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Abstract

Non-degenerate bilinear forms over fields of characteristic 2, in particular, non-symmetric ones, are classified with respect to various equivalences, and the Lie algebras preserving them are described. Although it is known that there are two series of distinct finite simple Chevalley groups preserving the non-degenerate symmetric bilinear forms on the space of even dimension, the description of simple Lie algebras related to the ones that preserve these forms is new. The classification of 1-forms is shown to be related to one of the considered equivalences of bilinear forms. A version of the above results for superspaces is also given.

§1 Introduction

1.1 Notations. The ground field $\mathbb{K}$ is assumed to be algebraically closed unless specified; its characteristic is denoted by p; we assume that $p = 2$ unless specified; vector spaces V are finite dimensional; $n = \dim V$. We often use the following matrices

$$J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad \Pi_n = \begin{cases} \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix} & \text{if } n = 2k, \\ \begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1 & 0 \\ 1_k & 0 & 0 \end{pmatrix} & \text{if } n = 2k+1, \end{cases} \tag{1.1}$$

$$S(n) = \operatorname{antidiag}_n(1, \dots, 1), \quad Z(2k) = \operatorname{diag}_k(\Pi_2, \dots, \Pi_2).$$

We call a square matrix *zero-diagonal* if it has only zeros on the main diagonal; let $ZD(n)$ be the space (Lie algebra if $p = 2$) of symmetric zero-diagonal $n \times n$-matrices.

For any Lie (super)algebra $\mathfrak{g}$, let $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$.

Let $\mathfrak{o}_I(n)$, $\mathfrak{o}_\Pi(n)$ and $\mathfrak{o}_S(n)$ be Lie algebras that preserve bilinear forms 1_n, Π_n and $S(n)$, respectively.

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1.2 Motivations. Recall that to any bilinear form B on a given space V one can assign its *Gram matrix* by abuse of notations also denoted by $B = (B_{ij})$: in a fixed basis $x_1, \dots, x_n$ of V we set

$$B_{ij} = B(x_i, x_j). \tag{1.2}$$

In what follows, we fix a basis of V and identify a bilinear form with its matrix. Two bilinear forms B and C on V are said to be *equivalent* if there exists an invertible linear operator $A \in GL(V)$ such that $B(x, y) = C(Ax, Ay)$ for all $x, y \in V$; in this case,

$$B = ACA^T \tag{1.3}$$

for the matrices of B, C and A in the same basis.

A bilinear form B on V is said to be *symmetric* if $B(v, w) = B(w, v)$ for any $v, w \in V$; it is *skew-symmetric* if $B(v, v) = 0$ for any $v \in V$.

Given a bilinear form B, let

$$L(B) = \{F \in \operatorname{End} V \mid B(Fx, y) + B(x, Fy) = 0\}.$$

be the Lie algebra that preserves B. If $p \neq 2$, some of the Lie algebras $L(B)$ are simple, for example, the orthogonal Lie algebras $\mathfrak{o}_B(n)$ that preserve non-degenerate symmetric forms and symplectic Lie algebras $\mathfrak{sp}_B(n)$ that preserve non-degenerate skew-symmetric forms.

If $p = 2$, either the derived algebras of $L(B)$ for non-degenerate forms B or their quotient modulo center are simple, so the canonical expressions of the forms B are needed as a step in classification of simple Lie algebras in characteristic 2 which is an open problem, and as a step in a version of this problem for Lie superalgebras, also open.

The problem of describing preserved bilinear forms has two levels: we can consider linear transformations (Linear Algebra) and arbitrary coordinate changes (Differential Geometry). In the literature, both levels are completely investigated, except for the case where $p = 2$.

More precisely, for $p = 2$, there are obtained rather esoteric results such as classifications of quadratic forms over skew fields [ET], and of analogs of hermitian forms in infinite dimensional spaces [Gr], whereas (strangely enough for such a classically formulated problem) the non-degenerate bilinear forms over fields were never classified, except for symmetric forms. Moreover, the fact that the non-split and split forms of the Lie algebras that preserve the symmetric forms are not always isomorphic was never mentioned (although known on the Chevalley group level), cf. the latest papers with reviews of earlier results [Br, Sh] and [GG].

Hamelink [H] considered simple Lie algebras over $\mathbb{K}$ but under too restrictive conditions (he considered only Lie algebras with a nonsingular invariant form) and so missed the fact that there are two types of orthogonal (or symplectic) simple Lie algebras.

The bilinear forms over fields of characteristic 2 also naturally appear in topological problems related to real manifolds, for example, in singularity theory: as related to "symplectic analogs of Weyl groups" and related bilinear forms over $\mathbb{Z}/2$, cf. [I].

We also consider the superspaces. The Lie superalgebras over $\mathbb{Z}/2$ were of huge interest in 1960s in relation with other applications in topology, see, e.g., [Ha, May]; lately, the interest comes back [V].

Let us review the known results and compare them with the new ones (§§3–8).

1.3 Known facts: The case $p \neq 2$. Having fixed a basis of the space on which bilinear or quadratic form is considered, we identify the form with its Gram matrix; this is understood throughout. Let me recall the known (both well known and not so well known) facts. First, recall the elementary Linear Algebra [Pra], [L]. Any nondegenerate bilinear form B on a finite dimensional space V can be represented as the sum $B = S + K$ of a symmetric and a skew-symmetric form. Classics investigated $B = S + K$ by considering it as a member of the *pencil* $B(\lambda, \mu) = \lambda S + \mu K$, where $\lambda, \mu \in \mathbb{K}$, and studying invariants of $B(\lambda, \mu)$, cf. [Ga].

Now let the form $B = B(x)$ depend on a parameter x running over a (super)manifold. Locally, there are obstructions to reducing non-degenerate 2-form $B(x)$ on a (super)manifold to the canonical expression. These obstructions are the Riemann tensor if $B(x)$ is symmetric (metric) and dB if $B(x)$ is a skew (differential) 2-form over $\mathbb{C}$ or $\mathbb{R}$; for these obstructions expressed in cohomological terms, see [LPS]. Analogous obstructions to local reducibility of bilinear forms to canonical expressions found here will be classified elsewhere.

Over $\mathbb{C}$, there is only one class of symmetric forms and only one class of skew-symmetric forms ([Pra]). For a canonical form of the matrix of the form B, one usually takes J_{2n} for the skew forms and 1_n, Π_n or $S(n)$ for symmetric forms, see [FH].

In order to have Cartan subalgebra of the orthogonal Lie algebra on the main diagonal (to have a split form of $\mathfrak{o}_B(n)$), one should take B of the shape Π_n or $S(n)$, not 1_n. Over $\mathbb{R}$, the Lie algebra might have no split form; for purposes of representation theory, it is convenient therefore to take its form most close to the split one.

Over $\mathbb{R}$, as well as over any ordered field, Sylvester's theorem states that the signature of the form is the only invariant ([Pra]).

Over an algebraically closed field $\mathbb{K}$ of characteristic $p \neq 2$, Ermolaev considered nondegenerate bilinear forms $B : V \times V \longrightarrow \mathbb{K}$, and gave the following description of the Lie algebras $L(B)$ (for details, see [Er]):

1.4 Statement ([Er])**.** *The Lie algebra $L(B)$ can not be represented as a direct sum of ideals (of type $L(B)$) if and only if all elementary divisors of the matrix B belong to the same point of the variety*

$$P = (\mathbb{K}^*/\{1, -1\}) \cup \{0\} \cup \{\infty\},$$

where $\mathbb{K}^*$ is the multiplicative group of $\mathbb{K}$. To each of the three different types of points in P (elementary divisors corresponding to 0, to ∞ or to a point of $\mathbb{K}^*/\{1, -1\}$), a series of Lie algebras $L(B)$ corresponds, and each of these algebras depends on a finite system of integer parameters.

1.5 Known facts: The case $p = 2$. 1) With any symmetric bilinear form B a quadratic form $Q(x) := B(x, x)$ is associated. The other way round, given a quadratic form Q, we define a symmetric bilinear form, called *the polar form* of Q, by setting

$$B_Q(x, y) = Q(x + y) - Q(x) - Q(y).$$

As we will see, the correspondence $Q \longleftrightarrow B_Q$ is not one-to-one and does not embrace non-symmetric forms.

Arf [Arf] has discovered *the Arf invariant* — an important invariant of non-degenerate quadratic forms in characteristic 2; for an exposition, see [D]. Two such forms are equivalent if and only if their Arf invariants are equal.

The Arf invariant, however, can not be used for classification of symmetric bilinear forms because one symmetric bilinear form can serve as the polar form for two non-equivalent (and having different Arf invariants) quadratic forms. Moreover, not every symmetric bilinear form can be represented as a polar form.

2) Albert [A] classified symmetric bilinear forms over a field of characteristic 2 and proved that

(1) two alternate symmetric forms (he calls a form B on V *alternate* if $B(x, x) = 0$ for every $x \in V$; here we call such forms *fully isotropic*) of equal ranks are equivalent;
(2) every non-alternate symmetric form has a matrix which is equivalent to a diagonal matrix;
(3) if $\mathbb{K}$ is such that every element of $\mathbb{K}$ has a square root, then every two non-alternate symmetric forms of equal ranks are equivalent.

3) Albert also gave some results on the classification of quadratic forms over a field $\mathbb{K}$ of characteristic 2 (considered as elements of the quotient space of all bilinear forms by the space of symmetric alternate forms). In particular, he showed that if $\mathbb{K}$ is algebraically closed, then every quadratic form is equivalent to exactly one of the forms

$$x_1x_{r+1} + \dots + x_rx_{2r} \quad \text{or} \quad x_1x_{r+1} + \dots + x_rx_{2r} + x_{2r+1}^2, \tag{1.4}$$

where $2r$ is the rank of the form.

4) Albert also considered *semi-definite* bilinear forms, i.e., symmetric forms, which are equivalent to forms whose matrix is of the shape

$$\begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}.$$

For $p = 2$, semi-definite forms constitute a linear space. In order not to have every non-alternate symmetric form semi-definite, one should take ground field $\mathbb{K}$ such that not every element of $\mathbb{K}$ has a square root. For this, $\mathbb{K}$ must be neither algebraically closed nor finite.

5) Skryabin [Sk] considered the case of the space V with a flag

$$\mathcal{F} : 0 = V_0 \subset V_1 \subset \dots \subset V_q = V,$$

and the equivalence of bilinear forms which, in addition to (1.3), *preserves* $\mathcal{F}$. He showed that under such equivalence the class of a (possibly, degenerate) skew-symmetric bilinear form is determined by parameters

$$n_{qr} = \dim(V_q \cap V_{r-1}^{\perp})/(V_q \cap V_r^{\perp} + V_{q-1} \cap V_{r-1}^{\perp})$$

for $q, r \geq 1$, where orthogonality is taken with respect to the form. This is true for any characteristic, but if $p = 2$, the skew forms do not differ from zero-diagonal symmetric ones.

1.6 The structure of the paper. In §2 we reproduce Albert's results on classification of symmetric bilinear forms with respect to the classical equivalence (1.3).

In §3 we consider other approaches to the classification of non-symmetric bilinear forms, select the most interesting one ("sociological") and describe the corresponding equivalence classes.

In §4 we classify bilinear forms on superspaces with respect to the classical and sociological equivalences.

In §5 we describe some relations between equivalences of bilinear forms and 1-forms.

In §6 we explicitly describe canonical forms of symmetric bilinear forms, related simple Lie algebras, and their Cartan subalgebras.

In §7 and §8, we give a super versions of §6 and §3, respectively.

1.7 Remarks. 1) In the study of simple Lie algebras over a field of characteristic $p > 0$, one usually takes an algebraically closed or sometimes finite ground field. Accordingly, these are the cases where bilinear or quadratic forms are to be considered first.

2) For quadratic forms in characteristic 2, we can also use Bourbaki's definition: q is quadratic if $q(ax) = a^2q(x)$, and $B(x, y) = q(x + y) - q(x) - q(y)$ is a bilinear form.

3) For some computations, connected with Lie algebras of linear transformations, preserving a given bilinear form (e.g., computations of Cartan prolongs), it is convenient to choose the form so that the corresponding Lie algebra has a Cartan subalgebra as close to the algebra of diagonal matrices as possible. It is shown in §6 that in the case of bilinear forms, equivalent to 1_n, over a space of even dimension, we need to take the form in one of the shapes:

$$\begin{pmatrix} 1_2 & 0 \\ 0 & S(n-2) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 0 & 1_{k-1} \\ 0 & 1_{k-1} & 0 \end{pmatrix}$$

The corresponding Cartan subalgebras consist of matrices of the following shape:

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \dots, a_{k-1}, a_{k-1}, \dots, a_1) \end{array}\right)$$

or

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \dots, a_{k-1}, a_1, \dots, a_{k-1}) \end{array}\right).$$

§2 Symmetric bilinear forms (Linear Algebra)

2.1 Theorem. *Let $\mathbb{K}$ be a field of characteristic 2 such that every element of $\mathbb{K}$ has a square root*[1]*. Let V be a n-dimensional space over $\mathbb{K}$.*

1) For n odd, there is only one equivalence class of non-degenerate symmetric bilinear forms on V.

2) For n even, there are two equivalence classes of non-degenerate symmetric bilinear forms, one contains 1_n and the other one contains $S(n)$.

Later we show that, if n is even, a non-degenerate bilinear form is equivalent to $S(n)$ if and only if its matrix is zero-diagonal.

Observe that the fact that the bilinear forms are not equivalent does not imply that the Lie (super)algebras that preserve them are not isomorphic; therefore the next Lemma is non-trivial.

Lemma. *1) The Lie algebras $\mathfrak{o}_I(2k)$ and $\mathfrak{o}_S(2k)$ are not isomorphic; the same applies to their derived algebras:*

2) $\mathfrak{o}_I^{(1)}(2k) \not\simeq \mathfrak{o}_S^{(1)}(2k)$;

3) $\mathfrak{o}_I^{(2)}(2k) \not\simeq \mathfrak{o}_S^{(2)}(2k)$.

[1]Since $a^2 - b^2 = (a - b)^2$ if $p = 2$, it follows that no element can have two distinct square roots.

2.2 Proof of Theorem 2.1. In what follows let E^{ij}, where $1 \leq i, j \leq n$, be a matrix unit, i.e., $(E^{i,j})_{kl} := \delta_{ik}\delta_{jl}$, and

$$T^{i,j} := I + E^{i,i} + E^{j,j} + E^{i,j} + E^{j,i}.$$

Note that the $T^{i,j}$ are invertible, and $T^{i,j} = T^{j,i} = (T^{i,j})^T$.

Note also, that any bilinear form B is equivalent to aB for any $a \in \mathbb{K}$ if $a \neq 0$. Indeed, since every element of $\mathbb{K}$ has a square root, $aB = (b1_n)B(b1_n)$, where $b \in \mathbb{K}$ is such that $b^2 = a$.

Now, let us first prove the following

2.3 Lemma. *Let B be a symmetric $n \times n$ matrix, and $\overline{B}$ be $n' \times n'$ matrix in the upper left corner of B, $n' < n$. Then, if $\overline{B}$ is invertible, B is equivalent to a matrix of the form*

$$\begin{pmatrix} \overline{B} & 0 \\ 0 & \widehat{B} \end{pmatrix}.$$

Proof. Let C be $(n - n') \times n'$ matrix in the lower left corner of B, and

$$M = \begin{pmatrix} 1_{n'} & 0 \\ C\overline{B}^{-1} & 1_{n-n'} \end{pmatrix}.$$

The matrix M is invertible because it is lower-triangular and has no zeros on the diagonal. Direct calculations show that the matrix MBM^T has the needed form. □

2.4 Lemma. *If B and C are $n \times n$ matrices, and*

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}; \qquad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

where B_1 and C_1 are equivalent $n' \times n'$ matrices, and B_2 and C_2 are equivalent $(n-n') \times (n-n')$ matrices, then B and C are equivalent.

Proof. If $M_1B_1M_1^T = C_1$ and $M_2B_2M_2^T = C_2$, and $M = \mathrm{diag}(M_1, M_2)$, then $MBM^T = C$. □

Let us fix terminology. A bilinear form B is said to be *fully isotropic* (Albert called them *alternate*) if the corresponding quadratic form $Q(x) = B(x, x)$ is identically equal to 0.

2.5 Lemma. *A bilinear form is fully isotropic if and only if its matrix is zero-diagonal.*

Proof. Let $e_1, \ldots, e_n$ be the basis in which matrix is taken. Then, if B is fully isotropic, $B_{ii} = B(e_i, e_i) = 0$. On the other hand, if the matrix of B is zero-diagonal, and $e = \sum_i c_ie_i$, then

$$B(e, e) = \sum_{i,j} B_{ij}c_ic_j = 2\sum_{i<j} B_{ij}c_ic_j = 0. \quad \square$$

Since a fully isotropic form can be equivalent only to a fully isotropic form, we have

2.6 Corollary. *If matrices A and B are symmetric and equivalent, and A is zero-diagonal, then B is zero-diagonal.*

Now, let us prove the following part of Theorem 2.1:

If $n = 2k$, any non-degenerate symmetric zero-diagonal matrix $B \in GL(n)$ is equivalent to the matrix $Z(2k)$.

We will prove a more general statement that will be needed later:

2.7 Lemma. *If B is a zero-diagonal $n \times n$ matrix (possibly, degenerate), then $r = \operatorname{rank} B$ is even, and B is equivalent to the matrix*

$$\tilde{Z}(n,r) = \begin{pmatrix} Z(r) & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. We will induct on n. In the cases $n = 1, 2$, the statement is evident.

If $B = 0$, the statement follows immediately. Otherwise, there exist i, j such that $B_{ij} \neq 0$, and B is equivalent to

$$C = (B_{i,j})^{-1} T^{2,j} T^{1,i} B T^{1,i} T^{2,j},$$

and $C_{12} = C_{21} = 1$, $C_{11} = C_{22} = 0$.

Then, by Lemma 2.3, C is equivalent to a matrix D of the form

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & D_1 \end{array}\right),$$

where D_1 is a $(n-2) \times (n-2)$ matrix. Since D is, by Corollary 2.6, symmetric and zero-diagonal, D_1 is also symmetric and zero-diagonal, and $\operatorname{rank} D_1 = r - 2$. Then, by the induction hypothesis, $r - 2$ is even, and D_1 is equivalent to $\tilde{Z}(n-2, r-2)$, and, by Lemma 2.4, D is equivalent to $\tilde{Z}(n,r)$. Therefore, B is equivalent to $\tilde{Z}(n,r)$. ☐

Now let us prove the following:

For n odd, any non-degenerate symmetric $n \times n$ matrix is equivalent to 1_n;
for n even, any non-degenerate symmetric $n \times n$ matrix which is not zero-diagonal is also equivalent to 1_n.

We will prove this by induction on n (simultaneously for n odd and even). For $n = 1$, the statement is evident.

Now, let n be even. If B is an invertible symmetric $n \times n$ matrix, $B_{ii} \neq 0$, then B is equivalent to $C = (B_{ii})^{-1} T^{1,i} B T^{1,i}$, and $C_{11} = 1$. Then, by Lemma 2.3, C is equivalent to matrix D of the form

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & D_1 \end{array}\right), \quad \text{where } D_1 \in \mathfrak{gl}(n-1). \tag{2.1}$$

Since B is symmetric and non-degenerate, D_1 is also symmetric and non-degenerate. Then, by induction hypothesis, D_1 is equivalent to 1_{n-1}, and, by Lemma 2.4, D is equivalent to 1_n, and B is also equivalent to 1_n.

If B is an invertible symmetric $n \times n$ matrix, and n is odd, then by Lemma 2.7, B has at least one non-zero element on the diagonal, and, similarly, we can show that B is equivalent to a matrix D of the form (2.1). Since B is symmetric and non-degenerate, D_1 is also symmetric and non-degenerate. Then, by induction hypothesis, D_1 is equivalent to either 1_{n-1} or $Z(n-1)$, and, by Lemma 2.4, B is equivalent to either 1_n or

$$\widehat{Z}(n) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Z(n-1) \end{array}\right).$$

Let M be a $n \times n$ matrix such that

$$M_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ or } j = 1, \\ & \text{or if } j = i, \\ & \text{or if } j = i + 1,\ i \text{ is odd}, \\ & \text{or if } j > i + 1; \\ 0 & \text{if } j = i + 1,\ i \text{ is even}, \\ & \text{or if } 1 < j < i. \end{cases}$$

Direct calculation shows, that $MM^T = \widehat{Z}(n)$, so $\widehat{Z}(n)$ is equivalent to 1_n, and B is equivalent to 1_n.

Now, to finish the proof of the theorem, we need to show that, for n even, 1_n and $Z(n)$ are not equivalent, which follows from Corollary 2.6. Theorem 2.1 is proved. □

2.8 Proof of Lemma 2.1. Let $C(\mathfrak{g})$ be the center of the Lie algebra $\mathfrak{g}$. We see that $1_n \in C(\mathfrak{o}_I(n))$, and $\dim C(\mathfrak{o}_I(n)) = 1$, because if $A \in \mathfrak{o}_I(n)$, and $A_{ii} \neq A_{jj}$, then $[A, E^{i,j} + E^{j,i}]_{ij} = A_{ii} + A_{jj} \neq 0$, and if $A_{ij} \neq 0$ for $i \neq j$, then $[A, E^{i,i}]_{ij} = A_{ij} \neq 0$. Since matrices from $\mathfrak{o}_I(n)^{(1)}$ are zero-diagonal ones, $\dim(C(\mathfrak{o}_I(n)) \cap \mathfrak{o}_I(n)^{(1)}) = 0$. At the same time,

$$1_n, \quad \sum_{i=1}^{k} E^{2i-1,2i}, \quad \sum_{i=1}^{k} E^{2i,2i-1} \in \mathfrak{o}_S(n); \qquad \text{and } 1_n \in C(\mathfrak{o}_S(n)),$$

but $[\sum_{i=1}^{k} E^{2i-1,2i}, \quad \sum_{i=1}^{k} E^{2i,2i-1}] = 1_n$, so $\dim(C(\mathfrak{o}_S(n)) \cap \mathfrak{o}_S(n)^{(1)}) \neq 0$, which shows that $\mathfrak{o}_I(n)$ and $\mathfrak{o}_S(n)$ are not isomorphic. Lemma is proved. □

§3 Non-symmetric bilinear forms (Linear algebra)

3.1 Non-symmetric bilinear forms: Discussion. If $p = 2$, there is no canonical way to separate symmetric part of a given bilinear form from its non-symmetric part, so in this subsection B is just a non-symmetric form. In this subsection I list several more or less traditional equivalences before suggesting (in the next subsection) the one that looks the best.

1) The standard definition (1.3). This equivalence is too delicate: there are too many inequivalent forms: the classification problem looks wild.

2) The idea of classics (see, e.g., [Ga]) was to consider the following equivalence of non-degenerate bilinear forms regardless of their symmetry properties. Observe that any bilinear form B on V can be considered as an operator

$$\widetilde{B} : V \longrightarrow V^* \qquad x \longmapsto B(x, \cdot). \tag{3.1}$$

If B is non-degenerate, then $\widetilde{B}$ is invertible. Two forms B and C are said to be *roughly equivalent*, if the operators $\widetilde{B}^{-1}\widetilde{B}^*$ and $\widetilde{C}^{-1}\widetilde{C}^*$ in V are equivalent; here * denotes the passage to the adjoint operator. This equivalence is, however, too rough: in the case of symmetric forms it does not differ fully isotropic and not fully isotropic forms, so *all* symmetric non-degenerate bilinear forms are roughly equivalent, for both odd- and even-dimensional V.

3) Leites suggested to call two bilinear forms B_1 and B_2 *Lie-equivalent* (we write $B_1 \simeq_L B_2$) if the Lie algebras that preserve them are isomorphic. This does reduce the number

of non-equivalent forms but only slightly as compared with (1.3) and no general pattern is visible, see the following Examples for $n = 2, 3, 4$. So this equivalence is also, as (1.3), too delicate.

Examples. $\underline{n = 2,\ \mathbb{K} = \mathbb{Z}/2.}$ In this case, there exist only two non-symmetric non-degenerate matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and they are equivalent.

$\underline{n = 2,\ \mathbb{K}\text{ infinite.}}$ In this case, there exist *infinitely many* equivalence classes of non-symmetric non-degenerate forms. For example, $\mathrm{antidiag}_2(1, a) \sim \mathrm{antidiag}_2(1, b)$ only if either $a = b$ or $ab = 1$. But all these classes are Lie-equivalent: any non-symmetric non-degenerate 2×2 matrix is only preserved by scalar matrices.

$\underline{n = 3,\ \mathbb{K} = \mathbb{Z}/2.}$ In this case, there exist 3 equivalence classes with the following representatives:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that the last two matrices are equivalent as forms over an extension of $\mathbb{Z}/2$ with 4 elements. All these matrices are Lie-equivalent — the corresponding Lie algebras are 2-dimensional and, since they contain 1_3, commutative.

$\underline{n = 3,\ \mathbb{K}\text{ infinite.}}$ In this case, again, there exist infinitely many equivalence classes.

Conjecture. *All symmetric non-degenerate* 3×3 *matrices are Lie-equivalent and the corresponding Lie algebras are* 2*-dimensional and commutative.*

$\underline{n = 4,\ \mathbb{K} = \mathbb{Z}/2.}$ In this case, there exist 8 equivalence classes with the following representatives:

$$B_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad B_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad B_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$B_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}; \quad B_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}; \quad B_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}; \quad B_8 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

The matrices in the pairs (B_1, B_4), (B_3, B_7), (B_5, B_6) are Lie-equivalent, so there are 5 Lie-equivalence classes.

$\underline{n = 4,\ \mathbb{K}\text{ infinite.}}$ Again, there exist infinitely many equivalence classes. There exist also at least 5 Lie-equivalence classes, described in the previous case.

3.2 A sociological approach to bilinear forms. Instead of considering non-symmetric forms individually, we can consider the quotient space $NB(n)$ of the space of all forms modulo the space of symmetric forms. We will denote the element of this quotient space with representative B, by $\{B\}$. We say that $\{B\}$ and $\{C\}$ are *equivalent*, if there exists an invertible matrix M such that

$$\{MBM^T\} = \{C\}, \text{ i.e., if } MBM^T - C \text{ is symmetric}$$

(this definition does not depend on the choice of representatives B and C).

Any $\{B\}$ has both degenerate and non-degenerate representatives: the representative with non-zero elements only above the diagonal (such representative is unique and characterizes $\{B\}$) is degenerate, and if we add the unit matrix to it, we get a non-degenerate representative of $\{B\}$.

Note that $\{B\}$ can be also characterized by the symmetric zero-diagonal matrix $B+B^T$. The rank of $B+B^T$ is said to be *the rank of* $\{B\}$. According to Lemma 2.7, it is always even. One can show that it is equal to doubled minimal rank of representatives of B.

3.3 Theorem. *The forms $\{B\}$ and $\{C\}$ are equivalent if and only of they have equal ranks.*

Proof. Let $\operatorname{rank}\{B\} = \operatorname{rank}\{C\}$, i.e., $\operatorname{rank}\,(B+B^T) = \operatorname{rank}\,(C+C^T)$. Since $B+B^T$ and $C+C^T$ are zero-diagonal, they are, according to Lemma 2.7, equivalent, i.e., there exists non-degenerate matrix M such that $M(B+B^T)M^T = C+C^T$. Then $MBM^T+C = (MBM^T+C)^T$ is symmetric, and $\{B\}$ is equivalent to $\{C\}$. These arguments are reversible. $\square$

So, we see that $NB(n)$ has $[n/2]+1$ equivalence classes consisting of elements with ranks $0, 2, \dots, 2[n/2]$. As the representatives of these classes we can take $\{\tilde{S}^{n,m}\}$, where $m = 0, \dots, [n/2]$ and where $\tilde{S}^{n,m}$ is $n\times n$ matrix such that

$$\tilde{S}^{n,m} = \begin{array}{c} \\ m \\ n-m \end{array} \begin{array}{c} \begin{array}{cc} n-m & m \end{array} \\ \begin{pmatrix} 0 & S(m) \\ 0 & 0 \end{pmatrix} \end{array}$$

The following definition of the linear transformations preserving an element of $NB(n)$ seems to be the most natural:

$$X \text{ preserves } \{B\} \text{ if } XB+BX^T \text{ is symmetric.} \tag{3.2}$$

Since

$$XB+BX^T+(XB+BX^T)^T = X(B+B^T)+(B+B^T)X^T,$$

X preserves $\{B\}$ if and only if X preserves $B+B^T$. Hence the transformations preserving $\{B\}$ do form a Lie algebra. One can check that they also form a Lie algebra if $p\neq 2$, and this algebra is the Lie algebra of transformations preserving the skew-symmetric representative of $\{B\}$.

The Lie algebra $\mathfrak{o}_{\tilde{S}^{n,m}}$ is spanned by the matrices

$$\left(\begin{array}{c|c|c} A & D & B \\ \hline 0 & E & 0 \\ \hline C & F & S(k)AS(k) \end{array}\right), \quad \begin{array}{c} \text{where } A\in\mathfrak{gl}(k);\ B, C\in\mathfrak{gl}(k) \text{ are} \\ \text{such that } B = S(k)BS(k) \text{ and } C = S(k)CS(k); \\ D \text{ and } F \text{ are } (n-2k)\times k \text{ matrices; } E\in\mathfrak{gl}(n-2k). \end{array}$$

This Lie algebra is isomorphic to the semi-direct sum (the ideal on the right)

$$(\mathfrak{o}_S(2m)\oplus\mathfrak{gl}(n-2m)) \niplus (R_{\mathfrak{o}}\otimes R^*_{\mathfrak{gl}}),$$

where $R_{\mathfrak{o}}$ and $R_{\mathfrak{gl}}$ are the spaces of the identity representations of $\mathfrak{o}_S(2m)$ and $\mathfrak{gl}(n-2m)$, respectively.

If $\operatorname{rank}\{B\} < n$, then the Lie algebra of linear transformations preserving $\{B\}$ is isomorphic to the Lie algebra of linear transformations preserving symmetric *degenerate* matrix $B+B^T$. So, it seems natural to call $\{B\}$ *non-degenerate* if and only if $\operatorname{rank}\{B\} = n$.

By Theorem 3.3, all non-degenerate elements of $NB(n)$ (they only exist if n is even) are equivalent. The Lie algebra of linear transformations preserving any of these forms is isomorphic to $\mathfrak{o}_S(2k)$.

§4 Bilinear forms on superspaces (Linear algebra)

4.1 Canonical expressions of symmetric bilinear forms on superspaces and the Lie superalgebras that preserve them.

For general background related to Linear Algebra in superspaces and proofs of the statements of this subsection, see [LS].

Speaking of superspaces we denote parity by Π, and superdimension by sdim. The operators and bilinear forms are represented by *supermatrices* which we will only consider here in the *standard format*. Recall only that to any bilinear form B on a given space V one can assign its *Gram matrix* also denoted $B = (B_{ij})$: in a fixed basis $x_1, \dots, x_n$ of V, we set (compare with (1.2); for $p = 2$, the sign disappears)

$$B_{ij} = (-1)^{\Pi(B)\Pi(x_i)} B(x_i, x_j). \tag{4.1}$$

In what follows, we fix a basis of V and identify a bilinear form with its matrix. Two bilinear forms B and C on V are said to be *equivalent* if there exists an invertible *even* linear operator $A \in GL(V)$ such that $B(x, y) = C(Ax, Ay)$ for all $x, y \in V$; in this case, $B = ACA^T$ for the matrices of B, C and A in the same basis.

Generally, the symmetry of the bilinear forms involves signs which leads to the notion of supertransposition of the corresponding Gram supermatrices, but for $p = 2$ the supertransposition turns into transposition.

Recall also that, over superspaces, the parity change sends symmetric forms to skew-symmetric and the other way round, so if $p \neq 2$, it suffices to consider only symmetric forms.

In super setting, over $\mathbb{C}$, there is only one class of non-degenerate even symmetric (ortho-symplectic) forms and only one class of non-degenerate odd symmetric (periplectic) forms in each superdimension. Over $\mathbb{R}$, the invariants of even symmetric forms are pairs of invariants of the restriction of the form onto the even and odd subspaces and only one class of odd symmetric (periplectic) forms.

In order to have Cartan subalgebra of the ortho-symplectic Lie superalgebra on the main diagonal (to have a split form of $\mathfrak{osp}(n|2m)$), one should take for the canonical form of B the expression

$$\begin{pmatrix} S(n) & 0 \\ 0 & J_{2m} \end{pmatrix}.$$

If $p = 2$, a given symmetric even form on a superspace can be represented as a direct sum of two forms on the even subspace and the odd subspace. For each of these forms, Theorem 2.1 is applicable. Lemma 2.4 is also applicable in this case, so every even symmetric non-degenerate form on a superspace of dimension $(n_{\bar{0}}|n_{\bar{1}})$ over a field of characteristic 2 is equivalent to a form of the shape (here: $i = \bar{0}$ or $\bar{1}$)

$$B = \begin{pmatrix} B_{\bar{0}} & 0 \\ 0 & B_{\bar{1}} \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 1_{n_i} & \text{if } n_i \text{ is odd;} \\ 1_{n_i} \text{ or } Z(n_i) & \text{if } n_i \text{ is even.} \end{cases}$$

Before we pass to Lie superalgebras preserving a bilinear form B, let us define the Lie superalgebras for $p = 2$. We do not know any general definition (the definition of squaring given below is hardly meaningful if the ground field $\mathbb{K}$ is finite), but for an algebraically closed $\mathbb{K}$ we are working with it is OK.

A *Lie superalgebra* for $p = 2$ is a superspace $\mathfrak{g}$ such that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is an $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by symmetry) and on $\mathfrak{g}_{\bar{1}}$ a *squaring* (roughly speaking, the halved bracket) is defined

$$x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar 1} \text{ and } a \in \mathbb{K} \tag{4.2}$$

Then the bracket of odd elements is defined as following:

$$[x, y] := (x + y)^2 - x^2 - y^2$$

The Jacobi identity for three odd elements is replaced by the following relation:

$$[x, x^2] = 0 \qquad \text{for any } x \in \mathfrak{g}_{\bar 1}.$$

The Lie superalgebra preserving B — by analogy with the orthosymplectic Lie superalgebras $\mathfrak{osp}$ in characteristic 0 we call it *ortho-orthogonal* and denote $\mathfrak{oo}(n_{\bar 0}|n_{\bar 1})$ — is spanned by the supermatrices which in the standard format are of the form

$$\begin{pmatrix} A_{\bar 0} & B_{\bar 0}C^TB_{\bar 1}^{-1} \\ C & A_{\bar 1} \end{pmatrix}, \quad \begin{array}{l} \text{where } A_{\bar 0} \in \mathfrak{o}_{B_{\bar 0}},\ A_{\bar 1} \in \mathfrak{o}_{B_{\bar 1}}, \text{ and} \\ C \text{ is arbitrary } n_{\bar 1} \times n_{\bar 0} \text{ matrix.} \end{array}$$

For an odd symmetric form B on a superspace of dimension $(n_{\bar 0}|n_{\bar 1})$ over a field of characteristic 2 to be non-degenerate, we need $n_{\bar 0} = n_{\bar 1} = k$, so the matrix of B is of the shape

$$\begin{pmatrix} 0 & \overline{B} \\ \overline{B}^T & 0 \end{pmatrix},$$

where $\overline{B}$ is a square invertible matrix. Let us take

$$M = \begin{pmatrix} 1_k & 0 \\ 0 & \overline{B}^{-1} \end{pmatrix},$$

then B is equivalent to

$$MBM^T = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix}.$$

This form is preserved by linear transformations with supermatrices in the standard format of the shape

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \quad \text{where } A \in \mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric } k \times k \text{ matrices.} \tag{4.3}$$

As over $\mathbb{C}$ or $\mathbb{R}$, the Lie superalgebra $\mathfrak{pe}(n)$ of supermatrices (4.3) (recall that $p = 2$) will be referred to as *periplectic*.

4.2 Non-symmetric forms on superspaces. If a non-symmetric form on a superspace is even, it can be again represented as a direct sum of two bilinear forms: one on the even subspace, and the other one on the odd subspace. These two forms can be independently transformed to canonical forms, see §3.

The situation with odd non-symmetric forms is more interesting. Such a form can be non-degenerate only on a space of superdimension $(k|k)$. In the standard format, the supermatrix of such a form has the shape

$$B = \begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix},$$

where A and C are invertible matrices. Let M be an invertible matrix such that $L = MC(A^T)^{-1}M^{-1}$ is the Jordan normal form of $C(A^T)^{-1}$. Then B is equivalent to

$$\begin{pmatrix} (M^T)^{-1}A^{-1} & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix} \begin{pmatrix} ((M^T)^{-1}A^{-1})^T & 0 \\ 0 & M^T \end{pmatrix} = \begin{pmatrix} 0 & 1_k \\ L & 0 \end{pmatrix}.$$

This expression (with L in the Jordan normal form) can be considered as a canonical form of a non-degenerate odd bilinear form.

Statement. *Two non-degenerate forms are equivalent if and only if they have equal canonical forms.*

§5 Relation with 1-forms (Differential geometry)

5.1 Notations. For $p > 0$, there are two types of analogs of polynomial algebra: the infinite dimensional ones and finite dimensional ones. The *divided power algebra* in indeterminates $x_1, \dots, x_m$ is the algebra of polynomials in these indeterminates, so, as space, it is

$$\mathcal{O}(m) = Span\{x_1^{(r_1)} \dots x_m^{(r_m)} \mid r_1, \dots, r_m \geq 0\}$$

with the following multiplication:

$$(x_1^{(r_1)} \dots x_m^{(r_m)}) \cdot (x_1^{(s_1)} \dots x_m^{(s_m)}) = \prod_{i=1}^{m} \begin{pmatrix} r_i + s_i \\ r_i \end{pmatrix} x_1^{(r_1+s_1)} \dots x_m^{(r_m+s_m)}.$$

For a *shearing parameter* $\underline{N} = (n_1, \dots, n_m)$, set

$$\mathcal{O}(m, \underline{N}) = Span\{x_1^{(r_1)} \dots x_m^{(r_m)} \mid 0 \leq r_i < p^{n_i}, i = 1, \dots, m\},$$

where $p^\infty = \infty$. If $n_i < \infty$ for all i, then $\dim \mathcal{O}(m, \underline{N}) < \infty$.

5.2 Matrices and 1-forms. Let B and B' be the matrices of bilinear forms on an n-dimensional space V over a field $\mathbb{K}$ of characteristic 2. Let $x_0, x_1, \dots, x_n$ be indeterminates as in sec. 5.1; set

$$\deg x_0 = 2, \deg x_1 = \dots = \deg x_n = 1.$$

We say that B and B' are *1-form-equivalent* if there exists a degree preserving transformation, i.e., a set of independent variables $x'_0, x'_1, \dots, x'_n$ such that

$$\deg x'_0 = 2, \quad \deg x'_1 = \dots = \deg x'_n = 1, \tag{5.1}$$

which are polynomials in $x_0, x_1, \dots, x_n$ in divided powers with shearing parameter

$$\underline{N} = (N_0, \dots, N_n) \text{ such that } N_i > 1 \text{ for every } i \text{ from } 1 \text{ to } n, \tag{5.2}$$

and such that

$$dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j = dx'_0 + \sum_{i,j=1}^{n} B'_{ij} x'_i dx'_j. \tag{5.3}$$

5.3 Lemma. *B and B' are 1-form-equivalent if and only if $\{B\}$ and $\{B'\}$ are equivalent.*

Proof. By (5.1), we have

$$x_0' = cx_0 + \sum_{i=1}^{n} A_{ii} x_i^{(2)} + \sum_{1\leq i<j\leq n} A_{ij} x_i x_j; \qquad x_i' = \sum_{j=1}^{n} M_{ij} x_j, \tag{5.4}$$

where $c \neq 0$ and M is an invertible matrix. Thanks to (5.3), comparing coefficients of dx_0 in the left- and right-hand sides, we get $c = 1$. Let A be a symmetric $n \times n$ matrix with elements A_{ij} for $i \leq j$ as in (5.4). Then

$$dx_0' + \sum_{i,j=1}^{n} B'_{ij} x_i' dx_j' = dx_0 + \sum_{i,j=1}^{n} A_{ij} x_i dx_j + \sum_{i,j,k,l=1}^{n} M_{ki} B'_{kl} M_{lj} x_i dx_j,$$

i.e., $B = M^T B' M + A$, so $\{B\}$ and $\{B'\}$ are equivalent. Since our arguments are invertible, the theorem is proved. $\square$

5.4 The case of odd indeterminates. Let us modify the definition of 1-form-equivalence for the super case where $x_1, \dots, x_n$ are all *odd*. In this case, we can only use divided powers with $\underline{N} = (N_0, 1, \dots, 1)$.

We say that B and B' are *1-superform-equivalent* if there exists a set of indeterminates $x_0', x_1', \dots, x_n'$, which are polynomials in $x_0, x_1, \dots, x_n$, such that

$$\Pi(x_0') = \bar{0},\ \Pi(x_1') = \dots = \Pi(x_n') = \bar{1}, \qquad \deg x_0' = 2,\ \deg x_1' = \dots = \deg x_n' = 1 \tag{5.5}$$

and

$$dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j = dx_0' + \sum_{i,j=1}^{n} B'_{ij} x_i' dx_j'. \tag{5.6}$$

Now, recall that the 1-form α on a superdomain M is said to be *contact* if it singles out a nonintegrable distribution in the tangent bundle TM and $d\alpha$ is non-degenerate on the fibers of this distribution; for details, see [GL] and [LPS].

5.5 Lemma. *The matrices B and B' are 1-superform-equivalent if and only if exist an invertible matrix M and a symmetric zero-diagonal matrix A such that*

$$B = MB'M^T + A. \tag{5.7}$$

Proof. It is analogous to the proof of Lemma 5.3. $\square$

Albert [A] considered the equivalence (5.7) as an equivalence of (matrices of) quadratic forms. In particular, he proved the following

5.6 Statement. *If $\mathbb{K}$ is algebraically closed, every matrix B is equivalent in the sense* (5.7) *to exactly one of the matrices*

$$Y(n,r) = \begin{pmatrix} 0_r & 1_r & 0 \\ 0_r & 0_r & 0 \\ 0 & 0 & 0_{n-2r} \end{pmatrix} \qquad or \qquad \widetilde{Y}(n,r) = \begin{pmatrix} 0_r & 1_r & 0 & 0 \\ 0_r & 0_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0_{n-2r-1} \end{pmatrix},$$

where $2r = rank(B + B^T)$. The corresponding quadratic form is non-degenerate if and only if either $n = 2r$, or $n = 2r + 1$ and the matrix is equivalent to $\widetilde{Y}(n,r)$.

If, in 1-form-equivalence, we consider divided powers with shearing parameter $\underline{N} = (N_0, 1, \dots, 1)$, it is the same as to consider 1-superform-equivalence.

5.7 Lemma. *Let $x_0, \dots, x_n$ be indeterminates, $\Pi(x_0) = \bar{0}$, $\Pi(x_1) = \dots = \Pi(x_n) = \Pi$. Then the 1-form on the $(n+1)$-dimensional superspace*

$$\alpha = dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j \tag{5.8}$$

is contact if and only if one of the following conditions holds:

1) $\Pi = \bar{0}$, and $\{B\}$ is non-degenerate, i.e., $n = rank(B + B^T)$ (this rank is always even);

2) $\Pi = \bar{1}$, and the quadratic form corresponding to B is non-degenerate, i.e., either $n = rank(B + B^T)$ or B is not zero-diagonal and $n = rank(B + B^T) + 1$.

Proof. From Theorem 3.3, Statement 5.6 ([A]), and Lemmas 5.3 and 5.5 we know that, if $\Pi = \bar{0}$, every symmetric bilinear form is 1-form-equivalent to one of the forms $\widetilde{S}(n, r)$, $n \geq 2r$, and, if $\Pi = \bar{1}$, every symmetric bilinear form is 1-superform-equivalent to one of the forms $Y(n, r)$, where $n \geq 2r$, or $\widetilde{Y}(n, r)$, where $n \geq 2r + 1$. Direct calculations show that if $\Pi = \bar{0}$, the 1-form (5.8) corresponding to $\widetilde{S}(n, r)$ is contact if and only if $n = 2r$; if $\Pi = \bar{1}$, then the 1-form (5.8) corresponding to $Y(n, r)$ is contact if and only if $n = 2r$ and the 1-form, corresponding to $\widetilde{Y}(n, r)$ is contact if and only if $n = 2r+1$. Since, by definition, two 1-forms that correspond to 1-(super)form-equivalent bilinear forms can be transformed into each other by a change of coordinates, we are done. $\square$

From this, we get the following:

Theorem. *The following are the canonical expressions of the odd contact forms:*

$$\alpha = dx_0 + \sum_{i=1}^{k} x_i dx_{k+i} \begin{cases} & \text{for } n = 2k \text{ and } x_1, \dots, x_n \text{ all even or all odd;} \\ +x_n dx_n & \text{for } n = 2k+1 \text{ and } x_1, \dots x_n \text{ odd.} \end{cases} \tag{5.9}$$

Remarks. 1) If $n > 1$ and $x_1, \dots x_n$ are odd, the 1-form $\alpha = dx_0 + \sum\limits_{i=1}^{n} x_i dx_i$ is not contact since (recall that $p = 2$)

$$\alpha = d\left(x_0 + \sum_{i<j} x_i x_j\right) + \left(\sum_{i=1}^{n} x_i\right) d\left(\sum_{i=1}^{n} x_i\right).$$

2) Let $p = 2$. Since there are two types of orthogonal Lie algebras if n is even, and orthogonal algebras coincide, in a sense, with symplectic ones, it seems natural to expect that there are also two types of the Lie algebras of hamiltonian vector fields (preserving I and S, respectively). Iyer investigated this question; for the answer, see [Iy].

Are there two types of contact Lie algebras corresponding to these cases? The (somewhat unexpected) answer is NO:

The classes of 1-(super)form-equivalence of bilinear forms which correspond to contact forms have nothing to do with classes of classical equivalence of symmetric bilinear forms. The 1-forms, corresponding to symmetric bilinear forms are exact if $x_1, \dots x_n$ are even, and are of rank ≤ 2 if $x_1, \dots x_n$ are odd.

Recall the the contact Lie superalgebra consists of the vector fields D that preserve the contact structure (nonintegrable distribution given by a contact form α hereafter in the form (5.9)). Such fields satisfy

$$L_D(\alpha) = F_D\alpha \ \text{ for some } F_D \in \mathbb{C}[t, p, q, \theta].$$

For any $f \in \mathbb{C}[t, p, q, \theta]$, we set (the signs here are important only for $p \neq 2$):

$$K_f = (1 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{5.10}$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator*, and H_f is the hamiltonian field with Hamiltonian f that preserves $d\alpha_1$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\left(\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}\right). \tag{5.11}$$

If one tries to build a contact algebra $\mathfrak{g}$ by means of a non-degenerate symmetric bilinear form B on the space V by setting (like it is done in characteristic 0) $\mathfrak{g}$ to be the generalized Cartan prolongation $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ (for the precise definition, see [Shch]), where the non-positive terms of $\mathfrak{g}$ are (here K_f is the contact vector field with the generating function f; for an exact formula, see, e.g., [GLS]):

$$\mathfrak{g}_i = \begin{cases} 0 & \text{if } i \leq -3;\\ \mathbb{K}\cdot K_1 & \text{if } i = -2\\ V & \text{if } i = -1\\ \mathfrak{o}_B(V)\oplus \mathbb{K}K_t & \text{if } i = 0\end{cases}$$

and where the multiplication is given by the formulas

$$\begin{aligned}
&[X, Y] = B(X, Y)K_1 \text{ for any } X, Y \in \mathfrak{g}_{-1};\\
&\mathfrak{o}_B(V) \text{ acts on } V \text{ via the standard action;}\\
&[\mathfrak{g}_0, \mathfrak{g}_{-2}] = 0;\\
&K_t \text{ acts as id on } \mathfrak{g}_{-1},\\
&[K_t, \mathfrak{o}_B(V)] = 0,
\end{aligned}$$

then the form B must be zero-diagonal one (because $0 = [X, X] = B(X, X)K_1$ for $X \in \mathfrak{g}_{-1}$).

One can also try to construct a Lie superalgebra in a similar way by setting $\Pi(\mathfrak{g}_{-1}) = \bar{1}$ and

$$X^2 = B(X, X)K_1 \text{ for any } X \in \mathfrak{g}_{-1}. \tag{5.12}$$

Let us realize this algebra by vector fields on a superspace of superdimension $(1|n)$ with basis $x_0, \dots, x_n$ such that

$$\Pi(x_0) = \bar{0}; \qquad \Pi(x_i) = \bar{1} \text{ for } 1 \leq i \leq n.$$

If $e_1, \dots, e_n$ is a basis of V and we set (here $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 0, \dots, n$):

$$K_1 = \partial_0; \qquad e_i = \partial_i + \sum_{j=1}^{n} A_{ij} x_j \partial_0 \text{ for } i = 1, \dots, n,$$

then, to satisfy relations (5.12), we need the following (here the Gram matrix B is taken in the basis $e_1, \dots, e_n$):

$$\begin{gathered} A_{ii} = B_{ii} \text{ for } 1 \leq i \leq n; \\ A_{ij} + A_{ji} = B_{ij} + B_{ji} \text{ for } 1 \leq i < j \leq n \end{gathered}$$

i.e., $A \in \{B\}$, where the equivalence class is taken with respect to zero-diagonal symmetric matrices.

These vector fields preserve the 1-form

$$\alpha = dx_0 + \sum_{i,j=1}^{n} A_{ij} x_i dx_j.$$

So, to get a contact Lie superalgebra in this way, one needs B to be non-symmetric with non-degenerate class $\{B\}$.

3) Lin [LinK] considered an n-parameter family of simple Lie algebras for $p = 2$ preserving in dimension $2n + 1$ the distribution given by the contact form

$$\alpha = dt + \sum_{i=1}^{n} \left((1 - a_i) p_i dq_i + a_i q_i dp_i \right), \text{ where } a_i \in \mathbb{K}.$$

Obviously, the linear change

$$t' = t + \sum a_i p_i q_i \quad \text{and identical on other indeterminates} \tag{5.13}$$

reduces α to the canonical form $dt + \sum\limits_{i=1}^{n} p_i dq_i$. So the parameters a_i can be eliminated. Although Lin mentioned the change (5.13) on p. 21 of [LinK], its consequence was not formulated and, seven years after, Brown [Br] reproduced Lin's misleading n-parameter description of $\mathfrak{k}(2n+1)$.

5.8 The case of indeterminates of different parities.

5.8.1 The case of an odd 1-form.

Let

$$\Pi(x_0) = \Pi(x_1) = \dots = \Pi(x_{n_{\bar 0}}) = \bar 0, \qquad \Pi(x_{n_{\bar 0}+1}) = \dots = \Pi(x_n) = \bar 1.$$

This corresponds to the following equivalence (we call it 1-superform-equivalence again) of *even* bilinear forms on a superspace V of superdimension $(n_{\bar 0}|n_{\bar 1})$, where $n_{\bar 1} = n - n_{\bar 0}$: two such forms B and B' are said to be 1-*superform-equivalent* if, for their supermatrices, we have (5.7), where $M \in GL(n_{\bar 0}|n_{\bar 1})$ and A is a symmetric even supermatrix such that the restriction of the bilinear form corresponding to it onto the odd subspace $V_{\bar 1}$ is fully isotropic. This means that, in the standard format of supermatrices,

$$B = \begin{pmatrix} B_{\bar 0} & 0 \\ 0 & B_{\bar 1} \end{pmatrix} \qquad \text{and} \qquad B' = \begin{pmatrix} B'_{\bar 0} & 0 \\ 0 & B'_{\bar 1} \end{pmatrix}$$

are 1-superform-equivalent if and only if (1) $B_{\bar{0}}$ and $B'_{\bar{0}}$ are 1-form-equivalent, and (2) $B_{\bar{1}}$ and $B'_{\bar{1}}$ are 1-superform-equivalent. (This also follows from the fact that the 1-form

$$dx_0 + \sum_{i,j=1}^{n_{\bar{0}}} A_{ij}x_i dx_j + \sum_{i,j=1}^{n_{\bar{1}}} B_{ij}x_{n_{\bar{0}}+i}dx_{n_{\bar{0}}+j}$$

is contact if and only if the forms $dx_0 + \sum_{i,j=1}^{n_{\bar{0}}} A_{ij}x_i dx_j$ and $dx_0 + \sum_{i,j=1}^{n_{\bar{1}}} B_{ij}x_{n_{\bar{0}}+i}dx_{n_{\bar{0}}+j}$ are contact on the superspaces of superdimension $(n_{\bar{0}}+1|0)$ and $(1|n_{\bar{1}})$, respectively.) Then, from (5.9) we get the following

Theorem. *The following are the canonical expressions for an odd contact form on a superspace:*

$$dt + \sum_{i=1}^{k} p_i dq_i + \sum_{j=1}^{l} \xi_i d\eta_i \begin{cases} & \text{for } n_{\bar{0}} = 2k \text{ and } n_{\bar{1}} = 2l, \\ +\theta d\theta & \text{for } n_{\bar{0}} = 2k \text{ and } n_{\bar{1}} = 2l+1, \end{cases}$$

where $t = x_0$; $p_i = x_i$, $q_i = x_{k+i}$ *for* $1 \leq i \leq k$; $\xi_i = x_{n_{\bar{0}}+i}$, $\eta_i = x_{n_{\bar{0}}+l+i}$ *for* $1 \leq i \leq l$; $\theta = x_n$ *for* $n_{\bar{1}} = 2l+1$.

5.8.2 The case of an even 1-form. Let $\Pi(x_0) = \bar{1}$. This corresponds to the following equivalence of *odd* bilinear forms on a superspace V of superdimension $(n_{\bar{0}}|n_{\bar{1}})$: two such forms B and B' are said to be 1-*superform-equivalent* if for their (super)matrices we have (5.7), where $M \in GL(n_{\bar{0}}|n_{\bar{1}})$ and A is a symmetric odd supermatrix. Then, since

$$\left(\begin{array}{c|c} 1_{n_{\bar{0}}} & 0 \\ \hline 0 & M \end{array}\right) \left(B + \left(\begin{array}{c|c} 0 & C \\ \hline C^T & 0 \end{array}\right)\right) \left(\begin{array}{c|c} 1_{n_{\bar{0}}} & 0 \\ \hline 0 & M^T \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline X(D+C^T) & 0 \end{array}\right)$$

$$\text{for } B = \left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array}\right),$$

any such B is equivalent to a form with a supermatrix of the shape (the indices above and to the left of the supermatrix are the sizes of the blocks)

$$\begin{array}{c} \\ n_{\bar{0}} \\ \hline r \\ n_{\bar{1}} - r \end{array} \begin{array}{c} \begin{array}{cc|c} r & n_{\bar{0}}-r & n_{\bar{1}} \end{array} \\ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 1_r & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array},$$

where $r = \operatorname{rank}(D + C^T)$. The corresponding form is contact if and only if $r = n_{\bar{0}} = n_{\bar{1}}$. Hence, we get the following somewhat unexpected result:

Theorem. *The following expressions for the canonical form of an even contact (pericontact) 1-form on a superspace of dimension* $(k|k+1)$ *are equivalent:*

$$d\tau + \sum_{i=1}^{k} \xi_i dq_i, \quad \text{or } d\tau + \sum_{i=1}^{k} q_i d\xi_i, \quad \text{or } d\tau + \sum_{i=1}^{l} \xi_i dq_i + \sum_{i=l+1}^{k} q_i d\xi_i,$$

where $\tau = x_0$, $\xi_i = x_{k+i}$, $q_i = x_i$ *for* $1 \leq i \leq k$.

§6 Canonical expressions of symmetric bilinear forms. Related simple Lie algebras

If we want to have a canonical expression of a non-degenerate bilinear form B such that the intersection of the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ or $\mathfrak{o}_B^{(2)}(n)$ with the space of diagonal matrices were of maximal possible dimension, we should take the following canonical forms of B. Each of the following subsections 6.1, 6.2, 6.3 contains two most convenient expressions of an equivalence class of bilinear forms.

6.1 $n = 2k + 1$.

6.1.1 If $B = S(2k+1)$, then $\mathfrak{o}_B(n)$ consists of the matrices, symmetric with respect to the side diagonal; it is convenient to express them in the block form

$$\begin{pmatrix} A & X & C \\ Y^T S(K) & z & X^T S(k) \\ D & Y & S(k)A^T S(k) \end{pmatrix}$$ where $A \in \mathfrak{gl}(k)$, C and D are symmetric with respect to the side diagonal, $X, Y \in \mathbb{K}^k$ are column-vectors, $z \in \mathbb{K}$.

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B(n)$, which have only zeros on the side diagonal; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ of maximal dimension is spanned by the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, 0, a_k, \dots, a_1).$$

6.1.2 If $B = \Pi_{2k+1}$, then $\mathfrak{o}_B(n)$ is spanned by the matrices

$$\begin{pmatrix} A & X & C \\ Y^T & z & X^T \\ D & Y & A^T \end{pmatrix}$$ where $A \in \mathfrak{gl}(k)$, C and D are symmetric, $X, Y \in \mathbb{K}^k$ are column-vectors, $z \in \mathbb{K}$.

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B$ such that C and D are zero-diagonal, $z = 0$; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ of maximal dimension is spanned by the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, 0, a_1, \dots, a_k).$$

6.2 $n = 2k$ and B equivalent to $S(2k)$.

6.2.1 If $B = S(2k)$, then $\mathfrak{o}_B(n)$ consists of the matrices, symmetric with respect to the side diagonal; it is convenient to express them in the block form

$$\begin{pmatrix} A & C \\ D & S(k)A^T S(k) \end{pmatrix}$$ where $A \in \mathfrak{gl}(k)$, C and D are symmetric with respect to the side diagonal.

The Cartan subalgebra of the related simple Lie algebra (it is described later) is spanned by the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, a_k, \dots, a_1) \text{ such that } a_1 + \dots + a_k = 0.$$

6.2.2 If $B = \Pi_{2k}$, then $\mathfrak{o}_B(n)$ is spanned by the matrices

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \text{ where } A \in \mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric.} \tag{6.1}$$

Observe that these matrices can be represented as $\Pi(2k)U$ or $V\Pi(2k)$, where U and V are symmetric.

The Cartan subalgebra of the related simple Lie algebra (it is described later) is spanned by the matrices

$$\mathrm{diag}_n(a_1, \ldots, a_k, a_1, \ldots, a_k) \text{ such that } a_1 + \cdots + a_k = 0.$$

6.3 $\boldsymbol{n = 2k}$ **and** $\boldsymbol{B}$ **equivalent to** $\boldsymbol{1_n}$**.** We get the greatest dimension of the intersection of the Cartan subalgebra with the space of diagonal matrices if the matrix of B is of any of the following shapes:

6.3.1 If $B = \begin{pmatrix} 1_2 & 0 \\ 0 & S(n-2) \end{pmatrix}$, then $\mathfrak{o}_B$ is spanned by the matrices

$$\begin{pmatrix} A & C \\ S(n-2)B^T & D \end{pmatrix} \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(2) \text{ is symmetric, } C \text{ is any } 2 \times (n-2) \text{ matrix,} \\ D \in \mathfrak{gl}(n-2) \text{ is symmetric with respect to the side diagonal.} \end{array}$$

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B(n)$ such that A is zero-diagonal, D has only zeros on the side diagonal; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ of greatest dimension is spanned by the matrices

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \ldots, a_{k-1}, a_{k-1}, \ldots, a_1) \end{array}\right).$$

6.3.2 If $B = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 0 & 1_{k-1} \\ 0 & 1_{k-1} & 0 \end{pmatrix}$, then $\mathfrak{o}_B(n)$ is spanned by the matrices

$$\begin{pmatrix} X & Y & Z \\ Z^T & A & C \\ Y^T & D & A^T \end{pmatrix} \quad \begin{array}{c} \text{where } X \in \mathfrak{gl}(2) \text{ is symmetric, } Y \text{ and } Z \text{ are of size } 2 \times (k-1)\ , \\ A \in \mathfrak{gl}(k-1),\ \ C, D \in \mathfrak{gl}(k-1) \text{ are symmetric.} \end{array}$$

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B(n)$ such that X, C and D are zero-diagonal; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ greatest dimension is spanned by the matrices

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \ldots, a_{k-1}, a_1, \ldots, a_{k-1}) \end{array}\right).$$

6.4 The derived algebras of $\mathfrak{o}_I(n)$.

Direct calculation shows that

$$\mathfrak{o}_I^{(1)}(n) = \begin{cases} 0 & \text{if } n = 1 \\ \{\lambda S(2) \mid \lambda \in \mathbb{K}\} & \text{if } n = 2; \end{cases} \qquad \mathfrak{o}_I^{(2)}(n) = 0 \text{ if } n \leq 2.$$

Lemma. *If $n > 2$, then*
i) $\mathfrak{o}_I^{(1)}(n) = ZD(n)$;
ii) $\mathfrak{o}_I^{(2)}(n) = \mathfrak{o}_I^{(1)}(n)$.

Proof. First, let us show that $\mathfrak{o}_I^{(1)}(n) \subset ZD(n)$. Indeed, if $A, A' \in \mathfrak{o}_I(n)$, then

$$[A, A']_{ii} = \sum_j A_{ij}A'_{ji} - A'_{ij}A_{ji} = 0$$

since A, A' are symmetric. So, matrices from $\mathfrak{o}_I^{(1)}(n)$ are zero-diagonal.

Let $F^{ij} = E^{ij} + E^{ji}$, where $1 \leq i, j \leq n$, $i \neq j$. These matrices are symmetric, so they are all in $ZD(n)$. Let us show that they also are in $\mathfrak{o}_I^{(1)}(n)$. Since, for $1 \leq i < j \leq n$, the matrices F^{ij} form a basis of $ZD(n)$, it follows that $ZD(n) \subset \mathfrak{o}_I^{(1)}(n)$, and it proves (i).

Direct calculation shows that if $1 \leq k \leq n$, $k \neq i, j$, then

$$[F^{ik}, F^{kj}] = F^{ij}, \tag{6.2}$$

so $F^{ij} \in \mathfrak{o}_I^{(1)}(n)$.

Moreover, once we have shown that $F^{ij} \in \mathfrak{o}_I^{(1)}(n)$, this computation also proves that $F^{ij} \in \mathfrak{o}_I^{(2)}(n)$. Since $\mathfrak{o}_I^{(2)}(n) \subset \mathfrak{o}_I^{(1)}(n) = ZD(n)$, it also proves (ii). □

Lemma. *If $n > 2$, then $\mathfrak{o}_I^{(1)}(n)$ is simple.*

Proof. Let $I \subset \mathfrak{o}_I^{(1)}(n)$ be an ideal, and $x \in I$ an element, such that its decomposition with respect to $\{F^{ij}\}$ contains F^{ab} with non-zero coefficient for some a, b. Let us note that

$$[F^{ij}, [F^{ij}, F^{kl}]] = \begin{cases} F^{kl} & \text{if } |\{i, j\} \cap \{k, l\}| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Let us define an operator $P_{F^{ab}} : \mathfrak{o}_I^{(1)}(n) \to \mathfrak{o}_I^{(1)}(n)$ as follows:

$$P_{F^{ab}} = \begin{cases} (\text{ad } F^{bc})^2(\text{ad } F^{ac})^2 \text{ for } c \neq a, b, \quad 1 \leq c \leq 3 & \text{if } n = 3; \\ \prod\limits_{1 \leq c \leq n,\ c \neq a,b} (\text{ad } F^{ac})^2 & \text{if } n > 3. \end{cases} \tag{6.4}$$

Then, from (6.3),

$$P_{F^{ab}} F^{cd} = \begin{cases} F^{cd} & \text{if } F^{cd} = F^{ab}; \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

So, $[F^{ac}, [F^{ac}, [F^{bc}, [F^{bc}, x]]]]$ is proportional (with non-zero coefficient) to F^{ab}, and $F^{ab} \in I$. Then, from (6.2), $F^{ib}, F^{ij} \in I$ for all i, j, $1 \leq i, j \leq n$, $i \neq j$, and $I = \mathfrak{o}_I^{(1)}(n)$. □

6.5 The derived Lie algebras of $\mathfrak{o}_\Pi(2n)$. Direct computations show that:

$\mathfrak{o}_\Pi^{(1)}(2) = \{\lambda \cdot 1_2 \mid \lambda \in \mathbb{K}\}$;
$\mathfrak{o}_\Pi^{(2)}(2) = 0$;
$\mathfrak{o}_\Pi^{(1)}(4) = \{$matrices of the shape (6.1) such that $B, C \in ZD(2)\}$;
$\mathfrak{o}_\Pi^{(2)}(4) = \{$matrices of $\mathfrak{o}_\Pi^{(1)}(4)$ such that $\mathrm{tr} A = 0\}$;
$\mathfrak{o}_\Pi^{(3)}(4) = \{\lambda \cdot 1_4 \mid \lambda \in \mathbb{K}\}$;
$\mathfrak{o}_\Pi^{(4)}(4) = 0$.

Lemma. *If $n \geq 3$, then*
i) $\mathfrak{o}_\Pi^{(1)}(2n) = \{$*matrices of the shape (6.1) such that* $B, C \in ZD(n)\}$*;*
ii) $\mathfrak{o}_\Pi^{(2)}(2n) = \{$*matrices of the shape (6.1) such that* $B, C \in ZD(n)$*, and* $\mathrm{tr}\ A = 0\}$*;*
iii) $\mathfrak{o}_\Pi^{(3)}(2n) = \mathfrak{o}_\Pi^{(2)}(2n)$.

Proof. Let M^1 and M^2 denote *conjectural* $\mathfrak{o}_\Pi^{(1)}(2n)$ and $\mathfrak{o}_\Pi^{(2)}(2n)$, respectively, as described in Lemma. First, let us prove that $\mathfrak{o}_\Pi^{(1)}(2n) \subset M^1$ and $\mathfrak{o}_\Pi^{(2)}(2n) \subset M^2$. Let

$$L = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}, \qquad L' = \begin{pmatrix} A' & B' \\ C' & A'^T \end{pmatrix} \in \mathfrak{o}(2n), \qquad \text{and} \quad L'' = [L, L'] = \begin{pmatrix} A'' & B'' \\ C'' & A''^T \end{pmatrix}.$$

Then, for any $i \in \overline{1, n}$, we have

$$B''_{ii} = \sum_{j=1}^{n} (A_{ij} B'_{ji} + B_{ij} A'_{ij} - A'_{ij} B_{ji} - B'_{ij} A_{ij}) = 0$$

since B,B' are symmetric. Analogically, $C''_{ii} = 0$, so $L'' \in M^1$. Hence, $\mathfrak{o}_\Pi^{(1)}(2n) \subset M^1$.

Now, if $L, L' \in \mathfrak{o}_\Pi^{(1)}(2n)$, then

$$\mathrm{tr} A'' = \sum_{i=1}^{n} A''_{ii} = \sum_{i,j=1}^{n} (A_{ij} A'_{ji} + B_{ij} C'_{ji} - A'_{ij} A_{ji} - B'_{ij} C_{ji}) = 0$$

since B, B', C, C' are symmetric and zero-diagonal. So, since $L'' \in \mathfrak{o}_\Pi^{(2)}(2n) \subset \mathfrak{o}_\Pi^{(1)}(2n) \subset M^1$, it follows that $L'' \in M^2$, and $\mathfrak{o}_\Pi^{(2)}(2n) \subset M^2$.

Let us introduce the following notations for matrices from $\mathfrak{o}_\Pi(2n)$:

F_1^{ij}, where $1 \leq i, j \leq n$, $i \neq j$, such that $\quad A = C = 0$, $B = E^{ij} + E^{ji}$;
F_2^{ij}, where $1 \leq i, j \leq n$, $i \neq j$, such that $\quad A = B = 0$, $C = E^{ij} + E^{ji}$;
G^{ij}, where $1 \leq i, j \leq n$, $i \neq j$, such that $\quad B = C = 0$, $A = E^{ij}$;
H^{ij}, where $1 \leq i, j \leq n$, $i \neq j$, such that $\quad B = C = 0$, $A = E^{ii} + E^{jj}$;
K_0 such that $\quad B = C = 0$, $A = E^{11}$;
K_1 such that $\quad A = C = 0$, $B = E^{11}$;
K_2 such that $\quad A = B = 0$, $C = E^{11}$;

Observe that F_1^{ij}, F_2^{ij}, G^{ij}, and H^{ij} span M^2; whereas M^2 and K_0 span M^1.

Direct computations give the following relations:

$$\begin{aligned} &\text{if } k \neq i, j, \text{ then } [H^{ik}, F_1^{ij}] = F_1^{ij}, \qquad [H^{ik}, F_2^{ij}] = F_2^{ij}, \qquad [H^{ik}, G^{ij}] = G^{ij}; \\ &[F_1^{ij}, F_2^{ij}] = H^{ij}; \\ &[K_1, K_2] = K_0. \end{aligned} \tag{6.6}$$

Since $F_1^{ij}, F_2^{ij}, G^{ij}, H^{ij}, K_1, K_2 \in \mathfrak{o}_\Pi(2n)$, it follows that $F_1^{ij}, F_2^{ij}, G^{ij}, H^{ij}, K_0 \in \mathfrak{o}_\Pi^{(1)}(2n)$. Hence, $M^1 \subset \mathfrak{o}_\Pi^{(1)}(2n)$, and $\mathfrak{o}_\Pi^{(1)}(2n) = M^1$. Relations (6.6) imply that $M^2 \subset [M^2, M^2]$, so $M^2 \subset [M^1, M^1] = \mathfrak{o}_\Pi^{(2)}(2n)$, and $\mathfrak{o}_\Pi^{(2)}(2n) = M^2$. Also, $M^2 \subset [M^2, M^2] = \mathfrak{o}_\Pi^{(3)}(2n)$, so $\mathfrak{o}_\Pi^{(3)}(2n) = M^2$. The lemma is proven. $\square$

Lemma. *If $n \geq 3$, then*

i) if n is odd, then $\mathfrak{o}_\Pi^{(2)}(2n)$ is simple;

ii) if n is even, then the only non-trivial ideal of $\mathfrak{o}_\Pi^{(2)}(2n)$ is the center $Z = \{\lambda \cdot 1_{2n} \mid \lambda \in \mathbb{K}\}$ (thus, $\mathfrak{o}_\Pi^{(2)}(2n)/Z$ is simple).

Proof. We use the notations of the previous Lemma. It follows from the relations

$$[F_1^{ij}, F_2^{ij}] = H^{ij};$$

$$[H^{ij}, X^{kl}] = \begin{cases} X^{kl} & \text{if } |\{i,j\} \cap \{k,l\}| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } X^{kl} = F_1^{kl}, F_2^{kl}, G^{kl};$$

that if an ideal I of $\mathfrak{o}_\Pi^{(2)}(2n)$ contains any of the elements F_1^{ij}, F_2^{ij}, then $I = \mathfrak{o}_\Pi^{(2)}(2n)$.

Let $1 \leq i, j, k \leq n$, $i \neq j \neq k \neq i$. Direct computation shows that the operators

$$P_{F_1^{ij}} = \mathrm{ad}F_1^{jk}\mathrm{ad}F_1^{ij}\mathrm{ad}F_2^{ij}\mathrm{ad}F_2^{jk}; \qquad P_{F_2^{ij}} = \mathrm{ad}F_2^{jk}\mathrm{ad}F_2^{ij}\mathrm{ad}F_1^{ij}\mathrm{ad}F_1^{jk}$$

on $\mathfrak{o}_\Pi^{(2)}(2n)$ act as follows: for X equal to one of the elements $F_1^{lm}, F_2^{lm}, H^{lm}, G^{lm}$,

$$P_{F_1^{ij}} X = \begin{cases} X & \text{if } X = F_1^{ij} \\ 0 & \text{otherwise} \end{cases}; \qquad P_{F_2^{ij}} X = \begin{cases} X & \text{if } X = F_2^{ij} \\ 0 & \text{otherwise} \end{cases}.$$

It follows from these two facts that any element of a non-trivial ideal I of $\mathfrak{o}_\Pi^{(2)}(2n)$ must not contain F_1^{ij}, F_2^{ij} in its decomposition with respect to the basis of $F_1^{ij}, F_2^{ij}, H^{ij}, G^{ij}$ — i.e., it must have the shape

$$\begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix}.$$

Then, for I to be an ideal, A must satisfy the following condition:

$$AB + BA^T = 0 \text{ for all } B \in ZD(n).$$

If A contains a non-zero non-diagonal entry A_{ij}, then

$$(A(E^{jk} + E^{kj}) + (E^{jk} + E^{kj})A^T)_{ik} = A_{ij} \neq 0$$

for $k \neq i, j$; if A contains two non-equal diagonal entries A_{ii} and A_{jj}, then

$$(A(E^{ij} + E^{ji}) + (E^{ij} + E^{ji})A^T)_{ij} = A_{ii} - A_{jj} \neq 0$$

So, A must be proportional to 1_n, and $1_{2n} \in \mathfrak{o}_\Pi^{(2)}(2n)$ if and only if n is even. $\square$

6.6 On simplicity of the derived algebras of $\mathfrak{o}_B(n)$. Let $\mathfrak{o}_B(n)$ be either $\mathfrak{o}_I(n)$ or $\mathfrak{o}_\Pi(n)$ or $\mathfrak{o}_S(n)$, the Lie algebras of linear transformations preserving bilinear forms 1_n or Π_n or $S(n)$, respectively.

Lemma. $\mathfrak{o}_B^{(1)}(n)$ *is a simple Lie algebra if* $n \equiv 1 \pmod 2$ *and* $n > 1$;
$\mathfrak{o}_B^{(2)}(n)/\mathfrak{center}$ *is a simple Lie algebra if* $n \equiv 0 \pmod 2$ *and* $n > 4$.

Proof. Follows from the fact that in the $\mathbb{Z}$-grading corresponding to the blocks of the matrix forms 6.1-6.3, the $\mathfrak{g}_0$-modules $\mathfrak{g}_{\pm 1}$ are irreducible, generate $\mathfrak{g}_\pm$, and $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$. □

§7 Canonical expressions of symmetric bilinear superforms. Related Lie superalgebras.

In this section we consider Lie superalgebras of linear transformations preserving bilinear forms on a superspace of superdimension $(n_{\bar 0}|n_{\bar 1})$ and their derived superalgebras. Since in the case where $n_{\bar 0} = 0$ or $n_{\bar 1} = 0$ these superalgebras are entirely even and do not differ from the corresponding Lie algebras, we do not consider this case.

As it was said in sec. 4.1, every even symmetric non-degenerate form on a superspace of superdimension $(n_{\bar 0}|n_{\bar 1})$ over a field of characteristic 2 is equivalent to a form of the shape (here: $i = \bar 0$ or $\bar 1$)

$$B = \begin{pmatrix} B_{\bar 0} & 0 \\ 0 & B_{\bar 1} \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 1_{n_i} & \text{if } n_i \text{ is odd;} \\ 1_{n_i} \text{ or } \Pi(n_i) & \text{if } n_i \text{ is even.} \end{cases} \tag{7.1}$$

The Lie superalgebra $\mathfrak{oo}_B(n_{\bar 0}|n_{\bar 1})$ preserving B is spanned by the supermatrices which in the standard format are of the shape

$$\begin{pmatrix} A_{\bar 0} & B_{\bar 0} C^T B_{\bar 1}^{-1} \\ C & A_{\bar 1} \end{pmatrix}, \quad \begin{array}{c} \text{where } A_{\bar 0} \in \mathfrak{o}_{B_{\bar 0}}(n_{\bar 0}),\ A_{\bar 1} \in \mathfrak{o}_{B_{\bar 1}}(n_{\bar 1}), \text{ and} \\ C \text{ is arbitrary } n_{\bar 1} \times n_{\bar 0} \text{ matrix.} \end{array} \tag{7.2}$$

In what follows we use the fact that in the case of matrices and supermatrices behave identically with respect to multiplication and Lie (super)bracket — i.e., if two square supermatrices of the same format are given, then the entries of their product or Lie (super)bracket do not depend on this format.

7.1 The derived Lie superalgebras of $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$.

Let B be of the shape (7.1) such that $B_i = 1_{n_i}$. We will denote Lie superalgebra preserving this form as $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$; this superalgebra consists of symmetric supermatrices.

Direct calculation shows that

$$\mathfrak{oo}_{II}^{(i)}(1|1) = \begin{cases} \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{K} \right\} & \text{if } i = 1, \\ \{a \cdot 1_{1|1} \mid a \in \mathbb{K}\} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

Lemma. *If* $n = n_{\bar 0} + n_{\bar 1} \geq 3$, *then*
i) $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ *consists of symmetric supermatrices of (super)trace* 0;
ii) $\mathfrak{oo}_{II}^{(2)}(n_{\bar 0}|n_{\bar 1}) = \mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$.

Proof. It was shown in the proof of Lemma 6.4 that a (super)bracket of any two symmetric matrices is zero-diagonal, so to prove that supermatrices from $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ have trace 0, we only need to prove this for the squares of odd symmetric supermatrices. If L is an odd matrix of the shape (7.2), then

$$\mathrm{tr}L^2 = \sum_{i=1}^{n_{\bar{0}}}(\sum_{j=1}^{n_{\bar{1}}} C_{ij})^2 + \sum_{j=1}^{n_{\bar{1}}}(\sum_{i=1}^{n_{\bar{0}}} C_{ij})^2 = 2\sum_{i=1}^{n_{\bar{0}}}\sum_{j=1}^{n_{\bar{1}}} C_{ij}^2 = 0.$$

Now let us introduce the following notations for matrices from $\mathfrak{oo}_{II}(n_{\bar{0}}|n_{\bar{1}})$:

$$F^{ij} = E^{ij} + E^{ji} \quad \text{for } 1 \leq i, j \leq n,\ i \neq j;$$
$$H^{ij} = E^{ii} + E^{jj} \quad \text{for } 1 \leq i \leq n_{\bar{0}},\ 1 \leq j \leq n_{\bar{1}}.$$

These matrices span the space of symmetric matrices with trace 0. As it was shown in the proof of Lemma 6.4, if $n \geq 3$, then the matrices F^{ij} generate themselves. Moreover, if $1 \leq i \leq n_{\bar{0}}$, $1 \leq j \leq n_{\bar{1}}$ (so that F^{ij} is odd), then $(F^{ij})^2 = H^{ij}$. So all the F^{ij}, H^{ij} lie in $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ for any i. □

Lemma. *If $n = n_{\bar{0}} + n_{\bar{1}} \geq 3$, then*

i) if n is odd, then $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ is simple;

ii) if n is even, then the only non-trivial ideal of $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ is the center $C = \{\lambda \cdot 1_n \mid \lambda \in \mathbb{K}\}$ (thus, $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})/C$ is simple).

Proof. Let us define operators $P_{F^{ab}}$ as in (6.4). Then, due to (6.5) and the fact that

$$P_{F^{ab}}H^{kl} = 0,$$

we can show in the same way as in the proof of Lemma 6.4 that if an ideal of $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ contains a non-diagonal matrix, it contains all the F^{ij}. Since all the H^{ij} are squares of odd F^{ij}, such an ideal is trivial.

So, any non-trivial ideal of $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ is diagonal. For a diagonal matrix X,

$$[X, F^{ij}] = (X_{jj} - X_{ii})F^{ij},$$

so all the elements of a non-trivial ideal must be proportional to 1_n, and $1_n \in \mathfrak{oo}_{II}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ if and only if n is even. □

7.2 The derived Lie superalgebras of $\mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$. Now let us consider the case where $n_{\bar{1}}$ is even and B is of the shape (7.1) such that $B_{\bar{0}} = 1_{n_{\bar{0}}}$, $B_{\bar{1}} = \Pi(n_{\bar{1}})$. (The case where $n_{\bar{0}}$ is even, $B_{\bar{0}} = \Pi(n_{\bar{0}})$, $B_{\bar{1}} = 1_{n_{\bar{1}}}$ is analogous to this one, so we will not consider it.) We will denote Lie superalgebra preserving this form by $\mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$; this superalgebra consists of supermatrices of the following shape:

$$\begin{pmatrix} A_{\bar{0}} & C^T\Pi(n_{\bar{1}}) \\ C & \Pi(n_{\bar{1}})A_{\bar{1}} \end{pmatrix}, \quad \begin{array}{l} \text{where } A_{\bar{0}}, A_{\bar{1}} \text{ are symmetric,} \\ C \text{ is an arbitrary } n_{\bar{1}} \times n_{\bar{0}} \text{ matrix.} \end{array} \tag{7.3}$$

Lemma. *i) $\mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$ consists of the matrices of the shape (7.3) such that $A_{\bar{0}}$ is zero-diagonal;*

ii) $\mathfrak{oo}_{I\Pi}^{(2)}(n_{\bar{0}}|n_{\bar{1}}) = \mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar{0}}|n_{\bar{1}})$.

Proof. Set $k_{\bar{1}} = n_{\bar{1}}/2$; let M be the *conjectural* space of $\mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$ as it is described in the Lemma. First, let us prove that $\mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}}) \subset M$. If

$$\begin{aligned} L &= \begin{pmatrix} A_{\bar{0}} & C^T\Pi(n_{\bar{1}}) \\ C & A_{\bar{1}} \end{pmatrix}, L' = \begin{pmatrix} A'_{\bar{0}} & C'^T\Pi(n_{\bar{1}}) \\ C' & A'_{\bar{1}} \end{pmatrix} \in \mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}}), \text{ and} \\ L'' &= [L, L'] = \begin{pmatrix} A''_{\bar{0}} & C''^T\Pi(n_{\bar{1}}) \\ C'' & A''_{\bar{1}} \end{pmatrix}, \end{aligned}$$

then

$$\begin{aligned} &(A''_{\bar{0}})_{ii} = ([A_{\bar{0}}, A'_{\bar{0}}] + C^T\Pi(n_{\bar{1}})C' - C'^T\Pi(n_{\bar{1}})C)_{ii} = \\ &\sum_{j=1}^{n_{\bar{0}}}((A_{\bar{0}})_{ij}(A'_{\bar{0}})_{ji} - (A'_{\bar{0}})_{ij}(A_{\bar{0}})_{ji}) + \sum_{j=1}^{k_{\bar{1}}}(C_{ji}C'_{j+k_{\bar{1}},i} + C_{j+k_{\bar{1}},i}C'_{ji} - C'_{ji}C_{j+k_{\bar{1}},i} + C'_{j+k_{\bar{1}},i}C_{ji}) = 0 \end{aligned}$$

since $A_{\bar{0}}, A'_{\bar{0}}$ are symmetric. Now, if L is odd (i.e., $A_{\bar{0}} = 0$, $A_{\bar{1}} = 0$), then

$$L = \begin{pmatrix} C^T\Pi(n_{\bar{1}})C & 0 \\ 0 & CC^T\Pi(n_{\bar{1}}) \end{pmatrix},$$

and

$$(C^T\Pi(n_{\bar{1}})C)_{ii} = \sum_{j=1}^{k_{\bar{1}}}(C_{ji}C_{j+k_{\bar{1}},i} + C_{j+k_{\bar{1}},i}C_{ji}) = 0.$$

Let us introduce the following notations for matrices from $\mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$:

F^{ij}, where $1 \le i, j \le n_{\bar{0}}$ and $i \ne j$, such that $C = 0$, $A_{\bar{1}} = 0$, $A_{\bar{0}} = E^{ij} + E^{ji}$;
G^{ij}, where $1 \le i \le n_{\bar{1}}$, $1 \le j \le n_{\bar{0}}$, such that $A_{\bar{0}} = 0$, $A_{\bar{1}} = 0$, $C = E^{ij}$;
H^i, where $1 \le i \le n_{\bar{1}}$, such that $A_{\bar{0}} = 0$, $C = 0$, $A_{\bar{1}} = \begin{cases} E^{i,i+k_{\bar{1}}} & \text{if } 1 \le i \le k_{\bar{1}}; \\ E^{i,i-k_{\bar{1}}} & \text{if } k_{\bar{1}} + 1 \le i \le n_{\bar{1}}; \end{cases}$
$I_{\bar{1}}$, such that $A_{\bar{0}} = 0$, $C = 0$, $A_{\bar{1}} = 1_{n_{\bar{1}}}$.

Direct calculations give the following relations:

$$\begin{aligned} &[G^{i1}, G^{k_{\bar{1}}+1,j}] = F^{ij}; \\ &[I_{\bar{1}}, G^{ij}] = G^{ij}; \\ &(G^{i1})^2 = H^i, \end{aligned}$$

so we get $F^{ij}, G^{ij}, H^i \in \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$.

Let us also denote by K the subalgebra of $\mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$, consisting of all matrices of the form (7.3), such that $A_{\bar{0}} = 0$, $C = 0$. Since $K \subset \mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$, it follows that $K^{(1)} \subset \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$. We also have $K \simeq \mathfrak{o}_{\Pi}(n_{\bar{1}})$, so, as it was shown in subsec. 6.5, $K^{(1)}$ consists of matrices from K, such that $A_{\bar{1}}$ has the shape (6.1) (even if $n_{\bar{1}} < 6$). Hence, $K^{(1)}, F^{ij}, G^{ij}, H^i$ span M, so $M \subset \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$, and $M = \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$. Since $F^{ij}, G^{ij}, H^{ij}, I_{\bar{1}} \in \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$, and $K \subset \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$ (because $K^{(1)}$ and H^i span K), we also get that $M \subset \mathfrak{oo}^{(2)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$, and $M = \mathfrak{oo}^{(2)}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}})$. $\square$

7.3 The derived Lie superalgebras of $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$. Now we consider the case where $n_{\bar 0}, n_{\bar 1}$ are even and B is of the shape (7.1) such that $B_i = \Pi(n_i)$. We set $k_{\bar 0} = n_{\bar 0}/2$, $k_{\bar 1} = n_{\bar 1}/2$. We will denote the Lie superalgebra preserving this form B by $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$; it consists of supermatrices of the following shape:

$$\begin{pmatrix} \Pi(n_{\bar 0})A_{\bar 0} & \Pi(n_{\bar 0})C^T\Pi(n_{\bar 1}) \\ C & \Pi(n_{\bar 1})A_{\bar 1} \end{pmatrix}, \quad \begin{matrix} \text{where } A_{\bar 0}, A_{\bar 1} \text{ are symmetric,} \\ C \text{ is an arbitrary } n_{\bar 1}\times n_{\bar 0} \text{ matrix.} \end{matrix} \tag{7.4}$$

Direct computation shows that

$$\mathfrak{oo}^{(i)}_{\Pi\Pi}(2|2) = \begin{cases} \{\text{matrices of the shape (7.4) such that } A_{\bar 0}, A_{\bar 1} \in ZD(2)\} & \text{if } i = 1, \\ \{\text{matrices of the shape (7.4) such that } \Pi(2)A_{\bar 0} = \Pi(2)A_{\bar 1} = \lambda\cdot 1_2\} & \text{if } i = 2, \\ \{\lambda \cdot 1_{2|2} \mid \lambda \in \mathbb{K}\} & \text{if } i = 3, \\ 0 & \text{if } i \geq 4. \end{cases}$$

Lemma. *If $n_{\bar 0} + n_{\bar 1} \geq 6$, then*

i) $\mathfrak{oo}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ consists of the matrices of the shape (7.4) such that $A_{\bar 0}, A_{\bar 1}$ are zero-diagonal;

ii) $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ consists of matrices from $\mathfrak{oo}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ such that

$$\sum_{i=1}^{n_{\bar 0}/2} (\Pi(n_{\bar 0})A_{\bar 0})_{ii} + \sum_{i=1}^{n_{\bar 1}/2} (\Pi(n_{\bar 1})A_{\bar 1})_{ii} = 0,$$

i.e., the "half-supertrace" of the matrix vanishes;

iii) $\mathfrak{oo}^{(3)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}) = \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$.

Proof. Let $\tilde B = \mathrm{diag}_2(\Pi(n_{\bar 0}), \Pi(n_{\bar 1}))$ be a (non-super) bilinear form on a space of dimension $n_{\bar 0} + n_{\bar 1}$. Denote:

$$M^k = \{L \in \mathfrak{gl}(n_{\bar 0}|n_{\bar 1}) \mid \text{ exists } L' \in \mathfrak{o}^{(k)}_{\tilde B}(n_{\bar 0}+n_{\bar 1}) \text{ such that } L_{ij} = L'_{ij} \text{ for all } i,j \in \overline{1, n_{\bar 0}+n_{\bar 1}}\};$$
$$N = \mathrm{Span}\{L^2 \mid L \in \mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}), L \text{ is odd }\}.$$

As it was noticed before, matrices and supermatrices behave identically with respect to multiplication and Lie (super)bracket. So we get the following inclusion:

$$M^i \subset \mathfrak{oo}^{(i)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}) \subset M^i + N.$$

Since $\tilde B$ is equivalent to $\Pi(n_{\bar 0}+n_{\bar 1})$, it follows from Lemma 6.5 that M^1, M^2, M^3 coincide with conjectural spaces $\mathfrak{oo}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}^{(3)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ as they are described in the lemma. So, to prove the lemma, it suffices to show that $N \subset M^2$. If

$$L = \begin{pmatrix} 0 & \Pi(n_{\bar 0})C^T\Pi(n_{\bar 1}) \\ C & 0 \end{pmatrix}$$

is an odd matrix from $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, then

$$L^2 = \begin{pmatrix} \Pi(n_{\bar 0})C^T\Pi(n_{\bar 1})C & 0 \\ 0 & C\Pi(n_{\bar 0})C^T\Pi(n_{\bar 1}) \end{pmatrix},$$

and

$$(C^T\Pi(n_{\bar 1})C)_{ii}=\sum_{j=1}^{k_{\bar 1}}C_{ji}C_{j+k_{\bar 1},i}+\sum_{j=k_{\bar 1}+1}^{n_{\bar 1}}C_{ji}C_{j-k_{\bar 1},i}=0;$$

similarly, $\Pi(n_{\bar 1})C\Pi(n_{\bar 0})C^T\Pi(n_{\bar 1})$ is zero-diagonal, so $N\subset M^1$. Now,

$$\begin{aligned}&\sum_{i=1}^{k_{\bar 0}}(\Pi(n_{\bar 0})C^T\Pi(n_{\bar 1})C)_{ii}+\sum_{l=1}^{k_{\bar 1}}(C\Pi(n_{\bar 0})C^T\Pi(n_{\bar 1}))_{ll}=\\&\sum_{i=1}^{k_{\bar 0}}(\sum_{j=1}^{k_{\bar 1}}C_{ji}C_{j+k_{\bar 1},i}+\sum_{j=k_{\bar 1}+1}^{n_{\bar 1}}C_{ji}C_{j-k_{\bar 1},i})+\sum_{l=1}^{k_{\bar 1}}(\sum_{m=1}^{k_{\bar 0}}C_{lm}C_{l,m+k_{\bar 0}}+\sum_{m=k_{\bar 0}+1}^{n_{\bar 0}}C_{lm}C_{l,m-k_{\bar 0}})=0,\end{aligned}$$

so $N\subset M^2$. □

Lemma. *If $n_{\bar 0}+n_{\bar 1}\geq 6$, then*
i) if $k=(n_{\bar 0}+n_{\bar 1})/2$ is odd, then $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is simple;
ii) if k is even, then the only non-trivial ideal of $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is $Z=\{\lambda\cdot 1_{n_{\bar 0}|n_{\bar 1}}\mid \lambda\in\mathbb{K}\}$, its center, and hence $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})/Z$ is simple.

Proof. Let $\iota:\mathfrak{gl}(n_{\bar 0}|n_{\bar 1})\to\mathfrak{gl}(n_{\bar 0}+n_{\bar 1})$ be a forgetful map that sends a supermatrix into the matrix with the same entries and superstructure forgotten. Since for $p=2$ matrices and supermatrices behave identically with respect to the Lie (super)bracket, $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ is a Lie algebra, and for any ideal $I\subset\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, $\iota(I)$ is an ideal in $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$.

Set

$$M=\begin{pmatrix}1_{k_{\bar 0}}&0&0&0\\0&0&1_{k_{\bar 0}}&0\\0&1_{k_{\bar 1}}&0&0\\0&0&0&1_{k_{\bar 1}}\end{pmatrix}.$$

Then, according to Lemma 6.5 and 7.3, the map $X\mapsto MXM^{-1}$ gives us an isomorphism between $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ and $\mathfrak{o}_{\Pi}(n_{\bar 0}+n_{\bar 1})$. So, according to Lemma 6.5, if k is odd, then $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is simple; if k is even, then the only non-trivial ideal of $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ is

$$\{\lambda\cdot M^{-1}1_{n_{\bar 0}+n_{\bar 1}}M=\lambda\cdot 1_{n_{\bar 0}+n_{\bar 1}}\mid\lambda\in\mathbb{K}\}=\iota(Z).$$

Thus, since ι is invertible, $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ can have a non-trivial ideal only if k is even, and this ideal must be equal to Z; direct computation shows that Z is indeed an ideal. □

7.4 The derived Lie superalgebras of $\mathfrak{pe}(k)$.

As it was shown in subsec. 4.1, any non-degenerate odd symmetric bilinear form on a superspace of dimension $(k|k)$ (if dimensions of the even and odd parts of the space are not equal, such a form does not exist) is equivalent to Π_{2k}.

The Lie superalgebra $\mathfrak{pe}(k)$ preserving this form consists of the supermatrices of the shape

$$\begin{pmatrix}A&C\\D&A^T\end{pmatrix},\quad\text{where } A\in\mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric } k\times k \text{ matrices.}\tag{7.5}$$

Direct computations show that:

$\mathfrak{pe}^{(1)}(1)=\{\lambda\cdot 1_2\mid\lambda\in\mathbb{K}\}$;

$\mathfrak{pe}^{(2)}(1) = 0$;
$\mathfrak{pe}^{(1)}(2) = \{$matrices of the shape (7.5) such that B and C are zero-diagonal$\}$;
$\mathfrak{pe}^{(2)}(2) = \{$matrices of $\mathfrak{pe}^{(1)}(2)$ such that $\mathrm{tr}A = 0\}$;
$\mathfrak{pe}^{(3)}(2|2) = \{\lambda \cdot 1_2 \mid \lambda \in \mathbb{K}\}$;
$\mathfrak{pe}^{(4)}(2) = 0$.

Lemma. *If $k \geq 3$, then*
i) $\mathfrak{pe}^{(1)}(k) = \{$*matrices of the shape (7.5) such that B and C are zero-diagonal*$\}$*;*
ii) $\mathfrak{pe}^{(2)}(k) = \{$*matrices of* $\mathfrak{pe}^{(1)}(k)$ *such that* $\mathrm{tr}A = 0\}$*;*
iii) $\mathfrak{pe}^{(3)}(k) = \mathfrak{pe}^{(2)}(k)$.

Proof. As it was noticed before, matrices and supermatrices behave identically with respect to multiplication and Lie (super)bracket. So, if we denote

$$M^i = \{L \in \mathfrak{gl}(k|k) \mid \text{ exists } L' \in \mathfrak{o}_\Pi(2k) \text{ such that } L_{ij} = L'_{ij} \text{ for all } i,j \in \overline{1,2k}\};$$
$$N^i = \mathrm{Span}\{L^2 \mid L \in \mathfrak{pe}^{(i-1)}(k), L \text{ is odd }\}, \quad \text{where } \mathfrak{pe}^{(0)}(k) = \mathfrak{pe}(k),$$

then we get the following inclusion:

$$M^i \subset \mathfrak{pe}^{(i)}(k) \subset M^i + N^i.$$

Now recall from the proof of Lemma 6.5 that M^1, M^2, M^3 coincide with conjectural spaces $\mathfrak{pe}^{(1)}(k)$, $\mathfrak{pe}^{(2)}(k)$, $\mathfrak{pe}^{(3)}(k)$ as they are described in the lemma. Notice also that $N^1 \subset M^1$ (since N^1 is an even subspace of $\mathfrak{pe}(k)$), so $\mathfrak{pe}^{(1)}(k) = M^1$. If

$$L = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$$

is an odd matrix from $\mathfrak{pe}^{(1)}(k)$ (so C and D are symmetric and zero-diagonal), then

$$L^2 = \begin{pmatrix} CD & 0 \\ 0 & DC \end{pmatrix},$$

and

$$\mathrm{tr}CD = \sum_{i,j=1}^{k} C_{ij}D_{ji} = 2 \sum_{1\leq i<j\leq n} C_{ij}D_{ij} = 0,$$

so $N^2 \subset M^2$, and $\mathfrak{pe}^{(2)}(k) = M^2$. Also, $N^3 \subset N^2 \subset M^2 = M^3$, and $\mathfrak{pe}^{(3)}(k) = M^3$. $\square$

§8 Canonical expressions of non-symmetric bilinear superforms. Related Lie superalgebras.

As it was shown in sec. 4.2, any even non-symmetric bilinear form B on a superspace in the standard format has the shape

$$\begin{pmatrix} B_{\bar 0} & 0 \\ 0 & B_{\bar 1} \end{pmatrix}.$$

This form is preserved by the matrices of the shape

$$\begin{pmatrix} A_{\bar 0} & C \\ D & A_{\bar 1} \end{pmatrix}, \text{ where } \begin{cases} A_{\bar 0} \text{ preserves the form } B_{\bar 0}, \\ A_{\bar 1} \text{ preserves the form } B_{\bar 1}, \\ CB_{\bar 1} + B_{\bar 0}D^T = 0, \\ DB_{\bar 0} + B_{\bar 1}C^T = 0. \end{cases}$$

Any odd non-degenerate non-symmetric bilinear form B on a superspace of dimension $(k|k)$ (non-degenerate odd form can exist only on a superspace with equal even and odd dimensions) is equivalent to a form of the shape

$$\begin{pmatrix} 0 & 1_k \\ J & 0 \end{pmatrix}, \text{ where } J \text{ is the Jordan normal form.}$$

Such a form is preserved by the matrices of the shape

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \text{ where } \begin{cases} AJ^T + J^T A = 0, \\ CJ + C^T = 0, \\ D + JD^T = 0. \end{cases}$$

Problem. *The (first or second) derived Lie superalgebra of $\mathfrak{pe}_B(k|k)$ is simple (perhaps, modulo center) only if J consists of 1×1 blocks. What is an explicit structure in other cases? Compare with Ermolaev's description [Er].*

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