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für Mathematik
in den Naturwissenschaften
Leipzig

On the Bott-Borel-Weil and Tolpygo theorems

by

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Preprint no.: 84 2006

ON THE BOTT-BOREL-WEIL AND TOLPYGO THEOREMS

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Abstract. An explicit basis for the cohomology of the maximal nilpotent subalgebra of each simple finite dimensional Lie algebra with trivial coefficients is offered.

1. Introduction

Let the ground field be $\mathbb{C}$ and let all spaces encountered be of finite dimension. Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{n}$ its maximal nilpotent subalgebra and M an irreducible $\mathfrak{g}$-module. The famous Bott-Borel-Weil theorem states that

$$\dim H^i(\mathfrak{n}; M) = |\{w \in W \mid l(w) = i\}|, \tag{1}$$

where W is the Weyl group of $\mathfrak{g}$ and $l(w)$ is the length of w, see [FH, OV]. In particular, (1) does not depend on M. For applications of the BBW theorem in physics, see, e.g., [W]; see also [GL3].

Attempts to describe the cohomology of $\mathfrak{n}$ with coefficients in $\mathfrak{n}$-modules were scanty; for their review and an extension of these results, see [T].

In these two cases, namely, where M is either (A) the trivial module $\mathbb{C}$ or (B) the adjoint one, the description of the BBW-type looks (and is) insufficient. Indeed, in these cases the space $H = \oplus H^i$ is naturally endowed, respectively, with a supercommutative superalgebra structure in case (A) or (in case (B)) with Lie superalgebra structure resembling the one determined by the Nijenhuis bracket. Even if we know all the dimensions (1), this does not describe the multiplication in the algebra; besides, even in the cases Tolpygo considered, these dimensions (1) are not described sufficiently explicitly, except for $i \leq 3$.

Our goal is the description of the algebra $H = \oplus H^i$ in terms of generators and defining relations. Such a description might be, however, rather complicated.

In case (B), such a description was offered for several simple Lie algebras and superalgebras $\mathfrak{g}$ in [LLS, GL1].

Still, if the elements of a basis of $H^*(\mathfrak{n}, \mathbb{C})$ are described in terms of the Weyl group sufficiently lucidly, the more complicated description in terms of generators and defining relations might be not needed *for computational purposes* (e.g., as the bracket of matrix units for $\mathfrak{gl}(n)$ suffices to determine the bracket in $\mathfrak{gl}(n)$, whereas the presentation in terms of Chevalley ([FH, GL4]) or Jacobson-Grozman-Leites ([GL2, LS]) or Sylvester-t'Hooft ([Sa]) generators, although needed in various problems, is much more complicated).

Here we elucidate the algebra structure of $H^{\bullet}(\mathfrak{n}; \mathbb{C})$ and describe, up to a sign, the supercommutative superalgebra structure in the space spanned by the elements of the Weyl group.

1991 *Mathematics Subject Classification.* 17B56; 17B60.

Key words and phrases. Lie superalgebras, Nijenhuis bracket, Bott-Borel-Weil theorem.

I am thankful to MPIMiS, Leipzig, and the International Max Planck Research School affiliated to it for financial support and most creative environment and to Dimitry Leites who raised the problem.

2. Results

Now we will describe the cohomology with coefficients in the trivial module for a maximal nilpotent subalgebra of any simple Lie algebra.

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, and Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Δ^+ and Δ^- be the sets of positive and negative roots; let $\mathfrak{n}^+$ and $\mathfrak{n}^-$ be the corresponding maximal nilpotent subalgebras of $\mathfrak{g}$. Let $\{e_\alpha \mid \alpha \in \Delta^+\}$ be a basis of $\mathfrak{n}^+$ such that e_α is of weight α with respect to $\mathfrak{h}$, and $\{f^\alpha \mid \alpha \in \Delta^+\}$ is the dual basis. Then for any $\alpha, \beta \in \Delta^+$, we have

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta} e_{\alpha+\beta}, \text{ where } N_{\alpha\beta} \in \mathbb{C} \setminus \{0\}, & \text{if } \alpha+\beta \in \Delta^+ \\ 0 & \text{if } \alpha+\beta \notin \Delta^+ \end{cases} \tag{2}$$

Let W be the Weyl group. For any $v \in W$, set

$$\Delta_v = \Delta^+ \cap v\Delta^- = \{\alpha \in \Delta^+;\ v^{-1}(\alpha) < 0\} \qquad \text{and} \qquad C_v = \bigwedge_{\alpha \in \Delta_v} f^\alpha. \tag{3}$$

Note that since the length of an element of the Weyl group is equal to the number of negative roots it maps into positive ones, the degree of C_v is equal to the length of v.

2.1. Lemma. *The set $\{C_v \mid v \in W\}$ is a basis of $H^*(\mathfrak{n}; \mathbb{C})$.*

Proof. A cochain is said to be *basic*, if it can be represented as a wedge product of some f^αs. Since the C_v are basic cochains, not proportional to one another, they are linearly independent.

Recall the definition of the differential in the cochain complex: for a given Lie algebra L and its module V, and for any $e \in L$, $\varphi_1, \dots, \varphi_n \in V^*$, we set

$$d(e \otimes \varphi_1 \wedge \dots \wedge \varphi_n) = de \wedge \varphi_1 \wedge \dots \wedge \varphi_n + \sum_{k=1}^{n} (-1)^k e \otimes \varphi_1 \wedge \dots \wedge d\varphi_k \wedge \dots \wedge \varphi_n. \tag{4}$$

Since we consider cochains with coefficients in the trivial module, we get the following:

$$d(\varphi_1 \wedge \dots \wedge \varphi_n) = \sum_{k=1}^{n} (-1)^k \otimes \varphi_1 \wedge \dots \wedge d\varphi_k \wedge \dots \wedge \varphi_n \tag{5}$$

From (2) we get that

$$df^\alpha = \sum_{\beta \in \Delta^+;\ \alpha-\beta \in \Delta^+;\ \beta \prec \alpha-\beta} N_{\beta,\alpha-\beta} f^\beta \wedge f^{\alpha-\beta}, \tag{6}$$

where $\prec$ is any complete ordering on Δ^+.

So we see that any basic cochain which appears with nonzero coefficient is the differential dc of a basic cochain c, it can be obtained by replacing some f^α in c by $f^\beta \wedge f^{\alpha-\beta}$ for some $\beta \in \Delta^+$ such that $\alpha - \beta \in \Delta^+$. But, by definition (3), for such α, β and any $v \in W$, if C_v contains f^α, then it also contains at least one of the elements f^β and $f^{\alpha-\beta}$ (because otherwise $v^{-1}(\beta) > 0$, $v^{-1}(\alpha-\beta) > 0$ and, therefore, $v^{-1}(\alpha) > 0$, and C_v can not contain f^α). Therefore, NO basic cochain can appear with nonzero coefficient in dC_v, and $dC_v = 0$.

Similarly, for any $v \in W$ and any $\alpha, \beta \in \Delta^+$ such that $\alpha - \beta \in \Delta^+$, if C_v does not contain f^α, then it can not contain f^β and $f^{\alpha-\beta}$ at the same time (otherwise $v^{-1}(\beta) < 0$, $v^{-1}(\alpha-\beta) < 0$, and therefore $v^{-1}(\alpha) < 0$, and hence C_v contains f^α). So C_v can not appear with nonzero coefficient in a differential of a cochain. Thus, no nontrivial linear combination of the C_v is a coboundary.

According to the BBW theorem, $\dim H^*(\mathfrak{n}; \mathbb{C}) = |W|$. Thus, $\{C_v \mid v \in W\}$ is indeed a basis of $H^*(\mathfrak{n}; \mathbb{C})$. $\square$

2.2. Lemma. *For any $v_1, v_2 \in W$, exactly one of the following two conditions holds:*
1) $C_{v_1} \wedge C_{v_2} = \pm C_v$ *for some* $v \in W$;
2) $C_{v_1} \wedge C_{v_2}$ *is a coboundary.*

Proof. Clearly, the two conditions can not hold simultaneously since C_v's are not coboundaries.

Since C_{v_1} and C_{v_2} are basic cochains, $C_{v_1} \wedge C_{v_2}$ is either 0 (and then it is a coboundary) or a basic cochain. Suppose that $C_{v_1} \wedge C_{v_2}$ is a basic cochain, which is not equal to $\pm C_v$ for any $v \in W$. Then it can be represented in the following way:

$$C_{v_1} \wedge C_{v_2} = \sum_{v \in W} a(v) C_v + B \quad \text{for some } a(v) \in \mathbb{C},\ B \in B^*(\mathfrak{n}, \mathfrak{n}).$$

Since, as it was shown in the proof of the previous Lemma, no C_v can appear with a non-zero coefficient in decomposition of a coboundary w.r.t basic cochains, we see that the coefficient at C_v in decomposition w.r.t basic cochains is equal to 0 for the left-hand side, and equal to $a(v)$ for the right-hand side. So all the $a(v)$'s are equal to 0, and $C_{v_1} \wedge C_{v_2} = B \in B^*(\mathfrak{n}, \mathfrak{n})$. □

2.3. Proposition. *For any $v \in W$, we have:*
1) *the weight of C_v is equal to $v\rho - \rho$;*
2) *any element from $C^*(\mathfrak{n}; \mathbb{C})$ of weight $v\rho - \rho$ is proportional to C_v.*

Proof. 1) By definition (3), the weight of C_v is equal to

$$-\sum_{\alpha \in \Delta^+ \cap v\Delta^-} \alpha = \frac{1}{2}\left(-\sum_{\alpha \in \Delta^+ \cap v\Delta^-} \alpha + \sum_{\alpha \in \Delta^- \cap v\Delta^+} \alpha\right) = \frac{1}{2}\left(-\sum_{\alpha \in \Delta^+ \setminus \Delta^+} \alpha + \sum_{\alpha \in v\Delta^+ \setminus v\Delta^+} \alpha\right) =$$

$$\frac{1}{2}\left(\sum_{\alpha \in v\Delta^+} \alpha - \sum_{\alpha \in \Delta^+} \alpha\right) = \frac{1}{2}\left(v \sum_{\alpha \in \Delta^+} \alpha - \sum_{\alpha \in \Delta^+} \alpha\right) = v\rho - \rho.$$

2) This statement is implicitly contained in [T]. At the moment I can not offer a short independent proof. □

Since the weight is additive on exterior products, we have proven the following

2.4. Statement. *If $C_{v_1} \wedge C_{v_2} = \pm C_v$ for some $v, v_1, v_2 \in W$, then*

$$(v_1\rho - \rho) + (v_2\rho - \rho) = v\rho - \rho, \qquad i.e., \qquad v_1\rho + v_2\rho = v\rho + \rho.$$

So from this statement and Proposition 2.3, we get the following multiplication rule of basic elements in $H^*(\mathfrak{n}; \mathbb{C})$:

1) *If, for some $v_1, v_2 \in W$, there is no $v \in W$ such that $v\rho = v_1\rho + v_2\rho - \rho$, then $C_{v_1} \wedge C_{v_2} = 0$.*
2) *If such v exists, then*
a) *if there is no $\alpha \in \Delta^+$ such that $v_1^{-1}\alpha < 0$ and $v_2^{-1}\alpha < 0$ (i.e., $C_{v_1} \wedge C_{v_2} \neq 0$), then $C_{v_1} \wedge C_{v_2} = \pm C_v$;*
b) *$C_{v_1} \wedge C_{v_2} = 0$ otherwise.*

Example. Let us consider $\mathfrak{g} = A_r = \mathfrak{sl}(r+1)$. In this case, the Weyl group is isomorphic to the group S_{r+1} of permutations of integers from 0 to r. Then we get the following:

2.5. Corollary. 1) *If for $s_1, s_2 \in S_{r+1}$, the numbers $s_1^{-1}(i)+s_2^{-1}(i)-i$, $0\le i\le r$, are not the integers from* 0 *to r in some order, then $C_{s_1}\wedge C_{s_2}=0$.*

2) *Otherwise, set*

$$s=\begin{pmatrix} 0 & 1 & \dots & r \\ s_1^{-1}(0)+s_2^{-1}(0) & s_1^{-1}(1)+s_2^{-1}(1)-1 & \dots & s_1^{-1}(r)+s_2^{-1}(r)-r \end{pmatrix}^{-1}.$$

If there are integers i, j such that $0\le i<j\le r$, $s_1^{-1}(i)>s_1^{-1}(j)$ and $s_2^{-1}(i)>s_2^{-1}(j)$, then $C_{s_1}\wedge C_{s_2}=0$; otherwise, $C_{s_1}\wedge C_{s_2}=\pm C_s$.

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