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by

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Global well-posedness for the Schrödinger equation coupled to a nonlinear oscillator

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Abstract. The Schrödinger equation with the nonlinearity concentrated at a single point proves to be an interesting and important model for the analysis of long-time behavior of solutions, such as the asymptotic stability of solitary waves and properties of weak global attractors. In this note, we prove global well-posedness of this system in the energy space H^1.

1 Introduction and main results

We are going to prove the well-posedness in H^1 for the nonlinear Schrödinger equation with the nonlinearity concentrated at a single point:

$$i\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}, \tag{1.1}$$

where the dots and the primes stand for the partial derivatives in t and x, respectively. The equation describes the Schrödinger field coupled to a nonlinear oscillator. This equation is a convenient playground for developing the tools for the analysis of long-time behavior of solutions to $U(1)$-invariant Hamiltonian systems with dispersion. In particular, this equation is a convenient model for studying the asymptotic stability and global attraction. Let us mention that for the Klein-Gordon equation with the nonlinearity of the same type the global attraction was addressed in [KK06a], [KK06b].

We assume that

$$F(\psi) = -\nabla_\psi U(\psi), \qquad \psi \in \mathbb{C}, \tag{1.2}$$

for some real-valued potential $U \in C^2(\mathbb{C})$, where ∇_ψ is the real derivative with respect to $(\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)$. Equation (1.1) is a Hamiltonian system with the Hamiltonian

$$\mathscr{H}(\psi) = \int_{\mathbb{R}} \frac{|\psi'(x)|^2}{2}\,dx + U(\psi(0)), \qquad \psi \in H^1 = H^1(\mathbb{R}). \tag{1.3}$$

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The Hamiltonian form of (1.1) is

$$\dot{\Psi} = JD\mathscr{H}(\Psi), \tag{1.4}$$

where

$$\Psi = \begin{bmatrix} \operatorname{Re}\psi \\ \operatorname{Im}\psi \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{1.5}$$

and $D\mathscr{H}$ is the Fréchet derivative on the Hilbert space H^1. The value of the Hamiltonian functional is conserved for classical finite energy solutions of (1.1). We assume that equation (1.1) possesses $U(1)$-symmetry, thus requiring that

$$U(\psi) = u(|\psi|^2), \qquad \psi \in \mathbb{C}. \tag{1.6}$$

It then follows that $F(0) = 0$ and $F(e^{is}\psi) = e^{is}F(\psi)$ for $\psi \in \mathbb{C}$, $s \in \mathbb{R}$, and that

$$F(\psi) = a(|\psi|^2)\psi, \qquad \psi \in \mathbb{C}, \qquad \text{where} \quad a(\cdot) = 2u'(\cdot) \in \mathbb{R}. \tag{1.7}$$

This symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. According to the Nöther theorem, the $U(1)$-invariance leads (formally) to the conservation of the charge, given by the functional

$$Q(\psi) = \frac{1}{2}\int_{\mathbb{R}} |\psi|^2\,dx. \tag{1.8}$$

We also assume that $U(\psi)$ is such that

$$U(z) \geq A - B|z|^2 \qquad \text{with some} \quad A \in \mathbb{R}, \quad B > 0. \tag{1.9}$$

We will show that equation (1.1) is globally well-posed in H^1. We will consider the solutions of class $\psi \in C_b(\mathbb{R}\times\mathbb{R})$. All the derivatives in equation (1.1) are understood in the sense of distributions.

Theorem 1.1 (Global well-posedness). *Let the conditions (1.2), (1.6) and (1.9) hold with $U \in C^2(\mathbb{C})$. Then*

(i) For any $\phi \in H^1(\mathbb{R})$, the equation for (1.1) with the initial data $\psi|_{t=0} = \phi$ has a unique solution $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.

(ii) The values of the charge and energy functionals are conserved:

$$Q(\psi(t)) = Q(\phi), \qquad \mathscr{H}(\psi(t)) = \mathscr{H}(\phi), \qquad t \in \mathbb{R}. \tag{1.10}$$

(iii) There exists $\Lambda(\phi) > 0$ such that the following a priori bound holds:

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1} \leq \Lambda(\phi) < \infty. \tag{1.11}$$

(iv) The map $\mathbf{U} : \psi(0) \mapsto \psi$ is continuous from H^1 to $L^\infty([0,T], H^1(\mathbb{R}))$, for any $T > 0$.

Theorem 1.2. *Under conditions of Theorem 1.1, $\psi \in C^{(1/4)}(\mathbb{R}\times\mathbb{R})$.*

Let us give the outline of the proof. We need a small preparation first: We show that, without loss of generality, it suffices to prove the theorem assuming that U is uniformly bounded together with its derivatives. Indeed, the a priori bounds on the L^∞-norm of ψ imply that the nonlinearity $F(z)$ may be modified for large values of $|z|$. Then we will prove the existence and uniqueness of the solution $\psi \in C_b(\mathbb{R} \times [0,\tau])$, for some $\tau > 0$. This is accomplished in Section 2.

In Section 3, we construct approximate solutions $\psi_\epsilon \in C_b(\mathbb{R}, H^1(\mathbb{R}))$ that are solutions to a regularized problem (the δ-function substituted by its smooth approximations ρ_ϵ, $\epsilon > 0$). On one hand, the approximate solutions have their energy and charge conserved. On the other hand, we will show in Section 4 that the approximate solutions converge to $\psi(x,t)$ uniformly for $|x| \le R$, $0 \le t \le \tau$.

In Section 5, we use the uniform convergence of approximate solutions to conclude that $\psi \in L^\infty([0,\tau], H^1(\mathbb{R}))$ and moreover that ψ could be extended to all $t \ge 0$. Then we show that the energy and the charge are conserved. We will use these conservations to extend the solution $\psi(x,t)$ for $t \in \mathbb{R}$. Then we prove that $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.

In Section 6, we study the Hölder continuity in time, showing that $\psi \in C^{(1/4)}(\mathbb{R} \times \mathbb{R})$.

2 Local well-posedness in C_b

Lemma 2.1. *A priori bound (1.11) follows from (1.9) and the energy and charge conservation (1.10).*

Proof. Let $A \in \mathbb{R}$, $B > 0$ be constants from (1.9). Then for $\psi \in H^1$,

$$(2B^2 + \tfrac{1}{2})Q(\psi) + \mathscr{H}(\psi) \ge (B^2 + \tfrac{1}{4})\|\psi\|_{L^2}^2 + \tfrac{1}{2}\|\psi'\|_{L^2}^2 + A - B|\psi(0)|^2$$

$$\ge (B^2 + \tfrac{1}{4})\|\psi\|_{L^2}^2 + \tfrac{1}{2}\|\psi'\|_{L^2}^2 + A - B^2\|\psi\|_{L^2}^2 - \tfrac{1}{4}\|\psi'\|_{L^2}^2 \ge \tfrac{1}{4}\|\psi\|_{H^1}^2 + A. \qquad (2.1)$$

The first inequality follows from (1.9), while the second one holds since $|\psi(0)|^2 \le B\|\psi\|_{L^2}^2 + \frac{1}{4B}\|\psi'\|_{L^2}^2$ by the Fourier representation $\psi(0) = \frac{1}{2\pi}\int \hat{\psi}(k)dk$. Obviously, (2.1) and (1.10) imply the inequality (1.11), with

$$\Lambda(\phi) = \sqrt{(8B^2 + 2)Q(\phi) + 4\mathscr{H}(\phi) - 4A}. \qquad (2.2)$$

□

Lemma 2.2. *Let us assume that Theorem 1.1 is true for the nonlinearities U that satisfy the following additional condition: for $k = 0, 1, 2$ there exist $U_k < \infty$ so that*

$$\sup_{z \in \mathbb{C}} |\nabla^k U(z)| \le U_k, \qquad k = 0,\ 1,\ 2. \qquad (2.3)$$

Then Theorem 1.1 is also true without this additional condition.

Proof. Fix a nonlinearity U that does not necessarily satisfy (2.3). For a particular initial data $\phi \in H^1(\mathbb{R})$ in Theorem 1.1, we choose $\widetilde{U}(z) \in C^2(\mathbb{C})$ so that $\widetilde{U}(z) = \widetilde{U}(|z|)$ for $z \in \mathbb{C}$

and $\widetilde{U}(z) = U(z)$ for $|z| \le \Lambda(\phi)$, where $\Lambda(\phi)$ is defined by (2.2). We can choose $\widetilde{U}$ so that it satisfies (1.9) with the same A, B as U does, and also satisfies the uniform bounds

$$\sup_{z\in\mathbb{C}} |\nabla^k \widetilde{U}(z)| < \infty, \qquad k = 0, 1, 2.$$

By the assumption of the Lemma, Theorem 1.1 is true for the nonlinearity $\widetilde{F} = -\nabla\widetilde{U}$ instead of $F = -\nabla U$. Hence, there is a unique solution $\psi(x,t) \in L^\infty(\mathbb{R}, H^1) \cap C_b(\mathbb{R}\times\mathbb{R})$ to the equation

$$i\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)\widetilde{F}(\psi(0,t)),$$

with $\psi|_{t=0} = \phi$. By Lemma 2.1, ψ satisfies the a priori bound (1.11) with $\Lambda(\phi)$ defined by (2.2). This bound implies that $|\psi(0,t)| \le \Lambda(\phi)$ for $t \in \mathbb{R}$. Therefore, $\widetilde{F}(\psi(0,t)) = F(\psi(0,t))$ for $t \in \mathbb{R}$, and $\psi(x,t)$ is also a solution to (1.1) with the nonlinearity $F = -\nabla U$. □

From now on, we shall assume in the proof of Theorem 1.1 that the bounds (2.3) hold true.

Lemma 2.3. *(i) Let $\phi \in H^1 := H^1(\mathbb{R})$. There exists $\tau > 0$ that depends only on U_2 in (2.3) so that there is a unique solution $\psi \in C_b(\mathbb{R}\times[0,\tau])$ to equation (1.1) with the initial data $\psi|_{t=0} = \phi$.*

(ii) The map $\phi \mapsto \psi$ is continuous from H^1 to $C_b(\mathbb{R}\times[0,\tau])$.

Proof. Let us denote the dynamical group for the free Schrödinger equation by

$$\boldsymbol{W}_t\phi(x) = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{i\frac{|x-y|^2}{2t}}\phi(y)\,dy, \qquad x\in\mathbb{R}. \tag{2.4}$$

For its Fourier transform, we have:

$$\mathcal{F}_{x\to k}[\boldsymbol{W}_t\phi(x)](k) = e^{ik^2t}\hat{\phi}(k), \qquad k\in\mathbb{R}. \tag{2.5}$$

Then the solution ψ to (1.1) with the initial data $\psi|_{t=0} = \phi$ admits the Duhamel representation

$$\psi(x,t) = \boldsymbol{W}_t\phi(x) = \boldsymbol{W}_t\phi(x) + \boldsymbol{Z}\psi(x,t), \tag{2.6}$$

where

$$\boldsymbol{Z}\psi(x,t) = -\int_0^t \boldsymbol{W}_s\delta(x)F(\psi(0,t-s))\,ds = -\int_0^t \frac{e^{i\frac{x^2}{2s}}}{\sqrt{2\pi s}}F(\psi(0,t-s))\,ds. \tag{2.7}$$

The Fourier representation (2.5) implies that $\boldsymbol{W}_t\phi(x) \in C_b(\mathbb{R}, H^1) \subset C_b(\mathbb{R}\times\mathbb{R})$. Further, we compute for ψ_1, $\psi_2 \in C_b(\mathbb{R}\times[0,\tau])$:

$$|\boldsymbol{Z}\psi_2(x,t) - \boldsymbol{Z}\psi_1(x,t)| \le \int_0^t \frac{|F(\psi_2(0,t-s)) - F(\psi_1(0,t-s))|}{\sqrt{2\pi s}}\,ds \tag{2.8}$$

$$\le U_2\sqrt{t}\sup_{0\le s\le t}|\psi_2(s) - \psi_1(s)|, \tag{2.9}$$

where we used (2.3) with $k = 2$. For definiteness, we set

$$\tau = \frac{1}{4U_2^2}. \tag{2.10}$$

Then the map $\psi \mapsto \mathbf{W}_t\phi + \mathbf{Z}\psi$ is contracting in the space $C_b(\mathbb{R} \times [0, \tau])$. It follows that equation (2.6) admits a unique solution $\psi \in C_b(\mathbb{R} \times [0, \tau])$, proving the first part of the theorem. The second part of the theorem also follows by contraction. □

3 Regularized equation

We proved that there is a unique solution $\psi(x, t) \in C([0, \tau] \times \mathbb{R})$. Now we are going to prove that $\psi \in L^\infty(\mathbb{R}_+, H^1)$ and moreover $\|\psi(t)\|_{H^1}$ is bounded uniformly in time.

Let us fix a family of functions $\rho_\epsilon(x)$ approximating the Dirac δ-function. We pick $\rho_1(x) \in C_0^\infty[-1, 1]$, nonnegative, and such that $\int_{\mathbb{R}} \rho_1(x)\,dx = 1$, and define

$$\rho_\epsilon(x) = \frac{1}{\epsilon}\rho_1\left(\frac{x}{\epsilon}\right), \qquad \epsilon \in (0, 1), \tag{3.1}$$

so that

$$\operatorname{supp}\rho_\epsilon(x) \subseteq [-\epsilon, \epsilon], \qquad \rho_\epsilon(x) \ge 0, \qquad \int_{\mathbb{R}} \rho_\epsilon(x)\,dx = 1.$$

Consider the smoothed equation with the "mean field interaction"

$$i\dot{\psi}(x, t) = -\Delta\psi(x, t) - \rho_\epsilon(x)F(\langle\rho_\epsilon, \psi(t)\rangle), \tag{3.2}$$

where

$$\langle\rho_\epsilon, \psi(t)\rangle = \langle\rho_\epsilon(\cdot), \psi(\cdot, t)\rangle = \int_{\mathbb{R}} \rho_\epsilon(x)\psi(x, t)\,dx.$$

Clearly, equation (3.2) is the Hamiltonian equation, with the Hamilton functional

$$\mathscr{H}_\epsilon(\psi) = \int \frac{|\nabla\psi|^2}{2}\,dx + U(\langle\rho_\epsilon, \psi\rangle). \tag{3.3}$$

The Hamiltonian form of (3.2) is (cf. (1.4))

$$\dot{\Psi}_\epsilon = JD\mathscr{H}_\epsilon(\Psi_\epsilon). \tag{3.4}$$

The solution ψ_ϵ to (3.2) with the initial data $\psi_\epsilon|_{t=0} = \phi$ admits the Duhamel representation

$$\psi_\epsilon(x, t) = \mathbf{W}_t\phi(x) + \mathbf{Z}_\epsilon\psi_\epsilon(x, t), \tag{3.5}$$

where

$$\mathbf{Z}_\epsilon\psi_\epsilon(x, t) = -\int_0^t \mathbf{W}_s\rho_\epsilon(x)F(\langle\rho_\epsilon, \psi_\epsilon(t - s)\rangle)\,ds. \tag{3.6}$$

Lemma 3.1 (Local well-posedness). *(i) For any $\epsilon \in (0,1)$, there exists $\tau_\epsilon > 0$ that depends on ϵ and on U_2 from (2.3) so that there is a unique solution $\psi_\epsilon \in C_b([0,\tau_\epsilon], H^1)$ to equation (3.2) with $\psi_\epsilon|_{t=0} = \phi$.*

(ii) For each $t \le \tau_\epsilon$, the map $\mathbf{U}_\epsilon(t) : \phi = \psi_\epsilon(0) \mapsto \psi_\epsilon(t)$ is continuous in H^1.

(iii) The values of the functionals $\mathscr{H}_\epsilon$ and Q are conserved in time.

Proof. (*i*) For $\psi_1,\ \psi_2 \in C_b([0,\tau_\epsilon], H^1)$, we compute:

$$\begin{aligned}
&\|\mathbf{Z}_\epsilon\psi_2(\cdot,t) - \mathbf{Z}_\epsilon\psi_1(\cdot,t)\|_{H^1}\\
&= \| \int_0^t \boldsymbol{W}_s\rho_\epsilon F(\langle\rho_\epsilon,\psi_2(t-s)\rangle) - F(\langle\rho_\epsilon,\psi_1(t-s)\rangle)\,ds\|_{H^1}\\
&\le \int_0^t \|\boldsymbol{W}_s\rho_\epsilon\|_{H^1}\,|F(\langle\rho_\epsilon,\psi_2(t-s)\rangle) - F(\langle\rho_\epsilon,\psi_1(t-s)\rangle)|\,ds.
\end{aligned}$$

The first factor under the integral sign is bounded uniformly for $0 < s \le t$:

$$\|\boldsymbol{W}_s\rho_\epsilon\|_{H^1_x} = \frac{1}{\sqrt{2\pi}}\left\|\sqrt{1+k^2}e^{ik^2s/2}\widehat{\rho}_\epsilon(k)\right\|_{L^2_k} = \|\rho_\epsilon\|_{H^1}.$$

Taking this into account, we get:

$$\begin{aligned}
\|\mathbf{Z}_\epsilon\psi_2(\cdot,t) - \mathbf{Z}_\epsilon\psi_1(\cdot,t)\|_{H^1} &\le \|\rho_\epsilon\|_{H^1}\int_0^t |F(\langle\rho_\epsilon,\psi_2(t-s)\rangle) - F(\langle\rho_\epsilon,\psi_1(t-s)\rangle)|\,ds\\
&\le tU_2\|\rho_\epsilon\|_{H^1}\sup_{s\in[0,t]}|\langle\rho_\epsilon,\psi_2(s)-\psi_1(s)\rangle|.
\end{aligned}$$

Therefore, the map $\psi \mapsto \boldsymbol{W}_t\phi + \mathbf{Z}_\epsilon\psi$ is contracting if we choose, for definiteness,

$$\tau_\epsilon = \frac{1}{4U_2\|\rho_\epsilon\|_{H^1}}. \tag{3.7}$$

(*ii*) The continuity of the mapping $\mathbf{U}_\epsilon(t)$ also follows from the contraction argument.

(*iii*) It suffices to prove the conservation of the values of $\mathscr{H}_\epsilon(\psi_\epsilon(t))$ and $Q(\psi_\epsilon(t))$ for $\phi \in H^2 := H^2(\mathbb{R})$ since the functionals are continuous on H^1. For $\phi \in H^2$, the corresponding solution belongs to the space $C_b([0,\tau_\epsilon], H^2)$ by the Duhamel representation (3.5). Then the energy and charge conservation follows by the Hamiltonian structure (3.4). Namely, the differentiation of the Hamilton functional gives by the chain rule,

$$\frac{d}{dt}\mathscr{H}_\epsilon(\Psi_\epsilon(t)) = \langle D\mathscr{H}_\epsilon(\Psi_\epsilon(t)), \dot{\Psi}_\epsilon(t)\rangle = \langle D\mathscr{H}_\epsilon(\Psi_\epsilon(t)), JD\mathscr{H}_\epsilon(\Psi_\epsilon(t))\rangle = 0 \tag{3.8}$$

since the Fréchet derivative $D\mathscr{H}_\epsilon(\Psi_\epsilon(t)) = -\Delta\Psi_\epsilon(\cdot,t) - \rho_\epsilon(\cdot)F(\langle\rho_\epsilon,\Psi_\epsilon(t)\rangle))$ belongs to $L^2(\mathbb{R})$ for $t \in [0,\tau_\epsilon]$. Similarly, the charge conservation follows by the differentiation,

$$\begin{aligned}
\frac{d}{dt}Q(\Psi_\epsilon(t)) &= \langle DQ(\Psi_\epsilon(t)), \dot{\Psi}_\epsilon(t)\rangle = \langle DQ(\Psi_\epsilon(t)), JD\mathscr{H}_\epsilon(\Psi_\epsilon(t))\rangle\\
&= \langle\Psi_\epsilon(x,t), J\Delta\Psi_\epsilon(x,t)\rangle - \langle\Psi_\epsilon(x,t), J\rho_\epsilon(x)F(\langle\rho_\epsilon,\Psi_\epsilon(t)\rangle)\rangle.
\end{aligned} \tag{3.9}$$

Here $\Psi_\epsilon(x,t), J\Delta\Psi_\epsilon(x,t)\rangle = \nabla\Psi_\epsilon(x,t), J\nabla\Psi_\epsilon(x,t)\rangle = 0$, and also

$$\begin{aligned}\langle\Psi_\epsilon(x,t), J\rho_\epsilon(x)F(\langle\rho_\epsilon,\Psi_\epsilon(t)\rangle)\rangle &= \int \Psi_\epsilon(x,t)\cdot[J\rho_\epsilon(x)F(\langle\rho_\epsilon,\Psi_\epsilon(t)\rangle)]\,dx \\ &= \langle\rho_\epsilon,\Psi_\epsilon(t)\rangle\cdot[JF(\langle\rho_\epsilon,\Psi_\epsilon(t)\rangle)] = 0. \end{aligned} \tag{3.10}$$

Here "·" stands for the real scalar product in $\mathbb{R}^2$, and $Z\cdot[JF(Z)] = 0$ for $Z\in\mathbb{R}^2$ since $F(Z) = a(|Z|)Z$ with $a(|Z|)\in\mathbb{R}$ by (1.7).

□

Corollary 3.2 (Global well-posedness). *(i) For any $\epsilon > 0$, $\epsilon \le 1$, there exists a unique solution $\psi_\epsilon \in C(\mathbb{R}, H^1)$ to equation (3.2) with $\psi_\epsilon|_{t=0} = \phi$.*

The H^1-norm of ψ_ϵ is bounded uniformly in time:

$$\sup_{t\in\mathbb{R}} \|\psi_\epsilon(t)\|_{H^1} \le \Lambda_\epsilon(\phi), \qquad t\in\mathbb{R}, \tag{3.11}$$

where

$$\Lambda_\epsilon(\phi) = \sqrt{(8B^2+2)Q(\phi) + 4\mathscr{H}_\epsilon(\phi) - 4A}. \tag{3.12}$$

(ii) For each $t \ge 0$, the map $\psi_\epsilon(0)\mapsto\psi_\epsilon(t)$ is continuous in H^1.

Proof. (*i*) The existence and uniqueness of the solution $\psi_\epsilon \in C_b([0,\tau_\epsilon], H^1)$ follow from Lemma 3.1 (*i*). The bound on the value of the H^1-norm of $\psi_\epsilon(t)$ is obtained as in Lemma 2.1. Namely, noting that

$$U(\langle\rho,\psi_\epsilon\rangle) \ge A - B\langle\rho,\psi_\epsilon\rangle^2 \ge A - B\sup_{x\in\mathbb{R}}|\psi_\epsilon|^2 \ge A - B^2\|\psi\|^2_{L^2} - \frac{1}{4}\|\psi'\|^2_{L^2}$$

and using the energy and charge conservation proved in Lemma 3.1 (*iii*), we conclude that

$$(2B^2+\tfrac{1}{2})Q(\phi) + \mathscr{H}_\epsilon(\phi) = (2B^2+\tfrac{1}{2})Q(\psi_\epsilon) + \mathscr{H}_\epsilon(\psi_\epsilon) \ge A + \tfrac{1}{4}\|\psi_\epsilon\|^2_{H^1},$$

so that

$$\|\psi_\epsilon\|^2_{H^1} \le (8B^2+2)Q(\phi) + 4\mathscr{H}_\epsilon(\phi) - 4A. \tag{3.13}$$

By (3.7), the time span τ_ϵ depends only on $\|\rho_\epsilon\|_{H^1}$ and U_2. Hence, the bound (3.11) allows us to extend the solution to $t\in[\tau_\epsilon, 2\tau_\epsilon]$. The bound (3.11) for $t\in[0,2\tau_\epsilon]$ follows from (3.13) by the energy and charge conservation proved in Lemma 3.1 (*iii*). We conclude by induction that the solution exists and the bound (3.11) holds for all $t\in\mathbb{R}$.

(*ii*) The continuity of the mapping $\boldsymbol{W}_\epsilon(t) : \psi_\epsilon(0)\mapsto\psi_\epsilon(t)$ for all $t\ge 0$ follows from its continuity for small times by dividing the interval $[0,t]$ into small time intervals.

□

4 Convergence of regularized solutions

Lemma 4.1. *Let τ and $\psi \in C_b(\mathbb{R} \times [0,\tau])$ be as in Lemma 2.3, and let $\psi_\epsilon \in C(\mathbb{R}_+, H^1)$ be as in Corollary 3.2. Then for any finite $R > 0$*

$$\psi_\epsilon(x,t) \underset{\epsilon\to 0}{\rightrightarrows} \psi(x,t), \qquad |x| \le R, \quad 0 \le t \le \tau. \tag{4.1}$$

Proof. We have

$$\psi_\epsilon(x,t) = \boldsymbol{W}_t \phi(x) + \int_0^t \boldsymbol{W}_s \rho_\epsilon(x) F(\langle \rho_\epsilon, \psi_\epsilon(t-s)\rangle)\, ds, \tag{4.2}$$

$$\psi(x,t) = \boldsymbol{W}_t \phi(x) + \int_0^t \boldsymbol{W}_s \delta(x) F(\psi(0,t-s))\, ds. \tag{4.3}$$

Taking the difference of these equations and regrouping the terms, we can write:

$$\begin{aligned} \psi_\epsilon(x,t) - \psi(x,t) = &\int_0^t \boldsymbol{W}_s \rho_\epsilon(x) \left(F(\langle \rho_\epsilon, \psi_\epsilon(t-s)\rangle) - F(\psi(0,t-s)) \right) ds \\ &+ \int_0^t [\boldsymbol{W}_s \rho_\epsilon(x) - \boldsymbol{W}_s \delta(x)] F(\psi(0,t-s))\, ds. \end{aligned} \tag{4.4}$$

Let us analyze the first term in the right-hand side of (4.4). It is bounded by

$$\begin{aligned} &\left| \int_0^t \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} \rho_\epsilon(y)\, dy\, ds \right| \sup_{0\le s\le t} |F(\langle \rho_\epsilon, \psi_\epsilon(s)\rangle) - F(\psi(0,s))| \\ &\quad \le \left| \int_0^t \frac{ds}{\sqrt{2\pi s}} \right| U_2 \sup_{|x|\le\epsilon,\, 0\le s\le t} |\psi_\epsilon(x,s) - \psi(x,s)| \\ &\quad \le \sqrt{\frac{2t}{\pi}} U_2 \sup_{|x|\le\epsilon,\, 0\le s\le t} |\psi_\epsilon(x,s) - \psi(x,s)| \\ &\quad \le \frac{1}{2} \sup_{|x|\le\epsilon,\, 0\le s\le t} |\psi_\epsilon(x,s) - \psi(x,s)|, \end{aligned} \tag{4.5}$$

where in the last inequality we used (3.7). Setting $M_{R,\tau} = \sup_{|x|\le R,\, 0\le t\le\tau} |\psi_\epsilon(x,t) - \psi(x,t)|$, we can rewrite (4.4) as

$$M_{R,\tau} \le \frac{1}{2} M_{R,\tau} + \sup_{|x|\le R,\, 0\le t\le\tau} \int_0^t [\boldsymbol{W}_s \rho_\epsilon(x) - \boldsymbol{W}_s \delta(x)] F(\psi(0,t-s))\, ds.$$

Therefore,

$$M_{R,\tau} \le 2 \sup_{|x|\le R,\, 0\le t\le\tau} \int_0^t \int \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\epsilon(y) - \delta(y)]\, dy\, F(\psi(0,t-s)) ds. \tag{4.6}$$

We claim that the right-hand side tends to zero as $\epsilon \to 0$. To prove this, we split the integral into two pieces:

$$I_1(\delta, \epsilon) = \int_\delta^t \int \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\epsilon(y) - \delta(y)]\, dy\, F(\psi(0, t-s)) ds, \tag{4.7}$$

$$I_2(\delta, \epsilon) = \int_0^\delta \int \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\epsilon(y) - \delta(y)]\, dy\, F(\psi(0, t-s)) ds, \tag{4.8}$$

where $\delta \in (0, t)$ is yet to be chosen. Let us analyze the term (4.7):

$$|I_1(\delta, \epsilon)| \le C U_0 \sup_{s \ge \delta, |x| \le R} \left| \int_{|y|<\epsilon} \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\epsilon(y) - \delta(y)]\, dy \right|. \tag{4.9}$$

Since $s \ge \delta > 0$ and $|x| \le R$, the function $\frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}}$ is Lipschitz in $y \in [-\epsilon, \epsilon]$, uniformly in all the parameters. Therefore,

$$\int_{\mathbb{R}} \frac{e^{i\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} [\rho_\epsilon(y) - \delta(y)]\, dy \to 0, \qquad \epsilon \to 0, \tag{4.10}$$

uniformly in the parameters. We conclude that

$$\lim_{\epsilon \to 0} I_1(\delta, \epsilon) = 0, \tag{4.11}$$

for any fixed $\delta > 0$. We then bound (4.8) uniformly by

$$I_2(\delta, \epsilon) \le C U_0 \int (\rho_\epsilon(y) + \delta(y))\, dy \int_0^\delta \frac{ds}{\sqrt{s}} \le C\sqrt{\delta},$$

with C independent of ϵ. Now apparently the right-hand side of (4.6) tends to zero as $\epsilon \to 0$. □

5 Well-posedness in energy space

Lemma 5.1 (Local well-posedness)**.** *There is a unique solution $\psi \in L^\infty([0,\tau], H^1(\mathbb{R})) \cap C_b(\mathbb{R} \times [0,\tau])$ to equation (1.1) with $\psi|_{t=0} = \phi$, where τ is as in (2.10).*

Proof. The unique solution $\psi \in C_b(\mathbb{R} \times [0,\tau])$ is constructed in Lemma 2.3. According to (3.11) and (4.1),

$$\|\psi(t)\|_{H^1} \le \liminf_{\epsilon \to 0} \|\psi_\epsilon(t)\|_{H^1} \le \Lambda(\phi), \qquad 0 \le t \le \tau. \tag{5.1}$$

□

Lemma 5.2. *The values of the functionals $\mathscr{H}$ and Q are conserved in time for $t \in [0,\tau]$.*

Proof. The convergence (4.1) and the bounds (3.11) imply that

$$Q(\psi(t)) = \frac{1}{2}\|\psi(t)\|_{L^2}^2 \le \frac{1}{2}\lim_{\epsilon\to 0}\|\psi_\epsilon(t)\|_{L^2}^2 = Q(\phi), \tag{5.2}$$

where we used the conservation of Q for the approximate solutions ψ_ϵ (Lemma 3.1). The same argument applied to the initial data $\psi|_{t=t_0}$ with any $t_0 \in (0,\tau)$ and combined with the uniqueness of the solution, allows to conclude that $Q(\psi(t))$ is monotonically non-increasing when time changes from 0 to τ. Instead, solving the Schrödinger equation backwards in time and using the uniqueness of solution, we can as well conclude that $Q(\psi(t))$ is monotonically non-decreasing when time changes from 0 to τ. This proves that $Q(\psi(t)) = \text{const}$ for $t \in [0,\tau]$.

To prove the conservation of $\mathscr{H}(\psi(t))$, we will need the relation

$$\lim_{\epsilon\to 0} U(\langle\rho_\epsilon,\psi_\epsilon\rangle) = U(\psi(0,t)). \tag{5.3}$$

This relation follows from continuity of the potential U and from

$$\lim_{\epsilon\to 0}\langle\rho_\epsilon,\psi_\epsilon(t)\rangle = \lim_{\epsilon\to 0}\langle\rho_\epsilon,(\psi_\epsilon(t)-\psi(t))\rangle + \lim_{\epsilon\to 0}\langle\rho_\epsilon,\psi(t)\rangle = \psi(0,t), \tag{5.4}$$

where $\lim_{\epsilon\to 0}\langle\rho_\epsilon,(\psi_\epsilon(t)-\psi(t))\rangle = 0$ since ψ_ϵ approaches ψ uniformly for $0 \le t \le \tau$ and $|x| \le R$ (including $x = 0$), while $\lim_{\epsilon\to 0}\langle\rho_\epsilon,\psi(t)\rangle = \psi(0,t)$ since ψ is continuous in x (due to the finiteness of H^1-norm of ψ that follows from (5.1)). We have:

$$\mathscr{H}(\psi(t)) = \frac{\|\nabla\psi(x,t)\|_{L^2}^2}{2} + U(\psi(0,t)) \le \lim_{\epsilon\to 0}\left\{\frac{\|\nabla\psi_\epsilon(x,t)\|_{L^2}^2}{2} + U(\langle\rho_\epsilon,\psi_\epsilon\rangle)\right\} = \mathscr{H}(\phi),$$

where we used the relation (5.3) and (4.1). We also used the conservation of the values of the functional $\mathscr{H}_\epsilon$ for the approximate solutions ψ_ϵ (see Lemma 3.1). Proceeding just as with $Q(\psi(t))$ above, we conclude that $\mathscr{H}(\psi(t)) = \text{const}$ for $0 \le t \le \tau$. □

Corollary 5.3 (Global well-posedness)**.** *There is a unique solution $\psi \in L^\infty([\mathbb{R}, H^1(\mathbb{R})) \cap C_b(\mathbb{R}\times\mathbb{R})$ to equation (1.1) with $\psi|_{t=0} = \phi$. The values of the functionals $\mathscr{H}$ and Q are conserved in time.*

Proof. The solution $\psi \in L^\infty([0,\tau], H^1)$ constructed in Lemma 5.1 exists for $0 \le t \le \tau$, where the time span τ defined in (2.10) depends only on U_2 from (2.3). Hence, the bound (1.11) at $t = \tau$ allows us to extend the solution ψ constructed in Lemma 5.1 to the time interval $[\tau, 2\tau]$. We proceed by induction. □

For the conclusion of Theorem 1.1, it remains to prove that $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$. This follows from the next two lemmas.

Lemma 5.4. $\psi \in C(\mathbb{R}, H^1_{weak}(\mathbb{R}))$.

Proof. Fix $f \in H^{-1}(\mathbb{R})$ and pick any $\delta > 0$. Since H^1 is dense in H^{-1}, there exists $g \in H^1(\mathbb{R})$ such that

$$\|f-g\|_{H^{-1}} < \frac{\delta}{4\Lambda(\phi)}, \tag{5.5}$$

where $\Lambda(\phi)$ given by (2.2) is the a priori bound on $\|\psi(t)\|_{H^1}$ proved in Lemma 2.1 on the grounds of the energy and the charge conservation for $\psi(t)$. Then

$$|\langle f,\psi(t)-\psi(t_0)\rangle| \le |\langle f-g,\psi(t)-\psi(t_0)\rangle| + |\langle g,\psi(t)-\psi(t_0)\rangle| \tag{5.6}$$

$$\le \|f-g\|_{H^{-1}}(\|\psi(t)\|_{H^1}+\|\psi(t_0)\|_{H^1}) + \|g\|_{H^1}\|\psi(t)-\psi(t_0)\|_{H^{-1}}. \tag{5.7}$$

By (5.5), the first term in the right-hand side of (5.7) is bounded by $\delta/2$. By (1.1), $\psi \in C(\mathbb{R}, H^{-1}(\mathbb{R}))$, hence the second term in the right-hand side of (5.7) becomes smaller than $\delta/2$ if t is sufficiently close to t_0. Since $\delta > 0$ was arbitrary, this proves that $\lim_{t\to t_0}\langle f,\psi(t)-\psi(t_0)\rangle = 0$. □

Proposition 5.5. *$\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.*

Proof. Let us fix $t_0 \in \mathbb{R}$ and compute

$$\lim_{t\to t_0}\|\psi(t)-\psi(t_0)\|^2_{H^1} = \lim_{t\to t_0}\left(\|\psi(t)\|^2_{H^1} - 2\langle\psi(t),\psi(t_0)\rangle_{H^1} + \|\psi(t_0)\|^2_{H^1}\right). \tag{5.8}$$

The relation

$$\|\psi(t)\|^2_{H^1} = 2(Q(\psi(t)) + H(\psi(t))) - 2U(\psi(0,t)),$$

together with the conservation of the energy and charge and the continuity of $\psi(0,t)$ for $t \in \mathbb{R}$ (see Corollary 5.3), shows that

$$\lim_{t\to t_0}\|\psi(t)\|^2_{H^1} = \|\psi(t_0)\|^2_{H^1}.$$

By Lemma 5.4, $\lim_{t\to t_0}\langle\psi(t),\psi(t_0)\rangle_{H^1} = \langle\psi(t_0),\psi(t_0)\rangle_{H^1}$. This shows that the right-hand side of (5.8) is equal to zero. □

Now Theorem 1.1 is proved.

6 Hölder regularity of solution

In this section, we prove Theorem 1.2.

Lemma 6.1. *If $\phi \in H^1$, then $\mathbf{W}_{(\cdot)}\phi(x) \in C^{(1/4)}[0,\tau]$, uniformly in $x \in \mathbb{R}$.*

Proof. Let t, $t' \in [0,\tau]$. We have by the Cauchy-Schwarz inequality:

$$|\mathbf{W}_{t'}\phi(x) - \mathbf{W}_t\phi(x)| \le C\left|\int e^{-ikx}\left(e^{i\frac{t'k^2}{2}} - e^{i\frac{tk^2}{2}}\right)\hat\phi(k)\,dk\right|$$

$$\le C\int \min(1,|t'-t|k^2)|\hat\phi(k)|\,dk \le C\left[\int_{\mathbb{R}}\frac{\min(1,|t'-t|k^2)^2}{1+k^2}\,dk\right]^{\frac12}\|\phi\|_{H^1}.$$

We bound the last integral as follows:

$$\int_{\mathbb{R}}\frac{\min(1,|t'-t|k^2)^2}{1+k^2}\,dk \le \int\limits_{|k|<|t'-t|^{-\frac12}}\frac{|t'-t|^2k^4}{1+k^2}\,dk + \int\limits_{|k|>|t'-t|^{-\frac12}}\frac{dk}{1+k^2} \le \text{const}\,|t'-t|^{\frac12}.$$

□

Lemma 6.2 (Regularity of $\psi(0,t)$)**.** *The unique solution $\psi \in C_b(\mathbb{R}\times[0,\tau])$ to equation (1.1) with the initial data $\psi|_{t=0} = \phi$ constructed in Lemma 2.3 satisfies*

$$\psi(0,\cdot) \in C^{(1/4)}[0,\tau].$$

Proof. Due to Lemma 6.1, it suffices to consider the regularity of $\mathbf{Z}\psi(0,t)$. For any t, $t' \in [0,\tau]$, $t' < t$, we have:

$$\mathbf{Z}\psi(0,t') - \mathbf{Z}\psi(0,t) = \int_0^t \left[\frac{F(\psi(0,s))}{\sqrt{2\pi(t'-s)}} - \frac{F(\psi(0,s))}{\sqrt{2\pi(t-s)}}\right] ds + \int_t^{t'} \frac{F(\psi(0,s))}{\sqrt{2\pi(t'-s)}}\, ds. \quad (6.1)$$

The first integral in the right-hand side of (6.1) is bounded by

$$C_1 \int_0^t \left|\frac{1}{\sqrt{t'-s}} - \frac{1}{\sqrt{t-s}}\right| ds \le C_2 |t'-t|^{1/2}.$$

The second integral in the right-hand side of (6.1) is also bounded by $C|t'-t|^{1/2}$. □

Lemma 6.3. *$\psi(x,\cdot) \in C^{(1/4)}(\mathbb{R})$, uniformly in $x \in \mathbb{R}$.*

Proof. We have the relation

$$\psi(x,t) = \boldsymbol{W}_0(t-t_0)\psi(x,t_0) + \int_0^{t-t_0} \frac{e^{i\frac{x^2}{2s}}}{\sqrt{2\pi s}} F(\psi(0,t-s))\, ds. \quad (6.2)$$

By Lemma 6.1, the first term in the right-hand side of (6.2), considered as a function of time, belongs to $C^{(1/4)}(\mathbb{R})$ (uniformly in $x \in \mathbb{R}$). The second term in the right-hand side of (6.2) is bounded by const $|t-t_0|^{1/2}$. This proves that $\psi(x,\cdot) \in C^{(1/4)}(\mathbb{R})$, uniformly in x. □

It remains to mention that the Hölder continuity in x follows from the inclusion $H^1(\mathbb{R}) \subset C^{(1/4)}(\mathbb{R})$. Theorem 1.2 is proved.

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