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Sylvester-'t Hooft generators of $sl(n)$ and $gl(n|n)$,
and relations between them

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Sylvester-'t Hooft generators of $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n|n)$, and relations between them

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Abstract

Among the simple finite dimensional Lie algebras, only $\mathfrak{sl}(n)$ possesses two automorphisms of finite order which have no common nonzero eigenvector with eigenvalue one. It turns out that these automorphisms are inner and form a pair of generators that allow one to generate all of $\mathfrak{sl}(n)$ under bracketing. It seems that Sylvester was the first to mention these generators, but he used them as generators of the associative algebra of all $n \times n$ matrices $Mat(n)$. These generators appear in the description of elliptic solutions of the classical Yang-Baxter equation, orthogonal decompositions of Lie algebras, 't Hooft's work on confinement operators in QCD, and various other instances. Here I give an algorithm which both generates $\mathfrak{sl}(n)$ and explicitly describes a set of defining relations. For close to simple (up to nontrivial center and outer derivations) Lie superalgebras, analogs of Sylvester generators exist only for $\mathfrak{gl}(n|n)$. The relations for this case are also computed.

§1 Introduction

Dealing with a given Lie algebra \mathfrak{g} and modules over it, especially when q -quantizing, we need a convenient *presentation* of \mathfrak{g} , i.e., a description in terms of generators and defining relations. Obviously, the basis elements qualify as generators, but there are too many of them. It is well-known [GL2] that

For any nilpotent Lie algebra \mathfrak{n} , the natural set of relations is a basis of $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = H_1(\mathfrak{n})$; relations between these generators can be described in terms of the basis of $H_2(\mathfrak{n})$. (1.1)

A simple Lie (super)algebra \mathfrak{g} (finite dimensional, Kac-Moody or of polynomial vector fields) is conventionally split into the sum $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of two maximal nilpotent subalgebras \mathfrak{n}_\pm (positive and negative) and the commutative Cartan subalgebra; the corresponding generators are called *Chevalley generators*; the relations between them are also known, cf. [GL3], [GLP]. They are numerous ($3n$ generators for a rank n algebra and $\sim n^2$ relations), but these relations are simple and therefore convenient.

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For comparison: for the simplest case, $\mathfrak{gl}(n)$, the matrix units are obvious generators, and the relations between them are simple, but far too numerous (n^2 generators and $\sim n^4$ relations).

Jacobson was, perhaps, the first to observe that every simple finite dimensional Lie algebra can be generated by just a *pair* of generators, but he did not specify his pairs, so no discussion of relations was made. Grozman and Leites [GL1] introduced a pair of generators associated with the principal embedding of $\mathfrak{sl}(2)$, and the relations between them are rather simple (at least, for computers). There are more generators similar to those Grozman and Leites had chosen, but experiments performed so far show that the ones Grozman and Leites considered are most convenient, and are related to various applications [GL2], [LS].

There are, however, certain pairs of generators indigenous only to the \mathfrak{sl} series, and only over an algebraically closed field, e.g. \mathbb{C} . Below, we describe such a pair of generators for $\mathfrak{sl}(n)$ and their analogs for $\mathfrak{gl}(n|n)$ and give relations between them.

Let $a = \exp\left(\frac{2i\pi}{n}\right)$ and define *Sylvester's generators* (also called *clock-and-shift* or *'t Hooft matrices*) to be

$$D = \text{diag}(1, a, a^2, \dots, a^{n-1}), \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.2)$$

Zachos [Z2] points out that

“apparently, Sylvester [S] was the first to study these (1.2) generators¹ of $\mathfrak{sl}(n)$; he worked them out for $\mathfrak{sl}(3)$ first, and called them “nonions” (after quaternions), and then generalized to $\mathfrak{sl}(n)$.”

They became popular in the 30s in the context of QM-around-the circle, i.e., on a discrete periodic lattice of N points, see [W]. That effort has continued to date, with the work of Schwinger, Santhanam, Tolar, Floratos, and others.

They also became popular among high-energy theorists, with the work of 't Hooft [tH], on order-disorder confinement operators in QCD, so that many in my end of the woods intriguingly call them “'t Hooft matrices”.

I have been using them every few years, starting from [FFZ] to identify cases of a Sine-algebra we found at that time with $\mathfrak{sl}(N)$, and also with the Moyal Bracket algebra [Moy] on a toroidal phase space; and hence take the $N \rightarrow \infty$ limit to get Poisson Brackets more directly than in Hoppe's first derivation [Ho] on a spherical phase space.

Our latest use of them was in our recent diversion, [FZ], on ring-indexed Lie algebras. They are apparently the most systematic basis for dealing with all $\mathfrak{sl}(N)$ s on an equal footing and taking naive $N \rightarrow \infty$ limits.”

¹More precisely, Sylvester used them as generators of an associative algebra, where they yield the algebra of $n \times n$ matrices $\text{Mat}(n)$. Having replaced the dot product by the bracket we endow the space of $\text{Mat}(n)$ with the structure of the Lie algebra $\mathfrak{gl}(n)$; having introduced parity in $\text{Mat}(n)$ by attributing parity to each basis vector (and hence to each row and column) and replacing the dot product by the superbracket we endow the superspace of $\text{Mat}(n; \text{Par})$, where Par is an ordered collection of parities, with the structure of the Lie superalgebra $\mathfrak{gl}(\text{Par})$. As generators of a Lie algebra, Sylvester's generators can only generate $\mathfrak{sl}(n)$, but not $\mathfrak{gl}(n)$.

For the passage from the notation of Zachos et al. to ours, observe that, e.g. in [FFZ], the authors generate $\mathfrak{gl}(n)$ from Sylvester's generators D, S (1.2) in the form

$$J_{(m_1, m_2)} = a^{m_1 m_2 / 2} D^{m_1} S^{m_2}$$

which are n^2 independent matrices labelled by two integers $0 \leq m_1, m_2 < n$. Under the bracket, the identity matrix $J_{(0,0)}$ spans the center. So dividing it out leaves $\mathfrak{sl}(n)$ with the bracket

$$[J_{(m_1, m_2)}, J_{(k_1, k_2)}] = -2i \sin\left(\frac{2\pi}{n}(m_1 k_2 - m_2 k_1)\right) J_{(m_1 + k_1, m_2 + k_2)}$$

Another important application of Sylvester's generators is the classical Yang-Baxter equation for a function taking values in a simple Lie algebra \mathfrak{g} . It turns out [BD1, BD2] that for this equation to have elliptic solutions, \mathfrak{g} has to possess two automorphisms of finite order which have no common nonzero eigenvector with eigenvalue 1. Sylvester's generators are such automorphisms for $\mathfrak{g} = \mathfrak{sl}(n)$; in fact, [BD1, BD2] prove that any \mathfrak{g} possessing such automorphisms must be isomorphic to $\mathfrak{sl}(n)$, and the elliptic solutions can be characterised by the images of Sylvester's generators (1.2) under this isomorphism. Also, they play a vital role in the study of orthogonal decompositions of Lie algebras [KKU, KT, FOS].

Finally, a more applied subject on which these generators have been used is hydrodynamics and the statistical theory of turbulent fluids and gases, in particular, the study of lattice models of inviscid fluids (Euler fluids), see, e.g., [MWC],[MW],[Ze].

The aim of this paper is to give an algorithm that generates $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n|n)$ from Sylvester's generators and which also produces a presentation for them. This presentation contains redundancies, but might be of interest for practical problems since it allows quick and easy computations in the adjoint representation. The main statements are the following ones.

Theorem 1.1. *Fix an integer $n \geq 2$. Then the matrices (1.2) are generators for $\mathfrak{sl}(n)$:*

$$\mathfrak{sl}(n) = \text{Span}(D, S, T_m^k \mid 1 \leq k, m \leq n, \text{ and } k \neq n \text{ for } m = 1, n, \text{ and } k \neq 1 \text{ for } m = n), \quad (1.3)$$

where for $1 \leq k, m \leq n$, we set

$$\begin{aligned} T_m^k &= (\text{ad } D)^{k-1}((\text{ad } S)^{m-1}((\text{ad } D(S))), \\ T_n^k &= \text{ad } S(T_{n-1}^k). \end{aligned}$$

A defining set of relations for generators (1.2) can be obtained in the following way. The relations

$$(\text{ad } D)^n(S) = (1 - a)^n S, \quad (1.4)$$

$$(\text{ad } D)^n((\text{ad } S)^{m-1}(\text{ad } D(S))) = (1 - a)^m (1 - a^m)^n (-1)^{m+1} (\text{ad } S)^{m-1}(\text{ad } D(S)) \quad (1.5)$$

$$\text{ad } S(T_{n-1}^1) = (1 - a)^n (-1)^n D, \quad (1.6)$$

$$\text{ad } S(T_{n-1}^n) = 0, \quad (1.7)$$

$$\text{ad } S(T_n^k) = (-1)^n (1 - a^k)^2 (1 - a^{n-1})^{k-1} (1 - a)^{n-k} T_1^k, \quad (1.8)$$

$$\text{ad } D(T_n^k) = 0 \quad (1.9)$$

prohibit generation of elements of order higher than n in both D and S . Besides them, for each T_m^k with $2 \leq m \leq n-1$, except for T_2^2 , $m-1$ relations have to hold, which can be written as

$$(\operatorname{ad} S)^{s_1}((\operatorname{ad} D)^{k-1}((\operatorname{ad} S)^{s_2}(T_1^1))) = \left(\frac{1-a^k}{1-a}\right)^{s_1} \left(\frac{1-a^{s_1}}{1-a^{m-1}}\right)^{k-1} T_m^k,$$

where $s_1 + s_2 = m-1$ and $s_1 = 1, 2, \dots, m-1$.

Theorem 1.2. Considered as $2n \times 2n$ supermatrices on a superspace with an alternating format (even, odd, even, odd, ...), (1.2) are generators for $\mathfrak{gl}(n|n)$:

$$\mathfrak{gl}(n|n) = \operatorname{Span}(D, S, T_m^k \mid 1 \leq k, m \leq 2n, \text{ and } k \neq 2n \text{ for } m=1 \text{ and } k \neq 1 \text{ for } m=2n)$$

with the same definition of T_m^k as in Thm. (1.1). A defining set of relations in this case are (3.1)-(3.5) and $m-1$ relations for each T_m^k with $2 \leq m \leq 2n-1$. These can be written as

$$(\operatorname{ad} S)^{s_1}(T_{s_2+1}^k) = \begin{cases} (-1)^{s_1}(a^{2k}-1)^{\frac{s_1-1}{2}} \frac{(a^k+1)}{1-a} \left(\frac{1-a^{s_1}}{1-a^{m-1}}\right)^{k-1} \left(-\frac{1-a}{1+a}\right)^{\frac{s_1-1}{2}} \tilde{T}_m^k & \text{for } s_1 \text{ odd and} \\ & s_2 \text{ even,} \\ (-1)^{s_1}(a^{2k}-1)^{\frac{s_1-1}{2}} \frac{(a^k-1)}{1-a} \left(\frac{1-a^{s_1}}{1-a^{m-1}}\right)^{k-1} \left(-\frac{1-a}{1+a}\right)^{\frac{s_1-1}{2}} \tilde{T}_m^k & \text{for } s_1, s_2 \text{ odd, or} \\ (-1)^{s_1} \frac{(a^{2k}-1)^{s_1/2}}{1-a} \left(\frac{1-a^{s_1}}{1-a^{m-1}}\right)^{k-1} \left(-\frac{1-a}{1+a}\right)^{s_1/2} \tilde{T}_m^k & \text{for } s_1, s_2 \text{ even,} \\ & \text{or } s_1 \text{ even, } s_2 \text{ odd,} \end{cases}$$

where again $s_1 + s_2 = m-1$ and $s_1 = 1, 2, \dots, m-1$.

In the following we show why the set of relations indicated in Thms. (1.1), (1.2) is a defining set. This will then automatically also deliver an upper bound on the number of independent relations for Sylvester's generators.

§2 Relations between Sylvester's generators for $\mathfrak{sl}(n)$

Setting

$$T_1^k := (\operatorname{ad} D)^k(S) \quad \text{for } k = 1, \dots, n-1$$

we obtain the matrices

$$T_1^k = (1-a)^k \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & a^k & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & \dots & a^{k(n-2)} \\ a^{k(n-1)} & 0 & \dots & 0 & 0 \end{pmatrix}$$

which are, clearly, all linearly independent. For $k=n$, we get the relation (1.4). Proceeding likewise, we generate a basis for $\mathfrak{sl}(n)$. We set

$$T_m^k = (\operatorname{ad} D)^{k-1}((\operatorname{ad} S)^{m-1}(T_1^1)) = (\operatorname{ad} D)^{k-1}((\operatorname{ad} S)^{m-1}((\operatorname{ad} D)(S)))$$

where $m = 1, \dots, n - 1$ and $k = 1, \dots, n$. In matrix form,

$$T_m^k = (1 - a)^m (1 - a^m)^{k-1} (-1)^{m+1} \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & 0 & a^k & 0 & \dots & 0 \\ \vdots & \vdots & & & & & & \vdots \\ 0 & \dots & & & & & & a^{k(n-m-1)} \\ a^{k(n-m)} & 0 & \dots & & & \dots & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \dots & a^{k(n-1)} & 0 & \dots & \dots & & 0 \end{pmatrix}$$

These are all the non-diagonal matrices needed for a basis of $\mathfrak{sl}(n)$. Their linear independence is easily checked. We also immediately read off the relation (1.5) for $k = n + 1$.

It remains to generate $n - 2$ diagonal matrices, which we do as follows:

$$T_n^k = \text{ad}S(T_{n-1}^k) = \text{ad}S((\text{ad}D)^{k-1}((\text{ad}S)^{n-2}(\text{ad}D(S)))) ,$$

where $k = 2, \dots, n - 1$, and we obtain the relations (1.6)-(1.9).

Since we have obtained $n^2 - 1$ linearly independent matrices, we have found a basis for $\mathfrak{sl}(n)$, see (1.3). One might, however, wish to describe $\mathfrak{sl}(n)$ as the quotient of the free Lie algebra generated by the two Sylvester generators modulo certain defining relations. Since we know that the matrices T_m^k span $\mathfrak{sl}(n)$, we know that any commutator of them must yield a relation. The relations stated above are merely those ones that are first encountered when we proceed through our chosen algorithm for the generation of the basis of $\mathfrak{sl}(n)$. To find out the number and an explicit realization of the minimal defining relations turns out to be quite a tough job, despite the seeming simplicity of the problem. P. Grozman was able to find those minimal relations for $n = 2, 3, 4$:

$$\begin{aligned} \underline{n=2} : & \quad (\text{ad } S)^2(D) = 4D, \quad (\text{ad } D)^2(S) = 4S; \\ \underline{n=3} : & \quad (\text{ad } S)^3(D) = -3(a - a^2)D, \quad (\text{ad } D)^3(S) = 3(a - a^2)S; \quad [T_2^1, T_1^2] = 0; \\ \underline{n=4} : & \quad (\text{ad } S)^4(D) = -4D, \quad (\text{ad } D)^4(S) = -4S, \quad [T_1^1, T_2^1] = [D, T_3^1], \\ & \quad [T_1^1, [T_2^1, T_3^1]] = -4T_1^2, \quad [T_1^3, T_3^1] = 0, \quad 2[T_2^1, T_1^3] = [T_1^2, [D, T_2^1]], \\ & \quad 2[T_1^2, T_3^1] = [T_2^1, [S, T_1^2]], \quad T_1^2 = [S, T_1^3] \quad [T_1^2, T_1^3] = 4T_2^1. \end{aligned} \tag{2.1}$$

with the help of *Mathematica* and his **SuperLie** package [Gr], but did not succeed to deduce from (2.1) a general formula. On the other hand, neither the number nor an explicit form of the *minimal* set of relations is of great practical importance when working with these generators. Rather one would like to have, e.g., formulae that describe the action of arbitrary products of the elements of $\mathfrak{sl}(n)$ in the adjoint representation. Such formulae will be given below and, additionally, a set of relations offered which contains redundancies, but which allows immediate reduction of an arbitrary expression of the form (with the X_i and Y being arbitrary elements of $\mathfrak{sl}(n)$)

$$\text{ad } X_1(\text{ad } X_2(\dots(\text{ad } X_q(Y))\dots))$$

to a linear combination of the basis elements produced by our algorithm.

By explicit calculation one first verifies that

$$[T_m^k, T_{m'}^{k'}] = \frac{(1 - a^m)^{k-1} (1 - a^{m'})^{k'-1}}{(1 - a^{m+m'})^{k+k'-1}} (a^{k'm'} - 1) T_{m+m'}^{k+k'}$$

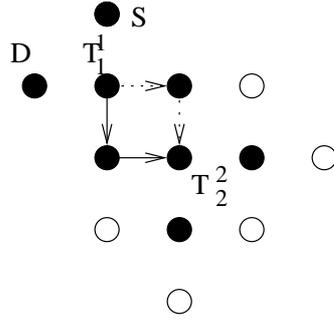


Figure 1: The two paths that generate T_2^2 in $\mathfrak{sl}(3)$

for any of the $T_m^k, T_m^{k'}$ defined above. Hereafter, $k + k'$ and $m + m'$ have to be understood mod n . This directly shows the following statement.

Lemma 2.1. *The result of the application of an arbitrary product of elements in the adjoint representation to a T_m^k depends up to a factor only on the number of S 's and D 's contained in these operators. That is,*

$$\text{ad } X_1(\text{ad } X_2(\cdots \text{ad } X_N(T_m^k)) \cdots) = C(k, k', m, m', n) T_{m+m'}^{k+k'} \quad (2.2)$$

where m_i and k_i is the number of S 's and D 's, respectively, contained in X_i ,

$$m' = \sum_i m_i, \quad k' = \sum_i k_i,$$

and $C(k, k', m, m', n)$ is a constant depending on all the indices.

Therefore we conclude that it suffices to check only relations between elements which are at most of degree n in both S 's and D 's. If we know all relations of this type, then any relation of a higher degree will follow from these and (1.4)-(1.9).

In order to find these relations, it is most convenient to visualise the generated basis as a grid of points. Fig. 1 shows the basis of $\mathfrak{sl}(3)$, starting from D and S in the upper left corner. Below them is $T_1^1 = [D, S]$. The other solid points are those that we generate with our algorithm by going only horizontally on each level, and vertically only along the left edge. The white points are those which are ruled out by the relations (1.4)-(1.9), i.e., they do not represent basis elements of $\mathfrak{sl}(3)$. Now, an arbitrary product of r -many $(\text{ad}D)$'s and s -many $(\text{ad}S)$'s applied to T_1^1 corresponds to a path on the grid starting at T_1^1 and reaching T_{s+1}^{r+1} , but one which will in general only produce a matrix proportional to T_{s+1}^{r+1} , with a factor $\neq 1$. A horizontal step of the path describes the action of $\text{ad}D$, a vertical one the action of $\text{ad}S$. In the picture, the solid line shows the way our algorithm went to generate T_2^2 , while the dotted lines show the alternative path, i.e.,

$$\text{solid line} \quad \Leftrightarrow \quad \text{ad}D(\text{ad}S(\text{ad}D(S))) \quad (2.3)$$

$$\text{dotted line} \quad \Leftrightarrow \quad \text{ad}S((\text{ad}D)^2(S)) \quad (2.4)$$

It is clear that any expression we have to examine can be expressed as a path from T_1^1 to some admissible T_m^k which only moves right and downwards (compare to Fig. 1). In general, there are

$$\binom{m+k-2}{k} \quad \text{paths from } T_1^1 \text{ to } T_m^k$$

However, we can rule out some of these. The algorithm always uses paths which run through all vertical steps first, then through all horizontal ones (called the algorithm path in what follows). A relation is obtained by running through any different path and comparing the result to what the algorithm path would have produced at this vertex.

Proposition 2.2. *For a path ending at T_s^r to yield an independent relation, it has to*

- *end with a vertical step if $s < n$,*
- *end with a horizontal step if $s = n$.*

Proof. Look at the $s < n$ case first. We know that at the vertex where the last vertical step ends, we will have produced a matrix proportional to the one that the algorithm path would have produced there (cf. Lemma 2.1). Thus, at this vertex we obtain a relation. But if it is followed by horizontal steps, these will then trivially also yield matrices proportional to those that the algorithm would have produced. Thus, the relations we can read off at these vertices are generated from the one obtained at the end of the last vertical step.

An analogous argument holds for $s = n$, except that at the last step of the algorithm there is a vertical step, so a path producing an independent relation cannot have a vertical step at its end. \square

Corollary 2.3. *Apart from those vertical steps which lie on the left edge of the grid, a path that leads to T_s^r and yields an independent relation for $s < n$ must contain all other vertical steps at its end. For $s = n$, the only path yielding a nontrivial relation is the algorithm path to T_n^{r-1} followed by a horizontal step.*

Proof. As a counterexample for the $s < n$ case, consider Fig. 2.

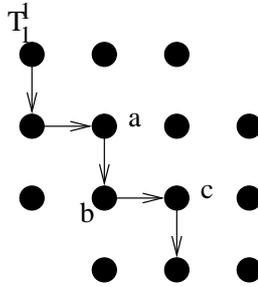


Figure 2: Example of a path ruled out by Corollary 2.3

Up to vertex a , it follows the algorithm path, then going to b will yield a relation. But proceeding further horizontally after b yields only dependent relations, as seen before.

In the $s = n$ case, we have seen in Lemma 2.1 that the last step of a path yielding a relation must be horizontal. Since going a horizontal step in the n -th row always gives zero (cf. (1.9)), a nontrivial path can only have exactly one horizontal piece at its end. So the second last step is always the last step of the algorithm to T_n^{r-1} , and therefore any other path leading to T_n^{r-1} followed by a horizontal step would trivially yield a result proportional to what the algorithm path followed by the horizontal step gives. The relations so obtained are precisely those of (1.9) \square

This reduces the number of possibly independent relations considerably: for any vertex T_m^k with $m < n$, there can now be at most $m - 1$ independent relations, which result from the paths leading there and having between zero and $m - 1$ vertical steps at their ends. For T_n^k , there can only be one relation. Among the relations thus obtained, there will still be redundancies, which are not obvious at first glance. To reveal them, one has to apply the Jacobi identity and other relations one has already obtained. As an example, look at T_2^2 in $\mathfrak{sl}(n)$ for $n \geq 3$. Two paths lead there, described in (2.3) and (2.4). Since they are both admissible in the sense of Proposition 2.2, one might think that we obtain a relation here between T_2^2 generated by the algorithm and the result of another path. However,

$$\text{ad}D(\text{ad}S(\text{ad}D(S))) = \text{ad}S((\text{ad}D)^2(S)) + \text{ad}(\text{ad}D(S))(\text{ad}D(S))$$

due to the Jacobi identity and the last term is of the form $\text{adz}(z) \equiv 0$. Therefore, the two paths *trivially* yield the same result, and we obtain no relation here. We will show now that there are no other interdependencies of this sort except the above one for T_2^2 .

Lemma 2.4. *It is impossible to trivially identify the result of two paths to a given T_s^r by rearranging them using the Jacobi identity, except for the case $r = s = 2$, where we have*

$$\text{ad}D(\text{ad}S(\text{ad}D(S))) = \text{ad}S((\text{ad}D)^2(S))$$

Proof. Any admissible path in the sense of Lemma 2.1 and its corollary is of the form

$$T_s^r = (\text{ad}S)^{s_2}((\text{ad}D)^{r-1}((\text{ad}S)^{s_1}(T_1^1))) \quad (2.5)$$

where $s_1 + s_2 = s - 1$. For $s_2 = 0$, we obtain the algorithm path. In order to show that two paths give the same result, we want to apply the Jacobi identity

$$\text{ad}(\text{ad}x(y))z = \text{ad}x(\text{ad}y(z)) - \text{ad}y(\text{ad}x(z))$$

in such a way that the left hand side of it becomes zero, i.e. is of the form $\text{adz}(z)$. This would rule out one of the relations these paths produce. We see immediately that for this to happen for adjoint operators x, y, z , the element z would have to contain as many D 's and S 's as x and y together. Looking at (2.5), which we would like to identify with $\text{ad}x(\text{ad}y(z))$, this implies $r = s$. We have to split (2.5) in two equally long subpaths, the head (including T_1^1) being z and the tail being $\text{ad}x(\text{ad}(y))$ and each containing $\frac{s}{2}$ -many D 's and S 's, implying s must be even.

For the case $r = s = 2$, we find that $z = T_1^1$, $x = \text{ad}S$ and $y = \text{ad}D$ meet these requirements.

Let now $r = s = 2n$, $n > 1$ and let x, y, z satisfy the above conditions. Then z represents the path of the algorithm to $T_{s/2}^{s/2}$ and $\text{ad}x(\text{ad}y)$ is of the form $(\text{ad}S)^{s/2}((\text{ad}D)^{s/2})$. But we see that it is impossible then to find x, y such that $\text{ad}y((\text{ad}x)(z))$ would again be an admissible path. \square

It is important to note that this still does not exclude all possible dependencies between the relations that various admissible paths yield. By clever rearrangement, it might still be possible to bring a bracket of two elements into a form which, when expanded into paths,

yields only a few admissible paths and several others which run over already excluded pieces. We could find no way to rule out all such possibilities. This seems only possible with the help of computers. But, as stated above, the minimal number might not be of practical interest. The preceding discussion still gives us an upper bound on the number of relations.

Theorem 2.5. *The number $R(n)$ of independent relations between Sylvester's generators is bounded from above by:*

$$R(n) \leq \begin{cases} 2 & \text{for } n = 2, \\ n^2 - 3 & \text{for } n \geq 3. \end{cases}$$

Proof. $n = 2$: See Fig. 3 for the diagram.

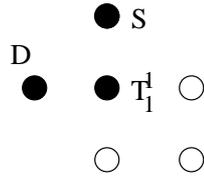


Figure 3: The grid of basis elements for $\mathfrak{sl}(2)$

The white dots are ruled out by the relations stated in the beginning, however the dot in the lower right corner is not independent here. Thus, the only relations are

$$(\text{ad}D)^2(S) = 4S, \quad (\text{ad}S)^2(D) = 4D.$$

which was also Grozman's result (2.1).

$n \geq 3$: As an example for the generic case, look at the $n = 5$ grid (Fig. 4).

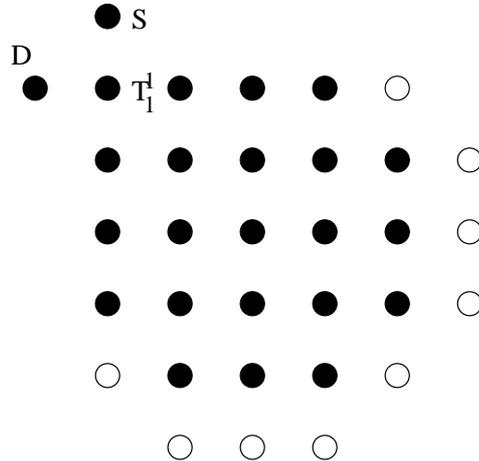


Figure 4: The grid of basis elements for $\mathfrak{sl}(5)$

We get here the following relations:

- 2 relations for the n -th powers of $\text{ad}D$ and $\text{ad}S$,
- $(n - 2)$ relations that limit the application of $\text{ad}D$ (the rightmost white dots),

- $(n - 2)$ relations that limit the application of $\text{ad } S$ (lowermost white dots),
- 1 relation corresponding to relation (1.7) (white dot in the lower right corner),
- $(n - 3)$ relations for the vertical paths from the first to the second row,
- $(n - 3)(n - 1)$ relations for the vertical paths between the second and third row, third and fourth row and so on down to the $(n - 1)$ st row,
- $(n - 2)$ relations for the horizontal paths in the n -th row.

This makes a total of $n^2 - 3$ relations. Lemma 2.1 and its corollary exclude the possibility that one of them is obtained by another by application of $\text{ad } D$ or $\text{ad } S$. Lemma 2.4 shows that none of them is a consequence of another via a rearrangement using the Jacobi identity. \square

We see that even for $n = 3$, the bound overestimates the exact number of relations. However, the number of relations found in the above manner is only of order $\sim n^2$, which can be expected to lie pretty close to the true behaviour of $R(n)$ so that the relative error will decrease for growing n . But the main advantage of our method is that it explicitly produces a *presentation* (albeit a redundant one): all relations can be directly read off from the grid representation of the basis of $\mathfrak{sl}(n)$.

§3 Relations between Sylvester's generators for $\mathfrak{gl}(n|n)$

Sylvester's generators can as well be used to generate a basis of $\mathfrak{gl}(n|n)$, and only for this "almost" simple (up to nontrivial center and outer derivations) finite dimensional Lie superalgebra, see [LSe]. It is most convenient to choose an alternating format for the superspace in which we express the supermatrices, i.e., if (e_1, e_2, \dots, e_n) is a basis of this vector space, let the e_{2k+1} be odd vectors and the e_{2k} be even ones for all k . This format has the advantage that we can use the same matrices D, S as above as Sylvester's generators, where now D is an even supermatrix and S an odd one. The result obtained below remains valid in any format, but looks nicest in the chosen one. To be able to compare the matrices obtained for the $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n|n)$ cases, we put a twiddle on the supermatrices: \tilde{D}, \tilde{S} .

As above, set

$$\tilde{T}_1^1 = [\tilde{D}, \tilde{S}]$$

which is now an odd supermatrix, but with the same entries as in the $\mathfrak{sl}(n)$ case. Likewise,

$$\tilde{T}_1^k = (\text{ad } \tilde{D})^{k-1}(\tilde{T}_1^1)$$

are all odd supermatrices, but look the same as in the $\mathfrak{sl}(n)$ case, and we find the analogue of relation (1.4) to be

$$(\text{ad } \tilde{D})^{2n}(\tilde{S}) = (1 - a)^{2n}\tilde{S}. \quad (3.1)$$

We follow the same algorithm as in the $\mathfrak{sl}(n)$ case, now using the superbracket: set

$$\tilde{T}_2^1 = [\tilde{S}, [\tilde{D}, \tilde{S}]] = [\tilde{S}, \tilde{T}_1^1]$$

which is the same matrix as in the $\mathfrak{sl}(n)$ case, except for the prefactor, which is now $(1 - a^2)$ instead of $-(1 - a)^2$. In general, for $k = 1, \dots, 2n$; $m = 1, \dots, 2n - 2$, we have

$$\begin{aligned} (\text{ad } \tilde{D})^{k-1}([\tilde{D}, \tilde{S}]) &= \tilde{T}_1^k = T_1^k \quad \text{for } k = 1, \dots, 2n - 1 \\ (\text{ad } \tilde{D})^{k-1}((\text{ad } \tilde{S})(\tilde{T}_m^1)) &= \tilde{T}_{m+1}^k = \begin{cases} -\left(-\frac{1+a}{1-a}\right)^{\frac{m+1}{2}} T_{m+1}^k & \text{for } m \text{ odd} \\ -\left(-\frac{1+a}{1-a}\right)^{\frac{m}{2}} T_{m+1}^k & \text{for } m \text{ even} \end{cases} \end{aligned}$$

so \tilde{T}_m^k is proportional to T_m^k . The $(1 + a)$ -factors stem from the application of anticommutators. One obtains the analogue of the relations (1.5) for $k = 2n + 1$ and $2 \leq m \leq 2n - 1$:

$$(\text{ad } \tilde{D})^{2n}((\text{ad } \tilde{S})^{m-1}((\text{ad } \tilde{D})(\tilde{S}))) = \left(-\frac{1+a}{1-a}\right)^{\frac{m+1}{2}} (1 - a^m)^{2n} (-1)^m (\text{ad } \tilde{S})^{m-1}(\text{ad } \tilde{D})(\tilde{S}). \quad (3.2)$$

For the diagonal basis elements, we set

$$\tilde{T}_n^k = \text{ad } \tilde{S}(\tilde{T}_{n-1}^k) \quad \text{for } 2 \leq k \leq n$$

and obtain the following relations:

$$\text{ad } \tilde{S}(\tilde{T}_{n-1}^1) = \left(-\frac{1+a}{1-a}\right)^n (-1)^{n+1} \tilde{D}, \quad (3.3)$$

$$\text{ad } \tilde{S}(\tilde{T}_n^k) = \left(-\frac{1+a}{1-a}\right)^{n-1} (1-a)(a^{2k} - 1)(1 - a^{2n-1})^{k-1} (-1)^n \tilde{T}_1^k. \quad (3.4)$$

Note that \tilde{T}_{2n}^{2n} is not zero here, but is proportional to the identity matrix. On any $(n|n)$ -dimensional superspace, the identity matrix is supertraceless, and therefore an element of $\mathfrak{sl}(n|n)$. Thus, no relation corresponds to (1.7) in the super case.

We have to add one more relation, which did not exist in the non-super case: the supercommutator of \tilde{S} with itself:

$$[\tilde{S}, \tilde{S}] = \frac{1}{(1+a)(1-a)^{2n-1}} \tilde{T}_2^{2n} \quad \text{for } n > 1. \quad (3.5)$$

For $n = 1$, this is not a relation, but really generates a new element, see Thm. (3.1).

Thinking of the set of basis elements again as a grid of points, we see that we have found relations of the same sort as in the $\mathfrak{sl}(n)$ case, with one exception: there is one more element, the one proportional to the identity matrix, represented by the rightmost dot in the last row.

One can again verify by explicit calculation that $[\tilde{T}_m^k, \tilde{T}_{m'}^{k'}]$ is proportional to $\tilde{T}_{m+m'}^{k+k'}$. This extends the validity of Lemma 2.1 to the super case. To find a bound for the number of relations again reduces to checking all paths from \tilde{T}_1^1 to the other \tilde{T}_m^k 's. This is done in the same way as before, it is clear that our algorithm proceeds on the same paths as in the $\mathfrak{sl}(n)$ case and that Prop. 2.2 and Cor. 2.3 also apply in the super case.

Also Lemma 2.4 generalises to the super case, now using the super Jacobi identity. But here we have to be careful about a specialty of the super case: supercommutators of elements with themselves do not necessarily vanish. Consider, for example, \tilde{T}_2^2 :

$$\text{ad } \tilde{D}(\text{ad } \tilde{S}(\text{ad } \tilde{D}(\tilde{S}))) = -\text{ad}(\text{ad } \tilde{S}(\tilde{D}))(\text{ad } \tilde{D}(\tilde{S})) - \text{ad } \tilde{S}((\text{ad } \tilde{D})^2(\tilde{S})). \quad (3.6)$$

Here, the first term on the right hand side does not vanish. Therefore the relation between the two paths to \tilde{T}_2^2 that we ruled out as being trivial in the $\mathfrak{sl}(n)$ case is nontrivial in the super case. Except for this fact, Lemma 2.4 remains valid.

Theorem 3.1. For $\mathfrak{gl}(n|n)$, the number $R(n)$ of independent relations between Sylvester's generators is bounded by

$$R(n) \leq \begin{cases} 4 & \text{for } n = 1, \\ (2n)^2 - 1 & \text{for } n > 1. \end{cases} \quad (3.7)$$

Proof. The $n = 1$ case differs from the $\mathfrak{sl}(2)$ case because of the relation

$$[\tilde{S}, \tilde{S}] = 2 \cdot \mathbb{1},$$

where $\mathbb{1}$ is the identity matrix. The basis elements of $\mathfrak{gl}(1|1)$ can be represented by the grid

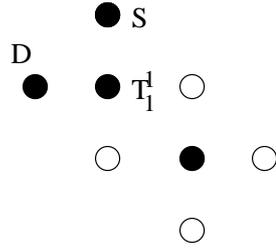


Figure 5: The grid of basis elements of $\mathfrak{gl}(1|1)$

in Fig 5. There are four relations:

$$[\tilde{D}, [\tilde{D}, \tilde{S}]] = 4\tilde{S} \quad (3.8)$$

$$[\tilde{S}, [\tilde{D}, \tilde{S}]] = 0 \quad (3.9)$$

$$[\tilde{D}, [\tilde{S}, \tilde{S}]] = 0 \quad (3.10)$$

$$[\tilde{S}, [\tilde{S}, \tilde{S}]] = 0 \quad (3.11)$$

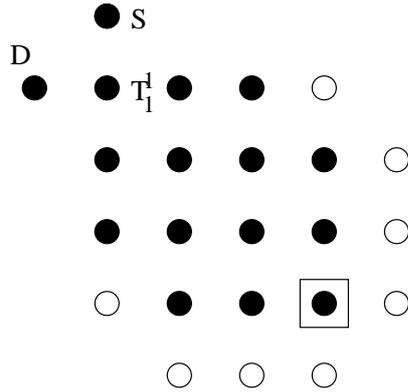


Figure 6: The grid of basis elements of $\mathfrak{gl}(2|2)$

For $n > 1$, the grid looks like in Fig. 6. Note that now there is one more black dot in the lower right corner which we generate from the dot above it. This provides one more relation. Another additional relation is obtained from the two paths to \tilde{T}_2^2 , which are now independent. Apart from this, the situation is identical to the non-super case. \square

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