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Chevalley's theorem for the complex crystallographic groups

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Abstract

We prove that, for the irreducible complex crystallographic Coxeter group W, the following conditions are equivalent:

a) W is generated by reflections;

b) the analytic variety X/W is isomorphic to a weighted projective space.

The result is of interest, for example, in application to topological conformal field theory. We also discuss the status of the above statement for other types of complex crystallographic group W and certain generalizations of the statement.

It is impossible to read this paper without first reading our paper [5] which contains all the notations and the data on affine root systems and complex crystallographic Coxeter groups. All the data needed on the modular functions theory is collected in §4.

Introduction

Let X be a connected complex variety, $W \subset \mathrm{Aut}X$ a discrete group of transformations of X. An element $w \in W$ is called a *reflection* if the set X^w of its fixed points is nonempty and $\mathrm{codim}_{\mathbb{C}} X^w = 1$. In many cases it turns out that if W is generated by reflections, then the quotient space X/W has a simple structure.

Examples. 1) Let $X = \mathbb{C}^l$ and let W be a finite group of linear transformations of X. Then the classical Chevalley's invariant theorem states that the following conditions a) – c) are equivalent:

a) W is generated by reflections;

b) $X/W \cong \mathbb{C}^l$;

c) the algebra of W-invariant polynomials on X is isomorphic to the polynomial algebra in l indeterminates.

2) If X is the upper half-plane and X/W is a compact, then W is generated by reflections if and only if $X/W \cong \mathbb{P}^1$.

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In this paper we will discuss the following situation: $X = V$ is an l-dimensional affine complex space, W a discrete group of affine transformations of V containing a lattice of rank $2l$ of translations; such a group W will be called a *complex crystallographic group.* Let us formulate an analogue of Chevalley's theorem for complex crystallographic groups.

0.1. The weighted projective space

The *weighted projective space* $\mathbb{P}(n_0, \dots, n_l)$ with exponents (weights) $n_0, \dots, n_l \in \mathbb{N}$ is the quotient space of $\mathbb{C}^{l+1} \setminus \{0\}$ modulo the $\mathbb{C}^*$-action determined by the formula

$$\mu_t(z_0, \dots, z_l) = (t^{n_0} z_0, t^{n_1} z_1, \dots, t^{n_l} z_l), \text{ for any } t \in \mathbb{C}^* \text{ and } (z_0, \dots, z_l) \in \mathbb{C}^{l+1} \setminus \{0\}.$$

In particular, $\mathbb{P}(1, \dots, 1)$ coincides with the projective space $\mathbb{P}^l$.

The main goal of this paper is to discuss the status of the following general statement.

Statement. *For any irreducible complex crystallographic group W, the following conditions are equivalent:*

a) *W is generated by reflections;*

b) *The quotient analytic variety X/W is isomorphic to a weighted projective space.*

This is an analogue of the Chevalley theorem. We will show that the "difficult" part of Chevalley's theorem, i.e., the implication a) $\Longrightarrow$ b), holds for a special kind of groups W, called *complex crystallographic Coxeter groups.* In fact, we will see that in this case a slightly stronger statement is true: t h e a l g e b r a o f θ - f u n c t i o n s a s s o c i a t e d t o t h i s s i t u a t i o n i s a p o l y n o m i a l a l g e b r a.

A weighted projective space can be a singular variety. An explanation of this fact in terms of complex crystallographic groups is due to the fact that even if W itself is generated by reflections, the stabilizers of some points might be not generated by reflections, and therefore the images of these points are singular points of the corresponding quotient space (cf. [10]).

We believe that the statement is true always but the case of Shephard-Todd type groups W requires additional investigation (and still is an open problem)

Note that the "easy" part of the Chevalley theorem, namely the implication b) $\Longrightarrow$ a), is proven in [27] in complete generality. Actually, the proof in [27] uses only topological arguments and it proves the following more general result.

Proposition. *Let X be a connected and simply-connected complex analytic manifold, W a discrete group of complex automorphisms of X and let $Z = X/W$ be the quotient analytic space.*

Then the following conditions are equivalent

a) *The group W is generated by reflections.*

b) *There exists a closed analytic subset $F \subset Z$ of codimension > 1 such that the complementary open subset $Z_0 = Z \setminus F$ is a nonsingular simply connected analytic variety.*

0.2. The condition b) of Statement 0.1 in algebraic terms

The geometric condition b) of Statement 0.1 can be formulated in an algebraic language. For this, we use θ-functions. Let a be a 1-cocycle on W with values in the group $\mathcal{O}^*(V)$

of invertible holomorphic functions on V. A holomorphic function f on V such that

$$wf = a_w f \text{ for all } w \in W$$

is called a *θ-function corresponding to the cocycle a.*

Consider now the graded algebra of θ-functions $\mathbb{A} = \bigoplus_{n=0}^{\infty} \mathbb{A}_n$, where $\mathbb{A}_n$ is the space of θ-functions corresponding to the cocycle a^n. The cocycle a is called *ample* if $\mathbb{A}$ has "plenty" elements (more exactly, if $\dim \mathbb{A}_n > c \cdot n^l$ for sufficiently large n and some constant c). In this case, $V/W \simeq \operatorname{Proj} \mathbb{A}$, see [2].

Here is a more geometric realization of θ-functions. Consider the trivial bundle $\Theta = V \times \mathbb{C}^*$ over V with the fiber $\mathbb{C}^*$. With the help of the cocycle a we lift the W-action to this bundle:

$$\rho_w^a(z, u) = (wz, a_w(wz)u) \text{ for any } w \in W,\ u \in \mathbb{C}^*,\ z \in V.$$

Then the algebra $\mathbb{A}$ of θ-functions is naturally realized as the algebra of W-invariant functions holomorphic on Θ and polynomial in $u \in \mathbb{C}^*$.

It is easy to verify that the quotient space Θ/W is isomorphic to $(\mathrm{Specm}\mathbb{A}) \setminus \{0\}$, where $\mathrm{Specm}\mathbb{A}$ is the spectrum of maximal ideals of $\mathbb{A}$, and 0 is the point of $\mathrm{Specm}\mathbb{A}$ corresponding to the ideal $\mathbb{A}_+ = \sum\limits_{n>0} \mathbb{A}_n$.

Together with Θ, consider also its universal covering $\widetilde{\Theta}$. For each element $w \in W$, the transformation ρ_w^a of Θ can be lifted in an infinite number of ways to a transformation of $\widetilde{\Theta}$. These coverings differ by elements of the monodromy group of $\widetilde{\Theta}$ over Θ; this monodromy group is isomorphic to $\mathbb{Z}$. We denote by $\widetilde{W}$ the group generated by all these liftings for all $w \in W$.

Clearly, $\widetilde{\Theta}/\widetilde{W} \cong \Theta/W$, the projection $\pi\colon \widetilde{W} \to W$ is an epimorphism and $\mathrm{Ker}\pi = \mathbb{Z}$ is the central subgroup of $\widetilde{W}$.

Statement. *Let W be a complex crystallographic group acting on the space V and let a be an ample 1-cocycle on W. Then the following conditions are equivalent:*

a) *$\widetilde{W}$ is generated by reflections;*

b) *The space $\Theta/W \cong \widetilde{\Theta}/\widetilde{W}$ is isomorphic to $\mathbb{C}^{l+1} \setminus \{0\}$;*

c) *The algebra $\mathbb{A}$ of θ-functions is a polynomial algebra in $f_0, \dots, f_l$, where $f_i \in \mathbb{A}_{n_i}$ and $n_i > 0$.*

0.3. Discussion

We cannot prove Statements 0.1 and 0.2 in full generality though we are quite sure that they are true. The most difficult part of Chevalley's Theorem is the "direct" one, i.e., implications

$$0.1\text{a}) \Longrightarrow 0.1\text{b}) \text{ and } 0.2\ \text{a}) \Longrightarrow 0.2\text{b}) \text{ and } 0.2\text{c}).$$

We only proved them for one class of complex crystallographic groups, namely the Coxeter type groups. For the proof of the "easy" part of Chevalley's Theorem, see, for example, [27].

If W is a complex crystallographic group, then the group of its linear parts, dW, is a finite linear group and if W is generated by reflections, then so is dW (for details, see [5], sec. 1.1, 1.2). Therefore the following two cases are possible:

1) dW is generated by real reflections, i.e., dW is a finite Coxeter group and then we say that W is a *Coxeter-type group*;

2) dW is not real (there is no basis in which it can be expressed by real matrices) and then W is called a *Shephard-Todd-type group*.

In this paper, the direct Chevalley's theorem is proved for all irreducible complex crystallographic Coxeter groups except (due to the method's shortcomings) for the series $W(D_l, \tau)$ connected with the affine root system D_l.

On the other hand, Kac and Peterson ([12]) proved Chevalley's theorem for all irreducible crystallographic Coxeter groups connected with classical affine root systems (see Remark 2.2 below).

Therefore, at present, Chevalley's theorem is proved for all irreducible complex crystallographic groups of Coxeter type.

Remarks. We have introduced complex crystallographic Coxeter groups [3] in connection with Macdonald's identities and related problems. The main results are published in [4], [3] and the detailed exposition is preprinted in [16], 2/1986-22, 1–66 (delivered at the Seminar in 1976). New interest in complex crystallographic Coxeter groups arose thanks to their applications: Dubrovin introduced Frobenius manifolds and applied complex crystallographic Coxeter groups in topological conformal field theory [B. Dubrovin, Preprint No. 89/94/FM, Internat. School Adv. Stud. (SISSA), Trieste; per bibl.], cf. [8], [9], [20]; for the elliptic case, Sheinman established that there is a one-to-one correspondence between the complex crystallographic Coxeter groups and Krichever-Novikov algebras of affine type [14], see [23] and [22].

Other related results: most close is that of Wirthmüller [29]. Looijenga [17], R. Friedman, J. Morgan, E. Witten [9], and K. Saito [19] introduced the concept of an *extended affine root system* and classified all elliptic root systems (2-extended affine root systems). Later I. Satake [20] defined the theta function associated to the elliptic root system of type $D_4^{(1,1)}$. The theta functions constructed in this paper are similar to those defined by T. Takebayashi [28]. For a review of related results, see also [24]–[27].

1 Cocycles on complex crystallographic Coxeter groups

1.1. Some standard facts on cocycles

Let G be a group, M a G-module. Denote by $C^1(G, M)$ an Abelian group of 1-cocycles on G with values in M, i.e., the group of mappings

$$a\colon G \longrightarrow M \quad (g \mapsto a_g) \quad \text{such that} \quad a_{g_1 g_2} = a_{g_1} + g_1 a_{g_2}.$$

To each element $m \in M$, there corresponds a 1-cocycle $a^m\colon g \mapsto gm - m$. Denote the group of such cocycles by $B^1(G, M)$.

The cocycles a and a' are called *homologic* to each other if $a - a' \in B^1(G, M)$. Set $H^1(G, M) = C^1(G, M)/B^1(G, M)$.

For the proof of the following **standard facts** on cocycles that we will need, see, e.g., Chapter IV in [7].

(i) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of G-modules, then the induced sequence

$$H^1(G, M') \longrightarrow H^1(G, M) \longrightarrow H^1(G, M'')$$

is exact.

(ii) If G is a finite group and M a uniquely divisible abelian group, then $H^1(G, M) = 0$.

(iii) Let H be a normal subgroup in G. Set $M^H := \{m \in M \mid Hm = m\}$. Then the naturally defined sequence

$$0 \longrightarrow C^1(G/H, M^H) \longrightarrow C^1(G, M) \longrightarrow C^1(H, M)$$

is exact.

(iv) If H is a normal subgroup in G trivially acting on M and $a \in C^1(G, M)$, then the restriction of a onto H is a G-equivariant homomorphism $H \longrightarrow M$ (i.e., $a_{h_1 h_2} = a_{h_1} + a_{h_2}$ and $a_{ghg-1} = g(a_h)$).

1.2. Specialization of the standard facts to complex crystallographic groups. Linear cocycles

Let W be a complex crystallographic group acting on a complex affine space V. Denote by $\mathcal{O}(V)$ the ring of holomorphic functions on V and define a W-action on it, by setting

$$(wf)(z) = f(w^{-1}z).$$

Denote by $\mathcal{O}^*$ the group of invertible elements of $\mathcal{O}(V)$ and set

$$C^1 = C^1(W, \mathcal{O}^*), \quad B^1 = B^1(W, \mathcal{O}^*), \quad H^1 = H^1(W, \mathcal{O}^*).$$

Let

$$\mathcal{L} = \exp \operatorname{Aff}(V) \subset \mathcal{O}^*,$$

where $\operatorname{Aff}(V)$ is the space of affine-linear functions on V. The cocycle $a \in C^1$ is called *linear* if $a_w \in \mathcal{L}$ for any $w \in W$.

Proposition. a) *Any cocycle $a \in C^1$ is homologic to a linear cocycle.*

b) *If U is a quadratic function on V, the quadratic part of U is dW-invariant and $f = \exp U \subset \mathcal{O}^*$, then the cocycle $a^f \in C^1$ is linear. Conversely, if $f \in \mathcal{O}^*$ and the cocycle a^f is linear, then f is of the above-mentioned form.*

Sketch of the proof. a) It suffices to verify that in the exact sequence

$$H^1(W, \mathcal{L}) \longrightarrow H^1(W, \mathcal{O}^*) \xrightarrow{\beta} H^1(W, \mathcal{O}^*/\mathcal{L})$$

the map β is zero.

Let $a \in C^1$, and let $a_T \in C^1(T, \mathcal{O}^*)$ be the restriction of a onto the subgroup of translations $T \subset W$. As follows from §1 in [18], the cocycle a_T is homologic to a linear one; therefore, replacing a by a homologous cocycle, we may assume that a_T is linear.

Therefore $\beta(a)|_T = 0$. Due to the standard fact (iii) (sec. 1.1) this implies

$$\beta(a) \in H^1(W/T, (\mathcal{O}^*/\mathcal{L})^T).$$

Notice that the map $\exp\colon \mathcal{O}(V)/\mathrm{Aff}(V) \longrightarrow \mathcal{O}^*/\mathcal{L}$ is an isomorphism, i.e., $\mathcal{O}^*/\mathcal{L}$ is a uniquely divisible group. Since W/T is finite, the standard fact (ii) (sec. 1.1) implies that $\beta(a) = 0$, as was required.

b) The first part of the statement is obvious. Let now $f \in \mathcal{O}^*$ be such that a^f is linear and $U = \log f$. Then, clearly, $wU - U \in \mathrm{Aff}(V)$ for any $w \in W$ implying that $\dfrac{\partial^2 U}{\partial z_i \partial z_j}$ is invariant with respect to the subgroup of translations T, i.e., $\dfrac{\partial^2 U}{\partial z_i \partial z_j}$ are constants since V/T is compact. Therefore U is a quadratic function. The condition $wU - U \in \mathrm{Aff}(V)$ shows that the quadratic part of U is dW-invariant. ■

Convention. *In what follows we will only consider*

a) irreducible complex crystallographic Coxeter groups;

b) linear cocycles.

1.3. Various facts concerning complex crystallographic Coxeter groups

Fix an affine root system S and a number τ such that $\operatorname{Im}\tau > 0$. Consider the group $W = W(S, \tau)$ (see [5], 3.1).

Let us describe the construction of W in detail. Let x_0 be a special point for S, let $V_{\mathbb{R}} \subset V$ be the real subspace from which V is recovered, W_0 the stabilizer of x_0. Denote by $T_{\mathbb{R}}$ and T_τ the subgroups in T consisting of translations in the direction $V_{\mathbb{R}}$ and $\tau^{-1}V_{\mathbb{R}}$, respectively. Fix a base $\alpha_0, \dots, \alpha_l$ of S so that $\alpha_1, \dots, \alpha_l$ is a base of the root system $R = \{\alpha \in S \mid \alpha(x_0) = 0\}$. Set

$$h_i = h_{\alpha_i} \quad \text{for } i = 1, \dots, l.$$

Lemma. a) $T_{\mathbb{R}} \cap T_\tau = 0$ *and* W_0 *is the normalizer of* $T_{\mathbb{R}}$ *and* T_τ*.*

b) $T_\tau = \tau^{-1}(\oplus \mathbb{Z} h_i)$*,* $T_{\mathbb{R}} = \oplus p_{\alpha_i} h_i$*, and* $T = T_\tau \oplus T_{\mathbb{R}}$*.*

c) $W_S = W_0 \cdot T_{\mathbb{R}}$ *so that* $W_S \cap T_\tau = 0$ *and* $W = W_S \cdot T_\tau$*.*

Proof. a) is clear.

b) As is shown in [5] (proof of Theorem 1.3)), $T = \oplus \mathfrak{a}_{\alpha_i} \cdot h_i$, where $\mathfrak{a}_\alpha = p_\alpha \mathbb{Z} + \tau^{-1}\mathbb{Z}$ implying b).

c) Set $W' = W_0 \cdot T_{\mathbb{R}}$. Clearly, $W' \subset W_S$ and $W' T_\tau = W_0 \cdot T_{\mathbb{R}} \cdot T_\tau = W_0 \cdot T = W$. Therefore W' is the stabilizer of $V_{\mathbb{R}}$. Hence, $W_S \subset W'$, i.e., $W_S = W' = W_0 \cdot T_{\mathbb{R}}$. ■

1.4. Normal cocycles

A cocycle $a \in C^1$ is said to be a *normal* one, if it is linear and trivial on the subgroup T_τ.

Lemma. *Any linear cocycle* $a = (a_w)$ *is homologic to a normal cocycle.*

Proof. (i) Let $L = L(V)$ be the linear space of translations. Define a homomorphism

$$\beta\colon \mathcal{L} \longrightarrow L^*, \quad \beta(f) = d\ln f.$$

The map $d\colon W \longrightarrow dW \subset AutL$ determines a W-action on L^* and we see that β is a W-module homomorphism.

Consider the cocycle $b = \beta(a) \in C^1(W, L^*)$. Since T_τ trivially acts on L, it follows that, due to the standard fact (iv) (sec. 1.1), the function

$$B(t, z) = b_t(z), \text{ where } t \in T_\tau \text{ and } z \in L$$

is $\mathbb{Z}$-linear in t, $\mathbb{C}$-linear in z, and W_0-invariant.

Let us extend B to a bilinear form $B\colon L \times L \longrightarrow \mathbb{C}$. This is possible since $L \cong T_\tau \otimes_{\mathbb{Z}} \mathbb{C}$ by Lemma 1.3. Clearly, B is W_0-invariant.

Since the W_0-action on L is irreducible, ([5], Prop. 1.2.2) all the W_0-invariant bilinear forms on L are proportional. Since W_0 is a Coxeter group, there exists a symmetric W_0-invariant form. Therefore B is symmetric. Set

$$f(z) = \exp\Big(\frac{1}{2}B(z, z)\Big).$$

Since B is W_0-invariant, it follows that the cocycle a^f is linear. Furthermore, it is easy to verify that the cocycle $b^f = \beta(a^f)$ on T_τ coincides with b (namely, $b^f_t(z) = B(t, z) = b_t(z)$). Therefore by replacing a by a homologous cocycle $a \cdot (a^f)^{-1}$, we may assume that $\beta(a)$ vanishes on the lattice T_τ.

(ii) The condition $\beta(a)|_{T_\tau} = 0$ means that $a_t \in \mathbb{C}^*$ for any $t \in T_\tau$. Let $t_1, \dots, t_l$ be a basis in T_τ, and therefore a basis in L. Determine a function $g \in L^*$ by setting $g(t_i) = \log a_{t_i}$ and let $f = \exp g$. Clearly, $f \in \mathcal{L}$. By replacing a by a homologous cocycle $a \cdot (a^f)^{-1}$ we may assume that $a_{t_i} = 1$ for $i = 1, \dots, l$. But then $a|_{T_\tau} = 1$, i.e., a is a normal cocycle. ■

1.5. Description of the group H^1 for any complex crystallographic Coxeter group W

More exactly, we will distinguish a subgroup $H^1_{ev} \subset H^1$ of *even* cohomology classes, and mainly study this subgroup. For a definition of complex crystallographic Coxeter groups, see [5].

Let $a \in C^1$ and $r \in \operatorname{Ref}(W)$ (see [5], sec. 1.2). Since $a_r \cdot r(a_r) = 1$, it follows that the value of a_r on the hyperplane $\pi(r)$ is equal to ± 1; denote this value by $\operatorname{sign}_r(a)$.

Therefore we see that from any reflection $r \in \operatorname{Ref}(W)$ we have recovered a homomorphism $\operatorname{sign}_r\colon C^1 \longrightarrow \{\pm 1\}$. It is easy to verify that

a) sign_r is constant on homologous cocycles,

b) if reflections r and r' are conjugate in W, then $\operatorname{sign}_r = \operatorname{sign}_{r'}$.

The cocycle $a \in C^1$ is called an *even* one, if $\operatorname{sign}_r(a) = 1$ for all $r \in \operatorname{Ref}(W)$; the group of even cocycles will be denoted by C^1_{ev}. Note that all cocycles in B^1 are automatically even. Set

$$H^1_{ev} = C^1_{ev}/B^1 \subset H^1.$$

In what follows we will see that $H^1_{ev} \cong \mathbb{Z}$. The elements of H^1/H^1_{ev} are described by the set $\{\operatorname{sign}_r \mid r \in \operatorname{Ref}(W)\}$. It is not difficult to verify that elements of H^1 realize any set of signs constant on classes of conjugate elements in $\operatorname{Ref}(W)$.

1.6. Description of the normal even cocycles

Let λ be a weight of a affine root system S (see [5], sec. 4.2), U a quadratic function on $V_{\mathbb{R}}$ representing λ. We will consider U as a quadratic function on a complex affine space V. By definition

$$r_\alpha U - U \in \mathbb{Z}\alpha \quad \text{for all } \alpha \in S.$$

Set $\nu = 2\pi i\tau$. Consider the function $a\colon W \longrightarrow \mathcal{L}$ (see 1.2) determined by the formula

$$a_w = \exp \nu(U - w_S U), \text{ where } w = t \cdot w_S \text{ for some } t \in T_\tau \text{ and } w_S \in W_S. \tag{1}$$

Clearly, a_w does not vary if we add a constant to U, therefore a only depends on λ and we will denote it by a^λ.

Theorem. a) *The function a^λ is a normal even cocycle for any weight λ;*

b) *The map $\lambda \mapsto a^\lambda$ determines an isomorphism of the group of weights Λ with the group of normal even cocycles.*

c) *A cocycle λ is homologic to* 0 *if and only if*[1] $\kappa(\lambda) = 0$ *(i.e., $\lambda \in \mathrm{Aff}(V)/\mathrm{Const}$).*

Corollary. $H^1_{ev} \cong \mathbb{Z}$.

Proof. a) Since W_S is generated by reflections, it follows that for any $w \in W_S$ the function $wU - U$ belongs to the lattice $\Gamma(S) \subset \mathrm{Aff}(V)$ generated by S.

Since $d\beta(h_\alpha) \in \mathbb{Z}$ for any $\alpha, \beta \in S$, we have $d(wU - U)(h_\alpha) \in \mathbb{Z}$.

Since $T_\tau = \tau^{-1}(\oplus \mathbb{Z} h_i)$ (by Lemma 1.3), the value of $d(wU - U)$ at any $t \in T_\tau$ belongs to $2\pi i\mathbb{Z}$. This means that all the functions a_w are invariant with respect to T_τ, i.e., $a_w \in \mathcal{L}^{T_\tau}$. Clearly, a is a cocycle on $W/T_\tau = W_S$ (a coincides with a^f, where $f = \exp \nu U$). Hence, a is a cocycle on the whole group W due to the standard fact (iii) from sec. 1.1.

Let us prove that a is an even cocycle. Consider the reflection r in the hyperplane

$$\pi(\alpha, k) = \{x \in V \mid \tau\alpha(x) = k\}.$$

Then $r = t \cdot r_\alpha$, where $t \in T_\tau$ is a translation by a vector $\tau^{-1} k h_\alpha$. Therefore

$$a_r = \exp \nu(U - r_\alpha U) = \exp \nu n_\alpha \alpha, \text{ where } n_\alpha \in \mathbb{Z}.$$

On the hyperplane $\pi(\alpha, k)$, the value of a_r is equal to

$$\exp(\tau^{-1}\nu k n_\alpha) = \exp(2\pi i k n_\alpha) = 1.$$

b) Let a be an even normal cocycle, $\alpha \in S$, $r = r_\alpha$ and $a_r = \exp \delta$, where $\delta \in \mathrm{Aff}(V)$. The condition $1 = a_{r^2} = a_r \cdot r a_r$ implies that $\delta + r\delta \in \mathbb{C}$, i.e., $\delta = p\alpha + q$, where $p, q \in \mathbb{C}$.

Furthermore, $a_r(\pi_\alpha) = 1$, so $q \in 2\pi i\mathbb{Z}$, i.e., we may assume that $q = 0$. Now, consider the reflection r' in the hyperplane $\pi(\alpha, 1)$; clearly, $r' = t \cdot r_\alpha$, where $t \in T_\tau$ is the translation by the vector $\tau^{-1} h_\alpha$. Then the value of the function $a_{r'} = a_r$ on the hyperplane $\pi(\alpha, 1)$ is equal to $1 = \exp(\tau^{-1} p)$ implying $p \in \mathbb{Z}\nu$.

Consider the basis $\alpha_0, \dots, \alpha_l$ of S. We have shown that $a_{r_i} = \exp(p_i \nu \alpha_i)$, where $p_i \in \mathbb{Z}$. Consider the cocycle a^λ, $\lambda = \sum p_i \lambda_i$, where the λ_i are the fundamental weights (see [5],

[1] For the definition of κ, see [5], sec. 2.5.

sec. 4.2). Then $a_w^\lambda = a_w$ for $w = t \in T_\tau$ and $w = r_i$ for $i = 0, \dots, l$. Since these elements generate W, it follows that $a = a^\lambda$.

c) Let $f \in \mathcal{O}^*$ be a function such that a^f is a normal cocycle. By Proposition 1.2 $f = \exp U$, where U is a quadratic function on V. Since $a^f|_{T_\tau} = 1$, it follows that U is an affine-linear function and $dU(t) \in 2\pi i\mathbb{Z}$ for any $t \in T_\tau$. Therefore

$$dU = \nu dU', \text{ where } dU'(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in S,$$

implying that λ is a weight ([5], sec. 2.5, 4.2), $\kappa(\lambda) = 0$ and $a^f = a^\lambda$.

Conversely, if λ is a weight such that $\kappa(\lambda) = 0$ and $U \in \mathrm{Aff}(V)$ a function representing it, then $a^\lambda = a^{\exp \nu U}$ is homologic to zero. ■

2 Main theorem and its corollaries

2.1. The algebra of θ-functions and the structure of V/W

Let W be a complex crystallographic group, $a \in C^1$ a cocycle. We denoted by $\Theta(a)$ the space of the θ-functions corresponding to a and consider the cocycles $a_k = k \cdot a$, where $k = 0, 1, \dots$ (we express the operation in the group C^1 additively; in other words, $(a_k)_w = (a_w)^k$). Set

$$\mathbb{A} = \bigoplus_{k=0}^{\infty} \mathbb{A}_k, \text{ where } \mathbb{A}_k = \Theta(a_k).$$

The multiplication of θ-functions determines a graded $\mathbb{C}$-algebra structure on $\mathbb{A}$.

Up to an isomorphism, the algebra $\mathbb{A}$ depends only on the cohomolgy class of a. Indeed, if $a' \sim a$, i.e., $a' = a + a^g$, where $g \in \mathcal{O}^*$, then the family of isomorphisms

$$\varphi_k \colon \mathbb{A}_k \longrightarrow \mathbb{A}'_k, \quad f \mapsto g^k \cdot f$$

gives an isomorphism of graded algebras $\varphi \colon \mathbb{A} \longrightarrow \mathbb{A}'$.

Statement. *Let a be an ample cocycle. Then the quotient space V/W is isomorphic, as an analytic space, to* $\mathrm{Proj}(\mathbb{A})$. *The isomorphism is determined by the formula $x \mapsto J_x$, where $J_x = \{f \in \mathbb{A} \mid f(x) = 0\}$ is an homogeneous ideal, an element of* $\mathrm{Proj}\mathbb{A}$.

Proof. Let $T \subset W$ be the subgroup of translations. If $W = T$, the statement follows from the analytic theory of Abelian varieties ([18]).

In the general case, consider the cocycle $a_T \in C^1(T, \mathcal{O}^*)$ obtained by restriction of a on T and denote by $C = \oplus C_k$ the algebra of θ-functions with respect to T and a_T. Clearly, a_T is an ample cocycle, hence $V/T \cong \mathrm{Proj}C$.

Determine the W-action on C_k by the formula

$$w(f) = a_w^{-k}(wf).$$

We should only verify the invariance of C_k with respect to this action. Indeed, for any $f \in C_k$, $t \in T$ and $w \in W$, we have

$$ta_w^{-k}(wf) = (ta_w^{-k})(twf) = (ta_w^{-k})(w(w^{-1}twf)) = (ta_w^{-k})(wa^k_{w^{-1}tw}f) =$$
$$(ta_w^{-k})(wa^k_{w^{-1}tw})(wf) = (ta_w^{-k})a^k_{tw}a_w^{-k}(wf) = a_t^k(a_w^{-k}(wf)),$$

i.e., $wf \in C_k$.

By definition, T acts trivially, so this action is, actually, a $dW = W/T$-action and the space C_k^{dW} of dW-invariant elements coincides with $\mathbb{A}_k$. Moreover, the elements $w \in dW$ are automorphisms of the algebra C, i.e., they determine an automorphism of $\mathrm{Proj} C = V/T$. Clearly, this automorphism coincides with the natural W-action on V/T. Therefore we see that

$$V/W \simeq (V/T)/dW \simeq (\mathrm{Proj} C)/dW \simeq \mathrm{Proj}\ C^{dW} \simeq \mathrm{Proj}\mathbb{A}. \quad \blacksquare$$

2.2. Formulation of the main theorem

Let W be a complex crystallographic Coxeter group. By Corollary 1.6 $H^1_{ev} \cong \mathbb{Z}$.

Theorem. (Main theorem). *One of the two generators of H^1_{ev} is an ample cohomology class. The corresponding to this class algebra $\mathbb{A}$ of θ-functions is isomorphic to the polynomial algebra $\mathbb{C}[f_0, \dots, f_l]$ in the indeterminates $f_i \in \mathbb{A}_{n_i}$, where $(n_0, \dots, n_l)$ is the set of exponents of W* (see [5], Remark 3.2).

We will prove this theorem in §3.

Remark. For classical root systems, Kac and Peterson ([12]) introduced canonical generators of $\mathbb{A}$. These generators are functions

$$\zeta_{\mathbb{A}_i} = \frac{\sigma_{\lambda_i+\rho}}{\sigma_\rho}$$

described in sec. 3.2 below.

Corollary. *The analytic space V/W is isomorphic to $\mathbb{P}(n_0, \dots, n_l)$.*

This corollary follows immediately from the Main theorem since $\mathrm{Proj}(\mathbb{C}[f_0, \dots, f_l])$ is the weighted projective space $\mathbb{P}(n_0, \dots, n_l)$ with exponents $n_0, \dots, n_l$.

Let us sharpen this corollary. On V/W, we determine a series of coherent analytic sheaves $\mathcal{A}_k$, where $k = 0, 1, \dots$, by setting

$$\Gamma(U, \mathcal{A}_k) = \{f \in \mathcal{O}(pr^{-1}(U)) \mid wf = a_w^k f \quad \text{for any } w \in W\},$$

where $pr\colon V \longrightarrow V/W$ is the natural projection, $U \subset V/W$ an open subset, $\mathcal{O}(pr^{-1}(U))$ the space of holomorphic functions on the open set $pr^{-1}(U)$, and $a \in C^1$ a cocycle representing an ample generator H^1_{ev}.

Further, on $\mathbb{P}(n_0, \dots, n_l)$, determine a series of coherent sheaves $\mathcal{A}'_k$ by setting

$$\Gamma(U, \mathcal{A}'_k) = \{f \in \mathcal{O}(pr^{-1}(U)) \mid f(\mu_t(z)) = t^k f(z) \quad \text{for any } t \in \mathcal{O}^*\},$$

where $pr\colon \mathbb{C}^{l+1} \setminus \{0\} \longrightarrow \mathbb{P}(n_0, \dots, n_l)$ is the natural projection, U a Zariski open subset of $\mathbb{P}(n_0, \dots, n_l)$ (see sec. 0.1).

Theorem 2.2 easily implies that the isomorphism $V/W \cong \mathbb{P}(n_0, \dots, n_l)$ determines an isomorphism of sheaves of graded algebras

$$\mathcal{A} = \oplus \mathcal{A}_k \xrightarrow{\sim} \mathcal{A}' = \oplus \mathcal{A}'_k.$$

Observe that if $\mathcal{A}$ is not generated by $\mathcal{A}_1$, then the sheaves $\mathcal{A}_k$ and $\mathcal{A}'_k$ are not necessarily locally free. In terms of the sheaves $\mathcal{A}_k$, this means that there exist $w \in W$ and $x \in V$ such that $wx = x$ but $a_w(x) \neq 1$. In terms of the sheaves $\mathcal{A}'_k$, this means that there exist $t \in \mathbb{C}^* \setminus \{1\}$ and $z \in \mathbb{C}^{l+1} \setminus \{0\}$ such that $\mu_t(z) = z$ and such t and z exist if not all the n_i are equal to 1.

2.3. Corollary of the Main theorem in terms of algebraic geometry

Let $W = W(S,\tau)$, let x_0 be a special point for S and $W_0 = W_{x_0}$ its stabilizer. Then $V/W \cong (V/T)/W_0$, and hence

$$(V/T)/W_0 \cong \mathbb{P}(n_0,\dots,n_l).$$

Since both these spaces are projective algebraic varieties, we may consider this isomorphism as an isomorphism of algebraic varieties (see [21]). Therefore we may formulate a corollary of Theorem 2.2 which does not appeal to complex geometry: in terms of algebraic geometry.

Let R be a finite irreducible reduced root system in the space L, let $Q \subset L$ be the lattice generated by h_α, where $\alpha \in R$. For an elliptic curve E, denote by E_R the abelian variety $Q \otimes_{\mathbb{Z}} E$ (this is the group isomorphic to the direct sum of $l = \dim L$ copies of E, so it is naturally endowed with the structure of an abelian variety).

In Q, consider the sublattice Q_S generated by the vectors corresponding to the short roots α. It is easy to verify that $[Q : Q_S] = p(R)$ (for the definition of $p(R)$ see [5], sec. 1.3). Let ε be a point of E of order $p(R)$. Denote by $E_{R,\varepsilon}$ the abelian variety obtained from E_R by factorization modulo the finite subgroup $Q_S \otimes \varepsilon$. The Weyl group W_R of R acts on the lattices Q and Q_S, and therefore it acts on the abelian varieties E_R and $E_{R,\varepsilon}$.

Corollary. *Algebraic varieties E_R/W_R and $E_{R,\varepsilon}/W_R$ are isomorphic to weighted projective varieties. The exponents $(n_0,\dots,n_l)$ of these projective varieties correspond to the affine root system $S(R,1)$ for E_R/W_R and to the system $S(R,p(R))$ (see [5], sec. 2.3) for $E_{R,\varepsilon}/W_R$.*

(See also [17], where this corollary is formulated for the systems of type $S(R,1)$ but the proof contains a considerable gap.)

Proof. Let $S = S(R,p)$, where $p = 1$ or $p(R)$, and let $W = W(S,\tau)$. Then $V/W = (V/T)/W_0$, where $W_0 = W_R$. Lemma 1.3 implies that the abelian variety V/T is of the following form:

$$V/T = \begin{cases} E'_R, & \text{where } E' = \mathbb{C}/(\tau^{-1}\mathbb{Z} \oplus \mathbb{Z}), \qquad \text{if } p = 1 \\ E''_{R,\varepsilon''}, & \text{where } E'' = \mathbb{C}/(\tau^{-1}\mathbb{Z} \oplus p\mathbb{Z}) \text{ and } \varepsilon'' \text{ is the image of } 1 \in \mathbb{C}, \quad \text{if } p = p(R). \end{cases}$$

Corollary 2.2 implies that $(V/T)/W_R \cong \mathbb{P}(n_0,\dots,n_l)$. Since any elliptic curve E is isomorphic to a curve of the form E' and any pair (E,ε) is isomorphic to a pair (E'',ε'') for some τ, it follows that E_R/W_R and $E_{R,\varepsilon}/W_R$ are weighted projective spaces. ∎

2.4. Remark. The main theorem can be molded in the following geometric form. Let a be a cocycle corresponding to the ample generator of H^1_{ev}. Consider the one-dimensional trivial bundle $Y = V \times \mathbb{C} \longrightarrow V$ and determine a W-action on it by the formula

$$w(x,u) = (wx, a_w(wx)u).$$

This action is, clearly, discrete.

Let $f_i \in \mathbb{A}_{n_i}$ be θ-functions considered in Theorem 2.2. On Y, determine functions g_i by setting

$$g_i(x, u) = f_i(x)u^{n_i}.$$

Clearly, the functions g_i are W-invariant, i.e., they determine the map $g\colon Y/W \longrightarrow \mathbb{C}^{l+1}$. This map sends the zero section $V/W \subset Y/W$ to 0. If we blow down this section into a point, then the space $\widetilde{Y/W}$ obtained is mapped isomorphically onto $\mathbb{C}^{l+1}$.

3 Proof of the Main theorem

3.1. A geometric realization of the algebra of θ-functions. θ-forms

Let $S = S(R, p)$ be an affine root system, τ a complex number such that $\operatorname{Im}\tau > 0$ and $W = W(S, \tau)$. Fix a basis $\alpha_0, \dots, \alpha_l$ of S (see [5], sec. 2.4).

Consider the fundamental weight λ_0 and denote by U_0 the corresponding quadratic function on V normed by the condition $U_0(x_0) = 0$, i.e.,

$$U_0(x) = \| x - x_0 \|^2 \quad \text{for } x \in V_{\mathbb{R}},$$

where $\| \cdot \|$ is the canonical metric on the space $V_{\mathbb{R}}$ (see [5], sec. 2.5).

Set $a = a^{\lambda_0} \in C^1$ (see sec. 1.6). By definition (1), if $w = tw_S$, where $t \in T_\tau$ and $w_S \in W_S$, then $a_{tw_S} = \exp \nu(w_S U_0 - U_0)$ for $\nu = 2\pi i\tau$. As follows from Theorem 1.6 and results of [5], sec. 4.2, the cohomology class determined by a is a generator in H^1_{ev}. We will show that

a is an ample cocycle and $\mathbb{A} \simeq \mathbb{C}[f_0, \dots, f_l]$, where $f_i \in \mathbb{A}_{n_i}$.

To prove this, it is convenient to give another geometric realization of $\mathbb{A}$. Consider the bundle $\Theta = V \times \mathbb{C}^* \longrightarrow V$ over V and determine a W-action on this bundle by setting

$$w(z, u) = (wz, a_w(wz) \cdot u), \quad \text{where } w \in W,\ z \in V,\ u \in \mathbb{C}^*.$$

In the space $\mathcal{O}(\Theta)$ of holomorphic functions on Θ, consider the subspace $\mathcal{O}_k(\Theta)$ consisting of functions of homogeneity degree k with respect to u, i.e., functions of the form $f(z)u^k$. It is easy to verify that the map $f \mapsto u^k f$ determines an isomorphism of $\mathbb{A}_k$ with the space $\mathcal{O}_k(\Theta)^W$ of W-invariant functions in $\mathcal{O}_k(\Theta)$. In what follows we will identify $\mathbb{A}_k$ with $\mathcal{O}_k(\Theta)^W$. In other words, we have realized $\mathbb{A}$ as the algebra of W-invariant functions holomorphic on Θ and polynomial in u.

Now, let $\Omega(\Theta)$ be the space of holomorphic forms of the highest degree on Θ and $\Omega_k(\Theta)$ the subspace of forms of homogeneity degree k in u, i.e., the forms of the shape

$$u^k f(z)\omega_0, \quad \text{where } \omega_0 = d\alpha_1 \dots d\alpha_l \frac{du}{u}.$$

Set

$$\Sigma = \oplus \Sigma_k, \quad \text{where } \Sigma_k = (\Omega_k(\Theta))^W.$$

Clearly, the space Σ of θ-forms is an $\mathbb{A}$-module.

Remark. If $\omega = u^k f(z)\omega_0 \in \Omega_k(\Theta)$, then $\omega \in \Sigma_k$ means that $wf = \det w \cdot a_w^k \cdot f$ for any $w \in W$. In other words, the space Σ_k determines a geometric realization of the space of "skew" θ-functions.

3.2. Bases of the spaces of θ-functions and θ-forms

Fix an integer $k > 0$ and set $\Lambda_k = \{\lambda \in \Lambda \mid \kappa(\lambda) = k\}$, see [5], sec. 4.3. Each weight $\lambda \in \Lambda_k$ can be expressed in the form $\lambda = k\lambda_0 + \delta_\lambda$, where $\delta_\lambda \in \mathrm{Aff}(V)/\mathrm{Const} \simeq L^*$. The correspondence

$$\lambda \longleftrightarrow \delta_\lambda$$

determines the bijection of Λ_k with the lattice

$$\Lambda_R = \{\delta \in L^* \mid \delta(h_i) \in \mathbb{Z}, \text{ where } i = 1, \dots, l\},$$

the lattice of weights of a finite root system R (see [5], sec. 2.1).

For each weight $\lambda \in \Lambda_k$, denote by U_λ the quadratic function on V corresponding to λ and normed by the condition $M_{(U_\lambda)} = 0$ (see [5], sec. 4.1). In other words,

$$U_\lambda = k \parallel x - x_\lambda \parallel^2 \text{ for } x \in V_\mathbb{R},$$

where x_λ is the center of U_λ. Clearly,

$$U_\lambda - kU_0 = \delta_\lambda + U_\lambda(x_0) \in \mathrm{Aff}(V).$$

For each weight $\lambda \in \Lambda_k$, consider the function

$$f_\lambda = u^k \exp \nu(U_\lambda - kU_0) \in \mathcal{O}_k(\Theta).$$

Let $\lambda \in \Lambda^+$, i.e., $\lambda = \sum k_i\lambda_i$, where $k_i \geq 0$. Determine the functions θ_λ and $\psi_\lambda \in \mathcal{O}_k(\Theta)$ by setting

$$\theta_\lambda = \sum_{w \in W_S} f_{w\lambda}, \qquad \psi_\lambda = \sum_{w \in W_S} \det w \cdot f_{w\lambda}.$$

Further on, set

$$\sigma_\lambda = \psi_\lambda \omega_0, \text{ where } \omega_0 = d\alpha_1 \dots d\alpha_l \cdot \frac{du}{u}. \tag{2}$$

Statement. *Series for θ_λ and ψ_λ converge. The functions $\{\theta_\lambda \mid \lambda \in \Lambda_k^+\}$ and the forms $\{\sigma_\lambda \mid \lambda \in \Lambda_k^+, \lambda \in \rho + \Lambda^+\}$, where $\rho = \lambda_0 + \dots + \lambda_l$, constitute bases in the spaces $\mathbb{A}_k$ and Σ_k, respectively.*

Convention. Hereafter "convergence" means the uniform convergence on compacts.

Corollary. *Σ is a free $\mathbb{A}$-module with generator σ_ρ. The functions $\left\{\frac{\sigma_{\rho+\lambda}}{\sigma_\rho} \mid \lambda \in \Lambda^+\right\}$ constitute a basis in $\mathbb{A}$.*

Indeed, the multiplication by σ_ρ determines an embedding

$$\mathbb{A}_k \longrightarrow \Sigma_{k+g},$$

where (see [5], sec. 4.1)

$$g = \kappa(\rho) = n_0 + \dots + n_l. \tag{3}$$

Statement 3.2 implies that

$$\dim \mathbb{A}_k = \#(\Lambda_k^+) = \dim \Sigma_{k+g}.$$

(Since V/W is compact, the case $k = 0$, where $\mathbb{A}_0 = \mathbb{C}$, should be considered separately.) Therefore the multiplication by σ_ρ defines an isomorphism of $\mathbb{A}$ with Σ.

Proof of Statement 3.2

(i) Denote by $\mathcal{O}_k^\tau(\Theta)$ the subspace of T_τ-invariant functions of $\mathcal{O}_k(\Theta)$.

Let us prove that the functions $\{f_\lambda \mid \lambda \in \Lambda_k\}$ constitute a basis of $\mathcal{O}_k^\tau(\Theta)$, i.e., any $f \in O_k^\tau(\Theta)$ can be uniquely expanded into a converging series $\sum c_\lambda f_\lambda$. Since

$$U_\lambda - kU_0 = \delta_\lambda + c,$$

where c is a constant, it follows that f_λ is proportional to $u^k \exp(\nu\delta_\lambda)$, and therefore we have to verify that the functions $\{\exp(\nu\delta_\lambda) \mid \lambda \in \Lambda_k\}$ constitute a basis of the space of T_τ-invariant functions of $\mathcal{O}(V)$.

Since the functions δ_λ run over the elements of the lattice Λ_R and $T_\tau = \tau^{-1}(\bigoplus_{i=1}^{l} \mathbb{Z}h_i)$ (by Lemma 1.3), it follows that

$$\{\nu\delta_\lambda \mid \lambda \in \Lambda_k\} = T_\tau^* = \{\delta \in L^* \mid \delta(T_\tau) \in 2\pi i\mathbb{Z}\}.$$

The theory of Fourier series implies that the functions $\{\exp\delta \mid \delta \in T_\tau^*\}$ constitute a basis of the space of T_τ-invariant functions of $\mathcal{O}(V)$.

(ii) Let K be any compact in Θ. It is easy to verify that

$$|U_\lambda(z) - k \parallel x_\lambda \parallel^2 | < C(1+ \parallel x_\lambda \parallel),$$

where $(z,u) \in K$ and where C only depends on K and k. Therefore

$$|f_\lambda(z,u)| < \widetilde{C} \exp(C \parallel x_\lambda \parallel) \exp(k\mathrm{Re}\ \nu \parallel x_\lambda \parallel^2).$$

where $\widetilde{C}$ and C depend on K and k. These inequalities immediately imply that the series $\sum_{\lambda\in\Lambda_k} |f_\lambda(z,u)|$ converges uniformly on K and the series for Θ_λ and ψ_λ also converge on K.

(iii) Clearly, the function θ_λ and the form σ_λ are T_τ-invariant. Since $wf_\lambda = f_{w\lambda}$ for $w \in W_S$, and $w\omega_0 = \det w \cdot \omega_0$, it follows that θ_λ and σ_λ are also W_S-invariant. Therefore $\theta_\lambda \in \mathbb{A}_k$ and $\sigma_\lambda \in \sum_k$.

If $f \in \mathbb{A}_k \subset \mathcal{O}_k^\tau(\Theta)$, then having expressed f in the form $f = \sum_{\lambda\in\Lambda_k} C_\lambda f_\lambda$, we see that C_λ only depends on the W_S-orbit of λ. Since W_S-orbits in Λ_k are labelled by elements of Λ_k^+ (see [5], sec. 4.3), it follows that the functions $\{\theta_\lambda \mid \lambda \in \Lambda_k^+\}$ constitute a basis of $\mathbb{A}_k$.

We similarly verify that the forms $\{\sigma_\lambda \mid \lambda \in \Lambda_k^+\}$ generate Σ_k. However, if the stabilizer of λ in the group W_S is nontrivial, then it is generated by reflections (see [6]), so that $\sigma_\lambda = 0$.

Therefore there remain only the forms σ_λ corresponding to the weights $\lambda \in \Lambda_k^+$ with trivial stabilizer. These weights are exactly of the form $\lambda = \sum k_i\lambda_i$, where $k_i > 0$, i.e., the weights $\lambda \in \rho + \Lambda^+$, where $\rho = \lambda_0 + \cdots + \lambda_l$. The corresponding forms σ_λ are linearly independent and constitute a basis of Σ_k. ■

3.3. Siegel's inner product

On the bundle $\Theta \longrightarrow V$ (see sec. 3.1), introduce an hermitian W-invariant metric. For this, we consider the projection along $\tau^{-1}L_{\mathbb{R}}$:

$$pr_\tau\colon V \longrightarrow V_{\mathbb{R}}, \quad pr_\tau(x + \tau^{-1}y) = x, \text{ where } x \in V_{\mathbb{R}} \text{ and } y \in L_{\mathbb{R}},$$

and determine the real-valued quadratic function on V by the formula

$$Q(z) = U_0(pr_\tau z).$$

Determine a metric in the fiber $\mathbb{C}^*$ over $z \in V$ by setting

$$\| u \|_z^2 = |u|^2 \exp\left(-(\nu + \bar{\nu})Q(z)\right).$$

It is subject to a direct verification that this metric is W-invariant. Denote by $B = B(\Theta)$ the fibration of unit balls: $B = \{(z,u) \in \Theta \mid \| u \|_z \leq 1\}$. Clearly, B is W-invariant. Now, define *Siegel's inner product* on Σ by setting

$$\langle \omega_1, \omega_2 \rangle = \int_{B/W} \omega_1 \bar{\omega}_2,$$

where $\omega_1 \bar{\omega}_2$ is considered as a volume form on Θ.

Proposition. *Let $\lambda, \mu \in \rho + \Lambda^+$, where $\rho = \lambda_0 + \cdots + \lambda_l$. Then $\langle \sigma_\lambda, \sigma_\mu \rangle = 0$ for $\lambda \neq \mu$ and*

$$\langle \sigma_\lambda, \sigma_\lambda \rangle = C|\tau|^{-2l}(\operatorname{Im}\tau)^{l/2},$$

where C is a constant depending only on S and the number $k = \kappa(\lambda)$.

Proof. In the following calculations C denotes a constant depending on S and $k = \kappa(\lambda)$ but independent of τ. This constant varies from formula to formula.

(i) Let φ be a smooth real function with compact support on $V_{\mathbb{R}}$ such that $\sum\limits_{w \in W_S} w\varphi \equiv 1$. We extend this function onto V by setting $\varphi(z) = \varphi(pr_\tau z)$. Clearly, if $\omega_1, \omega_2 \in \Sigma$, then

$$\langle \omega_1, \omega_2 \rangle = \int_{B/T_\tau} \varphi \omega_1 \bar{\omega}_2.$$

The Siegel inner product expressed in the above form can be extended to the space of T_τ-invariant forms of homogeneity degree of k in u.

(ii) Let $\lambda, \mu \in \Lambda$, $k = \kappa(\lambda) > 0$, $m = \kappa(\mu) > 0$. Let us calculate $\langle f_\lambda \omega_0, f_\mu \omega_0 \rangle$.

In Θ, introduce coordinates x, y, u by expressing each point $z \in V$ in the form $x + \tau^{-1} y$, where $x \in V_{\mathbb{R}}$, $y \in L_{\mathbb{R}}$, $u \in \mathbb{C}^*$. In these coordinates, the inner product is expressed by the integral over

$$\{(x, y, u) \mid x \in V_{\mathbb{R}},\ y \in L_{\mathbb{R}}/T_0,\ |u|^2 \leq \exp\left((\nu + \bar{\nu})U_0(x)\right)\},$$

where $T_0 = \tau T_\tau = \{\oplus \mathbb{Z} h_i\}$. It is easy to verify that

$$\omega_0 \bar{\omega}_0 = C\ D(\tau) dx dy \frac{du}{u} \frac{d\bar{u}}{\bar{u}},$$

where dx and dy are volume forms on $V_{\mathbb{R}}$ and $L_{\mathbb{R}}$ respectively, and $D(\tau) = (\operatorname{Im}(\tau^{-1}))^l$. Further,

$$f_\lambda(x + \tau^{-1} y) = f_\lambda(x) \exp(2\pi i \delta_\lambda(y)).$$

Therefore

$$\langle f_\lambda\omega_0, f_\mu\omega_0\rangle_\varphi = \\ CD(\tau) \int\limits_{(x,y,u)} \varphi(x) f_\lambda(x)\overline{f_\mu(x)} \exp\left(2\pi i(\delta_\lambda - \delta_\mu)(y)\right) \cdot u^{k-1} \cdot \bar{u}^{m-1} du d\bar{u} dx dy,$$

where $x = \kappa(\lambda)$, $m = \kappa(\mu)$. Integrating over u and y we see that the integral vanishes if either $k \neq m$ or $\delta_\lambda \neq \delta_\mu$.

Therefore we consider the case where $k = m$ and $\delta_\lambda = \delta_\mu$, i.e., $\lambda = \mu$. The integral over y is equal to the volume of $L_{\mathbb{R}}/T_0$ and the integral over u is equal to $C \exp k(\nu + \bar{\nu})U_0(x)$ (since the integral is taken over the domain $|u|^2 \leq C \exp(\nu + \bar{\nu})U_0(x)$). Therefore

$$\| f_\lambda\omega_0 \|_\varphi^2 = CD(\tau) \int\limits_{V_{\mathbb{R}}} \varphi(x) \exp(\nu + \bar{\nu})(U_\lambda - kU_0) \exp(\nu + \bar{\nu})(kU_0) dx = \\ CD(\tau) \int\limits_{V_{\mathbb{R}}} \varphi(x) \exp(\nu + \bar{\nu}) U_\lambda(x) dx.$$

(iii) Since $\langle f_\lambda\omega_0, f_\mu\omega_0\rangle = 0$ for $\lambda \neq \mu$, it follows that $\langle\sigma_\lambda, \sigma_\mu\rangle = \langle\sigma_\lambda, \sigma_\mu\rangle_\varphi = 0$ if $\lambda \neq \mu$ and

$$\langle\sigma_\lambda, \sigma_\lambda\rangle_\varphi = \sum_{w \in W_S} \langle f_{w\lambda}\omega_0, f_{w\lambda}\omega_0\rangle = \sum_{w \in W_S} CD(\tau) \int\limits_{V_{\mathbb{R}}} \varphi(x) \exp(\nu + \bar{\nu}) U_\lambda(wx) dx.$$

Let us replace wx by x, interchange the order of summation and integration; we see that

$$\langle\sigma_\lambda, \sigma_\lambda\rangle = CD(\tau) \int\limits_{V_{\mathbb{R}}} \exp(\nu + \bar{\nu}) U_\lambda(x) dx$$

because $\sum \varphi(wx) \equiv 1$. Since $U_\lambda(x) = k \| x - x_\lambda \|^2$, the integral is equal to $C(\nu + \bar{\nu})^{-l/2}$. Since $\nu + \bar{\nu} = -4\pi \mathrm{Im}\,\tau$, we have

$$\langle\sigma_\lambda, \sigma_\lambda\rangle = C(\mathrm{Im}\,\tau^{-1})^l \cdot (\mathrm{Im}\,\tau)^{-l/2} = C|\tau|^{-2l}(\mathrm{Im}\,\tau)^{-l/2}. \quad \blacksquare$$

3.4. The function $F_S(\tau)$

Let us prove now the Main Theorem 2.2. Given a set of functions $f = (f_0, \ldots, f_l)$, where $f_i \in \mathbb{A}_{n_i} \in \mathcal{O}(\Theta)$, set

$$J(f_0, \ldots, f_l) = df_0\, df_1\, \ldots\, df_l \in \Omega(\Theta).$$

If we prove that $J(f_0, \ldots, f_l) \neq 0$ for some set f, this would mean that the f_i are algebraically independent, i.e., the homomorphism $\mathbb{C}[f_0, \ldots, f_l] \longrightarrow \mathbb{A}$ is an embedding. But then from the formula for the dimension of $\mathbb{A}_k$ (see Statement 3.2 and [5], sec. 4.2, 4.3) we immediately deduce that this homomorphism is an isomorphism.

It is easy to verify that

$$J(f_0, \ldots, f_l) \in \Sigma_g, \text{ where } g = n_0 + \ldots + n_l,$$

i.e., $\dfrac{J(f_0,\dots,f_l)}{\sigma_\rho} \in \mathbb{C}$. Indeed, $J(f_0,\dots,f_l) \in \Omega^W(\Theta)$ because each differential df_i is a W-invariant 1-form on Θ of homogeneity degree n_i in u. Therefore the form $J(f_0,\dots,f_l)$ is of homogeneity degree $g = \sum n_i$ in u, i.e, $J(f_0,\dots,f_l) \in \Sigma_g$. To each weight $\mu \in \Lambda^+$, we assign a θ-function (see Statement 3.2)

$$\zeta_\mu = \frac{\sigma_{\mu+\rho}}{\sigma_\rho} \in \mathbb{A}_{\kappa(\mu)}.$$

To each set of weights $\widetilde{\mu} = (\mu_0,\dots,\mu_l)$ such that $\kappa(\mu_i) = n_i$, we assign the number

$$\varphi_{\widetilde{\mu}}(\tau) = \frac{J(\zeta_{\mu_0},\zeta_{\mu_1},\dots,\zeta_{\mu_l})}{\sigma_\rho}.$$

Clearly, $\varphi_{\widetilde{\mu}}$ analytically depends on the parameter τ. Set

$$F_S(\tau) = \sum_{\widetilde{\mu}} |\ \varphi_{\widetilde{\mu}}(\tau)|^2,$$

where $\widetilde{\mu}$ runs over the sets of the form $\widetilde{\mu} = (\mu_0,\dots,\mu_l)$ with $\mu_i \in \Lambda^+_{n_i}$. To prove the Main Theorem, it suffices to verify the following statement:

$$F_S(\tau) \neq 0 \quad \text{for any } \tau. \tag{4}$$

3.5. The behavior of $F_S(\tau)$ under isomorphisms of complex crystallographic Coxeter groups. The function $G_S(\tau)$

First, recall a construction from the linear algebra. Let $C_1,\dots,C_k,D$ be finite dimensional hermitian vector spaces, $J\colon C_1 \times \cdots \times C_k \longrightarrow D$ a multilinear map. Then J can be considered as an element of $(C_1 \otimes \cdots \otimes C_k)^* \otimes D$. The hermitian structures on the spaces C_i and D determine an hermitian structure on $(C_1 \otimes \cdots \otimes C_k)^* \otimes D$, and hence $\parallel J \parallel^2$ is defined. To calculate $\parallel J \parallel^2$ explicitly, we choose in each space C_i an orthonormal basis $\{e^i_j \mid j \in J_i\}$; then

$$\parallel J \parallel^2 = \sum_{(j_1,\dots,j_k)} \parallel J(e^1_{j_1}, e^2_{j_2},\dots,e^k_{j_k}) \parallel^2_D .$$

Let us apply this construction to the case where $D = \Sigma_g$ (for the definition of g, see sec. 3.3), the C_i are the spaces $\mathbb{A}_{n_0}, \mathbb{A}_{n_1},\dots,\mathbb{A}_{n_l}$, respectively, and $J(f_0,\dots,f_l) = df_0 \cdot \ldots \cdot df_l$.

The hermitian structure on Σ_g and $\mathbb{A}_k$ are determined by Siegel's inner product and the formula $\parallel f \parallel = \frac{\|f\sigma_\rho\|}{\|\sigma_\rho\|}$, respectively. Set $G_S(\tau) = \parallel J \parallel^2$.

Proposition. a) $G_S(\tau) = C_S|\tau|^{-2l}(\operatorname{Im}\tau)^{l/2}F_S(\tau)$, *where C_S is a constant depending only on S.*

b) *If $\varphi\colon W(S_1,\tau_1) \longrightarrow W(S_2,\tau_2)$ is an isomorphism of split complex crystallographic Coxeter groups* (see [5], Th. 3.2), *then $G_{S_1}(\tau_1) = G_{S_2}(\tau_2)$.*

Proof. a) Proposition 3.2 implies that the functions $\frac{\zeta_\mu}{\|\zeta_\mu\|}$ constitute an orthonormal basis in $\mathbb{A}$. We explicitly calculate $\| J \|^2$ in this basis and use the fact that $\| \zeta_\mu \|$ only depends on $\kappa(\mu)$ and $\| \sigma_\rho \|^2 = C|\tau|^{-2l}(\operatorname{Im}\tau)^{l/2}$; we obtain a).

b) Let us identify $W(S_1, \tau_1)$ with $W(S_2, \tau_2)$ by means of the isomorphism φ. Then we see that the cocycles $a_1, a_2 \in C^1$ are homologic to each other since they are ample and determine a generator of the group $H^1_{ev} \cong \mathbb{Z}$. Therefore $a_1 - a_2 = a^g$, where $g \in \mathcal{O}^*$.

By Proposition 1.2 $g = \exp U$, where U is a quadratic function on V with dW-invariant quadratic part; without loss of generality we may assume that $U(x_0) = 0$. Determine an isomorphism $\varphi\colon \Theta_1 \longrightarrow \Theta_2$ by the formula $(z, u) \mapsto (z, \exp U(z) \cdot u)$.

Since $a_2 = a_1 + a^g$, this isomorphism is compatible with the W-action on Θ_i for $i = 1, 2$. Let us identify Θ_1 with Θ_2 via this isomorphism and denote them by Θ. On Θ, there are two W-invariant metrics: $\| \cdot \|_1$ and $\| \cdot \|_2$ recovered from S_1 and S_2, respectively.

Each of these metrics is of the form $\| u \|_i^2 = |u|^2 \exp Q_i(z)$, where Q_i is a real-valued quadratic function on V (see sec. 3.3). This implies that the function $Q_2(z) - Q_1(z)$ is W-invariant, and therefore a constant.

Thus, the metrics $\| \cdot \|_i$ are proportional to each other; since they coincide on the fiber over $x_0 \in V$, they coincide everywhere.

The isomorphism $\varphi\colon \Theta_1 \cong \Theta_2$ allows us to identify the algebras of θ-functions and the spaces of θ-forms recovered from the cocycles a_1 and a_2. These identification identify as well the corresponding multilinear mappings

$$J_1, J_2 \colon \mathbb{A}_{n_0} \times \cdots \times \mathbb{A}_{n_l} \longrightarrow \Sigma_g.$$

Since $\varphi(B(\Theta_1)) = B(\Theta_2)$ because the metrics $\| \cdot \|_i$ coincide, we see that the formulas that determine Siegel's inner product imply that this identification preserves the inner products.

Since $\| J \|$ is defined with the help of the inner product, $\| J_1 \| = \| J_2 \|$, i.e., $G_{S_1}(\tau_1) = G_{S_2}(\tau_2)$. ■

3.6. The function $H_S(\tau)$

Set $p_i = p_{\alpha_i}$ $(i = 1, \dots, l)$, i.e., p_i is the least positive number such that $\alpha_i + p_i \in S$ (see [5], sec. 2.3). Set

$$\eta_S(\tau) = \prod_{1 \le i \le l} \eta(p_i \tau),$$

where η is Dedekind's η-function ([15], Ch. XI). The function $\eta_s(\tau)$ has the following properties (see [15]):

a) $\eta_s(\tau)$ is a nowhere vanishing on the upper half-plane H holomorphic function in τ;
b) $\eta_S(\tau) \sim q^{r(S)/24}$ as $\operatorname{Im}\tau \longrightarrow \infty$, where $q = \exp(2\pi i \tau)$ and $r(S) = \sum\limits_{1 \le i \le l} p_i$;
c) Set

$$f_S(\tau) = |\eta_S(\tau)|^2 (\operatorname{Im}\tau)^{l/2}.$$

Then a straightforward calculation (see [5], sec. 2.3, 2.6) shows that

$$f_S(\gamma\tau) = f_S(\tau) \ \text{ for any } \gamma \in \Gamma_0(p),$$

and

$$f_S(\gamma_p\tau) = f_{S^{\rm inv}}(\tau), \quad \text{where } p = p(S).$$

With the help of Th. 3.2 from [5] we can reformulate heading c) as follows:

c′) If $W(S_1, \tau_1) \cong W(S_2, \tau_2)$, then $f_{S_1}(\tau_1) = f_{S_2}(\tau_2)$.

Set

$$H_S(\tau) = G_S(\tau) \cdot f_S^{-1}(\tau).$$

By Proposition 3.5

$$H_S(\tau) = C_S F_S(\tau)|\tau|^{-2l}|\eta_S(\tau)|^{-2}.$$

Therefore $H_S(\tau)$ has the following properties:

A) $H_S(\tau) \in \mathcal{A}(H)$, i.e., $H_S(\tau)$ is the sum of squares of absolute values of holomorphic functions;

B) $H_S(\gamma\tau) = H_S(\tau)$ for $\gamma \in \Gamma_0(p)$, and $H_S(\gamma_p\tau) = H_{S^{\rm inv}}(\tau)$.

Property A) follows from the fact that $F_S \in \mathcal{A}(H)$ and property B) follows from the fact that $G_S(\gamma\tau) = G_S(\tau)$ and $G_S(\gamma_p\tau) = G_{S^{inv}}(\tau)$ thanks to Proposition 3.5 and Theorem 3.2 from [5].

In the next subsections we will show that the following asymptotic estimate holds:

$$\begin{aligned} H_S(\tau) = &O(|q|^{-2/r_p}) \text{ as } \operatorname{Im}\tau \longrightarrow \infty, \\ &\text{where } q = \exp(2\pi i\tau),\ r_1 = 3,\ r_2 = 4,\ r_3 = 6. \end{aligned} \tag{5}$$

Then the results of §4 imply that the function $H_S(\tau)$ satisfying A), B) and (5) does not vanish anywhere. This proves the non-vanishing of $F_S(\tau)$, and therefore the Main theorem (see sec. 3.4).

3.7. The asymptotic of $H_S(\tau)$

To prove estimate (5) above, let us use the formula

$$H_S(\tau) = C_S F_S(\tau)|\eta_S(\tau)|^{-2}|\tau|^{-2l}.$$

By definition, $F_S(\tau) = \sum |\varphi_{\widetilde{\mu}}|^2$, where the sum runs over all the sets $\widetilde{\mu} = (\mu_0, \dots, \mu_l)$ for $\mu_i \in \Lambda^+_{n_i}$. Therefore it suffices to verify that

$$|\varphi_{\widetilde{\mu}}| = O(|\tau|^l |q|^{r(S)/24 - 1/r_p}) \text{ for any } \widetilde{\mu}.$$

Clearly, if two weights in $\widetilde{\mu}$ coincide, i.e., $\mu_i = \mu_j$ for $i \neq j$, then $\varphi_{\widetilde{\mu}} = 0$.

Therefore it suffices to verify the estimate for the sets $\widetilde{\mu}$ without equal weights. The estimate of $|\varphi_{\widetilde{\mu}}|$ is a corollary of the following two statements. Let $\mathcal{C}$ be the chamber corresponding to the base $\alpha_0, \dots, \alpha_l$ of S (see [5], sec. 2.4).

3.7.1. Proposition. *If x belongs to the interior of $\mathcal{C}$ and*

$$\Delta\widetilde{\mu}(x) = \sum U_{\mu_i+\rho}(x) - (l+2)U_\rho(x),$$

then

$$|\varphi_{\widetilde{\mu}}| = O(|\tau|^l |q|^{\Delta\widetilde{\mu}(x)}).$$

For the set $\widetilde{\mu} = (\alpha_0, \dots, \alpha_l)$, this estimate is exact, i.e., $|\varphi_{\widetilde{\mu}}| \sim C|\tau|^l |q|^{\Delta\widetilde{\mu}(x)}$, where $C \neq 0$.

3.7.2. Proposition. *There exists $x \in \mathcal{C}$ such that $\Delta\widetilde{\mu}(x) > \frac{r(S)}{24} - \frac{1}{r_p}$ for any $\widetilde{\mu}$.*

Observe that Proposition 3.7.2 implies, in particular, that $F_S(\tau) \not\equiv 0$, i.e., $H_S(\tau) \not\equiv 0$.

3.8. Proof of Proposition 3.7.1

Recall the definition of $\varphi_{\widetilde{\mu}}$. For each weight $\lambda \in \Lambda$ with $\kappa(\lambda) = k > 0$, we set

$$f_\lambda(z) = u^k \cdot \exp \nu(U_\lambda(z) - kU_0(z)).$$

Further, for $\lambda \in \rho + \Lambda^+$, where $\rho = \lambda_0 + \cdots + \lambda_l$, we define a θ-function

$$\psi_\lambda = \sum_{w \in W_S} \det w \cdot f_{w\lambda}.$$

For $\mu \in \Lambda^+$, we define: $\zeta_\mu = \dfrac{\psi_{\mu+\rho}}{\psi_\rho}$.

Finally, we define a function $\varphi_{\widetilde{\mu}}$ on Θ by setting

$$\varphi_{\widetilde{\mu}} = \frac{d\zeta_{\mu_0} \cdot \ldots \cdot d\zeta_{\mu_l}}{\psi_\rho} \cdot \frac{1}{\omega_0}, \quad \text{where } \omega_0 = d\alpha_1 \ldots d\alpha_l \frac{du}{u}.$$

We will consecutively estimate all these functions and their differentials at $(x, 1) \in \Theta$.

To estimate the differentials, introduce a metric in the cotangent bundle over Θ by setting $\| \frac{du}{u} \| = 1$ and $\| \alpha \| = \| d\alpha \|$ for any $\alpha \in S$. Since

$$U_\lambda(z) - \kappa(\lambda)U_0(z) = \delta_\lambda(z) + c,$$

where $\delta_\lambda \in L^*$ is a linear function (see sec. 3.2) and c is a constant, it follows that

$$df_\lambda = \omega_\lambda \cdot f_\lambda, \qquad \text{where } \omega_\lambda = \nu d\delta_\lambda + \kappa(\lambda)\frac{du}{u}.$$

Furthermore, $|f_\lambda| = |q|^{e(\lambda)}$, where f_λ is evaluated at $(x, 1) \in \Theta$ and

$$e(\lambda) = U_\lambda(x) - kU_0(x).$$

Observe that if $\lambda \in \rho + \Lambda^+$, where $\rho = \lambda_0 + \cdots + \lambda_l$, then

$$e(w\lambda) = k \| wx_\lambda - x \|^2 - k \| x \|^2 > k \| x_\lambda - x \|^2 - k \| x \|^2 = e(\lambda)$$

for any $w \in W_S$ such that $w \neq 1$ (the inequality is strict since $x_\lambda \in \mathcal{C}$, see [5], sec. 2.4). This inequality easily implies that

$$\psi_\lambda = f_\lambda(1 + o(|q|^\varepsilon)), \quad d\psi_\lambda = f_\lambda(\omega_\lambda + o(|q|^\varepsilon))$$

for some positive ε. Since $\zeta_\mu = \frac{\psi_{\mu+\rho}}{\psi_\rho}$, it follows that

$$d\zeta_\mu = \frac{f_{\mu+\rho}}{f_\rho} \cdot (\omega_\mu + o(|q|^\varepsilon).$$

Finally, $\varphi_{\widetilde{\mu}} = \varphi_1 \cdot \varphi_2$, where

$$\varphi_1 = \prod \frac{f_{\mu_i+\rho}}{f_\rho^{l+2}}, \quad \varphi_2 = \frac{\prod \omega_{\mu_i} + o(|q|^\varepsilon)}{\omega_0}.$$

Since

$$\sum e(\mu_i + \rho) - (l+2)e(\rho) = \sum U_{\mu_i+\rho}(x) - (l+2)U_\rho(x) = \Delta\widetilde{\mu}(x),$$

it follows that $|\varphi_1| = |q|^{\Delta\widetilde{\mu}(x)}$.

On the other hand, clearly, $\prod \dfrac{\omega_{\mu_i}}{\omega_0} = C\tau^l$ and $C \neq 0$ for $\widetilde{\mu} = (\lambda_0, \dots, \lambda_l)$. ∎

3.9. Proof of Proposition 3.7.2

The estimate of Proposition 3.7.2 is rather rough; and this estimate is inapplicable to the affine root system D_l. For the other root systems, the proof will be carried through case-by-case checking. For the definition of g, see (3) in sec. 3.2. We use the data on classical affine root systems accumulated in the Table (at the end of the paper) and in the following easy to prove general statement:

(i) *Let $c_i > 0$ and $z_i \in V_\mathbb{R}$. Then the form $Q(x) = \sum c_i \parallel x - z_i \parallel^2$ is, actually, of the shape $Q(x) = C \parallel x - z \parallel^2 + c$, where $C = \sum c_i$, and $z = \frac{1}{C}\sum c_i z_i$.*

Recall that $U_{\lambda_i} = n_i \parallel x - x_i \parallel^2$, where x_i is a vertex of the chamber $\mathcal{C}$. Set $r = x_\rho$, so that $U_\rho(x) = g \parallel x - r \parallel^2$.

Then (i) implies that $r = \sum \dfrac{n_i x_i}{g}$, that is r is the center of the simplex $\mathcal{C}$. For any weight λ with $\kappa(\lambda) = k > 0$, we see that $x_{\lambda+\rho} = \dfrac{k x_\lambda + gr}{g+k}$. In particular, if x is of the form $\dfrac{ky + gr}{g+k}$, then

$$U_{\lambda+\rho}(x) = (g+k) \parallel x - x_{\lambda+\rho} \parallel^2 = (g+k) \parallel \frac{k}{g+k} \cdot (y - x_\lambda) \parallel^2 = \frac{k^2}{g+k} \parallel y - x_\lambda \parallel^2 .$$

<u>(ii) The case of nonclassical root systems S.</u> In this case

$$\Delta\widetilde{\mu}(r) = \sum U_{\mu_i+\rho}(r) > 0.$$

Therefore it suffices to verify that $\dfrac{r(S)}{24} \le \dfrac{1}{r_p}$. Since $r(S) + r(S^{\text{inv}}) = (p+1)l$ and for all nonclassical root systems $S^{\text{inv}} \cong S$, we see that $r(S) = \frac{1}{2}(p+1)l$.

Therefore we should verify that $\dfrac{(p+1)l}{48} \le \dfrac{1}{r_p}$. Indeed:

If $p = 1$, then $l \le 8$ implying $\dfrac{(p+1)l}{48} \le \dfrac{1}{3} \le \dfrac{1}{r_p}$;

if $p = 2$, then $l = 4$ (case $S(F_4, 2)$), so that $\dfrac{(p+1)l}{48} \le \dfrac{1}{4} \le \dfrac{1}{r_p}$;

if $p = 3$, then $l = 2$ (case $S(G_2, 3)$), so that $\dfrac{(p+1)l}{48} \le \dfrac{1}{6} \le \dfrac{1}{r_p}$.

(iii) The case of the classical root systems. In the Table we consider an l-dimensional Euclidean space L with an orthonormal basis $(\varepsilon_1, \dots, \varepsilon_l)$ and canonically identify L and L^*. Let us deduce several corollaries from the Table.

(α) Set $x_i' = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_i)$. Then x_i almost always coincides with x_i'; more exactly, $x_i = x_i'$ for $i = 0$, $i = l$ and any i with $n_i = 2$. Setting $y = \frac{1}{2}(x_0 + x_l)$ (i.e., $y = (\frac{1}{4}, \dots, \frac{1}{4})$) we, clearly, see that $\| x_i' - y \|^2 = \| y \|^2$ and $\| x_i - y \|^2 \ge \| y \|^2$ for all i.

(β) It is clear from Table that $r = (u_1, \dots, u_l)$, where $u_1, \dots, u_l$ is an arithmetic progression, $\frac{1}{2} \ge u_1 \ge u_2 > \cdots > u_l \ge 0$ and $u_1 + u_l \ge \frac{1}{2}$. This implies that $(y, r - y) \ge 0$; hence $\| r \|^2 \ge \| r - y \|^2 + \| y \|^2$, i.e.,

$$\| r - y \|^2 \le \| r \|^2 - \| y \|^2 .$$

(γ) Let us prove that $\| x_i - r \|^2 \le \| r \|^2$ for any i. This inequality is equivalent to the fact that $(2r - x_i, x_i) \ge 0$. If $x_i = x_i'$, the inequality holds because $u_1 + u_i \ge \frac{1}{2}$.

For the cases $x_i \ne x_i'$, this inequality easily follows from the Table.

(δ) Let us estimate the constant $A = \sum n_i \| x_i - r \|^2$. Thanks to (i) we have

$$\sum n_i \| x - x_i \|^2 = g \| x - r \|^2 + A. \tag{6}$$

Substituting $x = y$ into (6) we get (see (β))

$$\begin{aligned} A = & \sum n_i \| y - x_i \|^2 - g \| y - r \|^2 \ge \\ & \sum n_i \| y \|^2 - g(\| r \|^2 - \| y \|^2) = g(2 \| y \|^2 - \| r \|^2) \end{aligned}$$

(iv) Cases $S(A_l, 1)$ and $S(C_l, 1)$. Here $n_i = 1$ so $g = l + 1$ and there exists exactly one, up to a permutation, set $\widetilde{\mu} = (\lambda_0, \dots, \lambda_l)$.

Moreover, $r(S) = l$ and

$$\Delta \widetilde{\mu}(r) = \sum U_{\lambda_{i+p}}(r) = \sum (g+1) \| x_{\lambda_{i+p}} - r \|^2 = \frac{1}{g+1} \sum \| x_i - r \|^2 .$$

We have $\| x_i - r \|^2 = \| r \|^2$ for $S(A_l, 1)$ because there is an automorphism of $S(A_l, 1)$ sending any point x_i into $x_0 = 0$. Therefore (see [5], sec.4.4)

$$\Delta \widetilde{\mu}(r) = \frac{l+1}{g+1} \| r \|^2 = \frac{g \| r \|^2}{g+1} = \frac{1}{24} \cdot \frac{h+1}{g+1} r(S) = \frac{r(S)}{24} > \frac{r(S)}{24} - \frac{1}{r_p}.$$

For $S(C_l, 1)$, we have (see δ)

$$\Delta\widetilde{\mu}(r) = \frac{1}{g+1}\sum \| x_i - r \|^2 = \frac{A}{g+1} = \frac{g}{g+1}(2 \| y \|^2 - \| r \|^2) \geq$$
$$\frac{2g}{g+1}\cdot\frac{l}{16} - \frac{1}{24}\cdot\frac{l(2l+1)}{g+1} = \frac{l(3l+3-2l-1)}{24(l+2)} = \frac{l}{24} = \frac{r(S)}{24} > \frac{r(S)}{24} - \frac{1}{r_p}.$$

(v) Systems $S(B_l, 1)$, $S(B_l, 2)$, $S(C_l, 2)$. Let k be the total number of indices i such that $n_i = 1$. Clearly, $g = 2l + 2 - k$. Consider the set $\widetilde{\lambda} = (\lambda_0, \ldots, \lambda_l)$. Then (see sec. 3.9 (i))

$$\Delta\widetilde{\lambda}(x) \equiv \Delta\widetilde{\lambda}(r) = \sum (g+n_i) \| x_{\lambda_i+p} - r \|^2 = \sum \frac{n_i^2}{g+n_i} \| x_i - r \|^2 =$$
$$\frac{2}{g+2}\sum n_i \| x_i - r \|^2 + \left(\frac{1}{g+1} - \frac{2}{g+2}\right)\sum_1 \| x_i - r \|^2,$$

where $\sum_1$ is the sum over the i such that $n_i = 1$.

Since $\dfrac{1}{g+1} - \dfrac{2}{g+2} > -\dfrac{1}{g+2}$ and $\| x_i - r \|^2 \leq \| r \|^2$, we see that

$$\Delta\widetilde{\lambda}(x) \geq \frac{2}{g+2}A - \frac{k}{g+2} \| r \|^2 \geq \frac{1}{g+2}(4g \| y \|^2 - (2g+k) \| r \|^2).$$

Now, let $\widetilde{\mu} = (\mu_0, \ldots, \mu_l)$ be an arbitrary set of distinct weights. Rearranging the μ_i, if needed, we may assume that $\mu_i \neq \lambda_j$ for $i \neq j$. Let I be the set of indices such that $\mu_i \neq \lambda_i$. For any $i \in I$ we have $n_i = 2$ and $\mu_i = \lambda_p + \lambda_q$, where $n_p = n_q = 1$. In particular, $x_{\mu_i} = \frac{1}{2}(x_p + x_q)$.

We will say that the index i is *good* if $|p - q| \leq 1$ and *bad* otherwise; denote by I_g and I_b the sets of good and bad indices of I. Set $x = \dfrac{2y + gr}{g+2}$. If $\kappa(\mu) = 2$, then

$$U_{\mu+\rho}(x) = (g+2) \| x_{\mu+\rho} - x \|^2 = (g+2) \| \frac{2}{g+2}(x_\mu - y) \|^2 = \frac{4}{g+2} \| x_\mu - y \|^2 .$$

Therefore

$$\Delta\widetilde{\mu}(x) - \Delta\widetilde{\lambda}(x) = \sum_{i\in I} U_{\mu_i+\rho}(x) - U_{\lambda_i+\rho}(x) =$$
$$\frac{4}{g+2}\sum_{i\in I} \| x_{\mu_i} - y \|^2 - \| x_{\lambda_i} - y \|^2 .$$

If $i \in I$, then $n_i = 2$; hence, $x_{\lambda_i} = x_i'$ and $\| x_{\lambda_i} - y \|^2 = \| y \|^2$ (see step (iii)). It is easy to verify that if i is a good index, then $\| x_{\mu_i} - y \|^2 \geq \| y \|^2$ since all the coordinates of the vector $x_{\mu_i} = \frac{1}{2}(x_p + x_q)$ are half-integer. Therefore

$$\Delta\widetilde{\mu}(x) - \Delta\widetilde{\lambda}(x) \geq -\#(I_b)\cdot\frac{4}{g+2} \| y \|^2 .$$

Finally, we get

$$\Delta\widetilde{\mu}(x) \geq \frac{1}{g+2}(4(g-\#(I_b)) \parallel y \parallel^2 -(2g+k) \parallel r \parallel^2).$$

Consider consecutively all the systems (except $S(D_l,1)$).

(α) $S(B_l,1)$, where $l > 8$ (the cases $l \leq 8$ are considered as in (ii)). Then $g = 2l-1$, $k = 3$, $\#(I_b) \leq 2$, $\parallel y \parallel^2 = \frac{l}{32}$, $r(S) = l$;

$$\begin{aligned}\Delta\widetilde{\mu}(x) \geq \frac{1}{2l+1}\left(4(2l-3)\cdot\frac{l}{32} - \frac{(4l+1)(2l+1)l}{(2l-1)24}\right) = \\ ((2l-1)(2l-3)-(4l+1)(2l+1))\frac{l}{24(4l^2-1)} \geq \\ \frac{4l^2-30l}{4l^2\cdot 24}\cdot l \geq \frac{l}{24} - \frac{1}{3} = \frac{r(S)}{24} - \frac{1}{r_p}.\end{aligned}$$

(β) $S(B_l,2)$, $g = 2l$, $k = 2$, $\#(I_b) = 1$, $\parallel y \parallel^2 = \frac{l}{16}$, $r_S = 2l-1$;

$$\begin{aligned}\Delta\widetilde{\mu}(x) \geq \frac{1}{2l+2}\left(4(2l-1)\cdot\frac{l}{16} - (4l+2)\frac{(2l-1)(2l+1)}{24\cdot 2l}\right) = \\ \frac{(2l-1)}{l(l+1)48}(6l^2-4l^2-4l-1) > \frac{2l-1}{24} - \frac{1}{4} = \frac{r(S)}{24} - \frac{1}{r_p}.\end{aligned}$$

(γ) $S(C_l,2)$, $g = 2l$, $k = 2$, $\#(I_b) = 0$, $\parallel y \parallel^2 = \dfrac{l}{32}$, $r_S = l+1$;

$$\begin{aligned}\Delta\widetilde{\mu}(x) \geq \frac{1}{2l+2}\left(8l\cdot\frac{l}{32} - (4l+2)\frac{(l+1)(2l+1)l}{24\cdot 2l}\right) = \\ \frac{1}{l+1}\left(\frac{l^2-1}{8} + \frac{1}{8} - \frac{(2l+1)^2(l+1)}{48l}\right) \geq \frac{1}{48l}(2l^2-10l) = \frac{l+1}{24} - \frac{1}{4},\end{aligned}$$

because $\dfrac{1}{8(l+1)} > \dfrac{1}{48l}$.

4 Sertain automorphic functions on the upper half-plane

4.1. Upper half-plane

Let $H = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ be the upper half-plane. On H, the group $G = GL^+(2,\mathbb{R})$, where the superscript "+" singles out the subgroup of matrices with positive determinant, acts by the formula

$$\gamma(\tau) = \frac{a\tau+b}{c\tau+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group $PG = GL^+(2,\mathbb{R})/\mathbb{R}^*$ effectively acts on H.

4.2. Modular groups $\Gamma_0(p)$

Let $p = 1$, 2 or 3. Set

$$\gamma_p = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z},\ c \in p\mathbb{Z},\ ad - bc = 1 \right\}.$$

Clearly, $\gamma_1 \in \Gamma_0(1)$ and γ_p normalizes the subgroup $\Gamma_0(p)$, and it is also clear that in PG we have $\gamma_p^2 = 1$. Denote by $\hat{\Gamma}_0(p)$ the subgroup of PG generated by $\Gamma_0(p)$ and γ_p. It is known (see, e.g., [15], Ch. XI) that for $p = 1, 2, 3$ the group $\hat{\Gamma}_0(p)$ is isomorphic to the triangular Hecke group of signature $(0; 2, r_p, \infty)$, where $r_1 = 3$, $r_2 = 4$, $r_3 = 6$, i.e., $\hat{\Gamma}_0(p)$ is a subgroup of index 2 in the group generated by reflections with respect to the sides of the triangle with angles $(\frac{\pi}{2}, \frac{\pi}{r_p}, 0)$ in the sense of Lobachevsky geometry on H. This easily implies that

a) $H/\hat{\Gamma}_0(p)$ is simply connected and isomorphic to $\mathbb{C}$ since the one-point compactification $\overline{H/\hat{\Gamma}_0(p)}$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$;

b) The order of the stabilizer $\hat{\Gamma}_0(p)_\tau$ of τ is equal to

$$\begin{cases} 2 & \text{for the points of } \hat{\Gamma}_0(p)\text{-orbit of the vertex } \tau_1 \text{ of the right angle;} \\ r_p & \text{for the points of the } \hat{\Gamma}_0(p)\text{-orbit of the vertex } \tau_2 \text{ of the angle } \pi/r_p; \\ 1 & \text{for the other points.} \end{cases}$$

4.3. Modular invariant j_p

We will consider functions on H which vary in a certain way with respect to $\hat{\Gamma}_0(p)$ and be interested in the asymptotics of these functions as $\operatorname{Im}\tau \longrightarrow \infty$. The notations $f \sim g$, $f = o(g)$, and $f = O(g)$ mean, respectively, that $\frac{f}{g} \longrightarrow 1$ and $\frac{f}{g} \longrightarrow 0$ as $\operatorname{Im}\tau \longrightarrow \infty$, and $\lim\limits_{\operatorname{Im}\tau \longrightarrow \infty} \frac{f}{g} < \infty$.

Denote by $\mathcal{O}(H)$ the space of the holomorphic functions on H. If $f \in \mathcal{O}(H)$ and $\tau \in H$, then $\operatorname{ord}_\tau(f)$ denotes the order of the zero of f at τ.

Statement. *There exists a $\hat{\Gamma}_0(p)$-invariant function $j_p \in \mathcal{O}(H)$ such that*

a) *j_p takes each value at exactly one $\hat{\Gamma}_0(p)$-orbit in H and* $\operatorname{ord}_\tau(j_p - j_p(\tau)) = |\hat{\Gamma}_0(p)_\tau|$ *for any $\tau \in H$;*

b) *$j \sim q^{-1}$, where $q = \exp(2\pi i\tau)$.*

Proof. Since $H/\hat{\Gamma}_0(p)$ is simply connected, it easily follows that $H/\hat{\Gamma}_0(p)$ is isomorphic to $\mathbb{C}$. Let j be a $\hat{\Gamma}_0(p)$-invariant function on H that determines this isomorphism. By definition j satisfies condition a).

Since $\hat{\Gamma}_0(p)$ contains the transformation $\tau \longrightarrow \tau + 1$, it follows that $j = \sum\limits_{i \in \mathbb{Z}} c_i q^i$; since j takes each value at exactly one orbit and is not bounded as $\operatorname{Im}\tau \longrightarrow \infty$, we see that $c_i = 0$ for $i < -1$, and $c_{-1} \neq 0$. The function $j_p = \frac{j}{c_{-1}}$ satisfies conditions a) and b). ■

4.4. Statement. *There exists a function $\eta_p \in \mathcal{O}(H)$ such that*

a) *η_p does not vanish on H;*

b) *$|\eta_p|^2 \operatorname{Im}\tau$ is $\hat{\Gamma}_0(p)$-invariant;*

c) $|\eta_p| \sim |q|^{\frac{p+1}{24}}$.

Proof. For the Dedekind η-function, set

$$\eta_p(\tau) = \eta(\tau) \cdot \eta(p\tau).$$

Since η does not vanish anywhere, $|\eta| \sim |q|^{1/24}$ and $|\eta|^2(\operatorname{Im}\tau)^{\frac{1}{2}}$ is $\widehat{\Gamma}_0(1)$-invariant (see, e.g., [15], Ch. XI), it follows that η_p satisfies a)–c). ■

4.5. Main Lemma

Denote by $\mathcal{A}(H)$ the set of functions F on H which can be represented in the form $F = \sum |f_i|^2$, where $f_i \in \mathcal{O}(H)$.

Statement. *Given a nonzero $\Gamma_0(p)$-invariant function $F \in \mathcal{A}(H)$, suppose that*

$$F \cdot \gamma_p F = o(|q|^{-4/r_p}), \quad \text{where } r_1 = 3,\ r_2 = 4,\ r_3 = 6.$$

Then F does not vanish anywhere.

Proof. (i) If $F = \sum |f_i|^2$, where $f_i \in \mathcal{O}(H)$, we set

$$\operatorname{ord}_\tau(F) = \min_i \operatorname{ord}_\tau(f_i).$$

that is $\operatorname{ord}_\tau F$ is equal to the maximal m for which the function $\dfrac{F(s)}{|s-\tau|^{2m}}$ is bounded in a neighborhood of τ.

Lemma. *If $G \in \mathcal{A}(H)$ is a $\hat{\Gamma}_0(p)$-invariant function satisfying $G = o(|q|^{-2d})$, where $d > 0$, then either $G = 0$ or*

$$\sum \frac{\operatorname{ord}_{\tau_i}(G)}{\operatorname{ord}(\hat{\Gamma}_0(p))_{\tau_i}} < d \tag{7}$$

for any set of points $\tau_1, \dots, \tau_k$ belonging to different orbits of $\hat{\Gamma}_0(p)$.

Let us show how (7) implies the Statement. Suppose that $F(\tau) = 0$ for some $\tau \in H$. Set $G = F \cdot \gamma_p F$. Then $G = o(|q|^{-4/r_p})$.

Let us prove that $\dfrac{\operatorname{ord}_\tau(G)}{\operatorname{ord}(\hat{\Gamma}_0(p))_\tau} \geq \dfrac{2}{r_p}$, then the estimate (7) will lead to a contradiction.

If $\gamma_p(\tau) \in \Gamma_0(p)\tau$, then $\operatorname{ord}_\tau G \geq 2$ and $\operatorname{ord}(\hat{\Gamma}_0(p))_\tau \leq r_p$. Now, let $\gamma_p(\tau) \notin \Gamma_0(p)\tau$; in particular, $p > 1$. Then the stabilizer of τ in $\hat{\Gamma}_0(p)$ is contained in $\Gamma_0(p)$.

The description of stabilizers given in sec. 4.2 implies that $\operatorname{ord}(\hat{\Gamma}_0(p))_\tau = 1$ or p yielding

$$\frac{\operatorname{ord}_\tau(G)}{\operatorname{ord}(\hat{\Gamma}(p))_\tau} \geq \frac{1}{p} \geq \frac{2}{r_p}.$$

(ii) Let us prove the Lemma by induction on the number k of points. For this we prove at first that if $d = 0$ in Lemma, then the function G vanishes identically.

Let us identify $H/\hat{\Gamma}_0(p)$ with $\mathbb{C}$ and let G' be the function on $\mathbb{C}$ induced by G. The function G' is continuous and $G'(z) \longrightarrow 0$ as $\operatorname{Im} z \longrightarrow \infty$ since $G' = o(1)$. If u is any point of $\mathbb{C}$ over which the covering $H \longrightarrow \mathbb{C}$ is not ramified, then, in a neighborhood of u, the function G' can be represented in the form $G' = \sum |f_i|^2$, where the f_i are holomorphic functions. Therefore

$$\Delta G' = \sum \frac{\partial^2}{\partial \bar{z} \partial z} |f_i|^2 = \sum \left| \frac{\partial f_i}{\partial z} \right|^2 \geq 0,$$

where Δ is the Laplace operator. This easily implies that $G' = 0$ (see, e.g., [11]).

(iii) Now let $e = \mathrm{ord}_{\tau_k}(G)$ and $|\hat{\Gamma}_0(p)_{\tau_k}| = r$. The function $j_p - j_p(\tau_k)$ vanishes only on the orbit $\hat{\Gamma}(p)(\tau_k)$ and at the points of this orbit the zeros of this function are of multiplicity r (see sec. 4.3).

Therefore $f = (j_p - j_p(\tau))^{1/r} \in \mathcal{O}(H)$. Clearly, $G_1 = \dfrac{G}{|f|^{2e}} \in \mathcal{A}(H)$ and G_1 is $\hat{\Gamma}_0(p)$-invariant. We also have $\mathrm{ord}_{\tau_i}(G_1) = \mathrm{ord}_{\tau_i}(G)$ for $i < k$ and $G_1 = o(|q|^{-2d_1})$, where $d_1 = d - \dfrac{e}{r}$ (since $|f| \sim |q|^{-1/r}$). If $d_1 \leq 0$, then $G_1 = 0$ and $G = 0$. Otherwise applying (7) to G_1 and $\tau_1, \dots, \tau_{k-1}$ we obtain (7) for G and $(\tau_1, \dots, \tau_k)$. ■

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5 Table

In line x_i, there are i-many fractions $\frac{1}{2}$ in each column.

S	$S(B_l,1)$	$S(C_l,1)$	$S(D_l,1)$	$S(B_l,2)$	$S(C_l,2)$
α_0	$1-\varepsilon_1-\varepsilon_2$	$1-2\varepsilon_1$	$1-\varepsilon_1-\varepsilon_2$	$1-2\varepsilon_1$	$1-\varepsilon_1-\varepsilon_2$
α_i $1\leq i<l$	$\varepsilon_i-\varepsilon_{i+1}$	$\varepsilon_i-\varepsilon_{i+1}$	$\varepsilon_i-\varepsilon_{i+1}$	$2(\varepsilon_i-\varepsilon_{i+1})$	$\varepsilon_i-\varepsilon_{i+1}$
α_l	ε_l	$2\varepsilon_l$	$\varepsilon_{l-1}+\varepsilon_l$	$2\varepsilon_l$	$2\varepsilon_l$
n_0	1	1	1	1	1
n_1	1	1	1	2	1
n_i	2	1	2	2	2
n_{l-1}	2	1	1	2	2
n_l	1	1	1	1	1
x_0	$(0,\dots,0)$	$(0,\dots,0)$	$(0,\dots,0)$	$(0,\dots,0)$	$(0,\dots,0)$
x_1	$(1,0,\dots,0)$	$\frac{1}{2},0,\dots,0$	$1,0,\dots,0$	$\frac{1}{2},0,\dots,0$	$1,0,\dots,0$
x_i	$\frac{1}{2},\dots,\frac{1}{2},0,\dots,0$	$\frac{1}{2},\dots,\frac{1}{2},0,\dots,0$	$\frac{1}{2},\dots,\frac{1}{2},0,\dots,0$	$\frac{1}{2},\dots,\frac{1}{2},0,\dots,0$	$\frac{1}{2},\dots,\frac{1}{2},0,\dots,0$
x_{l-1}			$\frac{1}{2},\frac{1}{2},\dots,-\frac{1}{2}$		
g	$2l-1$	$l+1$	$2l-2$	$2l$	$2l$
h	$2l$	$2l$	$2l-2$	$2l$	$2l$
$r(S)$	l	l	l	$2l-1$	$l+1$
$U_\rho(0)=$ $g\parallel r\parallel^2$	$\frac{1}{24}l(2l+1)$	$\frac{1}{24}l(2l+1)$	$\frac{1}{24}l(2l-1)$	$\frac{1}{24}(2l+1)(2l-1)$	$\frac{1}{24}(2l+1)(l+1)$
r	$\frac{1}{2}\ \ 1,\frac{2l-3}{2l-1},\dots,\frac{1}{2l-1}$	$\frac{1}{2}\ \ \frac{l}{l+1},\frac{l-1}{l+1},\dots,\frac{1}{l+1}$	$\frac{1}{2}\ \ 1,\frac{l-2}{l-1},\dots,\frac{1}{l-1},0$	$\frac{1}{2}\ \ \frac{2l-1}{2l},\frac{2l-3}{2l},\dots,\frac{1}{2l}$	$\frac{1}{2}\ \ 1,\frac{l-1}{l},\dots,\frac{1}{l}$
$\frac{1}{r_p}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$