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Abstract

We provide a class of inequalities for detecting entanglements in multi-mode systems. Necessary conditions for fully separable, bi-separable and sufficient conditions for fully entangled states are explicitly presented.

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I. INTRODUCTION

Entanglement plays a key role in the rapidly developing field of quantum information processing. Consequently, the study of entanglement of bi- and multipartite systems has been the focus of research in quantum information theory. In recent years, increasing attention was paid to infinite dimensional systems, the so called continuous-variable (CV) systems. In particular, Gaussian state entanglement, a special case of CV systems, has aroused great interest[1–8]. The quantum teleportation network has been recently demonstrated experimentally with the use of fully symmetric three-mode Gaussian states[9]. Many efforts[2–5, 7, 8, 10–17] have been devoted to the study of detecting entanglement for CV systems.

It is of particular relevance to provide theoretical methods to determine the entanglement of a CV quantum system in quantum information processing and computation. Such an interest stems from the practical needs in the implementations of realistic information protocols. There has been some progress in the criteria for entanglement by uncertainty relations[10–15]. In [13] Hillery and Zubairy have provided a class of inequalities for detecting entanglement of two-mode states.

In this paper, we generalize the inequalities to the case of multi-mode states. We derive a class of inequalities for detecting the entanglement of multi-mode states and some inequalities to detect fully entangled states. These quantities and inequalities can be, in principle, measured in standard homodyne correlation experiments[18].

II. BIPARTITE ENTANGLED CONDITIONS FOR 3-MODE STATES

Firstly we consider the case of three-mode states. We focus on three-mode harmonic oscillators. The derived conditions can also be applied to radiation modes, motional states of trapped ions, and related systems.

Let operators a , b and c be the annihilation operators of the first (A), the second (B) and the third (C) mode, respectively. Set $L_1 = abc^\dagger + a^\dagger b^\dagger c$ and $L_2 = i(abc^\dagger - a^\dagger b^\dagger c)$. We have

$$[L_1, L_2] = 2i\{N_a N_b - N_a N_c - N_b N_c - N_c\}, \quad (1)$$

and

$$\begin{aligned}
(\Delta L_1)^2 + (\Delta L_2)^2 &= 4\langle N_a N_b N_c \rangle - 4|\langle abc^\dagger \rangle|^2 \\
&\quad + 2(\langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_a N_c \rangle + \langle N_c \rangle),
\end{aligned} \tag{2}$$

where $N_a = a^\dagger a$, $N_b = b^\dagger b$ and $N_c = c^\dagger c$. For a pure state that is $AB|C$ bi-separable with respect to the first, second and third mode, we have

$$\begin{aligned}
(\Delta L_1)^2 + (\Delta L_2)^2 &= 4\langle N_a N_b \rangle \langle N_c \rangle - 4|\langle ab \rangle \langle c^\dagger \rangle|^2 \\
&\quad + 2(\langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_a N_c \rangle + \langle N_c \rangle).
\end{aligned} \tag{3}$$

Noting that the Schwarz inequality implies that, for any state $|\Psi\rangle$, $|\langle a \rangle|^2 = |\langle \Psi | a | \Psi \rangle|^2 \leq |\langle \Psi | \Psi \rangle|^2 \cdot |a|^2 = \langle \Psi | a^\dagger a | \Psi \rangle = \langle N_a \rangle$, $|\langle ab \rangle|^2 \leq \langle N_a N_b \rangle$, for such a pure $AB|C$ separable state we get

$$(\Delta L_1)^2 + (\Delta L_2)^2 \geq 2(\langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_a N_c \rangle + \langle N_c \rangle). \tag{4}$$

Lemma 1 *For any density matrix*

$$\rho = \sum_k p_k \rho_k, \quad 0 < p_k \leq 1, \quad \sum_i p_i = 1, \tag{5}$$

and a variable S , we have

$$(\Delta S)^2 \geq \sum_k p_k (\Delta S_k)^2. \tag{6}$$

The lemma can be proved by the convexity. From the lemma, we have that the inequality (4) holds for $AB|C$ separable mixed states as well.

On the other hand the uncertainties ΔL_1 and ΔL_2 also satisfy

$$\begin{aligned}
(\Delta L_1)^2 + (\Delta L_2)^2 &\geq 2\Delta L_1 \Delta L_2 \geq |\langle [L_1, L_2] \rangle| \\
&= 2|\langle N_a N_c \rangle - \langle N_b N_c \rangle - \langle N_a N_c \rangle - \langle N_c \rangle|.
\end{aligned} \tag{7}$$

Comparing this result, which holds for any state, with inequality (4), which holds for any $AB|C$ separable states, we see that the right-hand side of inequality (7) is always less than or equal to that of inequality (4). Consequently, there may be states that violate inequality (4) while satisfying inequality (7). Thus we have the following theorem:

Theorem 2 *If the following inequality holds for a state ρ ,*

$$(\Delta L_1)^2 + (\Delta L_2)^2 < 2(\langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_a N_c \rangle + \langle N_c \rangle),$$

then the state is $AB|C$ entangled.

For a state $|\Psi\rangle = (|0\rangle_a|0\rangle_b|1\rangle_c + |0\rangle_a|1\rangle_b|0\rangle_c + |1\rangle_a|0\rangle_b|1\rangle_c + |1\rangle_a|1\rangle_b|0\rangle_c)/2$, we have

$$(\Delta L_1)^2 + (\Delta L_2)^2 < 2(\langle N_a N_c \rangle + \langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_c \rangle).$$

Therefore the theorem shows that the state $|\Psi\rangle$ is $AB|C$ entangled.

Comparing (2) with (4) we have the following corollary:

Corollary 3 *If the following inequality holds for a state ρ ,*

$$\langle N_a N_b N_c \rangle < |\langle abc^\dagger \rangle|^2, \quad (8)$$

then the state is $AB|C$ entangled.

Note that the Schwarz inequality implies that

$$|\langle abc^\dagger \rangle|^2 \leq \langle N_a N_b (N_c + 1) \rangle. \quad (9)$$

Formulae (8) and (9) suggest that there is a family of similar conditions for detecting entanglement, if one considers operator $a^m b^n (c^\dagger)^l$ instead of abc^\dagger . So we have the following theorem:

Theorem 4 *If the following inequality holds for a state ρ ,*

$$|\langle a^m b^n (c^\dagger)^l \rangle|^2 > \langle (a^\dagger)^m a^m (b^\dagger)^n b^n (c^\dagger)^l c^l \rangle, \quad (10)$$

where m, n and l are positive integers, then ρ is $AB|C$ entangled.

Proof: For a $AB|C$ separable pure state we have

$$\begin{aligned} |\langle a^m b^n (c^\dagger)^l \rangle|^2 &= |\langle a^m b^n \rangle|^2 |\langle c^l \rangle|^2 \leq \langle (a^\dagger)^m a^m (b^\dagger)^n b^n \rangle \langle (c^\dagger)^l c^l \rangle \\ &= \langle (a^\dagger)^m a^m (b^\dagger)^n b^n (c^\dagger)^l c^l \rangle. \end{aligned} \quad (11)$$

To show that formula (11) is valid for $AB|C$ separable mixed state (5), defining $A = a^m$, $B = b^n$ and $C = c^l$, we have

$$\begin{aligned} |\langle ABC^\dagger \rangle| &= \left| \sum_k p_k \text{tr}(\rho_k ABC^\dagger) \right| \leq \sum_k p_k |\text{tr}(\rho_k ABC^\dagger)| \\ &= \sum_k p_k |\text{tr}(\rho_k^{(ab)} AB) \text{tr}(\rho_k^{(c)} C^\dagger)| \leq \sum_k p_k \langle A^\dagger AB^\dagger B \rangle_k^{1/2} \langle C^\dagger C \rangle_k^{1/2}. \end{aligned} \quad (12)$$

From the convexity, we have that

$$\begin{aligned} |\langle ABC^\dagger \rangle| &\leq \left(\sum_k p_k \langle A^\dagger AB^\dagger B \rangle_k \langle C^\dagger C \rangle_k \right)^{1/2} \\ &= \left(\sum_k p_k \langle A^\dagger AB^\dagger BC^\dagger C \rangle_k \right)^{1/2} = \langle A^\dagger AB^\dagger BC^\dagger C \rangle^{1/2}, \end{aligned} \quad (13)$$

which shows that the inequality (11) does indeed hold for all $AB|C$ separable states. This ends the proof. \square

In analogy to L_1 and L_2 , we have the following theorems:

Theorem 5 *If any one of the following inequalities hold for a state ρ ,*

$$(\Delta J_1)^2 + (\Delta J_2)^2 < 2(\langle N_a N_c \rangle + \langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_a \rangle), \quad (14)$$

$$|\langle (a^\dagger)^m b^n c^l \rangle|^2 > \langle (a^\dagger)^m a^m (b^\dagger)^n b^n (c^\dagger)^l c^l \rangle, \quad (15)$$

where $J_1 = ab^\dagger c^\dagger + a^\dagger bc$, $J_2 = i(-ab^\dagger c^\dagger + a^\dagger bc)$, m , n and l are positive integers, then the state is entangled between A -party and BC -party.

This theorem can be directly proved by using the techniques in proving theorems 2 and 4.

We can Similarly obtain the following theorem for entangled states between B -party and AC -party:

Theorem 6 *If any one of the following inequalities hold for a state ρ ,*

$$(\Delta K_1)^2 + (\Delta K_2)^2 < 2(\langle N_a N_c \rangle + \langle N_a N_b \rangle + \langle N_b N_c \rangle + \langle N_b \rangle). \quad (16)$$

$$|\langle a^m (b^\dagger)^n c^l \rangle|^2 > \langle (a^\dagger)^m a^m (b^\dagger)^n b^n (c^\dagger)^l c^l \rangle, \quad (17)$$

where $K_1 = ab^\dagger c + a^\dagger bc^\dagger$ and $K_2 = i(ab^\dagger c - a^\dagger bc^\dagger)$, m , n and l are positive integers, then the state is $B|AC$ entangled.

In fact, there is another class of inequalities detecting the bipartite entangled states. We have the following theorem:

Theorem 7 *For a state ρ ,*

- if $|\langle a^m b^n c^l \rangle| > [\langle (a^\dagger)^m a^m \rangle \langle (b^\dagger)^n b^n (c^\dagger)^l c^l \rangle]^{1/2}$, then the state is $A|BC$ entangled;*
- if $|\langle a^m b^n c^l \rangle| > [\langle (b^\dagger)^n b^n \rangle \langle (a^\dagger)^m a^m (c^\dagger)^l c^l \rangle]^{1/2}$, then the state is $B|AC$ entangled;*
- if $|\langle a^m b^n c^l \rangle| > [\langle (a^\dagger)^m a^m (b^\dagger)^n b^n \rangle \langle (c^\dagger)^l c^l \rangle]^{1/2}$, then the state is $AB|C$ entangled.*

Proof: We prove the condition for $A|BC$ entanglement.

From Schwarz inequality, we have

$$|\langle a^m b^n c^l \rangle| \leq [\langle (a^\dagger)^m a^m \rangle \langle (b^\dagger)^n b^n \rangle \langle (c^\dagger)^l c^l \rangle]^{1/2} \quad (18)$$

for a pure $A|BC$ separable state.

For $A|BC$ separable mixed state $\rho = \sum_k p_k \rho_k$, where ρ_k is the density matrix corresponding to a pure $A|BC$ separable state, and p_k is the probability of ρ_k , we have

$$\begin{aligned} |\langle ABC \rangle|^2 &= \left| \sum_k p_k \text{tr}(\rho_k ABC) \right|^2 \leq \left(\sum_k p_k |\text{tr}(\rho_k ABC)| \right)^2 \\ &= \sum_{i,j} p_i p_j |\langle A \rangle_i \langle BC \rangle_i| |\langle C^\dagger B^\dagger \rangle_j \langle A^\dagger \rangle_j| \\ &\leq \sum_{i,j} p_i p_j (\langle A^\dagger A \rangle_i \langle B^\dagger B C^\dagger C \rangle_i \langle B^\dagger B C^\dagger C \rangle_j \langle A^\dagger A \rangle_j)^{1/2}, \end{aligned} \quad (19)$$

where $A = a^m$, $B = b^n$ and $C = c^l$. Set $\langle A^\dagger A \rangle_i = x_i$, $\langle B^\dagger B C^\dagger C \rangle_i = y_i$. (19) can be expressed as

$$|\langle ABC \rangle|^2 \leq \sum_k p_k^2 x_k y_k + 2 \sum_{i < j} p_i p_j (x_i y_i x_j y_j)^{1/2}. \quad (20)$$

While $\langle A^\dagger A \rangle \langle B^\dagger B C^\dagger C \rangle$ can be expressed as

$$\langle A^\dagger A \rangle \langle B^\dagger B C^\dagger C \rangle = \sum_k p_k^2 x_k y_k + \sum_{i < j} p_i p_j (x_i y_j + x_j y_i). \quad (21)$$

Noting that $x_i y_j + x_j y_i \geq 2(x_i y_i x_j y_j)^{1/2}$, we see that from (20) and (21) the inequality (18) holds for all $A|BC$ bipartite separable states. Hence if a state violates this inequality (18), it must be bipartite entangled between A-part and BC-part. The other conditions can be proved similarly. \square

For $m = n = l = 1$, formula (18) implies that for a $A|BC$ separable state

$$|\langle abc \rangle|^2 \leq \langle N_a \rangle \langle N_b N_c \rangle. \quad (22)$$

For an arbitrary state, we have $|\langle abc \rangle|^2 \leq [\langle N_a + 1 \rangle \langle N_b N_c \rangle]$. Therefore there may be states that do violate the inequality (22) and are $A|BC$ entangled at the same time.

III. FULLY ENTANGLED CONDITIONS FOR 3-MODE STATES

We first introduce the notions of fully separable 3-mode states and fully entangled ones in [19, 20]. A 3-mode state is fully separable if and only if the state can be described as

mixtures of $\rho_1 \otimes \rho_2 \otimes \rho_3$. A 3-mode entangled state is fully entangled if it is not bi-partite separable. We have our main theorem:

Theorem 8 *If the following inequality holds for a state ρ ,*

$$(\Delta K(\phi))^2 < 1, \quad (23)$$

where $K(\phi) = e^{i\phi} a^\dagger b^\dagger c^\dagger + e^{-i\phi} abc$, $\phi \in [0, 2\pi]$, then the state is fully entangled.

Proof:

$$\begin{aligned} (\Delta K(\phi))^2 &= e^{2i\phi} \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)^2 \rangle + e^{-2i\phi} \langle (abc - \langle abc \rangle)^2 \rangle \\ &\quad + \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)(abc - \langle abc \rangle) \rangle \\ &\quad + \langle (abc - \langle abc \rangle)(a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle) \rangle. \end{aligned} \quad (24)$$

As $e^{2i\phi} \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)^2 \rangle + e^{-2i\phi} \langle (abc - \langle abc \rangle)^2 \rangle$ is real, by using the Schwarz inequality

$$\begin{aligned} &|\langle (abc - \langle abc \rangle)^2 \rangle| \\ &\leq [\langle (abc - \langle abc \rangle)(a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle) \rangle \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)(abc - \langle abc \rangle) \rangle]^{1/2}, \end{aligned} \quad (25)$$

we obtain

$$\begin{aligned} &(\Delta K(\phi))^2 \\ &\geq -|\langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)^2 \rangle| - |\langle (abc - \langle abc \rangle)^2 \rangle| \\ &\quad + \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)(abc - \langle abc \rangle) \rangle + \langle (abc - \langle abc \rangle)(a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle) \rangle \\ &\geq -2[\langle (abc - \langle abc \rangle)(a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle) \rangle \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)(abc - \langle abc \rangle) \rangle]^{1/2} \\ &\quad + \langle (a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle)(abc - \langle abc \rangle) \rangle + \langle (abc - \langle abc \rangle)(a^\dagger b^\dagger c^\dagger - \langle a^\dagger b^\dagger c^\dagger \rangle) \rangle \\ &= [\langle (N_a + 1)(N_b + 1)(N_c + 1) \rangle - |\langle abc \rangle|^2]^{1/2} - [\langle N_a N_b N_c \rangle - |\langle abc \rangle|^2]^{1/2}. \end{aligned} \quad (26)$$

The inequality (26) is valid for all states. In particular, if the state is a fully separable pure state, (26) becomes

$$\begin{aligned} (\Delta K(\phi))^2 &\geq [\langle (N_a + 1) \rangle \langle (N_b + 1) \rangle \langle (N_c + 1) \rangle - |\langle abc \rangle|^2]^{1/2} \\ &\quad - [\langle N_a \rangle \langle N_b \rangle \langle N_c \rangle - |\langle abc \rangle|^2]^{1/2}. \end{aligned} \quad (27)$$

Denote $x_1 = \langle N_a \rangle$, $x_2 = \langle N_b \rangle$, $x_3 = \langle N_c \rangle$ and $z = |\langle abc \rangle|^2$. Let us find the minimum of the function

$$F(x_1, x_2, x_3) = \sqrt{(x_1 + 1)(x_2 + 1)(x_3 + 1) - z} - \sqrt{x_1 x_2 x_3 - z}, \quad (28)$$

in the region $x_1x_2x_3 \geq z \geq 0$.

There are two cases: 1. $F(x_1, x_2, x_3)$ has minimum in the region of interest. From $\partial F(x_1, x_2, x_3)/\partial x_i = 0$, $i = 1, 2, 3$, F acquires the minimum at $x_1 = x_2 = x_3 = x$,

$$F_{\min} = \sqrt{(x+1)^3 - z} - \sqrt{x^3 - z}. \quad (29)$$

As $(\sqrt{(x+1)^3 - z})^2 - (1 + \sqrt{x^3 - z})^2 \geq (x+1)^3 - (x^3 + 1 + 2\sqrt{x^3})$, we have $F_{\min} \geq 1$ in the region of interest.

2. $F(x_1, x_2, x_3)$ has no local minimum in the region of interest. Then the minimum of the function must lie on the boundary, $x_1x_2x_3 = z$ or at least one of the x_1 , x_2 and x_3 goes to infinity.

On the boundary $x_1x_2x_3 = z$ we have

$$F(x_1, x_2, x_3) = \sqrt{(x_1+1)(x_2+1)(x_3+1) - x_1x_2x_3} \geq 1. \quad (30)$$

When $x_1, x_2, x_3 \rightarrow \infty$, we obtain

$$\begin{aligned} F(x_1, x_2, x_3) &= \int_{x_1x_2x_3-z}^{(x_1+1)(x_2+1)(x_3+1)-z} \frac{1}{2\sqrt{u}} du \\ &\geq \frac{(x_1+1)(x_2+1)(x_3+1) - x_1x_2x_3}{2\sqrt{(x_1+1)(x_2+1)(x_3+1) - z}} \\ &\geq \frac{(x_1+1)(x_2+1)(x_3+1) - x_1x_2x_3}{2\sqrt{(x_1+1)(x_2+1)(x_3+1)}} \\ &= \frac{1}{2} \left[\sqrt{\frac{x_1+1}{(x_2+1)(x_3+1)}} + \sqrt{\frac{(x_2+1)(x_3+1)}{x_1+1}} \right. \\ &\quad \left. + \frac{x_1x_2 + x_1x_3 - 1}{\sqrt{(x_1+1)(x_2+1)(x_3+1)}} \right] \\ &\geq 1 + \frac{x_1x_2 + x_1x_3 - 1}{2\sqrt{(x_1+1)(x_2+1)(x_3+1)}}. \end{aligned} \quad (31)$$

When one of the x_1 , x_2 and x_3 goes to infinity, while the rest keep finite, the proof can be done similarly. Therefore, $F_{\min} \geq 1$ holds for any fully separable pure states. Finally, we have, from Lemma 1,

$$(\Delta K(\phi))^2 \geq 1 \quad (32)$$

for fully separable mixed states.

Actually inequality (32) is also true for any bi-separable pure states. For example, for a $A|BC$ separable pure state, we have

$$(\Delta K(\phi))^2 \geq [(\langle N_a + 1 \rangle \langle (N_b + 1)(N_c + 1) \rangle - |\langle abc \rangle|^2)^{1/2} - (\langle N_a \rangle \langle N_b N_c \rangle - |\langle abc \rangle|^2)^{1/2}]^2. \quad (33)$$

Setting $x_1 = \langle N_a \rangle$, $x_2 = \langle N_b N_c \rangle$, $x_3 = \langle N_b \rangle$, $x_4 = \langle N_c \rangle$ and $z = |\langle abc \rangle|^2$, we define a function

$$\begin{aligned} G(x_1, x_2, x_3, x_4, z) &= \sqrt{(x_1 + 1)(x_2 + x_3 + x_4 + 1) - z} - \sqrt{x_1 x_2 - z} \\ &\geq \sqrt{(x_1 + 1)(x_2 + 1) - z} - \sqrt{x_1 x_2 - z}, \end{aligned} \quad (34)$$

which is greater than or equal to 1 in the region of interest. From lemma 1, we know that (32) is also valid for any bi-partite separable mixed states. Therefore violation of inequality (32) implies full entanglement. This ends the proof. \square

As an example we consider the GHZ state, $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. For this state, we have $(\Delta K)^2 = (\langle K^2 \rangle) - (\langle K \rangle)^2 = 0$, where $K = K(\phi = 0)$. Thus the GHZ state is fully entangled.

In analogy to (11), it is possible to find other relations satisfied by separable states. For example, in the case of pure fully separable states we have that

$$|\langle abc \rangle| = |\langle a \rangle \langle b \rangle \langle c \rangle| \leq [\langle N_a \rangle \langle N_b \rangle \langle N_c \rangle]^{1/2}. \quad (35)$$

However, (35) is not always true for some fully separable mixed states. In fact, for a tripartite pure state, we can detect a fully entangled state with respect to all three bipartite decompositions. If all three bipartite decompositions are entangled, the tripartite state is fully entangled.

Here is another criterion for detecting pure fully entangled states.

Theorem 9 *If a pure state violates inequality (35) and satisfies the following relations simultaneously,*

$$|\langle a \rangle \langle bc \rangle|^2 \leq \langle N_a \rangle \langle N_b \rangle \langle N_c \rangle, \quad (36)$$

$$|\langle ab \rangle \langle c \rangle|^2 \leq \langle N_a \rangle \langle N_b \rangle \langle N_c \rangle, \quad (37)$$

$$|\langle ac \rangle \langle b \rangle|^2 \leq \langle N_a \rangle \langle N_c \rangle \langle N_b \rangle, \quad (38)$$

then the pure state is fully entangled.

Proof: Without loss of generality, we suppose that a $A|BC$ separable state satisfies the above three inequalities and the inequality (35), $|\langle abc \rangle| > [\langle N_a \rangle \langle N_b \rangle \langle N_c \rangle]^{1/2}$ simultaneously. However for a $A|BC$ separable state, $|\langle abc \rangle| = |\langle a \rangle \langle bc \rangle|$. This contradicts with (36), $|\langle a \rangle \langle bc \rangle|^2 \leq \langle N_a \rangle \langle N_b \rangle \langle N_c \rangle$. This ends the proof. \square

We can extend these inequalities to general forms.

Corollary 10 *If a state satisfies the following relations simultaneously,*

$$|\langle a^m b^n c^l \rangle| > [\langle (a^\dagger)^m a^m \rangle \langle (b^\dagger)^n b^n \rangle \langle (c^\dagger)^l c^l \rangle]^{1/2}, \quad (39)$$

$$|\langle a^m b^n \rangle \langle c^l \rangle|^2 \leq \langle (a^\dagger)^m a^m \rangle \langle (b^\dagger)^n b^n \rangle \langle (c^\dagger)^l c^l \rangle, \quad (40)$$

$$|\langle a^m \rangle \langle b^n c^l \rangle|^2 \leq \langle (a^\dagger)^m a^m \rangle \langle (b^\dagger)^n b^n \rangle \langle (c^\dagger)^l c^l \rangle, \quad (41)$$

$$|\langle a^m c^l \rangle \langle b^n \rangle|^2 \leq \langle (a^\dagger)^m a^m \rangle \langle (c^\dagger)^l c^l \rangle \langle (b^\dagger)^n b^n \rangle, \quad (42)$$

where m, n and l are positive integers, we can conclude that the state is fully entangled.

Consider the state

$$|\psi\rangle = N_-(\alpha, \beta, \gamma)(|\alpha, \beta, \gamma\rangle - |-\alpha, -\beta, -\gamma\rangle), \quad (43)$$

where the normalization

$$N_- = \left[2 \left(1 - e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)} \right) \right]^{-1/2}, \quad (44)$$

$|\alpha, \beta, \gamma\rangle$ is the coherent state. For this state we have

$$\begin{aligned} |\langle a^\dagger (b)^2 c \rangle| &= \left| \frac{1}{2(1 - e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)})} (\alpha^* \langle \alpha, \beta, \gamma | + \alpha^* \langle -\alpha, -\beta, -\gamma |) \right. \\ &\quad \left. (\beta^2 \gamma |\alpha, \beta, \gamma\rangle + \beta^2 \gamma |-\alpha, -\beta, -\gamma\rangle) \right| \\ &= \left| \frac{2\alpha^* \beta^2 \gamma (1 + e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)})}{2(1 - e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)})} \right| = |\alpha^* \beta^2 \gamma| \coth(|\alpha|^2 + |\beta|^2 + |\gamma|^2) \end{aligned} \quad (45)$$

and

$$\begin{aligned} &(\langle a^\dagger a b^\dagger b^\dagger b b c^\dagger c \rangle)^{1/2} \\ &= \left(\frac{1}{2(1 - e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)})} (\alpha^* \beta^* \beta^* \gamma^* \langle \alpha, \beta, \gamma | - \alpha^* \beta^* \beta^* \gamma^* \langle -\alpha, -\beta, -\gamma |) \right. \\ &\quad \left. (\alpha \beta^2 \gamma |\alpha, \beta, \gamma\rangle - \alpha \beta^2 \gamma |-\alpha, -\beta, -\gamma\rangle) \right)^{1/2} \\ &= \left(\frac{2|\alpha|^2 |\beta|^4 |\gamma|^2 (1 - e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)})}{2(1 - e^{-2(|\alpha|^2 + |\beta|^2 + |\gamma|^2)})} \right)^{1/2} = |\alpha| |\beta|^2 |\gamma|. \end{aligned} \quad (46)$$

Eq. (46) is clearly less than Eq. (45) for all nonzero parameters. Hence, the state is $A|BC$ entangled from inequality (15). It is easily shown that for this state $|a^2b^\dagger c|^2 > \langle a^\dagger a^\dagger a a N_b N_c \rangle$, $|a^2 b c^\dagger|^2 > \langle N_a N_b c^\dagger c^\dagger c c \rangle$. Therefore we conclude from (17) and (11) that this state is not only $AB|C$ entangled but also $B|AC$ entangled: this state is fully entangled.

IV. MULTI-MODE STATE

The methods employed above for 3-mode states can be extended to n -mode states. Consider n modes whose annihilation operators are a_1, a_2, \dots and a_n , respectively. For a state that is separable between m -mode and $(n - m)$ -mode, we have that

$$|\langle a_1 a_2 \cdots a_m a_{m+1}^\dagger \cdots a_n^\dagger \rangle|^2 \leq \langle N_1 N_2 \cdots N_n \rangle, \quad (47)$$

$$|\langle a_1 a_2 \cdots a_m a_{m+1} \cdots a_n \rangle|^2 \leq \langle N_1 N_2 \cdots N_m \rangle \langle N_{m+1} \cdots N_n \rangle. \quad (48)$$

Therefore, if the inequalities (47) or (48) are violated, the state is entangled between m -mode and $(n - m)$ -mode. We also have that for a general $m|(n - m)$ separable state

$$|\langle a_1^{l_1} a_2^{l_2} \cdots a_m^{l_m} \cdots (a_n^\dagger)^{l_n} \rangle|^2 \leq \langle (a_1^\dagger)^{l_1} a_1^{l_1} (a_2^\dagger)^{l_2} a_2^{l_2} \cdots (a_m^\dagger)^{l_m} a_m^{l_m} \rangle, \quad (49)$$

$$|\langle a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \rangle|^2 \leq \langle (a_1^\dagger)^{l_1} a_1^{l_1} (a_2^\dagger)^{l_2} a_2^{l_2} \cdots (a_m^\dagger)^{l_m} a_m^{l_m} \rangle \langle (a_{m+1}^\dagger)^{l_{m+1}} a_{m+1}^{l_{m+1}} \cdots (a_n^\dagger)^{l_n} a_n^{l_n} \rangle. \quad (50)$$

Denote $K(\phi) = e^{i\phi} a_1 a_2 \cdots a_n + e^{-i\phi} a_1^\dagger a_2^\dagger \cdots a_n^\dagger$, we have that for a fully separable state

$$(\Delta K(\phi))^2 \geq 1. \quad (51)$$

In analogy to formula (32), (51) holds for any separable states. Hence if the inequality (51) is violated, the multi-mode state must be fully entangled.

V. CONCLUSIONS

We have studied inseparability conditions for multi-mode states by presenting a series of inequalities. These conditions provide, in principle, measurable tests of entanglement, in the sense that all of the quantities appeared in the inequalities can be measured experimentally. The results in section 2 (theorems 4-7) have simple forms and can easily applied. They coincide with the ones in [10, 14, 15] that can be obtained by extending the methods used on non-positivity of partial transposition (NPT) in [14, 15]. In many practical cases, such

as teleportation network, we firstly need to detect the fully entanglement of a state. Our theorem 8-9 could be applied for such purpose, as the quantities in the inequalities can be measured experimentally. For dealing with the entanglement of Gaussian states there are already very nice results [4, 5, 7, 8]. Our theorems 8-9 give some results which are derived for general states. Hence they could be applied for detecting entanglement of non-Gaussian states.

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