X^+	0	0
∂_3	X^-	X^+	$\frac{1}{c}H$

Since $[\mathfrak{g}_1,\mathfrak{g}_{-1}]=\mathfrak{sl}(2)$, and $\mathfrak{g}_{\pm 1}$ are irreducible $\mathfrak{sl}(2)$-modules, the Cartan prolongation $Br(2;0,1,c):=\mathop{\oplus}\limits_{i\geq -1}\mathfrak{g}_i$ is a simple Lie algebra whose positive part is generated by $\mathfrak{g}_1$ and (if $n>1$) $\mathfrak{g}_2$. It seems, $Br(2;0,1,c)$ is a new simple Lie algebra, more precisely, it is a deformation of the nonstandard Hamiltonian algebra $\mathfrak{h}(2:\underline{(1,n)};\omega)$ preserving the form $\omega=\exp(x)dx\wedge du$, and considered in [BKK] in nonstandard grading $\deg x=0$, $\deg y=1$.

$\underline{Br(3)}$: Basis (of $Br(3)_\pm$):

(64)
$$\begin{aligned}
&x_1,\ x_2,\ x_3;\\
&[x_1,x_2],\quad [x_2,x_3];\\
&[x_3,[x_3,x_2]],\quad [x_3,[x_2,x_1]];\\
&[x_3,[x_3,[x_1,x_2]]];\\
&[[x_2,x_3],[x_3,[x_1,x_2]]];\\
&[[x_3,[x_1,x_2]],[x_3,[x_2,x_3]]];\\
&[[x_3,[x_2,x_3]],[x_3,[x_3,[x_1,x_2]]]];\\
&[[x_3,[x_2,x_3]],[[x_2,x_3],[x_3,[x_1,x_2]]]];\\
&[[x_3,[x_3,[x_1,x_2]]],[[x_2,x_3],[x_3,[x_1,x_2]]]].
\end{aligned}$$

The non-positive terms of the $\mathbb{Z}$-gradings in terms of $\mathfrak{g}_0$-modules are (underlined are the dimensions of the irreducible $\mathfrak{g}_0$-modules)

i_0	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-3}$	$\mathfrak{g}_{-4}$
1	$Br(2) \Subset \mathbb{K}h_1$	$\underline{8}$	$\underline{1}$	–	–
2	$(\mathfrak{sl}(2) \ni \mathbb{K}h_2) \ni \mathfrak{hei}(2;3;\underline{1})$	$\underline{2} \otimes \underline{3}$	$\underline{1} \otimes \underline{3}$	$\underline{2} \otimes \underline{1}$	–
3	$\mathfrak{gl}(3)$	$\underline{3}$	$\underline{3}$	$\underline{1}$	$\underline{3}$

The last line coincides with DY$_-$, see (46); the first line shows that $Br(3)$ is a partial prolong of $(\mathfrak{sp}(10)_-, Br(2) \Subset \mathbb{K}h_1)$ in the contact grading of $\mathfrak{sp}(10)$.

Theorem. *The defining relations between the positive (or negative) generators are as follows:*

$$\begin{aligned}
&[x_1, x_3] = 0;\\
&\mathrm{ad}^2 x_2(x_1) = 0, \quad \mathrm{ad}^2 x_2(x_3) = 0;\\
&\mathrm{ad}^3 x_3(x_2) = 0;\\
&[[x_3,[x_3,x_2]],[[x_3,[x_2,x_1]],[x_3,[x_3,x_2]]]] = 0.
\end{aligned} \tag{65}$$

The last non-Serre relation resembles relations for Lie superalgebras with Cartan matrix, cf. [GL1].

The Brown algebras given by tridiagonal $r \times r$ Cartan matrices of type (35) of larger size seem to be of infinite dimension, though $\dim \oplus_{|i| \leq n}$ grows rather slow as $n \longrightarrow \infty$, at least, for $r = 4, 5$; **conjecturally, these algebras (for $r = 4, 5$) are of polynomial growth.**

$\underline{\mathfrak{g} = \mathrm{DY}}$. Here $\mathfrak{g}_0 = \mathfrak{gl}(3)$ (the weights are given with respect to the h_i). Let $\mathfrak{g}_-$ be realized by vector fields as follows:

(66)

$\mathfrak{g}_0$	$(-1,0,1)\quad u_3\partial_1 + u_3^2\partial_5 - (u_2u_3+u_4)\partial_6 - u_2u_3^2\partial_7 + (u_2u_3u_5+u_4u_5+u_{10})\partial_8 - (u_2u_3u_4+u_4^2)\partial_9 - u_3^2u_4\partial_{10},$
	$(-1,1,0)\quad u_2\partial_1 + u_4\partial_5 - u_2^2\partial_6 + u_2^2u_3\partial_7 + (u_2^2u_5+u_9)\partial_8 - u_2^2u_4\partial_9 + (u_2^2u_3^2+u_4^2)\partial_{10},$
	$(0,-1,1)\quad u_3\partial_2 - u_3^2\partial_4 - u_5\partial_6 + u_5^2\partial_8 - u_{10}\partial_9,$
	$(0,1,-1)\quad u_2\partial_3 - u_2^2\partial_4 - u_6\partial_5 + u_6^2\partial_8 + u_9\partial_{10},$
	$(1,-1,0)\quad -u_1\partial_2 + u_5\partial_4 + u_1^2\partial_6 - u_1^2u_3\partial_7 - u_1^2u_5\partial_8 + (u_1^2u_4 - u_8)\partial_9 - (u_1^2u_3^2+u_5^2)\partial_{10},$
	$(1,0,-1)\quad u_1\partial_3 + (u_1u_2+u_6)\partial_4 + u_1^2\partial_5 - u_1^2u_2\partial_7 - (u_1^2u_2^2+u_6^2)\partial_9 - (u_1^2u_2u_3 + u_1^2u_4 - u_5u_6 - u_8)\partial_{10},$
	$h_1 = u_1\partial_1 + u_5\partial_5 + u_6\partial_6 + u_7\partial_7 - u_8\partial_8 + u_9\partial_9 + u_{10}\partial_{10},$
	$h_2 = u_2\partial_2 + u_4\partial_4 + u_6\partial_6 + u_7\partial_7 + u_8\partial_8 - u_9\partial_9 + u_{10}\partial_{10},$
	$h_3 = u_3\partial_3 + u_4\partial_4 + u_5\partial_5 + u_7\partial_7 + u_8\partial_8 + u_9\partial_9 - u_{10}\partial_{10}$

(67)

$\mathfrak{g}_{-1}$	$\partial_1 - u_2\partial_6 + u_3\partial_5 + (u_2u_3 - u_4)\partial_7 + (u_2u_5+u_7)\partial_8 + (u_2u_3^2 - u_2u_4)\partial_{10},$ $\quad \partial_2 - u_3\partial_4 - u_5\partial_7 + u_7\partial_9, \quad \partial_3 - u_6\partial_7 + u_7\partial_{10}$
$\mathfrak{g}_{-2}$	$\partial_4 + u_6\partial_9 - u_5\partial_{10}, \quad \partial_5 - u_6\partial_8, \quad \partial_6$
$\mathfrak{g}_{-3}$	∂_7
$\mathfrak{g}_{-4}$	$\partial_8, \quad \partial_9, \quad \partial_{10}$

Remark. The above realization of $\mathrm{DY}(\underline{N})$ shows that $\underline{N} \in \mathbb{N}^{10}$. Representation (42) shows that $\underline{N} \in \mathbb{N}^{10}$ depends on at least 3 parameters. The concealed parameter we found for $\mathrm{BY}(\underline{N})$ urges to investigate this question for $\mathrm{DY}(\underline{N})$. Computer experiments intended to reveal the number of parameters in $\mathrm{DY}(\underline{N})$ depends on are in the process.

Suppose we do not have [S] or [Sk] to consult, but wish to describe DY in a form similar to $\mathrm{Me}(\underline{N})$: as a sum of $W(3,\underline{N})$ and its modules. The lines deg $= -4$ through -1 in table (68) give us weights in terms of the identity $\mathfrak{gl}(V)$-module V. It is clear that deg $= -3$ corresponds to the volume forms and either line -4 or line -2 should correspond to $W(3,\underline{N})$ which should lie in even degrees. A few experiments approve just one scenario. Line -4 gives us the highest weight of $W(3,\underline{N})_{-1}$ in a nonstandard grading:

$$\mathrm{weight}(\partial_3) = (-1,-1,-2) \Longrightarrow \mathrm{weight}(x_1) = (2,1,1) \Longrightarrow \mathrm{weight}(vol) = (1,1,1),$$

and hence the highest weights and the corresponding vectors of the following components are

$$\begin{aligned}
\mathfrak{g}_{-1}: &\quad \mathrm{weight}(du_1 du_2) = (3,3,2) \cong (0,0,-1),\\
\mathfrak{g}_{-2}: &\quad \mathrm{weight}(du_1 \cdot vol) = (3,2,2) \cong (0,-1,-1),\\
\mathfrak{g}_{-3}: &\quad \mathrm{weight}(vol^{-1}) = (-1,-1,-1),\\
\mathfrak{g}_{-4}: &\quad \mathrm{weight}(\partial_3) = (-1,-1,-2).
\end{aligned}$$

The subalgebra of $\mathfrak{g} = \mathrm{DY}$ generated by $\mathfrak{g}_-$ and $\mathfrak{g}_1$ (the underlined components in table (68)) is, clearly, isomorphic to $Br(3)$. But the new generator of degree 2 and weight $(0,0,2)$ generates, together with $Br(3)$, a larger algebra.

Below are the dimensions and **highest** weights of the components of $\mathrm{DY}^{(1)}$ as $\mathrm{DY}_0 = \mathfrak{gl}(3)$-modules. The terms of DY not contained in $\mathrm{DY}^{(1)}$ are marked in parentheses in "dim"column.

Recall (42) and (45); below $i, j, k, l = 1, 2, 3$.

(68)

deg	dim	highest weights of components	basis (for all possible indices)
-4	3	$\underline{(-1,-1,-2)}$	∂_i
-3	1	$\underline{(-1,-1,-1)}$	v^{-1}
-2	3	$\underline{(0,-1,-1)}$	$du_i v$
-1	3	$\underline{(0,0,-1)}$	$du_i du_j$
0	9	$\underline{(1,0,-1)}, \underline{(0,0,0)}$	$u_i \partial_j$
1	3	$\underline{(1,0,0)}$	$u_i v^{-1}$
2	9	$(2,0,0), \underline{(1,1,0)}$	$u_i du_j v$
3	8	$(2,1,0) \supset \underline{(1,1,1)}$	$\omega = \sum f_{ij} du_i du_j \mid d\omega = 0, \deg f_{ij} = 1$
4	18	$(3,1,0), \underline{(2,1,1)}$	$u_i u_k \partial_j$
5	6	$(3,1,1)$	$u_i u_j v^{-1}$
6	18	$(4,1,1),\ (3,2,1)$	$u_i u_j du_k v$
7	15	$(4,2,1)$	$\omega = \sum f_{ij} du_i du_j \mid d\omega = 0, \deg f_{ij} = 2$
8	21	$(4,3,1),\ (4,2,2)$	$u_i u_k u_l \partial_j$
9	7	$(4,3,2)$	$u_i u_k u_l v^{-1}$
10	21	$(5,3,2),\ (4,4,2)$	$u_i u_k u_l du_j v$
11	15	$(5,4,2)$	$\omega = \sum f_{ij} du_i du_j \mid d\omega = 0, \deg f_{ij} = 3$
12	18	$(5,5,2), (5,4,3)$	$f(u)\partial_j \mid \deg f = 4$
13	6	$(5,5,3)$	$f(u)v^{-1} \mid \deg f = 4$
14	18	$(6,5,3),\ (5,5,4)$	$f(u)du_j v \mid \deg f = 4$
15	8(+3)	$(6,5,4)$	$\omega = \sum f_{ij} du_i du_j \mid d\omega = 0, \deg f_{ij} = 4$
16	9	$(6,6,4),\ (6,5,5)$	$f(u)\partial_j \mid \deg f = 5$
17	3	$(6,6,5)$	$f(u)v^{-1} \mid \deg f = 5$
18	9	$(7,6,5),\ (6,6,6)$	$f(u)du_j v \mid \deg f = 5$
19	3	$(7,6,6)$	$\omega = \sum f_{ij} du_i du_j \mid d\omega = 0, \deg f_{ij} = 5$
20	3	$(7,7,6)$	$f(u)\partial_j \mid \deg f = 6$
21	1	$(7,7,7)$	$f(u)v^{-1} \mid \deg f = 6$
22	3	$(8,7,7)$	$f(u)du_j v \mid \deg f = 6$

The modules with such highest weights are irreducible if $\mathrm{Char}\mathbb{K} = 0$; but since $\mathrm{Char}\mathbb{K} = 3$, some of these components are reducible.

$\underline{\mathfrak{g} = \mathrm{BY}}$. Here $\mathfrak{g}_0 = \mathfrak{gl}(3)$ (the weights are given with respect to the h_i). Let $\mathfrak{g}_-$ be realized as follows:

(69)

$\mathfrak{g}_0$	$(-1,0,1)\quad u_3\partial_1 + u_3^2\partial_5 + (u_2u_3 - u_4)\,\partial_6 - u_2u_3^2\partial_7$ $(-1,1,0)\quad -u_2\partial_1 + u_4\partial_5$ $(0,-1,1)\quad -u_3\partial_2 + u_3^2\partial_4 + (u_1u_3 + u_5)\,\partial_6 - u_1u_3^2\partial_7;$ $h_1 = u_1\partial_1 + u_5\partial_5 + u_6\partial_6 + u_7\partial_7,$ $h_2 = u_2\partial_2 + u_4\partial_4 + u_6\partial_6 + u_7\partial_7,$ $h_3 = u_3\partial_3 + u_4\partial_4 + u_5\partial_5 + u_7\partial_7;$ $(0,1,-1)\quad -u_2\partial_3 + u_2^2\partial_4 + (u_1u_2 + u_6)\,\partial_5 - u_1u_2^2\partial_7,$ $(1,-1,0)\quad -u_1\partial_2 + u_5\partial_4,$ $(1,0,-1)\quad -u_1\partial_3 + (-u_1u_2 + u_6)\,\partial_4 - u_1^2\partial_5 + u_1^2u_2\partial_7$
$\mathfrak{g}_{-1}$	$-\partial_1 - u_2\partial_6 - u_3\partial_5 + u_4\partial_7, \quad -\partial_2 + u_3\partial_4 + u_1\partial_6 + u_5\partial_7, \quad -\partial_3 + u_6\partial_7$
$\mathfrak{g}_{-2}$	$\partial_4 + u_1\partial_7, \quad \partial_5 + u_2\partial_7, \quad \partial_6 + u_3\partial_7$
$\mathfrak{g}_{-3}$	∂_7

Consider the Tanaka-Shchepochkina prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$. As $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is a direct sum of two submodules, $\mathfrak{g}_1'$ and $\mathfrak{g}_1''$ with lowest weights $(0,0,1)$ and $(-1,1,1)$, respectively. As algebra, $\mathfrak{g}_1$ generates 224-dimensional algebra $\mathfrak{g}_+$ of height 14 and relations up to degree 6 (for comparison: the defining relations of $\mathfrak{g}_+$ for simple vectorial Lie algebras are of degree 2 (and 3 for the Hamiltonian series), cf. [GLP]). Observe that even so ugly and seemingly impossible to use relations are sometimes useful since they are explicit.

Let BY be the algebra generated by $\mathfrak{g}_-$ and $\mathfrak{g}_1$. Its dimension is 240. The Tanaka-Shchepochkina prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ has, however, 4 elements more than BY: one, of degree 9 and weight $\underline{\underline{(3,3,3)}}$ and three more elements of degree 12 whose weights are $\underline{\underline{(6,3,3)}}$, $\underline{\underline{(3,6,3)}}$, and $\underline{\underline{(3,3,6)}}$. These four elements are outer derivatives of $(\mathfrak{g}_-, \mathfrak{g}_0)_*$; so there are four linearly independent traces on $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ and $\mathrm{BY} = (\mathfrak{g}_-, \mathfrak{g}_0)_*^{(1)}$.

We have

$$[\mathfrak{g}_1', \mathfrak{g}_{-1}] = \mathfrak{g}_0, \quad [\mathfrak{g}_1', \mathfrak{g}_1'] = 0.$$

Let BY' be the algebra generated by $\mathfrak{g}_-$ and $\mathfrak{g}_1'$. Its dimension is 19. It is not simple: the part $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$ is an ideal.

We also have

$$[\mathfrak{g}_1'', \mathfrak{g}_{-1}] = \mathfrak{sl}(3).$$

Let BY'' be the algebra generated by $\mathfrak{g}_-$ and $\mathfrak{g}_1''$. It is the special subalgebra of BY; its dimension is 78. The element of weight $\underline{\underline{(3,3,3)}}$ is its outer derivative; together with $\mathfrak{g}_-$, it generates $\mathrm{BY}'' := \mathrm{SBY}$; the three other outer derivatives of BY are also divergence free and belong to $(\mathfrak{g}_-, \mathfrak{sl}(3))_*$, the complete prolong of $(\mathfrak{g}_-, \mathfrak{sl}(3))$.

Here are the dimensions and the **highest** weights of the components of BY (recall that $\mathrm{BY}_0 = \mathfrak{gl}(3)$):

deg	dim	weights of components	deg	dim	weights of components
-3	1	$(-1,-1,-1)$	6	26	$(4,1,1),(3,2,1),(3,2,1)$
-2	3	$(0,-1,-1)$	7	24	$(4,2,1),(3,3,1),(3,2,2)$
-1	3	$(0,0,-1)$	8	24	$(4,3,1),(4,2,2),(3,3,2)$
0	9	$(1,0,-1),(0,0,0)$	9	19	$(5,2,2),(4,3,2),\underline{\underline{(3,3,3)}}$
1	9	$(1,1,-1),(1,0,0)$	10	18	$(5,3,2),(4,3,3)$
2	18	$(2,1,-1),(1,1,0)$	11	9	$(5,3,3),(4,4,3)$
3	16	$(2,1,0),(2,1,0)$	12	11	$(4,4,4)$, $\underline{\underline{(6,3,3)}}$, $\underline{\underline{(3,6,3)}}$, $\underline{\underline{(3,3,6)}}$
4	24	$(3,1,0),(2,2,0),(2,1,1)$	13	3	$(4,4,4)$
5	24	$(3,2,0),(3,1,1),(2,2,1)$	14	3	$(5,5,4)$

The dimensions of the respective components of $(\mathfrak{g}_-, \mathfrak{sl}(3))_*$ (same of BY'' only differ in dimensions 6 and 9; they are paranthesized) are:

deg	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
dim	1	3	3	8	6	15	7	15	6	11 (8)	3	3	1 (0)

The dimension of degree 3 here looks like a mistake if one compares with deg $= 3$ line table for BY: but the point is that one component of weight $(2,1,0)$ contains a 1-dimensional submodule $(1,1,1)$ while the other component of weight $(2,1,0)$ contains a submodule of dimension 7.

As spaces, and $\mathbb{Z}/2$-graded Lie algebras, we have (hence $\dim \mathrm{SBY}(\underline{N}) = 3^{|\underline{N}|+1} + 1$)

$$
\begin{aligned}
&\mathrm{SBY}(\underline{N}) := \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1} \simeq S(3;\underline{N}) \oplus \mathcal{O}(3;\underline{N}) \\
&\mathrm{SBY}(\underline{N})^{(1)} \simeq S(3;\underline{N})^{(1)} \oplus \mathcal{O}'(3;\underline{N}), \qquad \dim \mathrm{SBY}(\underline{N})^{(1)} = 3^{|\underline{N}|+1} - 3.
\end{aligned} \tag{70}
$$

The multiplication of functions is given by

$$
[f,g] = \left(\frac{\partial f}{\partial u_1}\frac{\partial g}{\partial u_2} - \frac{\partial f}{\partial u_2}\frac{\partial g}{\partial u_1} \right) \frac{\partial}{\partial u_3} + cycle(123). \tag{71}
$$

Let us now figure out what is $\mathrm{BY}(\underline{\tilde N})$ and what is its $\underline{\tilde N}$. It is not difficult to see that, as space, $\mathrm{BY}(\underline{\tilde N})$ is:

$$
\mathrm{BY}(\underline{\tilde N}) \simeq W(3;\underline{N}) \oplus \mathcal{O}(3;\underline{N})_{\mathrm{div}} \oplus W(3;\underline{N})_{-\mathrm{div}} \oplus Z^2(3;\underline{N}) \tag{72}
$$

and we accordingly set:

$$
\begin{aligned}
&\deg u^{\underline{r}} vol = 2|\underline{r}| - 3, \qquad &&\deg u^{\underline{r}}\partial_i = 2|\underline{r}| - 2, \\
&\deg u^{\underline{r}}\partial_i v^{-1} = 2|\underline{r}| + 1, \qquad &&\deg u^{\underline{r}} du_i du_j = 2|\underline{r}| + 4.
\end{aligned} \tag{73}
$$

The structure of $\mathrm{BY}(\underline{\tilde N})$, as $W(3;\underline{N})$-module, is remarkable:

$$
\begin{aligned}
&\mathrm{BY}(\underline{\tilde N}) \simeq W(3;\underline{N}) \oplus Z^2(3;\underline{N}) \oplus R, \quad \text{where} \\
&0 \longrightarrow W(3;\underline{N})_{-\mathrm{div}} \longrightarrow R \longrightarrow \mathcal{O}(3;\underline{N})_{\mathrm{div}} \longrightarrow 0
\end{aligned} \tag{74}
$$

is an exact sequence.

The multiplication is easy to describe in terms of the vector fields that constitute the Tanaka-Schepochkina prolongation, but is too bulky. To describe the multiplication in $\mathrm{BY}(\underline{\tilde N})$ in terms of constituents (74), especially the $W(3;\underline{N})$-action on R, observe that the basis of the complementary module to $W(3;\underline{N})_{-\mathrm{div}}$ can be selected canonically if we restrict the $W(3;\underline{N})$-action on R to $WS(3;\underline{N})$-action or even $\mathfrak{gl}(3)$-action. These complementary elements will be denoted by "fv".

$$\begin{aligned}
&[\text{“}fv\text{”}, \text{“}gv\text{”}] = (df dg)v^{-1};\\
&[\text{“}fv\text{”}, Dv^{-1}] = fD;\\
&[\text{“}fv\text{”}, \omega^2] = f\omega^2 v \in W(3;\underline{N})_{-\mathrm{div}};\\
&[D_1 v^{-1}, D_2 v^{-1}] = [D_1, D_2]v \in Z^2(3;\underline{N});\\
&[Z^2(3;\underline{N}),\ Z^2(3;\underline{N})] = 0, \qquad [Z^2(3;\underline{N}),\ W(3;\underline{N})_{-\mathrm{div}}] = 0.
\end{aligned}\tag{75}$$

The description (72) makes an impression that the parameter $\underline{\tilde{N}}$ in $\mathrm{BY}(\underline{\tilde{N}})$ depends on a 3-dimensional $\underline{N}$. Computer experiments show, however, that without restrictions on $\underline{\tilde{N}}$ the complete prolong depends **not on three parameters but on four**:

$$\underline{\tilde{N}} = (1,1,1,N_4,N_5,N_6,N_7).\tag{76}$$

At the moment we can not explain the meaning of the fourth parameter. This result invites to investigate also $\mathrm{DY}(\underline{\tilde{N}})$ lifting all restrictions on $\underline{\tilde{N}}$.

$\underline{\mathfrak{g} = \mathrm{MY}}$. Here $\mathfrak{g}_0 = \mathfrak{gl}(3)$ (the weights are given with respect to the h_i). Let $\mathfrak{g}_-$ be realized by vector fields as follows:

(77)

$\mathfrak{g}_0$	$(-1,0,1)$	$u_3\partial_1 + 2u_3^2\partial_5 + (u_2u_3 + 2u_4)\,\partial_6$	$h_1 = u_1\partial_1 + u_5\partial_5 + u_6\partial_6$
	$(-1,1,0)$	$2u_2\partial_1 + 2u_2^2\partial_6 + u_4\partial_5$	$h_2 = u_2\partial_2 + u_4\partial_4 + u_6\partial_6$
	$(0,-1,1)$	$u_3\partial_2 + u_3^2\partial_4 + 2u_5\partial_6$	$h_3 = u_3\partial_3 + u_4\partial_4 + u_5\partial_5$
	$(0,1,-1)$	$2u_2\partial_3 + 2u_2^2\partial_4 + u_6\partial_5$	
	$(1,-1,0)$	$2u_1\partial_2 + 2u_1^2\partial_6 + u_5\partial_4$	
	$(1,0,-1)$	$2u_1\partial_3 + (2u_1u_2 + u_6)\,\partial_4 + u_1^2\partial_5$	

$\mathfrak{g}_{-1}$	$\partial_1,\ \partial_2 + u_1\partial_6,\ \partial_3 - u_1\partial_5 + u_2\partial_4$
$\mathfrak{g}_{-2}$	$\partial_4,\ \partial_5,\ \partial_6$

Consider the Tanaka-Shchepochkina prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$. As $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is a direct sum of two submodules, $\mathfrak{g}_1'$ and $\mathfrak{g}_1''$ with lowest weights $(0,0,1)$ and $(-1,1,1)$, respectively. The space $\mathfrak{g}_1$ generates an algebra of height 11. Here are the dimensions and **highest** weights of the components of MY:

deg	dim	weights of components	deg	dim	weights of components
-2	3	$\underline{(0,0,-1)}$	-1	3	$\underline{(1,0,0)}$
0	9	$\underline{(1,0,-1)}, \underline{(0,0,0)}$	1	9	$(1,1,-1),(1,0,0)$
2	18	$(2,1,-1),(1,1,0)$	3	18	$(2,2,-1),(2,1,0)$
4	21	$(3,2,0),(3,1,1)$	5	21	$(3,2,0),(3,1,1)$
6	18	$(4,1,1),(3,2,1)$	7	18	$(4,2,1),(3,2,2)$
8	9	$(4,2,2),(3,3,2)$	9	9	$(4,3,2),(3,3,3)$
10	3	$(4,3,3)$	11	3	$(4,4,3)$

Now, consider partial Tanaka-Shchepochkina prolongs $(\mathfrak{g}_-, \mathfrak{g}_0)_*$. As $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is a direct sum of two submodules, $\mathfrak{g}_1'$ and $\mathfrak{g}_1''$ with lowest weights $(0,0,1)$ and $(-1,1,1)$, respectively.

We have

$$[\mathfrak{g}_1', \mathfrak{g}_{-1}] = \mathfrak{g}_0, \quad [\mathfrak{g}_1', \mathfrak{g}_1'] = 0.$$

Let MY' be the algebra generated by $\mathfrak{g}_-$ and $\mathfrak{g}'_1$. We have (given are the **highest** weights):

deg	dim	weights of components	deg	dim	weights of components
1	3	$(1,0,0)$	2	3	$(1,1,0)$

So $\mathrm{MY}' \simeq \mathfrak{o}(7)$.

Let MY'' be the algebra generated by $\mathfrak{g}_-$ and $\mathfrak{g}''_1$. Its negative part consists of the same components as for MY,

$$[\mathfrak{g}''_1, \mathfrak{g}_{-1}] = \mathfrak{sl}(3),$$

and hence MY'' is isomorphic to SMY, the divergence-free subalgebra, and $\dim \mathrm{SMY} = 77$.

$\underline{\mathfrak{g} = \mathrm{Er}}$. We have $\mathfrak{g}_0 = \mathfrak{sl}(2) \ni \mathfrak{hei}(2;3;\underline{1})$; we realize the vital components $\mathfrak{g}_i$ for $i = -1, 0, 1$ by vector fields as follows (a realization in terms of (57) is indicated in parentheses, where $D_i = \frac{\partial}{\partial u_i}$); the weights are given with respect to $B-A$ and $A+B$ from $\mathfrak{g}_0$; in order not to mix the indeterminates u_1 and u_2 of realization (57), we denote the new three indeterminates the x_i, although they generate the algebra of divided powers:

$\mathfrak{g}_{-1}$	∂_1, ∂_3 (this is 1), ∂_2
$\mathfrak{g}_0$	$(-2,0) : x_2\partial_1$, $(-1,-1) : x_2\partial_3 + x_3\partial_1$ (this is u_2) $(0,0) : A := x_1\partial_1 - x_3\partial_3$ (this is u_1D_1), $B := x_2\partial_2 - x_3\partial_3$ (this is u_2D_2), $(1,-1) : x_1\partial_3 - x_3\partial_2$ (this is u_1), $(2,0) : x_1\partial_2$

$\mathfrak{g}_1$	$(-3,1) : -x_2^2\partial_1$, $(-2,0) : x_2^2\partial_3 + x_2x_3\partial_1$, $(-1,1) : x_1x_2\partial_1 - x_2^2\partial_2$, $x_1x_2\partial_1 - x_2x_3\partial_3 - x_3^2\partial_1$, $(0,0) : -x_1x_2\partial_3 - x_1x_3\partial_1 + x_2x_3\partial_2$, $(1,1) : -x_1^2\partial_1 + x_1x_2\partial_2$, $-x_1^2\partial_1 + x_1x_3\partial_3 - x_3^2\partial_2$, $(2,0) : -x_1^2\partial_3 + x_1x_3\partial_2$, $(3,1) : -x_1^2\partial_2$

The other components of the complete prolong are also computed; Er_1 is irreducible as a Er_0-module, it generates the codimension 1 subalgebra of $(\mathrm{Er}_-, \mathrm{Er}_0)_+$; the dimensions of the components of degree 1, 2, 3 are 9, 6 and 2, respectively; $\dim \mathrm{Er} = 26$.

Frank algebras. The algebras $\mathrm{Br}(2,a)$ (deformations of $Br(2)$), $\mathfrak{sp}(4)$ and $\mathrm{Fr}(n)$ are partial prolongs with the same non-positive part as $K(3;(1,1,n))$. The generator of $\mathfrak{sp}(4)_1$ is tq, the generator of $\mathrm{Fr}(n)_1$ is given above, and

$$\mathrm{Br}(2,a)_1 = Span(\alpha q^2p + qt,\ \alpha qp^2 - pt) \ \text{ for } \alpha = \frac{a-1}{a+1} \text{ and } \alpha \neq 1, 2.$$

The partial prolong of $\left(\underset{i\leq 0}{\oplus} K(3;(1,1,n))_i\right) \oplus \mathrm{Fr}(n)_1$ coincides with $\mathrm{Fr}(n)$ described componentwise in [S]; the generators of the positive part are $z_1 = pq^2 + qt$ and $z_2 = q^2t$, and for $n > 1$, conjecturally, $z_3, \ldots, z_{i+2} = t^{3^i} - p^2q^2t^{3^i-2}$ for $1 \leq i < n$).

The relations for $n = 1$ are ($x_1 = p^2$):

$$\begin{aligned}
\deg = 1: \quad & [x_1, [x_1, z_1]] = 0, \\
\deg = 2: \quad & [x_1, [x_1, [x_1, z_2]]] = 0, \\
\deg = 3: \quad & [z_1, z_2] = 0, \quad [z_1, [z_1, [x_1, z_1]]] = 0, \quad [[x_1, z_1], [x_1, [x_1, z_2]]] = 0, \\
\deg = 4: \quad & [z_2, [z_1, [x_1, z_1]]] + [z_2, [x_1, z_2]] = 0, \\
& [[x_1, z_1], [[x_1, z_1], z_2]] + [z_2, [x_1, [x_1, z_2]]] = 0, \\
\deg = 5: \quad & [z_2, [[x_1, z_1], z_2]] = 0, \quad [[x_1, z_2], [[x_1, z_1], z_2]] = 0, \\
\deg = 6: \quad & [z_2, [z_2, [x_1, z_2]]] = 0, [[x_1, z_2], [z_2, [x_1, z_2]]] = 0, \\
& [[x_1, [x_1, z_2]], [z_2, [x_1, z_2]]] = 0
\end{aligned}$$

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