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by

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Abstract

We classify (up to an isomorphism in the category of affine groups) the complex crystallographic groups Γ generated by reflections and such that $d\Gamma$, its linear part, is a Coxeter group, i.e., $d\Gamma$ is generated by "real" reflections of order 2.

Introduction

Let X be a connected complex manifold. An automorphism of X is said to be a *reflection* if the set of its fixed points is a non-empty subset of X of complex codimension 1.

The groups generated by reflections with the discrete action on "nice" complex manifolds are the source of various tempting conjectures. For instance, in many cases the quotient space with respect to the action of such a group possesses a simple structure from the point of view of algebraic geometry.

The classification of finite linear groups generated by reflections being completed ([10], [4]; for an excellent review, see [9]), it is natural to study the actions of reflection groups on Hermitian symmetric spaces of constant holomorphic curvature.

The first on the agenda is the zero curvature space $\mathbb{C}^\ell$.

Let V be a finite dimensional affine space over $\mathbb{C}$ and L the linear space of translations of V. We denote by $\mathrm{Aut}(V)$ the group of affine transformations of V.

A discrete subgroup $\Gamma \subset \mathrm{Aut}(V)$ is said to be a *complex crystallographic group* if its translation subgroup $T = \Gamma \cap L$ is a lattice of full rank in L.

By definition, a *complex crystallographic* **reflection** *group* is a complex crystallographic group generated by (complex) reflections.

If Γ is a complex crystallographic reflection group, then the group $d\Gamma \subset GL(L)$ of linear parts of elements of Γ is a finite linear reflection group.

In this note we present a classification of complex crystallographic reflection groups Γ such that the linear part $d\Gamma$ is a Coxeter group, i.e., it is generated by "real" reflections of order 2 (we call them *complex crystallographic Coxeter groups*). Namely, we show that any such group is isomorphic to a product of irreducible complex crystallographic Coxeter groups and classify all such irreducible groups.

The results of our classification are contained in Theorems 1.3 and 1.4 and, in other terms, in Theorems 3.1 and 3.2.

It seems to us that the statements here are much more instructive and suggestive than the proofs. The classification in terms of affine root system (Theorems 3.1 and 3.2) gives us a particular satisfaction. To make this important part of our work more lucid, we have summarized in §2 most of the known to us relevant facts about affine root systems.

Among the new[1] results, we mention here the invariant description of the dual root system and of the weight lattice for the affine root systems. The rest of §2 consists of Macdonald's results [8] represented in a form convenient to us (see also [6]).

The classification (especially Theorems 1.3 and 1.4) implies that complex crystallographic Coxeter groups have moduli, i.e., the groups admit deformations that holomorphically depend on one variable. That is why the object of our investigation is closely related to the classical automorphic forms, Cartan's domains of type IV, Macdonald's identities and many other wonders.

If Γ is a complex crystallographic reflection group such that the group $d\Gamma$ is not of Coxeter type, i.e., is a Shephard–Todd group, then it is easy to prove the rigidity of the group Γ. Due to this fact the classification of such groups is more simple, see [9]. Observe that, together with the results of V. L. Popov [9], our results complete the classification of affine complex reflection groups.

The results were announced in [1] and the details were deposited in [2]. Delivered at Leites' Seminar on Supersymmetries in 1976, these details were preprinted in [7], v. 2. Recently our results became of interest in topological field theory, see, e.g., [5].

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§1 Complex crystallographic Coxeter groups

1.1. Complex crystallographic groups

Let V be a finite dimensional affine space over $\mathbb{C}$ and L the linear space of translations of V. Consider the group $\mathrm{Aut}(V)$ of affine tranformations of V and denote by $d : \mathrm{Aut}(V) \longrightarrow GL(L)$ the morphism which to every affine transformation g assigns its linear part dg. The kernel of d is naturally isomorphic to the group L of translations.

Definition. Let Γ be a discrete subgroup in $\mathrm{Aut}(V)$. The group Γ is said to be a *complex crystallographic group* if the subgroup $T = \Gamma \cap L$ is of full rank in L.

Let Γ be a complex crystallographic group. Then the action of Γ on V has a compact quotient and admits an invariant Hermitian metric H (we always assume that the metric H is translation invariant). This follows from the fact that the group $d\Gamma := \Gamma/T \subset GL(L)$ is finite, and hence preserves some Hermitian positive definite form on L.

[1] At least they were new when discovered, in 1975–76; in English the proofs were only preprinted as [7].

Conversely, if Γ is discrete subgroup of $\mathrm{Aut}(V)$ which preserves an Hermitian metric and such that the space V/Γ is compact, then Bieberbach's theorem implies that Γ is a complex crystallographic group.

Let Γ and Γ' be complex crystallographic groups acting in V and V', respectively. By definition, a *morphism* $\Gamma \longrightarrow \Gamma'$ is an isomorphism $\varphi : V \longrightarrow V'$ of complex affine spaces such that the induced map $\Gamma \longrightarrow \Gamma'$ given by $w \mapsto \varphi w \varphi^{-1}$ is an isomorphism.

A point $x \in V$ is said to be *special* (with respect to Γ) if the map $d : \Gamma_x \longrightarrow d\Gamma$ is an isomorphism (here: Γ_x is the stabilizer of x). Equivalently, x is special if $\Gamma = T \cdot \Gamma_x$, cf. sec. 2.3.

The complex crystallographic group Γ possessing a special point is said to be *splittable.* The choice of a splittable point defines a *splitting* of Γ.

A pair (Γ, x) consisting of a complex crystallographic group Γ and its splitting is called a *split* complex crystallographic group; morphisms of split complex crystallographic groups are assumed to preserve splitting.

Given a splitting $x \in V$, we will usually identify $L(V)$ with V by setting $l \mapsto l + x$.

Remark. Clearly, isomorphisms of complex crystallographic groups transform special points into special points. Moreover, it is easy to verify that if x and x' are special points for a complex crystallographic group Γ and $\varphi : V \longrightarrow V$ is a translation which sends x to x', then φ is an automorphism of the complex crystallographic group Γ. Thus, two split complex crystallographic groups are isomorphic if and only if they are isomorphic as complex crystallographic groups.

1.2. The complex crystallographic Coxeter groups

Let r be an affine transformation of the space V. It is easy to see that r is a reflection if and only if the fixed point set of r is a hyperplane. This hyperplane is called the *mirror* of r and is denoted by $\pi(r)$.

For any transformation group Γ of V, denote:

$\mathrm{Ref}(\Gamma)$, the set of reflections belonging to Γ;

$\Pi(\Gamma)$, the set of hyperplanes which are mirrors of these reflections.

A complex crystallographic group Γ generated by reflections is said to be a *complex crystallographic reflection group* (or briefly, a *ccr-group*). Clearly, for any ccr-group, the finite group $d\Gamma$ of linear transformations of the space $L(V)$ is generated by reflections.

If $d\Gamma$ is a Coxeter group (i.e., in some basis it is represented by real matrices), then Γ will be called a *complex crystallographic Coxeter group* (or briefly, a *ccc-group*). Clearly, in a complex crystallographic Coxeter group, all reflections are of order two.

We will see that, for any ccc-group Γ, the finite group $d\Gamma$ is always a Weyl group. For this reason we will use notation W for complex crystallographic Coxeter groups.

1.2.1. Lemma. *Each complex crystallographic Coxeter group W is splittable.*

Proof. Set $R' = d\mathrm{Ref}(W)$. Clearly, the subset $R' \subset \mathrm{Ref}(dW)$ satisfies the following conditions:

a) R' is invariant with respect to inner automorphisms of the group dW;

b) R' generates dW.

Since dW is a finite group generated by reflections, properties a) and b) imply that $R' = \mathrm{Ref}(dW)$, i.e., any reflection $s \in dW$ belongs to R' (see, for example, [4]).

By assumption, dW is a Coxeter group. Let us choose a system of positive roots for the group dW and consider the corresponding system of simple reflections $s_1, \ldots, s_\ell$ in dW (see [3]). Choose reflections $r_1, \ldots, r_\ell \in \mathrm{Ref}(W)$ such that $dr_i = s_i$. Then the mirrors $\pi(r_i)$ intersect at some point x, since they are parallel to transversal mirrors $\pi(s_i)$. Clearly, $r_i \in W_x$; hence, $s_i \in dW_x$.

Since the reflections s_i generate the group dW, it follows that $dW_x = dW$, i.e., x is a special point for W. ■

1.2.2 Decomposition of a given ccc-group into the product of irreducible ccc-groups

The *product* of complex crystallographic groups W' and W'' is naturally defined as the group $W' \times W''$ acting on the space $V' \times V''$. A complex crystallographic group is said to be *irreducible* if it is not isomorphic to any non-trivial product of complex crystallographic groups.

Proposition. *If a complex crystallographic Coxeter group W is irreducible, then the representation of the group dW in the space $L(V)$ is irreducible.*

Proof. Let L_1 be a non-trivial dW-invariant subspace in $L(V)$ and $L_2 = L_1^\perp$, the orthogonal complement of L_1.

Since dW is a Coxeter group, it is the product of two Coxeter groups, (H_1, L_1) and (H_2, L_2).

Consider the space $V_2 = V/L_1$ as an affine space associated with the vector space L_2.

Consider a reflection $r \in W$ and its image $dr \in dW$.

There are two possibilities:

(i) $dr \in H_1$. In this case r acts trivially on V_2.
(ii) $dr \in H_2$. In this case the action of r on V_2 is a reflection.

Let (W_2, V_2) be a group generated by reflections $\{r \in \mathrm{Ref}(W) \mid dr \in H_2\}$. It is clear, that the group W_2 acts as a complex reflection group on V_2; moreover W_2 acts trivially on V_1.

In a similar way we construct a reflection group (W_1, V_1).

Since W is generated by W_1 and W_2 it follows that $W = W_1 \times W_2$. And we have a natural morphism $V \to V_1 \times V_2$ compatible with the action of $W = W_1 \times W_2$. The group W is cocompact. This implies that the groups W_1 and W_2 are also cocompact. ■

1.3. A construction of irreducible complex crystallographic Coxeter groups

Fix an irreducible reduced root system R; we consider R as a finite system of linear functionals on a complex linear space L (see [3]). For any $\alpha \in R$, let $h_\alpha \in L$ be the corresponding coroot.

Let W_0 denote the Weyl group of R; fix a W_0-invariant Hermitian metric on L. We know (see [3]) that R contains roots of not more than two distinct lengths (long and short: l_l and l_s); let $p = p(R)$ be $\left(\dfrac{l_l}{l_s}\right)^2$ (i.e., $p = 1, 2, 3$).

Let us assign to every root $\alpha \in R$ a cocompact lattice $\mathfrak{a}_\alpha \in \mathbb{C}$ in such a way that

(i) If $|\alpha| = |\beta|$, then $\mathfrak{a}_\alpha = \mathfrak{a}_\beta$;
(ii) If $|\alpha| < |\beta|$, then $\mathfrak{a}_\beta \subset \mathfrak{a}_\alpha$ is a sublattice of index $\leq p$.

A root system R together with assignment $\alpha \longrightarrow \mathfrak{a}_\alpha$ for any $\alpha \in R$ of such lattices is said to be an *equipped root system* $\mathfrak{a} = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R\}$.

Let V be the affine space with special point $x_0 = 0$ corresponding to the linear space L. For any pair $(\alpha \in R and \tau \in \mathfrak{a}_\alpha)$, consider the hyperplane

$$\pi(\alpha, \tau) = \{z \in V \mid \alpha(z) = \tau\}.$$

Let

$$\Pi(\mathfrak{a}) \text{ be the set of all hyperplanes of the form } \pi(\alpha, \tau).$$

Denote by $W(\mathfrak{a})$ the group generated by the reflections in hyperplanes from $\Pi(\mathfrak{a})$. We will consider $W(\mathfrak{a})$ as the split ccc-group with splitting point x_0.

Theorem. a) *The group $W = W(\mathfrak{a})$ is an irreducible complex crystallographic Coxeter group and x_0 is its special point. Moreover, $\Pi(W) = \Pi(\mathfrak{a})$.*

b) *Any irreducible split complex crystallographic Coxeter group is isomorphic to a group of the form $W(\mathfrak{a})$.*

For the proof, see sec. 1.5.

1.4. Classification of complex crystallographic Coxeter groups

We wish to describe complex crystallographic Coxeter groups up to an isomorphism. Lemma 1.2.1 and Remark 1.1 show that it suffices to classify *split* complex crystallographic Coxeter groups. By Theorem 1.3 any irreducible split complex crystallographic Coxeter group is isomorphic to a group of the form $W(\mathfrak{a})$. Hence, it suffices to describe all isomorphism between these groups.

Let $\mathfrak{a} = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R\}$ and $\mathfrak{a}' = \{(R', \mathfrak{a}'_\beta) \mid \beta \in R'\}$ be two equipped root systems. A *similitude* of $\mathfrak{a}$ with $\mathfrak{a}'$ is a pair (ψ, λ), where $\psi : R \longrightarrow R'$ is the root system isomorphism and $\lambda \in \mathbb{C}^*$ is such that $\lambda \mathfrak{a}_\alpha = \mathfrak{a}'_{\psi(\alpha)}$ for any $\alpha \in R$.

A similitude (ψ, λ) defines an isomorphism of split complex crystallographic Coxeter groups $\varphi : W(\mathfrak{a}) \longrightarrow W(\mathfrak{a}')$ from the expression

$$\varphi = \lambda\psi : L \longrightarrow L.$$

For any equipped root system $\mathfrak{a} = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R\}$, define the *dual equipped root system* $\mathfrak{a}^{inv} = \{(R^{inv}, \mathfrak{a}_\alpha^{inv}) \mid \alpha \in R\}$ in the same space L by assigning to any root $\alpha \in R$ the root α^{inv} and the lattice $\mathfrak{a}_\alpha^{inv}$ as follows (see sec. 1.3):

if α is a long root, then $\alpha^{inv} = \alpha$ and $\mathfrak{a}_\alpha^{inv} = \mathfrak{a}_\alpha$,

if α is a short root, then $\alpha^{inv} = p\alpha$ and $\mathfrak{a}_\alpha^{inv} = p\mathfrak{a}_\alpha$, where $p = p(\mathfrak{a}) = [\mathfrak{a}_s : \mathfrak{a}_l]$.

It is clear that $p(\mathfrak{a}^{inv}) = p(\mathfrak{a})$ and if $p = 1$, then $\mathfrak{a}^{inv} = \mathfrak{a}$.

It is quite straightforward that $\Pi(\mathfrak{a}^{inv}) = \Pi(\mathfrak{a})$, hence, $W(\mathfrak{a}^{inv}) = W(\mathfrak{a})$. In particular, any similitude (ψ, λ) of $\mathfrak{a}^{inv}$ with $\mathfrak{a}'$ defines an isomorphism $\varphi : W(\mathfrak{a}) = W(\mathfrak{a}^{inv}) \xrightarrow{\sim} W(\mathfrak{a}')$.

Theorem. *Any isomorphism $\varphi : W(\mathfrak{a}) \longrightarrow W(\mathfrak{a}')$ of split complex crystallographic Coxeter groups is of the form $\varphi = \lambda\psi$, where (ψ, λ) is either a similitude of $\mathfrak{a}$ with $\mathfrak{a}'$ or a similitude of $\mathfrak{a}^{inv}$ with $\mathfrak{a}'$. The similitude (ψ, λ) is defined uniquely up to replacement of (ψ, λ) by $(-\psi, -\lambda)$.*

1.5.1. Proof of Theorem 1.3

Proof of Theorem 1.3a). (i) Let $\mathfrak{a} = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R\}$ be an equipped root system, W_0 the Weyl group of R. Clearly, W_0 normalizes T (see 1.1) and TW_0 is contained in $W(\mathfrak{a})$. Since TW_0 contains all the reflections $r(\alpha, \tau)$, we see that $TW_0 = W(\mathfrak{a})$.

(ii) Let $\alpha_1, \dots, \alpha_\ell$ be the base (system of simple roots) of R. Denote: $T_\alpha = \{\mathfrak{a}_\alpha h_\alpha\}$. Let us prove that the group $T' = \oplus T_\alpha$ coincides with T. Each root $\alpha \in R$ is of the form $\alpha = w\alpha_i$, where $w \in W_0$, and $T_{w\alpha_i} = wT_{\alpha_i}$; and hence it suffices to verify that T' is W_0-invariant. Since reflections r_α, where α is a simple root, generate W_0, it suffices to verify that $r_\alpha(T') \subset T'$.

If $\tau \in \mathfrak{a}_\beta$, then

$$(1 - r_\alpha)(\tau h_\beta) = (\alpha(h_\beta)\tau)h_\alpha \in \mathfrak{a}_\alpha h_\alpha$$

by property of any equipped root systems. Hence, $(1 - r_\alpha)T_\beta \subset T_\alpha$ implying $r_\alpha T' \subset T'$.

(iii) It follows from (ii) that T is a lattice in L of full rank. Since $W = T \ltimes W_0$, where W_0 is a finite group, we deduce that W is a complex crystallographic group. Since W is generated by reflections and $dW \cong W_0$, it follows that W is a complex crystallographic Coxeter group.

Now, let us prove that $\Pi(W) \subset \Pi(\mathfrak{a})$. Let $r \in \mathrm{Ref}(W)$. It is straightforward that r has the unique decomposition as $r = tr_\alpha$, with $\alpha \in R$ and $t \in T_\alpha$. Without loss of generality we may assume that α is a simple root. Then step (ii) implies that $t \in T_\alpha$, and hence $\pi(r) \in \Pi(\mathfrak{a})$.

Proof of Theorem 1.3b). Let W be a complex crystallographic Coxeter group, $x_0 \in V$ its special point, $L = L(V)$ and $W_0 = W_{x_0}$.

(i) Since W_0 is a Coxeter group, there exists a real W_0-invariant subspace $L_{\mathbb{R}}$ such that $L = \mathbb{C} \otimes_{\mathbb{R}} L_{\mathbb{R}}$. Multiplying $L_{\mathbb{R}}$ by a number $\lambda \in \mathbb{C}^*$ we may assume that $T \cap L_{\mathbb{R}} \neq 0$. This follows from the fact that, for any reflection $r \in R(W)$, the intersection of each of the groups T and $L_{\mathbb{R}}$ with the one-dimensional space $L_r := (1 - r)L$ is non-zero. Indeed,

$$T \supset (1 - r)T \neq 0 \quad \text{ and } L_{\mathbb{R}} \supset (1 - r)L_r \neq 0.$$

Denote:

$$N = T \cap L_{\mathbb{R}}.$$

Clearly, N is a W_0-invariant discrete subgroup of $L_{\mathbb{R}}$. Since the W_0-action on $L_{\mathbb{R}}$ is irreducible, we have $L_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$.

(ii) Let $r \in R(W_0)$. Set

$$N_r = N \cap L_r.$$

Since

$$0 \neq (r - 1)N \subset N_r \subset (r - 1)L_{\mathbb{R}},$$

we have $N_r \cong \mathbb{Z}$. Let h be any generator of N_r and α the functional on L such that

$$\alpha(\pi(r)) = 0 \text{ and } \alpha(h) = 2. \tag{1}$$

For all pairs (r, h), denote by R the set of all functionals α satisfying (1). For the functional $\alpha \in R$ corresponding by means of (1) to the pair (r, h), we will write

$$r_\alpha = r, \quad h_\alpha = h, \quad \pi_\alpha = \pi(r), \quad N_\alpha := N_r, \quad T_\alpha := T_r = T \cap L_r.$$

By definition

$$r_\alpha(x) = x - \alpha(x)h_\alpha \text{ and } r_\alpha(f) = f - f(h_\alpha)\alpha \text{ for any } x \in L, \ f \in L^*.$$

(iii) Let us prove that R is a root system in $L^*_{\mathbb{R}}$ with the Weyl group W_0. Since the lattice N is W_0-invariant, so is R. Hence, it suffices to verify that $\beta(h_\alpha) \in \mathbb{Z}$ for any $\alpha, \beta \in R$. Indeed,

$$\beta(h_\alpha) = \frac{1}{2}\beta((1 - r_\beta)h_\alpha) \in \frac{1}{2}\beta(N_\beta) = \mathbb{Z}.$$

For each root $\alpha \in R$, set

$$\Pi_\alpha = \{\pi \in \Pi(W) \mid \pi \parallel \pi_\alpha\}.$$

Define a subset of $\mathbb{C}$:

$$\mathfrak{a}_\alpha := \{\alpha(\pi) \mid \pi \in \Pi_\alpha\}.$$

Lemma 1.2.2 implies that $\mathfrak{a}_\alpha = \{\tau \in \mathbb{C} \mid \tau h_\alpha \in T_\alpha\}$, hence $\mathfrak{a}_\alpha$ is a rank 2 lattice in $\mathbb{C}$.

If $|\alpha| = |\beta|$, then $\beta = w\alpha$ for some $w \in W_0$. But then $h_\beta = wh_\alpha$ and $T_\beta = wT_\alpha$, so $\mathfrak{a}_\beta = \mathfrak{a}_\alpha$.

Let now $|\beta|^2 = p|\alpha|^2$, where $p = 2, 3$. By replacing the root α by a root of the same length, we may assume that

$$\beta(h_\alpha) = p \text{ and } \alpha(h_\beta) = 1.$$

Since

$$(1 - r_\alpha)T_\beta \subset T_\alpha, \quad (1 - r_\beta)T_\alpha \subset T_\beta, \quad (1 - r_\alpha)h_\beta = h_\alpha, \quad (1 - r_\beta)h_\alpha = ph_\beta,$$

it follows that $p\mathfrak{a}_\alpha \subset \mathfrak{a}_\beta \subset \mathfrak{a}_\alpha$. Furthermore, by construction, $\mathfrak{a}_\alpha \cap \mathbb{R} = \mathfrak{a}_\beta \cap \mathbb{R} = \mathbb{Z}$, so $\mathfrak{a}_\beta \neq p\mathfrak{a}_\alpha$. Thus, $\mathfrak{a} = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R\}$ is an equipped root system.

(iv) By definition, $\Pi(\mathfrak{a}) = \Pi(W)$, implying $W = W(\mathfrak{a})$. ■

1.5.2. Proof of Theorem 1.4

Let $\varphi : V \longrightarrow V'$ be a group isomorphism of $W(\mathfrak{a})$ and $W(\mathfrak{a}')$. Since φ transforms the fixed special point into the fixed one, we may assume that φ is a linear operator $\varphi : L \longrightarrow L'$.

Fix a long root $\beta \in R$. Since φ is an isomorphism, $\varphi(\pi_\beta) \in \Pi(W(\mathfrak{a}'))$. Hence there is a root $\beta' \in R'$, such that $\pi_{\beta'} = \varphi(\pi_\beta)$. This means that

$$\lambda\varphi(h_\beta) = h_{\beta'} \text{ for some } \lambda \in \mathbb{C}^*.$$

Since $W_0(h_\beta)$ generates $L_{\mathbb{R}}$ over $\mathbb{R}$, we see that $\lambda\varphi(L_{\mathbb{R}}) = L'_{\mathbb{R}}$.

Having identified $L_{\mathbb{R}}$ with $L'_{\mathbb{R}}$ by means of $\lambda\varphi$ and having replaced $\mathfrak{a}_{\alpha'}$ by $\lambda\mathfrak{a}_{\alpha'}$ we may assume that from the very beginning

$$L = L', \quad L_{\mathbb{R}} = L'_{\mathbb{R}}, \quad \varphi = \mathrm{id}, \quad W = W(\mathfrak{a}) = W(\mathfrak{a}'), \quad \beta' = \beta \in R'.$$

Let us show that either $\mathfrak{a}' = \mathfrak{a}$ or $\mathfrak{a}' = \mathfrak{a}^{inv}$. Clearly, since $r_\alpha \in W(\mathfrak{a}')$, it follows that $\alpha' = \lambda\alpha \in R'$ for some $\lambda > 0$; moreover, for this λ, we have

$$\mathfrak{a}_{\alpha'} = \alpha'(\Pi_\alpha) = \lambda\alpha(\Pi_\alpha) = \lambda\mathfrak{a}_\alpha.$$

If $R' = R$, then $\mathfrak{a}' = \mathfrak{a}$. Let now $R' \neq R$. Since R and R' are W_0-invariant, have the same number of roots, and $\beta \in R'$, it follows that none of the short roots $\alpha \in R$ belongs to R'. Let α be a short root such that $\alpha(h_\beta) = 1$.

Let us prove that $\mathfrak{a}_\alpha \neq \mathfrak{a}_\beta$. Indeed, let $\lambda > 0$ be such that $\alpha' = \lambda\alpha \in R'$. Then, since

$$\lambda = \alpha'(h_\beta) = 0, \quad \pm 1, \quad \pm p(R),$$

it follows that $\lambda = p(R) > 1$. Hence, $\mathfrak{a}_{\alpha'} = p\mathfrak{a}_\alpha$, where $p = p(R)$. If $\mathfrak{a}_\alpha = \mathfrak{a}_\beta$, then $\mathfrak{a}_{\alpha'} = p\mathfrak{a}_\alpha = p\mathfrak{a}_\beta = p\mathfrak{a}_{\beta'}$ contradicting the definition of equipped root systems.

Thus, $\mathfrak{a}_\alpha \neq \mathfrak{a}_\beta$, i.e., $p(\mathfrak{a}) = p > 1$. Consider the system R^{inv}. We see that $\beta^{inv} = \beta$ and $\alpha^{inv} = p\alpha$ belong to R'. By W_0-invariance this implies $R^{inv} = R'$. As above, we deduce that $\mathfrak{a}^{inv} = \mathfrak{a}'$. ■

§2 Affine root systems

Let V be a real ℓ-dimensional affine space, $L = L(V)$ the linear space of translations of V. Let $\mathrm{Aff}(V)$ be the space of affine-linear functions on V, i.e., polynomial functions of degree ≤ 1. Let $\mathrm{Const} \subset \mathrm{Aff}(V)$ be the subspace of constant functions. The action of the group L on V and on $\mathrm{Aff}(V)$ is given by the formulas

$$t(x) = x + t, \quad (t\varphi)(x) = \varphi(x - t) \text{ for any } x \in V \text{ and } \varphi \in \mathrm{Aff}(V).$$

For any $\varphi \in \mathrm{Aff}(V)$, let $\tilde{\varphi}$ denote the linear part of φ, i.e., $\tilde{\varphi} \in L^*$. We will assume that V is endowed with a Euclidean space structure. Then L and L^* are also endowed with a Euclidean space structure. We define a semi-norm on $\mathrm{Aff}(V)$ by setting $|\varphi| = |\tilde{\varphi}|$.

2.1. Root systems

For any non-constant function $\alpha \in \mathrm{Aff}(V)$, denote by

$$\pi_\alpha = \{x \in V \mid \alpha(x) = 0\}$$

the hyperplane corresponding to α and by r_α the reflection in π_α. Let $h_\alpha \in L$ be the vector orthogonal to π_α and such that $\tilde{\alpha}(h_\alpha) = 2$. Then r_α is defined by the formulas

$$r_\alpha(x) = x - \alpha(x)h_\alpha, \qquad r_\alpha(\varphi) = \varphi - \tilde{\varphi}(h_\alpha)\alpha, \text{ for any } x \in V,\ \varphi \in \mathrm{Aff}(V).$$

A *root system* is a finite subset $R \subset L^* \setminus \{0\}$ satisfying the following condition:

$$r_\alpha(\beta) \in R \text{ and } \beta(h_\alpha) \in \mathbb{Z} \text{ for any } \alpha, \beta \in R. \tag{2}$$

Denote by W_R the group generated by reflections r_α, where $\alpha \in R$. This group is called the *Weyl group* of the root system R.

The root system R is said to be *irreducible* if W_R irreducibly acts on L and R is said to be *reduced* if $R \cap 2R = \emptyset$.

For any root $\alpha \in R$, set

$$\alpha^\vee = \begin{cases} p(R)\alpha & \text{if } \alpha \text{ is short,} \\ \alpha & \text{if } \alpha \text{ is long.} \end{cases}$$

The system $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is called the *dual root system*. Observe that $(R^\vee)^\vee = p(R)R$.

2.2. Affine root systems

An *affine root system* is a subset $S \subset \mathrm{Aff}(V) \setminus \mathrm{Const}$ satisfying the following conditions:

a) if $\alpha, \beta \in S$, then $r_\alpha(\beta) \in S$ and $\tilde{\beta}(h_\alpha) \in \mathbb{Z}$;

b) the group W_S generated by reflections r_α, where $\alpha \in S$, discretely acts on V and the quotient space V/W_S is compact.

The group W_S is called the *Weyl group* of the affine root system S.

For an affine root system S, set $dS = \{\tilde{\alpha} \mid \alpha \in S\}$. Then dS is a root system.

An affine root system S is called *irreducible* if so is dS, *reduced* if $S \cap 2S = \emptyset$ and *completely reduced* if dS is reduced.

Denote by $\Gamma(S)$ the subgroup in $\mathrm{Aff}(V)$ generated by S. The group $\Gamma(S)$ always contains a non-zero constant function. The system S is said to be *normalized* if $\Gamma(S) \cap \mathrm{Const} = \mathbb{Z} \cdot 1$. Any irreducible system S may be multiplied by a constant to make it normalized (see [8]).

In what follows, all affine root systems S are assumed to be irreducible and normalized.

2.3. Special points

A point $x \in V$ is said to be *special* for S if $S_x = \{\alpha \in S \mid \alpha(x) = 0\}$ is isomorphic to dS. Note, that this definition differs somewhat from the definition in [8]. Any completely reduced system S has a special point x (see [8]). Usually, to identify V with L and S_x with dS, we fix a special point x.

Let R be an irreducible reduced root system, let p be either 1 or $p(R)$. We construct an affine root system $S = S(R, p)$ as follows. For any root $\alpha \in R$, define p_α by setting:

$$p_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is short,} \\ p & \text{if } \alpha \text{ is long.} \end{cases}$$

Observe that $V \simeq L$ and set

$$S = \{\alpha + p_\alpha l \mid \alpha \in R,\ \ l \in \mathbb{Z}\} \subset \mathrm{Aff}(V).$$

It is easy to verify that S is a normalized affine root system, $dS = R$ and 0 is a special point for S.

Conversely, let S be an irreducible completely reduced normalized affine root system, and fix x_0, a special point for S. Let $R = S_{x_0} = dS$ and let $p = p(S)$ be the least positive number such that $S + p = S$. Then $p = 1$, 2, or 3 and $S = S(R, p)$.

For $S = S(R, p)$, define the root system S^{inv} as follows. Set

$$p_\alpha^{\text{inv}} = \frac{p}{p_\alpha}, \quad \alpha^{\text{inv}} = p_\alpha^{\text{inv}}\alpha, \quad R^{\text{inv}} = \{\alpha^{\text{inv}} \mid \alpha \in R\},$$

$$S^{\text{inv}} = S(R^{\text{inv}}, p) = \{\alpha^{\text{inv}} + p_\alpha^{\text{inv}} l \mid \alpha^{\text{inv}} \in R^{\text{inv}},\ l \in \mathbb{Z}\}.$$

If $p = 1$, then $S^{\text{inv}} = S$ and if $p > 1$, then $R^{\text{inv}} = R^\vee$.

These statements follow from Macdonald's results [8]. In Macdonald's terms, we have, up to a similitude,

$$S(R, 1) \sim S(R), \qquad S(R^\vee, p(R)) \sim (S(R))^\vee,$$

$$S(R)^{\text{inv}} \sim S(R), \qquad (S(R)^\vee)^{\text{inv}} \sim S(R^\vee)^\vee.$$

2.4. Bases and chambers

For the proof of statements of this subsection, see [8].

Let S be an irreducible affine root system. Fix a *chamber* C, i.e., a connected component of the set $V \setminus \underset{\alpha \in S}{\cup} \pi_\alpha$. Then C is an open simplex (and the closure $\bar{C}$ of C is a closed simplex). Denote the vertices of $\bar{C}$ by $x_0, x_1, \dots, x_\ell$.

Let $B(S) := B(S, C)$ be the set of indecomposable roots $\alpha \in S$ such that $\alpha(C) > 0$ and let π_α be the facets of C. The set $B(S)$ consists of $\ell + 1$ elements $\alpha_0, \dots, \alpha_\ell$ numbered so that $\alpha_i(x_j) = 0$ for $i \neq j$. Let S^+ be the subset of positive roots.

The set $B(S)$ is called a *base* of S; the elements of $B(S)$ are called *simple roots*. Set $h_i = h_{\alpha_i}$ and $r_i = r_{\alpha_i}$. The reflections r_i are called *simple reflections*. Here are several properties of simple roots and simple reflections:

a) The reflections r_i generate W_S.

b) $W_S\bar{C} = V$.

c) If $wC = C$, then $w = 1$. Moreover,

$$|x - y| \leq |x - wy| \ \text{ for any } x \in C,\ y \in \bar{C},\ w \in W_S$$

and the equality is only attained when $wy = y$.

d) Any indecomposable root $\alpha \in S$ can be represented in the form $w\alpha_i$, where $w \in W_S$ and $\alpha_i \in B(S)$.

e) The roots α_i are linearly independent and any root $\alpha \in S$ is representable in the form $\sum k_i\alpha_i$, where $k_i \in \mathbb{Z}$, and, for all i, either $k_i \geq 0$ (then α is said to be a *positive root*) or $k_i \leq 0$ (then α is said to be a *negative root*).

Example. Let R be an irreducible reduced root system, $\alpha_1, \dots, \alpha_\ell$ the system of simple roots in R. Consider a root $\varphi \in R$ such that $\varphi(h_{\alpha_i}) \geq 0$ for $i = 1, \dots, \ell$. Each orbit of the group W_R in R has exactly one such root; choose among them the long and short roots φ_l and φ_s respectively.

The set $\{\alpha_0 = 1 - \varphi_l, \alpha_1, \dots, \alpha_\ell\}$ is a base of $S(R, 1)$.

The set $\{\alpha_0 = 1 - \varphi_s, \alpha_1, \dots, \alpha_\ell\}$ is a base of $S(R, p(R))$.

The chamber C is singled out by conditions $\alpha_i(x) > 0$, where $i = 0, 1, \dots, \ell$.

2.5. Dual affine root systems

Let S be an irreducible normalized affine root system. Denote by $Q = Q_S(V)$ the space of quadratic functions U on V satisfying

$$(wU - U) \in \mathrm{Aff}(V) \text{ for any } w \in W_S$$

and set

$$\bar{Q} = Q/\mathrm{Const}.$$

Clearly, $Q \supset \mathrm{Aff}(V)$. By assigning to any $U \in Q$ the corresponding quadratic form $\tilde{U}$ on L we establish an isomorphism of $Q/\mathrm{Aff}(V)$ with the space of dW-invariant quadratic forms on L.

Since the dW-action on L is irreducible, all dW-invariant quadratic forms on L are proportional to a fixed form B that determines the metric on V. The proportionality coefficient

$$\kappa(U) = -\frac{\tilde{U}}{B}$$

(here the minus sign is chosen to simplify the subsequent formulas) determines a functional κ on the space Q; the kernel of κ coincides with $\mathrm{Aff}(V)$.

Let $\alpha \in S$ and $U \in Q$. Then $U - r_\alpha U \in \mathrm{Aff}(V)$ is a multiple of α since both vanish on the hyperplane π_α.

Define the linear functional $\alpha^\vee \in Q^*$ by setting

$$\alpha^\vee(U) = \frac{U - r_\alpha U}{\alpha} \in \mathbb{R}.$$

Observe that, on the subspace $\mathrm{Aff}(V) \subset Q$, the functional $\alpha^\vee$ is given by the formula

$$\alpha^\vee(\varphi) = \tilde{\varphi}(h_\alpha).$$

Since $\alpha^\vee(\mathrm{Const}) = 0$, we may consider $\alpha^\vee$ as a functional on $\bar{Q} = Q/\mathrm{Const}$. Set

$$S^\vee = \{\alpha^\vee \mid \alpha \in S\} \subset \bar{Q}^*.$$

Set

$$V^\vee \simeq \{U \in \bar{Q} \mid \kappa(U) = 1\}.$$

The spaces $L(V^\vee)$ and $\mathrm{Aff}(V^\vee)$ are naturally identified with L^* and $\bar{Q}^*$, respectively. We have

$$r_\alpha(U) = U - \alpha^\vee(U)\tilde{\alpha} \text{ for any } U \in V^\vee.$$

This implies that $S^\vee \subset \bar{Q}^* = \mathrm{Aff}(V^\vee)$ is an affine root system on $V^\vee$.

It is easy to verify that the root system $S^\vee$ is irreducible (since $W_{S^\vee} = W_S$). Hence by replacing the form B that determines the metric on V by the form λB for some $\lambda > 0$ we may assume that $S^\vee$ is normalized (see sec. 2.2). The normalization condition uniquely determines the form λB. The corresponding metric is called the *canonical metric associated with the affine root system S* and denoted by $\|\cdot\|$.

In the sequel we will always assume that the Euclidean metric on V is canonical and κ is defined with respect to this metric.

§3 Complex crystallographic Coxeter groups and affine root systems

We have classified complex crystallographic Coxeter groups in terms of equipped root systems. Let us now describe another classification: in terms of affine root systems.

Let S be an irreducible completely reduced normalized affine root system in a real affine space $V_{\mathbb{R}}$ (see sec. 1.2). Consider S as a system of affine-linear functions on the complexification V of the space $V_{\mathbb{R}}$. Fix $\tau \in \mathbb{C}$ such that $\operatorname{Im} \tau > 0$. For any $\alpha \in S$ and $k \in \mathbb{Z}$, set

$$\pi(\alpha, k) = \{z \in V \mid \tau\alpha(z) = k\};$$

$$\Pi(S, \tau) = \{\text{all the hyperplanes } \pi(\alpha, k)\};$$

$$W(S, \tau) = \text{the group generated by the reflections in hyperplanes } \pi(\alpha, k).$$

3.1. Theorem. a) *$W(S,\tau)$ is a complex crystallographic Coxeter group, and $\Pi(S,\tau) = \Pi(W(S,\tau))$. If $x_0 \in V_{\mathbb{R}}$ is a special point for S, then it is a special point for $W(S,\tau)$, too.*

b) *Any irreducible complex crystallographic Coxeter group W is isomorphic to a group of the form $W(S,\tau)$.*

To describe groups of the form $W(S,\tau)$ up to an isomorphism, introduce several notations. Fix a special point x_0 for the system S which defines the splitting of $W(S,\tau)$. With the help of this point we will identify $L_{\mathbb{R}}$ with $V_{\mathbb{R}}$ and L with V. Define the number $p = p(S)$ (= 1, 2 or 3) and the affine root system S^{inv} as in sec. 1.3. In the group $GL^+(2;\mathbb{R})$ of 2×2 matrices with positive determinant, consider the element $\gamma_p = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ and the subgroup

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, \quad c \in p\mathbb{Z}, \quad ad - bc = 1 \right\}.$$

It turns out that $W(S,\tau) \cong W(S',\tau')$ if and only if, for some $\gamma \in \Gamma_0(p)$,

$$\begin{array}{lll} \text{either} & S' \cong S & \text{and } \tau' = \gamma\tau \\ \text{or} & S' \cong S^{inv} & \text{and } \tau' = \gamma(\gamma_p\tau), \end{array} \tag{3}$$

where

$$\gamma\tau = \frac{a\tau + b}{c\tau + d} \quad \text{for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{R}).$$

Let us make the statement (3) more precise.

3.2. Theorem. a) *If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$, then the operator*

$$\varphi_\gamma : L \longrightarrow L, \quad \varphi_\gamma(z) = \frac{z}{a + b\tau^{-1}}$$

defines an isomorphism of split complex crystallographic Coxeter groups

$$\varphi_\gamma : W(S,\tau) \tilde{\longrightarrow} W(S, \gamma\tau).$$

The operator

$$\varphi_{\gamma_p} : L \longrightarrow L, \qquad \varphi_{\gamma_p}(z) = \tau z$$

determines an isomorphism of split complex crystallographic Coxeter groups

$$\varphi_{\gamma_p} : W(S,\tau) \xrightarrow{\sim} W(S^{inv}, \gamma_p\tau).$$

b) *Any isomorphism* $\varphi : W(S_1,\tau_1) \longrightarrow W(S_2,\tau_2)$ *of split complex crystallographic Coxeter groups factorizes into the composition*

$$W(S_1,\tau_1) \xrightarrow{\varphi'} W(S',\tau_2) \xrightarrow{\psi} W(S_2,\tau_2),$$

where ψ *is an isomorphism induced by the isomorphism of affine root systems* $\psi : S' \longrightarrow S_2$ *and, for some* $\gamma \in \Gamma_0(p)$,

$$\begin{array}{llll} \textit{either} & S' = S_1, & \tau_2 = \gamma(\tau_1), & \varphi' = \varphi_\gamma \\ \textit{or} & S' = S_1^{inv}, & \tau_2 = \gamma(\gamma_p(\tau_1)), & \varphi' = \varphi_\gamma\varphi_{\gamma_p}. \end{array}$$

Remark. In sec. 4.2 we will assigne to any affine root system S the set of numbers $(n_0, n_1, \dots, n_\ell)$. It is easy to verify that for systems S and $S^{\rm inv}$ these sets coincide:

$$\begin{aligned} &S^{\rm inv} \simeq S \text{ if } S \not\simeq S(B_\ell, 2) \text{ or } S(C_\ell, 2), \\ &S^{\rm inv}(B_\ell, 2) \simeq S(C_\ell, 2) \end{aligned}$$

and, for these systems,

$$(n_0, \dots, n_\ell) = (1, 1, 2, 2, \dots, 2),$$

see Appendix 1 to [8]; the numbers n_i for the systems $S(B_\ell, 2)$ and $S(C_\ell, 2)$ are the numbers assigned there to Dynkin diagrams of the systems $C_\ell \simeq S(B_\ell, 2)^\vee$ and $C_\ell \simeq S(C_\ell, 2)^\vee$. Therefore, Theorem 3.2 implies that, for the group $W = W(S,\tau)$, the set $\{n_0, \dots, n_\ell\}$ depends only on W.

3.3.1. Proof of Theorem 3.1

Proof of 3.1a). Let x_0 be a special point for S and $R = \{\alpha \in S \mid \alpha(x_0) = 0\}$. If we identify V with $L = L(V)$ taking x_0 for the origin 0, we see that R is a finite irreducible reduced root system.

Let us construct an equipped root system $\mathfrak{a} = \mathfrak{a}(S,\tau) = \{R, \mathfrak{a}_\alpha \mid \alpha \in R\}$ such that $\Pi(S,\tau) = \Pi(\mathfrak{a})$. Due to sec.2.3 S is of the form

$$S(R,p) = \{\alpha + p_\alpha l \mid \alpha \in R,\ l \in \mathbb{Z}\}, \ \text{ where } p = 1 \text{ or } p(R).$$

Set

$$\mathfrak{a}_\alpha = p_\alpha\mathbb{Z} + \tau^{-1}\mathbb{Z}.$$

It is clear that $\mathfrak{a} = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R\}$ is an equipped root system (such that $p(\mathfrak{a}) = p(S)$). The hyperplanes from $\Pi(S,\tau)$ are described by equations of the form

$$\tau(\alpha(z) + p_\alpha l) = k, \qquad \text{where} \qquad l, k \in \mathbb{Z}$$

or, equivalently, by equations

$$\alpha(z) = -p_\alpha l + \tau^{-1}k.$$

This means that $\Pi(S,\tau) = \Pi(\mathfrak{a})$.

Heading a) follows now from Theorem 1.3a) and the equality $\Pi(S,\tau) = \Pi(\mathfrak{a})$.

Proof of 3.1b). Let W be an indecomposable complex crystallographic Coxeter group. Then by Theorem 1.3b), $W \cong W(\mathfrak{a})$. First, consider the case $p(\mathfrak{a}) = 1$, i.e., the case where all the lattices $\mathfrak{a}_\alpha$ coincide. Let $\{\lambda, \mu\}$ be a basis of $\mathfrak{a}_\alpha$. By multiplying $\mathfrak{a}_\alpha$ by λ^{-1} (this replaces $W(\mathfrak{a})$ by an isomorphic group) we may assume that $\lambda = 1$. Furthermore, by replacing, if necessary, μ by $-\mu$ we may assume that $\operatorname{Im}\mu > 0$. Thus,

$$\mathfrak{a}_\alpha = \mathbb{Z} + \tau^{-1}\mathbb{Z}, \text{ where } \tau = -\mu^{-1} \text{ and } \tau \text{ belongs to upper half plane } H.$$

Hence $\mathfrak{a} = \mathfrak{a}(S,\tau)$, where $S = S(R,1)$.

If $p = p(\mathfrak{a}) = 2,\ 3$, then there are lattices of the two types: $\mathfrak{a}_s$ and $\mathfrak{a}_l$ such that $p\mathfrak{a}_s \subset \mathfrak{a}_l \subset \mathfrak{a}_s$, where the inclusions are strict.

Since p is prime, there is a basis (λ,μ) of $\mathfrak{a}_s$, such that $\{p\lambda, \mu\}$ is a basis of $\mathfrak{a}_l$. The same arguments as above enable us to assume that $\mathfrak{a}_s = \mathbb{Z} + \tau^{-1}\mathbb{Z}$ and $\mathfrak{a}_l = p\mathbb{Z} + \tau^{-1}\mathbb{Z}$. Then $\mathfrak{a} = \mathfrak{a}(S,\tau)$, where $S = S(R,p)$.

Thus, we have shown that $\Pi(\mathfrak{a})$ is isomorphic to $\Pi(S,\tau)$ for suitable S and τ, so that $W \simeq W(\mathfrak{a}) \simeq W(S,\tau)$.

3.3.2. Proof of Theorem 3.2

(i) Let $S = S(R,p)$. Then $S^{\mathrm{inv}} = S(R^{\mathrm{inv}}, p)$, where

$$R^{\mathrm{inv}} = \{\alpha^{\mathrm{inv}} \mid \alpha \in R\} \text{ for } \alpha^{\mathrm{inv}} = p_\alpha^{\mathrm{inv}}\alpha,\ \ p_\alpha^{\mathrm{inv}} = \tfrac{p}{p_\alpha}$$

(see sec. 2.3). At the proof of Theorem 3.1a) we constructed an equipped root system

$$\mathfrak{a}(S,\tau) = \{(R, \mathfrak{a}_\alpha) \mid \mathfrak{a}_\alpha = p_\alpha\mathbb{Z} + \tau^{-1}\mathbb{Z}\}$$

with the property

$$\Pi(S,\tau) = \Pi(\mathfrak{a}(S,\tau)).$$

Define the inverse equipped system

$$\mathfrak{a}^{\mathrm{inv}}(S,\tau) = \{(R^{\mathrm{inv}}, \mathfrak{a}_\alpha^{\mathrm{inv}}) \mid \alpha \in R \quad \text{and} \quad \mathfrak{a}_\alpha^{\mathrm{inv}} = p_\alpha^{\mathrm{inv}}\mathfrak{a}_\alpha = p\mathbb{Z} + p_\alpha^{\mathrm{inv}}\tau^{-1}\mathbb{Z}\};$$

it has the same property. On the other hand, we have

$$\mathfrak{a}(S^{\mathrm{inv}}, \gamma_p(\tau)) = \{(R^{\mathrm{inv}}, \mathfrak{a}'_\alpha) \mid \alpha \in R \text{ and } \mathfrak{a}'_\alpha = p_\alpha^{\mathrm{inv}}\mathbb{Z} + \gamma_p(\tau)^{-1}\mathbb{Z} = p_\alpha^{\mathrm{inv}}\mathbb{Z} + p\tau\mathbb{Z}\}.$$

(ii) Let $S = S(R,p)$. Then

$$\mathfrak{a}(S,\tau) = \{(R, \mathfrak{a}_\alpha) \mid \alpha \in R \text{ and } \mathfrak{a}_\alpha = p_\alpha\mathbb{Z} + \tau^{-1}\mathbb{Z}\}.$$

Let us describe all the triples (λ, τ, τ'), where $\lambda \in \mathbb{C}^*$ and $\tau, \tau' \in H$, such that

$$(\mathrm{id}, \lambda) : \mathfrak{a}(S,\tau) \simeq \mathfrak{a}(S,\tau').$$

i.e.,

$$\lambda(p_\alpha\mathbb{Z}+\tau^{-1}\mathbb{Z}) = p_\alpha\mathbb{Z}+\tau'^{-1}\mathbb{Z} \qquad \text{for any } \alpha. \tag{4}$$

Since the values of p_α are 1 and p, the condition (4) is equivalent to the following conditions:

$$\lambda = a+b\tau^{-1} \text{ and } \lambda(\tau')^{-1} = c+d\tau^{-1}, \quad \text{where} \quad a,b,c,d\in\mathbb{Z},\ c\in p\mathbb{Z},\ ad-bc=\pm 1.$$

This implies that

$$\tau' = \frac{a-b\tau^{-1}}{c+d\tau^{-1}} = \frac{a\tau+b}{c\tau+d} = \gamma(\tau) \ \text{ for } \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\operatorname{Im}\tau>0$ and $\operatorname{Im}\tau'>0$, we see that $ad-bc=1$, and hence $\gamma\in\Gamma_0(p)$.

Conversely, to each pair (γ,τ), where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ and $\tau\in H$, there corresponds the triple (λ,τ,τ'), where $\lambda = a+b\tau^{-1}$ and $\tau'=\gamma(\tau)$.

(iii) It follows from (ii) that the multiplication by $\lambda^{-1}=(a+b\tau^{-1})^{-1}$ determines an isomorphism $\varphi_\gamma : W(S,\tau)\longrightarrow W(S,\gamma(\tau))$ for any $\gamma\in\Gamma_0(p)$. This and (i) imply Theorem 3.2a).

(iv) Let $S_i = S(R_i,p_i)$ for $i=1,2$ and let $\varphi : W(S_1,\tau_1)\longrightarrow W(S_2,\tau_2)$ be an isomorphism. By Theorem 1.4 φ is determined by either

(1) the similitude (ψ,λ) of $\mathfrak{a}(S_1,\tau_1)$ with $\mathfrak{a}(S_2,\tau_2)$

or by

(2) the similitude (ψ,λ) of $\mathfrak{a}^{inv}(S_1,\tau_1)$ with $\mathfrak{a}(S_2,\tau_2)$.

In case (1), ψ defines the isomorphism $\psi : R_1\longrightarrow R_2$, hence, the isomorphism $\psi : S_1\longrightarrow S_2$, because

$$p(S_1) = p(\mathfrak{a}(S_1,\tau_1)) = p(\mathfrak{a}(S_2,\tau_2)) = p(S_2).$$

Having corrected φ with the help of this isomorphism, we may assume that $S_1=S_2$, $\psi=\mathrm{id}$ and φ is the multiplication by λ. Step (ii) implies that $\varphi=\varphi_\gamma$ for some $\gamma\in\Gamma_0(p)$.

In case (2), consider the homomorphism $\varphi' = \varphi_0(\varphi_{\gamma_p})^{-1}$ determined by the similitude of $\mathfrak{a}(S_1^{\mathrm{inv}},\gamma_p(\tau_1))$ with $\mathfrak{a}(S_2,\tau_2)$. Hence, φ' is of the form $\psi\circ\varphi_\gamma$ for some $\gamma\in\Gamma_0(p)$. Thus, $\varphi=\psi\circ\varphi_\gamma\circ\varphi_{\gamma_p}$. ∎

§4 Appendix

The results collected in this Appendix will be needed in the sequel to this paper, but are also of independent interest.

4.1. Root systems and their dual

Fix a quadratic function $U\in Q\setminus\operatorname{Aff}(V)$. Clearly, U takes its extremal value (the maximal for $\kappa(U)>0$, the minimal for $\kappa(U)<0$) at some uniquely defined point of V. Denote this point by x_U and call it the *center* of U. Set $M_U=U(x_U)$. By definition

$$U(x) = -\kappa(U)\parallel x-x_U\parallel^2 + M_U.$$

Observe that x_U only depends on the image of U in $\bar{Q}$, so x_U is defined for any $U \in \bar{Q} \setminus L^*$. The map $U \mapsto x_U$ defines an affine-linear isomorphism $V^\vee \cong V$. If we identify V with $V^\vee$ with respect to this isomorphism, then, for any $\alpha \in S$, the function $\alpha^\vee$ is defined by the formula

$$\alpha^\vee = \| h_\alpha \|^2 \alpha.$$

Indeed,

$$\alpha^\vee(x_U) = \alpha^\vee(U) = \frac{\| x - r_\alpha(x_U) \|^2 - \| x - x_U \|^2}{\alpha(x)} = \frac{\| x_U - r_\alpha(x_U) \|^2}{\alpha(x_U)} = \alpha(x_U) \| h_\alpha \|^2 .$$

Example. Let S be completely reduced. Then $S = S(R, p)$, and we may set

$$B(S) = \alpha_0 = 1 - \varphi,\ \alpha_1,\ \ldots,\ \alpha_\ell,$$

where $\alpha_1, \ldots, \alpha_\ell$ is a base (system of simple roots) of R (see sec. 2.4). Since $W_S = W_{S^\vee}$, we see that the roots $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee$ form a base of $S^\vee$. Since $\alpha_i^\vee = \| h_\alpha \|^2 \alpha_i$ and $S^\vee$ is normalized, it follows that the canonical metric on V is defined by the condition

$$\| h_{\alpha_0} \| = \| h_\varphi \| = 1.$$

Equivalently, $\| h_\alpha \|^2 = \frac{1}{p}$ for any long root α. The above formulas imply that

$$S^\vee := S(R, p)^\vee = S\left(\frac{1}{p} R^\vee, \frac{p(R)}{p}\right)$$

and the canonical metrics for S and $S^\vee$ coincide (with respect to the described identification of V with $V^\vee$).

4.2. The fundamental weights

Let S be a completely reduced root system. We call $\lambda \in \bar{Q}$ a *weight* If $\alpha^\vee(\lambda) \in \mathbb{Z}$ for any $\alpha \in S$; let Λ be the group of all weights. Let λ be a weight such that $\kappa(\lambda) > 0$. Denote by $U_\lambda \in Q$ the quadratic function representing the weight λ and normalized by the condition $U_\lambda(x_\lambda) = 0$.

Fix a base $B(S) = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ of S. Then Λ is defined by the system of equations

$$\Lambda = \{\lambda \in \bar{Q} \mid \alpha_i^\vee(\lambda) \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, \ell\}.$$

Indeed, this system of equations is invariant with respect to simple reflections, hence, with respect to W_S, and, since $W_S(B(S)) = S$, this system is equivalent to the initial system

$$\Lambda = \{\lambda \in \bar{Q} \mid \alpha^\vee(\lambda) \in \mathbb{Z} \text{ for any } \alpha \in S\}.$$

Thus, the weights $\lambda_0, \ldots, \lambda_\ell$ defined by the equations $\alpha_i^\vee(\lambda_j) = \delta_{ij}$ form a base of Λ. These weights are called *fundamental*. Define the numbers n_i, where $i = 0, 1, \ldots, \ell$, from the condition

$$\sum n_i \alpha_i^\vee = 1.$$

Then sec. 3.2 implies that $n_0 = 1$ and, for $i > 0$, the numbers n_i are determined from the decomposition

$$\varphi^\vee = \sum n_i \alpha_i^\vee.$$

In particular, all the n_i are positive integers. The equation $\sum n_i \alpha_i^\vee = 1$ on $V^\vee$ turns on $\bar{Q}$ into equality of two functionals:

$$\sum n_i \alpha_i^\vee = \kappa.$$

This fact gives another expression for the n_i:

$$n_i = \kappa(\lambda_i).$$

Set

$$\rho = \sum \lambda_i \tag{5}$$

and

$$g = \kappa(\rho) = \sum n_i. \tag{6}$$

These numbers, by definition, coincide with Macdonald's parameters of $S^\vee$, see [8], except that Macdonald's definition of g depends on the metric. As a result, Macdonald's g differs from ours by the factor $\dfrac{\alpha_0^\vee}{\alpha_0}$.

4.3. Lattices and orbits in root systems

In L^*, consider the lattice Λ_R of weights of the root system R:

$$\Lambda_R = \{\lambda \in L^* \mid \lambda(h_\alpha) \in \mathbb{Z} \text{ for any } \alpha \in R\}.$$

Let us identify L^* with the subspace $\mathrm{Aff}(V)/\mathrm{Const} \subset \bar{Q}$, and thus consider Λ_R as a subgroup of $\bar{Q}$. Since $\lambda(h_\alpha) = \alpha^\vee(\lambda)$ for $\lambda \in L^*$, it follows that $\Lambda_R = L^* \cap \Lambda$.

Let us prove that $\Lambda = \mathbb{Z}\lambda_0 \oplus \Lambda_R$. For this, to each weight λ we assign the functional $\delta_\lambda \in L^*$ by the formula

$$\delta_\lambda = \lambda - \kappa(\lambda)\lambda_0.$$

Clearly, δ_λ lies in the subspace $L^* \subset \bar{Q}$ since $\kappa(\delta_\lambda) = 0$, and δ_λ may be viewed as the differential at x_0 of the quadratic function U corresponding to λ. Since $\delta_\lambda \in \Lambda$, it follows that $\delta_\lambda \subset \Lambda_R$ yielding the decomposition desired:

$$\lambda = \kappa(\lambda)\lambda_0 + \delta_\lambda.$$

It is easy to verify that $\delta_{\lambda_1}, \dots, \delta_{\lambda_\ell} \in \Lambda_R$ are fundamental weights of R.

Further on, denote by Λ^+ the group generated by $\lambda_0, \dots, \lambda_\ell$. For any $k > 0$, set

$$\Lambda_k = \{\lambda \in \Lambda \mid \kappa(\lambda) = k\} = k\lambda_0 + \Lambda_R$$

and

$$\Lambda_k^+ := \Lambda_k \cap \Lambda^+ = \{\lambda = k_i\lambda_i \in \Lambda_k \mid k_i \in \mathbb{Z}^+\}.$$

Clearly, the set Λ_k is W_S-invariant and the set Λ_k^+ is finite.

Lemma. *Each W_S-orbit in Λ_k has exactly one element of Λ_k^+.*

Proof. It is easy to verify that, for any function $U \in Q$ such that $\kappa(U) > 0$, the sign of $\alpha^\vee(U)$ coincides with the sign of $\alpha(x_U)$. Hence if $\lambda \in \Lambda_k$, then $\lambda \in \Lambda_k^+$ if and only if $x_U \in \bar{C}$. Since $\lambda \in \Lambda_k$ is defined by its center x_λ and since $\bar{C}$ is a fundamental domain for W_S (see sec. 2.4), Lemma follows. ■

4.4. The Freudenthal-de Vries formula. Corollaries

Let R be an irreducible reduced root system, let p be equal to either 1 or $p(R)$ and $S = S(R, p)$ (see sec. 2.3). Let t_l and t_s be the number of long and short roots among simple roots $\alpha_1, \dots, \alpha_\ell$; hence, $t_l + t_s = \ell$. (Then the total number of long and short roots in R is equal to ht_l and ht_s, respectively, see [3], [12].) Set

$$r(S) = \sum_{1 \le i \le \ell} p_{\alpha_i} = pt_l + t_s = \ell + (p-1)t_l.$$

Define a quadratic form on L:

$$U_R(x) = \sum_{\alpha \in R} \alpha^2(x).$$

Set (compare with (5))

$$\sigma = \frac{1}{2} \sum_{\alpha \in R^+} \alpha,$$

where R^+ is the set of positive roots in R (with respect to $\alpha_1, \dots, \alpha_\ell$).

Claim. *Let h be the Coxeter number of R. We have*

$$U_\rho = \frac{\ell}{4r(S)} U_R - \sigma + \frac{1}{24}(h+1)r(S).$$

Proof. Set

$$S^0 = \{\alpha \in S \mid 0 \le \alpha(C) \le p_{\tilde{\alpha}}\}.$$

(recall that $\tilde{\alpha} = d\alpha$ It is clear that if $\alpha \in S^0$, then $p_{\tilde{\alpha}} - \alpha \in S^0$ and $\alpha + mp_{\tilde{\alpha}} \notin S^0$ for $m \ne 0$. On V, consider the quadratic function

$$U^0(x) = \frac{1}{4} \sum_{\alpha \in S^0} \frac{(\alpha(x))^2}{p_{\tilde{\alpha}}}.$$

It is easy to verify that

$$r_i U^0 = U^0 + \alpha_i \text{ for } i = 0, 1, \dots, \ell.$$

In other words, $U^0 = U_\rho + c$, where c is a constant. Clearly, $S^0 = \{\alpha, p_\alpha - \alpha \mid \alpha \in R^+\}$. Hence,

$$U^0 = \frac{1}{2} \sum_{\alpha \in R^+} \frac{\alpha^2}{p_\alpha} - \sigma + c = U_\rho + c,$$

where c is a constant. For any quadratic form Q on L, set

$$\text{tr } Q = \sum Q(l_i),$$

where $\{l_1, \ldots, l_\ell\}$ is a basis in L orthonormal with respect to the canonical metric $\tilde{U}^0$ (see 2.5). Clearly,

$$\operatorname{tr}\alpha^2 = \frac{4}{\| h_\alpha \|^2} = \begin{cases} 4p & \text{if } \alpha \text{ is a long root,} \\ \dfrac{4p}{p(R)} & \text{if } \alpha \text{ is a short root.} \end{cases}$$

Since the forms $\tilde{U}^0$, U_R and $\tilde{U}_\rho$ are $W(R)$-invariant, they are proportional. We have

$$\begin{aligned} &\operatorname{tr}\tilde{U}^0 = \ell, \\ &\operatorname{tr}U_R = 4h\left(t_l + \frac{t_s}{p(R)}\right); \\ &\operatorname{tr}\tilde{U}_\rho = h\left(t_l + \frac{pt_s}{p(R)}\right). \end{aligned}$$

This implies that

$$g = \frac{\tilde{U}_\rho}{\tilde{U}^0} = \frac{h}{\ell}\left(t_l + \frac{p}{p(R)t_s}\right),$$

$$\frac{\tilde{U}_\rho}{U_R} = \frac{\frac{1}{4p}\left(t_l + \frac{p}{p(R)}t_s\right)}{t_l + \frac{t_s}{p(R)}} = \frac{\ell}{4r(S)}.$$

Thus,

$$U_\rho = \frac{\ell}{4r(S)}U_R - \sigma + c,$$

where c is a constant. It only remains to show that

$$U_\rho(0) = \frac{h+1}{24}r(S). \tag{7}$$

Let U be a positive definite form on L and $r \in L$. Set

$$U_r(x) = U(x - r).$$

It is easy to verify that

$$U_r(x) = \frac{1}{4}\hat{U}(dU_r(x)),$$

where $\hat{U}$ is the quadratic form on L^* dual to U and $dU_r(x)$ is the differential of U_r at x.

Applying this fact in our case, we see that

$$U_\rho(0) = \frac{r(S)}{\ell}\Phi_R(\sigma),$$

where Φ_R is the form on L^* dual to U_R. Due to the Freudenthal–de Vries formula (see [8]) we have

$$\Phi_R(\sigma) = \frac{1}{24}(h+1)\ell$$

implying (7). ■

References

[1] Bernstein J. N., Schwarzman O. W. Chevalley's theorem for complex crystallographic Coxeter groups. (Russian) Funktsional. Anal. i Prilozhen. 12 (1978), no. 4, 79–80. (English translation: Functional Anal. Appl. 12 (1978), no. 4, 308–309 (1979))

[2] Bernstein J., Schwazman O., Complex crystallographic groups generated by reflections and Macdonald's identity. VINITI deposition Jan. 3, 1977, Dep. n. 28–77 (in Russian)

[3] Bourbaki N. *Lie groups and Lie algebras. Chapters* $4-6$. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics. Springer-Verlag, Berlin, 2002. xii+300 pp.

[4] Cohen A.M. Finite complex reflection groups, Ann. Sci. École. Norm. Sup.(4) 9 (1976), 379–436 pp.

[5] Dubrovin B. Geometry of 2D topological field theories. In: Donagi R., Dubrovin B., Frenkel E., Previato E. *Integrable systems and quantum groups.* Lectures given at the First 1993 C.I.M.E. Session held in Montecatini Terme, June 14–22, 1993. Edited by M. Francaviglia and S. Greco. Lecture Notes in Mathematics, 1620. Fondazione C.I.M.E.. [C.I.M.E. Foundation] Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 1996,120–348

[6] Kac V. G., *Infinite Dimensional Lie Algebras.* 3rd ed. Cambridge Univ. Press, Cambridge, 1992

[7] Leites D. (ed.) *Seminar on supermanifolds, 1977–1990*, Reports of Stockholm University, 1986–90, vv. 1–34.

[8] Macdonald I. G. Affine root systems and Dedekind' s η-function, Inv. Math. 15 (1972), 91–143

[9] Popov V. L. *Discrete complex reflection groups.* Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, 15. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1982. 89 pp.

[10] Shephard G. C., Todd J. A. Finite unitary reflection groups, Canad. J. Math. 6 (1954), 274–304

[11] Springer T. A. Regular elements of finite reflection groups, Inv. Math. 25 (1974), 159–198

[12] Steinberg R. *Lectures on Chevalley groups*, Yale Univ., 1967.