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**Regularity of domain walls in magnetic nanowires**

by


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# Regularity of domain walls in magnetic nanowires

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**Abstract**

This paper investigates regularity properties of energy minimising 180 degree domain walls in magnetic nanowires. For small radii we prove uniform bounds in $C^{1,\beta}$. Uniform bounds imply convergence in strong norms and are important for the step from statics to dynamics. First we show the convergence of minimisers in $H^1$ and a bound on the rate of convergence of the minimal energies. Then we use the notion of almost minimisers to prove $C^{0,\alpha}$ estimates and the Morrey-Campanato approach to get uniform $C^{1,\beta}$ estimates.

## 1 Introduction

In recent numerical studies on switching modes in magnetic nanowires two different reversal modes have been observed. For thin wires the magnetisation looks smooth (transverse mode), while for thick wires the magnetisation forms a singularity (vortex mode). Forster et al. [3] have suggested that the reversal modes correspond to static domain walls that minimise the energy. Numeric simulations of domain wall profiles support this idea [10] and in a previous paper we have investigated it rigorously [7], establishing a crossover between two scaling regimes of the energy. This crossover corresponds to the change of scaling from transverse walls to vortex walls. Moreover we have shown that for $R \to 0$ the energy minimising domain walls converge to a domain wall that is smooth and constant on each cross section. However, since the convergence is only in a topology that comes from the energy bounds, we do not know whether the domain walls are smooth for finite radii and whether we have convergence in stronger norms. This question is not only of intrinsic interest but also crucial in the step from statics to dynamics [8].

We work in the framework of micromagnetism. This is a mesoscopic continuum theory that assigns a nonlocal nonconvex energy to each magnetisation $m$ from the domain $\Sigma \subset \mathbb{R}^3$ to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Experimentally observed ground states correspond to minimisers of the micromagnetic energy functional $E$. When appropriately rescaled, for a soft magnetic material without external field this energy is

$$E(m) = \int_\Sigma |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 \qquad \text{where } \Delta u = \operatorname{div} m \ \text{ in } \mathbb{R}^3. \tag{1}$$

We consider magnetisations where the domain is an infinite cylinder

$$\Sigma(R) := \mathbb{R} \times D_R := \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^2 : |y| < R\right\}.$$

Since we are interested in 180 degree domain walls we have to prescribe the limits at $\pm\infty$. We define

$$\chi : \mathbb{R} \to [-1, 1], \qquad m \mapsto \tanh(x),$$
$$\mathcal{M}_l(R) := \left\{m : \Sigma(R) \to \mathbb{S}^2 \; : \; m - \chi \vec{e}_x \in H^1(\Sigma)\right\}. \tag{2}$$

In [7] we have shown that for every radius $R > 0$ there exists a minimiser in $\mathcal{M}_l(R)$. In this paper we show that for small radii these minimisers are uniformly bounded in $C^{1,\beta}$ for $\beta < \frac{1}{8}$. In general, we cannot expect such good regularity since for thick wires the examples of domain walls with low energy are vortex walls that are not even continuous [7]. So the arguments rely crucially on the fact that the wires are thin. In this paper we refine some of the results of [7]. We prove convergence of the minimisers in $H^1$ and show a bound on the rate of convergence of the minimal energies.

The regularity of minimisers of the micromagnetic energy has been studied independently by Carbou [1] and Hardt and Kinderlehrer [6]. Carbou investigates critical points of the micromagnetic energy in two and three dimensions using the Euler-Lagrange equation. He finds that critical points in $H^1(D_1, \mathbb{S}^2)$ are smooth, while critical points in $H^1(B_1, \mathbb{S}^2)$ are smooth away from a set of one dimensional Hausdorff measure zero. Here $D_1 \subset \mathbb{R}^2$ denotes the unit disc and $B_1 \subset \mathbb{R}^3$ denotes the unit ball.

Hardt and Kinderlehrer use the fact that on small scales the exchange energy is the dominant part of the micromagnetic energy. Using the notion of *almost-minimisers* they show how the stray field energy can be treated as a lower order perturbation. They find that minimisers of the micromagnetic energy functional on bounded, sufficiently regular domains are smooth away from a discrete set $Z$.

Thei moreover show that the set $Z$ is empty if the exchange energy of $m$ is small enough. Here "small enough" depends on the domain. This implies that energy minimising domain walls are smooth for small $R > 0$, but we do not get uniform bounds. However, uniform bounds are necessary to show the convergence of minimisers in stronger norms. In this paper we focus on the estimates that control the dependence of the bounds on the radius. We will not repeat arguments that have been given in similar form elsewhere and refer to [9] for details on how they are implemented here.

## 1.1 Outline of the paper

In Section 2 we summarise the results of [7] that are relevant for this paper.

In Section 3 we show the convergence of minimisers in $H^1$.

In Section 4 we prove an upper bound on the rate of convergence of the minimal energies.

In Section 5 we give a short introduction to the ideas and definitions we are using to prove regularity results.

In Section 6 we present the major steps to get $C^{0,\alpha}$ estimates for almost-minimisers without proofs since the arguments are essentially the same as in [6].

In Section 7 we use the Euler-Lagrange equations for $m^R$ and apply the Morrey-Campanato approach to regularity (cf. [4]) to prove uniform $C^{1,\beta}$ regularity. To get a bound on $\|m^R\|_{C^{1,\beta}(\Sigma(R))}$ that is uniform in $R$, we show uniform bounds on integrals of the form $\frac{1}{R^\gamma}\int_{B_R(a)}|\nabla m|^2$. The proof relies on the bound for the rate of convergence of the minimal energies.

# 2 Preliminaries

## 2.1 The main results about domain walls in magnetic nanowires

Summarising the results of [7] we discuss the question of existence of energy minimising domain walls, the scaling of the energy and the shape of the optimal wall profile. First we note that there is a simple characterisation for magnetisations with finite energy.

**Theorem 1.** *For $m : \Sigma(R) \to \mathbb{S}^2$ we have $E(m) < \infty$ if and only if one of the four maps $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma(R))$.*

Thus $\mathcal{M}_l(R)$ as defined in (2) includes all magnetisations with finite energy and $\lim_{x\to\pm\infty} m(x,y) = \pm\vec{e}_x$. Analogously to $\mathcal{M}_l(R)$ we define the following classes of domain walls

$$\begin{aligned} \mathcal{T}_l(R) &:= \{m \in \mathcal{M}_l(R) \mid m \text{ is constant on each cross section}\}, \\ \mathcal{V}_l(R) &:= \left\{ m \in \mathcal{M}_l(R) \;\middle|\; \begin{array}{l} m_y(x,y_1,y_2) \text{ is parallel to } (-y_2,y_1), \\ |m_y| \text{ depends only on } x \text{ and } |y| \end{array} \right\} \end{aligned}$$

and the infima of the energy in all of these classes

$$E_X(R) := \inf_{m\in X} E(m), \qquad \text{for } X = \mathcal{M}_l(R),\ \mathcal{T}_l(R),\ \mathcal{V}_l(R).$$

In fact, all infima are minima.

**Theorem 2 (Existence).** *For each radius $R > 0$ there exist minimisers of the energy $E$ in $\mathcal{M}_l(R)$, $\mathcal{T}_l(R)$ and $\mathcal{V}_l(R)$.*

The energy of the optimal wall profile scales like $E_{\mathcal{T}_l}$ when the radius goes to zero and scales like $E_{\mathcal{V}_l}$ for radius to infinity.

**Theorem 3 (Energy scaling).** *There exist constants $c$, $C$ such that*

$$\begin{aligned} &\text{for } R \le 2: & cR^2 &\le E_{\mathcal{M}_l}(R) \le E_{\mathcal{T}_l}(R) \le CR^2, \\ &\text{for } R > 2: & cR^2\sqrt{\ln(R)} &\le E_{\mathcal{M}_l}(R) \le E_{\mathcal{V}_l}(R) \le CR^2\sqrt{\ln(R)}. \end{aligned}$$

*Neither $E_{\mathcal{T}_l}$ nor $E_{\mathcal{V}_l}$ has the optimal scaling in the opposite regime: There exists a constant $\tilde{c}$ such that for all $R \in \mathbb{R}^+$ we have*

$$E_{\mathcal{T}_l}(R) \ge \tilde{c}R^{\frac{8}{3}} \quad \text{and} \quad E_{\mathcal{V}_l}(R) \ge \tilde{c}R.$$

This shows that transverse walls are energetically favourable for small radii and vortex walls are energetically favourable for big radii.

To capture the essence of the energy minimising problem for small radii, we use the notion of Γ-convergence as described in [2]. We rescale and make the following definition:

**Definition 4.** (i) After rescaling, the energy functional of the *full variational problem for* $R \in \mathbb{R}^+$ is

$$\frac{1}{R^2}E(m) \;=\; \frac{1}{R^2}\int_{\Sigma(R)} |\nabla m|^2 + |\nabla u|^2 \quad \text{where } \Delta u = \operatorname{div} m.$$

The admissible set is

$$\mathcal{M}(R) \;=\; \left\{m\colon \Sigma(R) \to \mathbb{S}^2 |\; E(m,R) < \infty\right\}.$$

For each admissible function $m \in \mathcal{M}(R)$ we set

$$\acute{m}\colon \Sigma(1) \to \mathbb{S}^2, \qquad \acute{m}\left(x, \frac{y}{R}\right) := m(x,y). \tag{3}$$

(ii) The energy functional for the *reduced variational problem* is

$$E_{\text{red}}(m) \;:=\; \pi\|\partial_x m\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2}\|m_y\|^2_{L^2(\mathbb{R})}.$$

The admissible set is

$$\mathcal{M}(0) \;=\; \{m\colon \mathbb{R} \to \mathbb{S}^2 |\; E_{\text{red}}(m) < \infty\}.$$

(iii) We use the following notion of convergence: Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers that converges to zero, let $m^n \in \mathcal{M}(R_n)$ and let $m^0 \in \mathcal{M}(0)$. We say the sequence $(m^n)_{n\in\mathbb{N}}$ converges to $m^0$ if

- $\nabla_y \acute{m}^n$ converges to 0 strongly in $L^2(\Sigma(1))$ and
- $\partial_x \acute{m}^n$ converges to $\partial_x m^0$ weakly in $L^2(\Sigma(1))$ and
- $\acute{m}^n$ converges to $m^0$ strongly in $L^2_{\text{loc}}(\overline{\Sigma}(1))$.

**Theorem 5** (**Γ-convergence**)**.** *The reduced variational problem (Definition 4 (ii)) is the* Γ*-limit of the full variational problem (Definition 4 (i)) with respect to the convergence stated in Definition 4 (iii).*

The minimiser of the reduced problem can be calculated explicitly.

**Lemma 6.** *The minimiser of* $E_{\text{red}}$ *in*

$$\mathcal{M}_l(0) \;:=\; \{m \in \mathcal{M}(0) : m - \chi \in H^1(\mathbb{R})\}$$

*is unique up to translation and rotation. It is given by*

$$m^{\text{red}}\colon \mathbb{R} \to \mathbb{S}^2, \quad x \mapsto \left(\tanh\left(\tfrac{x}{\sqrt{2}}\right),\; \frac{1}{\cosh\left(\tfrac{x}{\sqrt{2}}\right)},\; 0\right),$$

*and its energy is* $\sqrt{8}\pi$*.*

Since Γ-convergence implies convergence of the minimisers and convergence of the minimal energies we have the following asymptotic result.

**Theorem 7.** *Let $m^{\mathrm{red}}$ be as in Lemma 6. For each positive sequence $(R_n)_{n\in\mathbb{N}}$ converging to zero and each sequence of minimisers $m^n \in \mathcal{M}_l(R_n)$, the rescaled energy $\frac{1}{R_n^2}E(m^n, R_n)$ converges to $E_{\mathrm{red}}(m^{\mathrm{red}}) = \sqrt{8}\pi$. Moreover, there is a sequence of translations $T^n$ such that a subsequence of $(T^n(m_n))_{n\in\mathbb{N}}$ converges, up to a rotation, to $m^{\mathrm{red}}$ in the sense of Definition 4 (iii).*

## 2.2 Useful theorems for the calculation of the micromagnetic energy

Naming different parts of the micromagnetic energy, for $m : \Sigma(R) \to \mathbb{R}^3$ we set

$$E_{ex}(m) := \int_{\Sigma(R)} |\nabla m|^2, \qquad E_H(m) \quad := \int_{\mathbb{R}^3} |\nabla u|^2 \quad \text{where } \Delta u = \operatorname{div} m.$$

The divergence of $m$ consists of two parts: the *body charges* $\rho$ in the interior of the cylinder and the *surface charges* $\sigma$, the divergence that comes from the normal component of the magnetisation on the surface,

$$\rho(p) = \begin{cases} -\operatorname{div} m(p) & \text{if } p \in \Sigma, \\ 0 & \text{otherwise,} \end{cases} \qquad \sigma(p) = m \cdot \vec{e}_\nu \text{ for all } p \in \partial\Sigma.$$

Define $u_\rho$, $u_\sigma$ as the solutions of $\Delta u_\rho = \rho$, $\Delta u_\sigma = \sigma$ and set

$$E_{\rho\rho}(m) := \int_{\mathbb{R}^3} |\nabla u_\rho|^2, \quad E_{\sigma\sigma}(m) := \int_{\mathbb{R}^3} |\nabla u_\sigma|^2, \quad E_{\rho\sigma}(m) := \int_{\mathbb{R}^3} \nabla u_\rho \cdot \nabla u_\sigma.$$

If $m : \Sigma(R) \to \mathbb{R}^3$ is constant on each cross section, symmetry considerations imply $E_{\rho\sigma}(m) = 0$ [7, Lemma 7] and we can formulate the following Lemma:

**Lemma 8.** *If $m : \Sigma(R) \to \mathbb{R}^3$ is constant on each cross section then*

$$E_H(m) = E_{\rho\rho}(m) + E_{\sigma\sigma}(m).$$

In this case we can calculate $E(m)$ using a Fourier multiplier [7, Thm. 8].

**Theorem 9 (Estimates via Fourier multipliers).**
*(i) Let $m_y \in L^2(\Sigma(R), \{0\} \times \mathbb{R}^2)$ be a function that is constant on each cross section and let $\hat{m}_y : \mathbb{R} \to \{0\} \times \mathbb{R}^2$ be the Fourier transform of $m_y(\cdot, 0)$. Then*

$$E_{\sigma\sigma}(m_y) \;=\; R^2 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 g_F(\xi R)\, d\xi.$$

*Here $g_F$ is a positive smooth function, monotonously decreasing in $|t|$ with $g_F(0) = \frac{\pi}{2}$. Moreover, we have the relation*

$$\frac{\pi}{2} - g_F(t) \;\le\; \frac{\pi}{2} t^2 |\ln(t)| \qquad \text{for } |t| \le \frac{1}{2}. \tag{4}$$

*(ii) Let $m_x\colon \Sigma(R) \to \mathbb{R}$ be a function that is constant on each cross section with $\rho := \partial_x m_x \in L^2(\Sigma(R))$ and let $\hat\rho : \mathbb{R} \to \mathbb{R}$ be the Fourier transform of $\rho(\cdot, 0)$. Then*

$$E_{\rho\rho}(m_x \vec{e}_x) \;=\; R^4 \int_{\mathbb{R}} |\hat\rho(\xi)|^2 h_F(\xi R)\, d\xi.$$

*Here $h_F$ is a positive smooth function on $\mathbb{R} \setminus \{0\}$ with*

$$h_F(t) \le \begin{cases} \pi |\ln(|t|)| & \text{for } |t| \le \frac{1}{2}, \\ \frac{\pi}{t^2} & \text{for } |t| \ge \frac{1}{2}, \end{cases} \qquad h_F(t) \ge \frac{\pi}{2} |\ln(|t|)| \quad \text{for } |t| \le 1. \tag{5}$$

For general $m : \Sigma(R) \to \mathbb{R}^3$ we set

$$\overline{m}\colon \Sigma(R) \to \mathbb{R}^3, \quad (x, y) \mapsto \int_{D_R} m(x, y')\, dy', \tag{6}$$

$$\tilde m\colon \Sigma(R) \to \mathbb{R}^3, \quad (x, y) \mapsto m(x, y) - \overline{m}(x, y). \tag{7}$$

**Lemma 10.** *For $m : \Sigma(R) \to \mathbb{R}^3$ we have*

$$E_{ex}(\tilde m) + E_{ex}(\overline{m}) \;=\; E_{ex}(m), \tag{8}$$

$$\int_{D_R} |\overline{m}(x, y)|^2 + |\tilde m(x, y)|^2\, dy \;=\; \int_{D_R} |m(x, y)|^2\, dy \qquad \text{for all } x \in \mathbb{R}, \tag{9}$$

$$16 R^2\, \|\nabla_y \tilde m(x, \cdot)\|^2_{L^2(D_R)} \;\ge\; \|\tilde m(x, \cdot)\|^2_{L^2(D_R)} \qquad \text{for all } x \in \mathbb{R}. \tag{10}$$

*Proof.* The first two equations can be shown by direct calculation, for details see [9, Lemma 2.18]. The last equation is an instance of the Poincaré inequality [5, p. 164]. □

The following Lemma is a direct consequence of the fact that, for $m \in H^1(\Sigma)$ and $\Delta u = \operatorname{div} m$, we have $\|\nabla u\|_{L^2(\mathbb{R}^3)} \le \|m\|_{L^2(\Sigma)}$.

**Lemma 11.** *Let $f, g : \Sigma \to \mathbb{R}^3$ such that $E_H(f), E_H(g) < \infty$. Then*

$$|E_H(f) - E_H(g)| \;\le\; E_H(f - g) + 2\sqrt{E_H(f) E_H(f - g)}. \tag{11}$$

*In particular, if $\|f - g\|_{L^2(\Sigma)} < \infty$,*

$$|E_H(f) - E_H(g)| \;\le\; \|f - g\|^2_{L^2(\Sigma)} + 2\|f - g\|_{L^2(\Sigma)} \sqrt{E_H(f)}. \tag{12}$$

# 3 Convergence of minimisers in $H^1$

To show that the minimisers converge not only in the sense of Definition 4 (iii) but also in $H^1$ we have to refine the results of [7]. For magnetisations that are constant on each cross section we can use the Fourier multiplier Theorem 9 to get bounds on the energy.

**Lemma 12.** *(i) Let $m_y \in H^1(\Sigma(R), \{0\} \times \mathbb{R}^2)$ be a map that is constant on each cross section and let $R$ be so small that $-\ln(R) \ge 1$. Then*

$$0 \;\le\; \frac{1}{2} \|m_y\|^2_{L^2(\Sigma(R))} - E_{\sigma\sigma}(m_y) \;\le\; 3R^2\, |\ln(R)|\, \|m_y\|^2_{H^1(\Sigma(R))}.$$

*(ii) Let $m_x : \Sigma(R) \to \mathbb{R}$ be constant on each cross section with $\rho := \partial_x m_x \in L^2(\Sigma(R))$ and let $R$ be so small that $-\ln(R) \geq 1$. Then, for $\acute{m}_x \vec{e}_x$ as in (3) we have*

$$0 \;\leq\; E_{\rho\rho}(m_x\vec{e}_x) \;\leq\; 5R^4\,|\ln(R)|\,E(\acute{m}_x\vec{e}_x).$$

*Proof.* (i) Theorem 9 and the equality $\pi R^2\|\hat{m}_y\|^2_{L^2(\mathbb{R})} = \|m_y\|^2_{L^2(\Sigma(R))}$ directly implie the lower bound. For the upper bound, Theorem 9 yields the relation

$$\begin{aligned}
&\frac{1}{2}\|m_y\|^2_{L^2(\Sigma(R))} - E_{\sigma\sigma}(m_y)\\
&\leq \underbrace{-\frac{\pi}{2}R^2\int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{m}_y(\xi)|^2\,\ln(|\xi R|)\xi^2R^2\,d\xi}_{A} \;+\; \underbrace{\frac{\pi}{2}R^2\int_{\mathbb{R}\setminus[-\frac{1}{2R},\frac{1}{2R}]} |\hat{m}_y(\xi)|^2\,d\xi}_{B}\,.
\end{aligned}$$

Using the identity $-\ln(|\xi R|) = -\ln(|\xi|) - \ln(R)$ and the fact that $-\ln(|\xi|)$ is negative for $|\xi| > 1$ we have

$$\begin{aligned}
A \;&=\; -\frac{\pi}{2}R^2\int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{m}_y(\xi)|^2\ln(|\xi|)\xi^2R^2\,d\xi - \frac{\pi}{2}R^2\int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{m}_y(\xi)|^2\ln(R)\xi^2R^2\,d\xi\\
&\leq\; \frac{\pi}{2}R^4\int_{-1}^{1} |\hat{m}_y(\xi)|^2\,|\ln(|\xi|)|\,\xi^2\,d\xi + \frac{\pi}{2}R^4\,|\ln(R)|\int_{\mathbb{R}} |\hat{m}_y(\xi)|^2\xi^2\,d\xi\\
&\leq\; \frac{\pi}{2}R^4\|\hat{m}_y\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2}R^4\,|\ln(R)|\,\|\partial_x m_y(\cdot,0)\|^2_{L^2(\mathbb{R})}\\
&\leq\; \frac{1}{2}R^2\,|\ln(R)|\,\|m_y\|^2_{H^1(\Sigma(R))}.
\end{aligned}$$

The second summand can be bounded by

$$B \;\leq\; \frac{\pi}{2}R^2\int_{\mathbb{R}\setminus[-\frac{1}{2R},\frac{1}{2R}]} |\hat{m}_y(\xi)|^2(2\xi R)^2\,d\xi \;\leq\; 2\pi R^4\|\partial_x m_y(\cdot,0)\|^2_{L^2(\mathbb{R})},$$

so we have

$$0 \;\leq\; \frac{1}{2}\|m_y\|^2_{L^2(\Sigma(R))} - E_{\sigma\sigma}(m_y,R) \;\leq\; 3R^2\,|\ln(R)|\,\|m_y\|^2_{H^1(\Sigma(R))}.$$

(ii) To show the second statement we note that (5) implies

$$\pi\int_{-1}^{1} |\hat{\rho}(\xi)|^2\,|\ln(\xi)|\,d\xi \;\leq\; 2E_{\rho\rho}(\acute{m}_x\vec{e}_x).$$

Using (5) once more we calculate

$$E_{\rho\rho}(m_x\vec{e}_x) \;\leq\; \underbrace{\pi R^4\int_{-\frac{1}{2R}}^{\frac{1}{2R}} -|\hat{\rho}(\xi)|^2\,\ln(|\xi R|)\,d\xi}_{A} + \underbrace{\pi R^4\int_{\mathbb{R}\setminus[-\frac{1}{2R},\frac{1}{2R}]} |\hat{\rho}(\xi)|^2\frac{1}{|\xi R|^2}\,d\xi}_{B}$$

$$\begin{aligned}
A \;&\leq\; \pi R^4|\ln(R)|\int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{\rho}(\xi)|^2\,d\xi + \pi R^4\int_{-1}^{1} |\hat{\rho}(\xi)|^2\,|\ln(|\xi|)|\,d\xi\\
&\leq\; R^4|\ln(R)|\|\partial_x\acute{m}_x\|^2_{L^2(\Sigma(1))} + 2R^4E_{\rho\rho}(\acute{m}_x\vec{e}_x)\\
B \;&\leq\; \pi R^4\int_{\mathbb{R}} 4|\hat{\rho}(\xi)|^2\,d\xi \;=\; 4R^4\|\partial_x\acute{m}_x\|^2_{L^2(\Sigma(1))}.
\end{aligned}$$

Thus $E_{\rho\rho}(m_x\vec{e}_x, R) \leq 5R^4\, |\ln(R)|\, E(\acute{m}_x\vec{e}_x)$. $\square$

One property of Γ-convergence in lower semicontinuity. In our case this means: if the sequence $(R^n)_{n\in\mathbb{N}}$ converges to 0, and if $(m^n)_{n\in\mathbb{N}}, m^n \in \mathcal{M}(R_n)$ converges in the sense of Definition 4 to $m^0 \in \mathcal{M}(0)$ then $E^{\mathrm{red}}(m^0) \leq \frac{1}{R_n^2}E(m^n)$. The next Lemma refines this statement.

**Lemma 13.** *Let $m^n \in \mathcal{M}(R_n)$ with $\lim_{n\to\infty} R_n = 0$ and assume that $\frac{1}{R_n^2}E(m^n)$ is bounded by some number $C$. If $(m^n)_{n\in\mathbb{N}}$ converges in the sense of Definition 4 to $m^{\lim} \in \mathcal{M}(0)$ then we have*

$$\lim_{n\to\infty} \frac{1}{R_n^2}\left(\|\partial_x m^n\|^2_{L^2(\Sigma(R_n))} - \|\partial_x m^n - \partial_x m^{\lim}\|^2_{L^2(\Sigma(R_n))}\right) \;=\; \pi\|\partial_x m^{\lim}\|^2_{L^2(\mathbb{R})},$$

$$\lim_{n\to\infty} \frac{1}{R_n^2}\left(E_H(m^n) - \frac{1}{2}\left\|\overline{m}_y^n - m_y^{\lim}\right\|^2_{L^2(\Sigma(R))}\right)$$
$$= \lim_{n\to\infty} \frac{1}{R_n^2}\left(E_{\sigma\sigma}(\overline{m}^n) - \frac{\pi}{2}\left\|\overline{m}_y^n - m_y^{\lim}\right\|^2_{L^2(\Sigma(R))}\right) \;=\; \frac{\pi}{2}\left\|m_y^{\lim}\right\|^2_{L^2(\mathbb{R})}.$$

*Proof.* We calculate

$$\begin{aligned}
&\frac{1}{R_n^2}\|\partial_x m^n\|^2_{L^2(\Sigma(R_n))} - \frac{1}{R_n^2}\|\partial_x m^n - \partial_x m^{\lim}\|^2_{L^2(\Sigma(R_n))} \\
&= \frac{2}{R_n^2}\left(\int_{\Sigma(R_n)} \partial_x m^n \cdot \partial_x m^{\lim}\right) - \frac{1}{R_n^2}\|\partial_x m^{\lim}\|^2_{L^2(\Sigma(R_n))} \\
&= \underbrace{2\left(\int_{\Sigma(1)} (\partial_x \acute{m}^n - \partial_x \acute{m}^{\lim})\partial_x m^{\lim}\right)}_{(*)} + \pi\|\partial_x m^{\lim}\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

By assumption, $(*)$ converges to 0, so we obtain

$$\lim_{n\to\infty} \frac{1}{R_n^2}\left(\|\partial_x m^n\|^2_{L^2(\Sigma(R_n))} - \|\partial_x m^n - \partial_x m^{\lim}\|^2_{L^2(\Sigma(R_n))}\right) \;=\; \pi\|\partial_x m^{\lim}\|^2_{L^2(\mathbb{R})}.$$

We now show the second equation. The Poincaré inequality (10) yields

$$E_H(\tilde{m}) \;\leq\; \|\tilde{m}^n\|^2_{L^2(\Sigma(R_n))} \;\leq\; 16R_n^2\|\nabla_y m\|^2_{L^2(\Sigma(R_n))} \;\leq\; 16R_n^4 C.$$

Using this estimate and Lemma 11 we can calculate

$$\begin{aligned}
&\left|\frac{1}{R_n^2}E_H(m^n) - \frac{1}{R_n^2}E_H(\overline{m}^n, R_n)\right| \\
&\quad\leq \frac{1}{R_n^2}\left(E_H(\tilde{m}^n) + 2\sqrt{E_H(\overline{m}^n)}\,\sqrt{E_H(\tilde{m}^n)}\right) \\
&\quad\leq \frac{1}{R_n^2}\left(16CR_n^4 + 2\sqrt{2E_H(m^n) + 2E_H(\tilde{m}^n)}\,\sqrt{16CR_n^4}\right) \\
&\quad\leq 16CR_n^2 + 2\sqrt{2CR_n^2 + 32CR_n^4}\,4\sqrt{C}.
\end{aligned}$$

Since $E_H(\overline{m}) = E_{\sigma\sigma}(\overline{m}) + E_{\rho\rho}(\overline{m})$ (Lemma 8) and $\lim_{n\to\infty}\frac{1}{R_n^2}E_{\rho\rho}(\overline{m}) = 0$ (Lemma 12 (ii)) we have

$$\lim_{n\to\infty}\left(\frac{1}{R_n^2}E_H(m^n) - \frac{1}{R_n^2}E_{\sigma\sigma}(\overline{m}^n, R_n)\right) \;=\; 0. \tag{13}$$

To calculate $\lim_{n\to\infty} E_{\sigma\sigma}(\overline{m}^n)$ we use the Fourier multiplier of Theorem 9. We have

$$\begin{aligned} E_{\sigma\sigma}(\overline{m}^n) &= \int_{\mathbb{R}} g_F(R_n\xi)\, |\hat{\overline{m}}_y^n(\xi)|^2 \, d\xi \\ &= \underbrace{\int_{\mathbb{R}} g_F(R_n\xi) \left( |\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \right) \, d\xi}_{a_n} \\ &\quad + \underbrace{2 \int_{\mathbb{R}} g_F(R_n\xi)\, \hat{m}_y^{\lim}(\xi) \left( \hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi) \right) \, d\xi}_{b_n} . \end{aligned}$$

Here $g_F$ is a continuous function with $\frac{\pi}{2} = g_F(0) \geq g_F(t)$ for all $t \in \mathbb{R}$.

Considering the first summand $a_n$ we have, for every $t > 0$, the relation

$$\begin{aligned} 0 &\leq \frac{\pi}{2} \|\hat{m}^{\lim}\|_{L^2(\mathbb{R})}^2 + \liminf_{n\to\infty} \left( \frac{\pi}{2} \|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 - a_n \right) \\ &\leq \frac{\pi}{2} \|\hat{m}^{\lim}\|_{L^2(\mathbb{R})}^2 + \limsup_{n\to\infty} \left( \frac{\pi}{2} \|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 - a_n \right) \\ &\leq \limsup_{n\to\infty} \underbrace{\int_{-t}^{t} (g_F(0) - g_F(R_n\xi)) \left( |\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \right) \, d\xi}_{a_{n,1}} \\ &\quad + \limsup_{n\to\infty} \underbrace{\frac{\pi}{2} \int_{\mathbb{R}\setminus[-t,t]} |\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \, d\xi}_{a_{n,2}} . \end{aligned}$$

Because of (13) and the relation $\|\partial_x \overline{m}_y^n\|_{L^2(\Sigma(R_n))}^2 \leq C R_n^2$ the term

$$\frac{1}{R_n^2} \left( E_{\sigma\sigma}(\overline{m}_y^n) + \|\partial_x \overline{m}_y^n\|_{L^2(\Sigma(R_n))}^2 \right)$$

is uniformly bounded. Therefore, with [7, Lemma 10], $\|\hat{\overline{m}}_y^n\|_{L^2(\mathbb{R})}$ is uniformly bounded. This implies for every $t > 0$

$$\limsup_{n\to\infty} a_{n,1} \;\leq\; \limsup_{n\to\infty} \, (g_F(0) - g_F(R_n t)) \left( 3\|\hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 + 2\|\hat{\overline{m}}_y^n\|_{L^2(\mathbb{R})}^2 \right) \;=\; 0.$$

Regarding $a_{n,2}$, we have the relation

$$\begin{aligned} a_{n,2} &\leq \frac{\pi}{2} \int_{\mathbb{R}\setminus[-t,t]} \frac{\xi^2}{t^2} \left( |\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \right) \\ &\leq \frac{\pi}{2} \int_{\mathbb{R}} \left( \frac{3\xi^2}{t^2} |\hat{m}^{\lim}(\xi)|^2 + \frac{2\xi^2}{t^2} |\hat{\overline{m}}^n(\xi)|^2 \right) \\ &= \frac{3\pi}{2t^2} \|\partial_x m^{\lim}\|_{L^2(\mathbb{R})}^2 + \frac{1}{t^2 R^2} \|\partial_x \overline{m}^n\|_{L^2(\Sigma(R))}^2 \;\leq\; \frac{3}{t^2} C. \end{aligned}$$

Since $t$ was arbitrary we obtain

$$\lim_{n\to\infty} \left( a_n - \frac{\pi}{2} \|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 \right) \;=\; \frac{\pi}{2} \|\hat{m}^{\lim}\|_{L^2(\mathbb{R})}^2 .$$

We now show that the summand $b_n$ converges to zero. Since $\|\hat{\overline{m}}_y^n\|_{L^2(\mathbb{R})}$ is bounded, the sequence $(f_n)_{n\in\mathbb{N}}$,

$$f_n\colon \mathbb{R} \to \mathbb{R}, \qquad \xi \mapsto g_F(\xi R_n)(\hat{\overline{m}}^n(\xi) - \hat{m}^{\lim}(\xi))$$

converges weakly, up to a subsequence. Since $\lim_{n\to\infty} \overline{m}_y^n(\cdot,0) - m^{\lim} = 0$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, the only possible limit of the sequence $(f_n)_{n\in\mathbb{N}}$ is 0. In particular, this implies $\lim_{n\to\infty} b_n = 0$. Therefore we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{R_n^2}\left(E_{\sigma\sigma}(\overline{m}^n) - \|\overline{m}_y^n - m_y^{\lim}\|^2_{L^2(\Sigma(R_n))}\right) \\
&\quad = \lim_{n\to\infty}\left(a_n - \left\|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\right\|^2_{L^2(\mathbb{R})}\right) \;=\; \frac{\pi}{2}\left\|m_y^{\lim}\right\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

as claimed. □

Together with the convergence result for minimal energies of Theorem 7 this lemma implies that minimisers converge in $H^1$ up to a subsequence.

**Lemma 14.** *Let $m^{\mathrm{red}}$ be as in Lemma 6, let $(R_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let $m^n \in \mathcal{M}_l(R_n)$ be a sequence of minimisers converging to $m^{\mathrm{red}}$ in the sense of Definition 4 (iii). Then*

$$\lim_{n\to\infty} \frac{1}{R_n}\|m^n - m^{\mathrm{red}}\|_{H^1(\Sigma(R))} \;=\; 0.$$

*Proof.* Lemma 13 implies

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{R_n^2} E(m^n) = E_{\mathrm{red}}(m^{\mathrm{red}}) + \lim_{n\to\infty} \frac{1}{R_n^2}\Big(&\|\overline{m}_y^n - m_y^{\mathrm{red}}\|^2_{L^2(\Sigma(R_n))} + \\
&\|\partial_x m^n - \partial_x m^{\mathrm{red}}\|^2_{L^2(\Sigma(R_n))} + \|\nabla_y m^n\|^2_{L^2(\Sigma(R))}\Big)
\end{aligned}$$

Since by Theorem 7 $\lim_{R_n^2\to 0} E(m^n) = E_{\mathrm{red}} = \sqrt{8}\pi$ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{R_n}\|\overline{m}_y^n - m_y^{\mathrm{red}}\|_{L^2(\Sigma(R_n))} \;&=\; 0, \\
\lim_{n\to\infty} \frac{1}{R_n}\|\partial_x m^n - \partial_x m^{\mathrm{red}}\|_{L^2(\Sigma(R_n))} \;&=\; 0, \\
\lim_{n\to\infty} \frac{1}{R_n}\|\nabla_y m^n\|_{L^2(\Sigma(R_n))} \;&=\; 0. \qquad (14)
\end{aligned}$$

Combining (14) and the Poincaré inequality (10) implies

$$\lim_{n\to\infty} \frac{1}{R_n}\|\tilde{m}^n\|_{L^2(\Sigma(R_n))} \;=\; 0.$$

Fix $x_0$ such that $|m_x^{\mathrm{red}}(x)| > \frac{1}{2}$ for $x \in \mathbb{R}\setminus[-x_0,x_0]$. By assumption we have

$$\lim_{n\to\infty} \frac{1}{R_n}\|m^n - m^{\mathrm{red}}\|_{L^2([-x_0,x_0]\times D_{R_n})} \;=\; 0.$$

Thus it remains to show

$$\lim_{n\to\infty} \frac{1}{R_n}\|\overline{m}_x^n - m_x^{\mathrm{red}}\|_{L^2(\Sigma(R_n)\setminus[-x_0,x_0]\times D_{R_n})} \;=\; 0.$$

Since

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{R_n^2} \int_{\Sigma(R_n)} \left| |m_y^n|^2 - |m_y^{\mathrm{red}}|^2 \right| \\
\leq\ & \lim_{n\to\infty} \frac{1}{R_n^2} \left\| m_y^n - m_y^{\mathrm{red}} \right\|_{L^2(\Sigma(R_n))} \left( \left\| m_y^n - m_y^{\mathrm{red}} \right\|_{L^2(\Sigma(R_n))} + 2 \left\| m_y^{\mathrm{red}} \right\|_{L^2(\Sigma(R_n))} \right) \\
=\ & 0
\end{aligned}
$$

and

$$
\begin{aligned}
\left| |m_y^n|^2 - |m_y^{\mathrm{red}}|^2 \right| &= \left| 1 - |m_x^n|^2 - (1 - |m_x^{\mathrm{red}}|^2) \right| \\
&= \left| -|\tilde{m}_x^n|^2 - |\overline{m}_x^n|^2 + |m_x^{\mathrm{red}}|^2 \right| \\
&\geq -|\tilde{m}_x^n|^2 + \left| (|\overline{m}_x^n| + |m_x^{\mathrm{red}}|)(|\overline{m}_x^n| - |m_x^{\mathrm{red}}|) \right| \\
&\geq -|\tilde{m}_x^n|^2 + (|\overline{m}_x^n| - |m_x^{\mathrm{red}}|)^2,
\end{aligned}
$$

the functions $|\overline{m}_x^n|$ converge in $L^2(\mathbb{R})$ to $|m_x^{\mathrm{red}}|$.

Now (14) implies that they converge also in $H^1(\mathbb{R})$, and with the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ they converge in $L^\infty(\mathbb{R})$. In particular, there is $n_0$ such that for all $n > n_0$ we have $\left| |\overline{m}_x^n| - |m_x^{\mathrm{red}}| \right| < \frac{1}{2}$. This implies that the functions $\mathrm{sign}(\overline{m}_x^n)|_{]-\infty,-x_0]}$ and $\mathrm{sign}(\overline{m}_x^n)|_{[x_0,\infty[}$ are constant. Now the fact that $m^n$ converges in the sense of Definition 4 (iii) to $m^{\mathrm{red}}$ implies

$$
\mathrm{sign}(\overline{m}_x^n)|_{\mathbb{R}\setminus]-x_0,x_0[} \;=\; \mathrm{sign}(m_x^{\mathrm{red}})|_{\mathbb{R}\setminus]-x_0,x_0[}
$$

and we have

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{R_n} \|\overline{m}_x^n - m_x^{\mathrm{red}}\|_{L^2(\Sigma(R_n)\setminus[-x_0,x_0]\times D_{R_n})} \\
&\quad= \lim_{n\to\infty} \frac{1}{R_n} \||\overline{m}_x^n| - |m_x^{\mathrm{red}}|\|_{L^2(\Sigma(R_n)\setminus[-x_0,x_0]\times D_{R_n})} \;=\; 0.
\end{aligned}
$$

□

# 4 The rate of convergence of the minimal energies

Theorem 7 states that

$$
\lim_{R\to 0} \frac{1}{R^2} E_{\mathcal{M}_l}(R) - \sqrt{8}\,\pi \;=\; 0.
$$

In this section we prove an upper bound on the difference $|\frac{1}{R^2} E_{\mathcal{M}_l}(R) - \sqrt{8}\pi|$ in terms of $R$. Let $m^R$ be a minimiser of $E$ in $\mathcal{M}_l(R)$. In the first lemma we consider $\|\tilde{m}^R\|_{L^2(\Sigma)}$. We already know from Lemma 14 that

$$
\lim_{R\to 0} \frac{1}{R^2} \|\tilde{m}^R\|_{L^2(\Sigma(R))} \;\leq\; \lim_{R\to 0} \frac{4}{R} \|\nabla_y \tilde{m}^R\|_{L^2(\Sigma(R))} \;=\; 0.
$$

Using the fact that $m^R$ is a minimiser, we can improve this estimate.

**Lemma 15.** *There exist positive constants $R_0$, $C$ such that for all $R \le R_0$*

$$\left\| \frac{\overline{m}^R}{|\overline{m}^R|} - \overline{m}^R \right\|_{L^2(\Sigma(R))} \le \left\|\tilde{m}^R\right\|_{L^2(\Sigma(R))}, \tag{15}$$

$$\left\| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right\|^2_{L^2(\Sigma(R))} - \left\|\partial_x \overline{m}^R\right\|^2_{L^2(\Sigma(R))} \le \frac{1}{4}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma(R))}, \tag{16}$$

$$E\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right) - E(m^R) \le 64R^2\left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma(R))} - \frac{1}{2}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma(R))}, \tag{17}$$

$$\left\|\tilde{m}^R\right\|_{L^2(\Sigma(R))} \le CR^3, \qquad \left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma(R))} \le CR^2. \tag{18}$$

*Proof.* Let $R_0$ be so small that for all $R \le R_0$ the following inequalities hold:

$$\inf_{x\in\mathbb{R}} |\overline{m}^R(x)| \ge \frac{1}{2}, \quad \frac{48}{R}E(m^R) \le \frac{1}{4}, \quad 64R^2 \le \frac{1}{4}, \quad E(m^R) \le 16R^2.$$

Estimate (15) is the result of the following calculation. In the last step we use (9).

$$\left\| \frac{\overline{m}^R}{|\overline{m}^R|} - \overline{m}^R \right\|^2_{L^2(\Sigma(R))} = \int_{\Sigma(R)} \left(1-|\overline{m}^R|\right)^2 \le \int_{\Sigma(R)} 1-\left|\overline{m}^R\right|^2 = \left\|\tilde{m}^R\right\|^2_{L^2(\Sigma(R))}.$$

To prove (16) we first show that $1-|\overline{m}^R|^2$ is small, using (9) and the Poincaré inequality (10):

$$\begin{aligned}
1-\left|\overline{m}^R\right|^2 &= \frac{1}{|D_R|}\int_{D_R}\left|\tilde{m}^R\right|^2 \le \frac{1}{|D_R|}\int_{\mathbb{R}}\left|\partial_x\int_{D_R}\left|\tilde{m}^R\right|^2\right| \\
&\le \frac{2}{|D_R|}\left\|\tilde{m}^R\right\|_{L^2(\Sigma(R))}\left\|\partial_x\tilde{m}^R\right\|_{L^2(\Sigma(R))} \\
&\le \frac{8R}{|D_R|}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma(R))} \le \frac{3}{R}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma(R))}.
\end{aligned} \tag{19}$$

Moreover we have

$$\partial_x\overline{m}^R = \partial_x\left(\left|\overline{m}^R\right|\frac{\overline{m}^R}{|\overline{m}^R|}\right) = \left|\overline{m}^R\right|\,\partial_x\frac{\overline{m}^R}{|\overline{m}^R|} + \left(\partial_x|\overline{m}^R|\right)\frac{\overline{m}^R}{|\overline{m}^R|},$$

and since $\partial_x\frac{\overline{m}^R}{|\overline{m}^R|}$ is perpendicular to $\frac{\overline{m}^R}{|\overline{m}^R|}$ this yields

$$\begin{aligned}
&\left\|\partial_x\frac{\overline{m}^R}{|\overline{m}^R|}\right\|^2_{L^2(\Sigma(R))} - \left\|\partial_x\overline{m}^R\right\|^2_{L^2(\Sigma(R))} \le \int_{\Sigma(R)}(1-|\overline{m}^R|^2)\left|\partial_x\frac{\overline{m}^R}{|\overline{m}^R|}\right|^2 \\
&= \int_{\Sigma(R)}(1-|\overline{m}^R|^2)\left|\frac{\partial_x\overline{m}^R}{|\overline{m}^R|} - \frac{\partial_x|\overline{m}^R|\,\overline{m}^R}{|\overline{m}^R|^2}\right|^2 \le 4\int_{\Sigma(R)}(1-|\overline{m}^R|^2)\left|\frac{\partial_x\overline{m}^R}{|\overline{m}^R|}\right|^2.
\end{aligned}$$

Using (19), the assumption $|\overline{m}| \ge \frac{1}{2}$, and the bound $\frac{48}{R}E(m^R) \le \frac{1}{4}$, we get

$$\begin{aligned}
&\left\|\partial_x\frac{\overline{m}^R}{|\overline{m}^R|}\right\|^2_{L^2(\Sigma(R))} - \left\|\partial_x\overline{m}^R\right\|^2_{L^2(\Sigma(R))} \le \frac{48}{R}\|\nabla\tilde{m}^R\|^2_{L^2(\Sigma(R))}\int_{\Sigma(R)}|\partial_x\overline{m}^R|^2 \\
&\le \frac{48}{R}\|\nabla\tilde{m}^R\|^2_{L^2(\Sigma(R))}E(m^R) \le \frac{1}{4}\|\nabla\tilde{m}^R\|^2_{L^2(\Sigma(R))}.
\end{aligned}$$

We consider (17). With (8) we have

$$
\begin{aligned}
&E\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right) - E(m^R) \\
&= \underbrace{\left\|\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}\right\|^2_{L^2(\Sigma(R))} - \left\|\partial_x \overline{m}^R\right\|^2_{L^2(\Sigma(R))} - E_{ex}(\tilde{m}^R)}_{A} + \underbrace{E_H\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right) - E_H(m^R)}_{B}.
\end{aligned}
$$

For the first summand we have

$$
A \;\le\; -\frac{3}{4}\|\nabla \tilde{m}\|^2_{L^2(\Sigma(R))}.
$$

For the second summand we use (12), (15), (10) and the assumptions $64R^2 \le \frac{1}{4}$, $E(m^R) \le 16R^2$. We calculate

$$
\begin{aligned}
B \;&\le\; \left\|\frac{\overline{m}^R}{|\overline{m}^R|} - m^R\right\|^2_{L^2(\Sigma(R))} + 2\sqrt{E_H(m^R)}\left\|\frac{\overline{m}^R}{|\overline{m}^R|} - m^R\right\|_{L^2(\Sigma(R))} \\
&\le\; 4\left\|\tilde{m}^R\right\|^2_{L^2(\Sigma(R))} + 4\sqrt{E_H(m^R)}\left\|\tilde{m}^R\right\|_{L^2(\Sigma(R))} \\
&\le\; 64R^2\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma(R))} + 16R\sqrt{E(m^R)}\left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma(R))} \\
&\le\; \frac{1}{4}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma(R))} + 64R^2\left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma(R))}.
\end{aligned}
$$

Adding the summands yields (17).

Now the fact that $m^R$ is the minimiser of $E(m^R)$ in $\mathcal{M}_l(R)$ implies (18): We have

$$
0 \;\le\; E\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right) - E(m^R),
$$

and thus $\|\nabla\tilde{m}^R\|_{L^2(\Sigma(R))} \le 128R^2$. The bound on $\|\tilde{m}^R\|_{L^2(\Sigma(R))}$ follows from the Poincaré inequality (10). □

We now determine an upper bound on the rate of convergence of the minimal energies $E_{\mathcal{M}_l}$.

**Theorem 16.** *There exists $C, R_0 > 0$ such that for all $R \le R_0$ we have*

$$
\left|\frac{1}{R^2}E_{\mathcal{M}_l}(R) - \sqrt{8}\,\pi\right| \;\le\; CR^2|\ln(R)|.
$$

*Proof.* Let $m^{\mathrm{red}}$ be as in Lemma 6 and let $R \le R_0$ where $R_0$ as in Lemma 15. By definition $E\left(m^R\right) \le E\left(m\right)$ for all $m \in \mathcal{M}_l(R)$, so in particular $E\left(m^R\right) \le E\left(m^{\mathrm{red}}\mathbb{1}_{\Sigma(R)}\right)$. Moreover, since $E_{\mathrm{red}}\left(m^{\mathrm{red}}\right) = \sqrt{8}\,\pi$ (Lemma 6) we have

$$
\begin{aligned}
\frac{1}{R^2}E\left(m^R\right) - \sqrt{8}\,\pi \;&\le\; \frac{1}{R^2}E\left(m^{\mathrm{red}}\mathbb{1}_{\Sigma(R)}\right) - E_{\mathrm{red}}\left(m^{\mathrm{red}}\right) \\
&= \frac{1}{R^2}E_{\rho\rho}\left(m^{\mathrm{red}}\mathbb{1}_{\Sigma(R)}\right) + \frac{1}{R^2}E_{\sigma\sigma}\left(m^{\mathrm{red}}\mathbb{1}_{\Sigma(R)}\right) - \frac{\pi}{2}\|m^{\mathrm{red}}\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

So Lemma 12 implies

$$\frac{1}{R^2}E\left(m^R\right)-\sqrt{8}\,\pi \;\le\; \frac{1}{R^2}E_{\rho\rho}\left(m^{\mathrm{red}}\mathbb{1}_{\Sigma(R)}\right) \;\le\; 5R^2\,|\ln(|R|)|\;E_{\rho\rho}(m^{\mathrm{red}}\mathbb{1}_{\Sigma(1)}).$$

On the other hand we can use (17) and Lemma 12 (i) to get the estimate

$$\begin{aligned}
&\sqrt{8}\,\pi-\frac{1}{R^2}E\left(m^R\right)\\
&\le\; E_{\mathrm{red}}\left(\frac{\overline{m}^R(\cdot,0)}{|\overline{m}^R(\cdot,0)|}\right)-\frac{1}{R^2}E\left(m^R\right)\\
&=\; E_{\mathrm{red}}\left(\frac{\overline{m}^R(\cdot,0)}{|\overline{m}^R(\cdot,0)|}\right)-\frac{1}{R^2}E\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right)+\frac{1}{R^2}E\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right)-\frac{1}{R^2}E\left(m^R\right)\\
&\le\; \frac{\pi}{2}\left\|\frac{\overline{m}^R(\cdot,0)}{|\overline{m}^R(\cdot,0)|}\right\|^2_{L^2(\mathbb{R})}-\frac{1}{R^2}E_{\sigma\sigma}\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right)+64\|\nabla\tilde{m}^R\|_{L^2(\Sigma(R))}\\
&\le\; 3\pi R^2\,|\ln(R)|\,\left\|\frac{\overline{m}^R_y(\cdot,0)}{|\overline{m}^R(\cdot,0)|}\right\|^2_{H^1(\mathbb{R})}+64\left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma(R))}.
\end{aligned}$$

Since (15) and (16) imply that $\left\|\frac{\overline{m}^R_y(\cdot,0)}{|\overline{m}^R(\cdot,0)|}\right\|_{H^1(\mathbb{R})}$ is uniformly bounded, with (18) we see that there exists $C>0$ such that

$$\sqrt{8}\,\pi-\frac{1}{R^2}E\left(m^R\right)\;\le\; CR^2\,|\ln(R)|.$$

□

# 5 Introduction to regularity

In this section we give a short introduction to the ideas and definitions that we are using to prove regularity results. First we extend functions defined on $\Sigma(R)$ to a larger domain and discuss properties of this extension. Then we show how scaled $L^2$-estimates can be used to prove regularity results.

In the rest of this paper we will sometimes denote $\vec{e}_x$, $\vec{e}_{y_1}$, $\vec{e}_{y_2}$ by $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ and the derivatives $\partial_x$, $\partial_{y_1}$, $\partial_{y_2}$ by $\partial_1$, $\partial_2$, $\partial_3$. We will do this in order to write sums like $\sum_i \partial_i f\vec{e}_i$ in a compact way. Moreover, $Df$ denotes the derivative of a function $f$, and $D^n f$ denotes the $n^{th}$ derivative of $f$.

We will frequently use the following well known estimates.

**Lemma 17.** *(i) For $u\in H^1(B_\rho,R^n)$, $n\in\mathbb{N}$ we have*

$$\left\|u-\langle u\rangle_{B_\rho}\right\|_{L^2(B_\rho)}\;\le\;8\rho\|\nabla u\|_{L^2(B_\rho)}. \tag{20}$$

*(ii) For $u\in H^1(B_\rho,R^n)$, $n\in\mathbb{N}$ and $v\in R^n$ we have*

$$\int_{B_\rho}\left|u-\langle u\rangle_{B_\rho}\right|^2\;\le\;\int_{B_\rho}|u-v|^2\,. \tag{21}$$

## 5.1 Extending functions

We need uniform bounds on $m^R$ not only in the interior of $\Sigma(R)$ but up to the boundary. There are two strategies to get such estimates. Either we argue first in the interior and then locally at the boundary, or we extend the functions globally to a function on a larger domain and work in the interior of the large domain. The first strategy is more flexible since it works in arbitrary sufficiently regular domains. It is applied by Hardt and Kinderlehrer [6]. However, since we are considering only cylinders, it is simpler to extend functions $f\colon \Sigma(R) \to \mathbb{R}$ by reflection on the boundary to functions $f^*\colon \Sigma(\frac{3}{2}R) \to \mathbb{R}$, even if this means that we have to generalise the notion of almost-minimisers (see Definition 18 and the remark following it).

For $x, y \in \Sigma(\frac{3}{2}R)$, $\Omega \subset \Sigma(\frac{3}{2}R)$ we set

$$(x,y)^* := \begin{cases} (x,y) & \text{if } |y| \le R \\ \Big(x, (2R - |y|)\frac{y}{|y|}\Big) & \text{otherwise,} \end{cases} \qquad \Omega^* := \{p^* \mid p \in \Omega\}, \tag{22}$$

and for $f\colon \Sigma \to \mathbb{R}^n$ we define

$$f^*\colon \Sigma(\tfrac{3}{2}R) \to \mathbb{R}^n, \quad p \mapsto f(p^*).$$

If $f$ is continuous, $f^*$ is continuous and $f \in W^{1,q}(\Sigma)$ implies $f^* \in W^{1,q}(\frac{3}{2}\Sigma)$ for all $1 \le p \le \infty$. Moreover, if $\partial_\nu f = 0$ on $\partial\Sigma$ and $f \in W^{k,q}(\Sigma)$ for $k \in \{2,3\}, 1 \le p \le \infty$, then $f^* \in W^{k,q}(\frac{3}{2}\Sigma)$.

We now discuss the relation between derivatives and integrals of $f$ and derivatives and integrals of $f^*$. Setting

$$\kappa\colon \Sigma(\tfrac{3}{2}R) \to \mathbb{R}, \qquad (x,y) \mapsto \frac{|y|}{|y^*|},$$

we have for $\Omega \subset \Sigma(\frac{3}{2}R) \setminus \Sigma(R)$ the relation

$$\int_\Omega \frac{1}{\kappa} f^* = \int_{\Omega^*} f. \tag{23}$$

For the cylindrical coordinate system $(x, r, \phi)$, let $\vec{e}_x$, $\vec{e}_r = \vec{e}_r(p)$, $\vec{e}_\phi = \vec{e}_\phi(p)$ be the canonical unit vectors. We define the matrix-valued function $A^R\colon \Sigma(\frac{3}{2}R) \setminus (\mathbb{R} \times \{0\}) \to \mathbb{R}^{3\times 3}$, setting

$$A^R(p)\vec{e}_x = \vec{e}_x, \quad A^R(p)\vec{e}_r = \vec{e}_r, \quad A^R(p)\vec{e}_\phi = \kappa\vec{e}_\phi.$$

Then $A^R$ is symmetric, and in the cartesian coordinate system we have

$$A^R(p)\, \nabla f^*(p) \;=\; \nabla f(p^*), \tag{24}$$

We derive an elliptic equation in divergence form for $f^*$. If $f^*$ is regular enough, using (23) and (24) for every $\phi \in C_c^\infty(\Sigma(R))$ we have

$$\begin{aligned} \int\limits_{\Sigma\left(\frac{3}{2}R\right)\setminus\Sigma(R)} \operatorname{div}\left(\frac{1}{\kappa}\left(A^R\right)^2 \nabla f^*\right)\, \phi^* &= -\int\limits_{\Sigma\left(\frac{3}{2}R\right)\setminus\Sigma(R)} \frac{1}{\kappa}\left(A^R\nabla f^*\right)\cdot\left(A^R\nabla\phi^*\right) \\ &= -\int_{\Sigma(R)} \nabla f \cdot \nabla\phi \;=\; \int_{\Sigma(R)} \Delta f\, \phi, \end{aligned}$$

so (23) yields for $p \in \Sigma(\frac{3}{2}R) \setminus \overline{\Sigma}(R)$ the equation

$$\operatorname{div}\left(\frac{1}{\kappa}\left(A^R(p)\right)^2 \nabla f^*(p)\right) \;=\; \frac{1}{\kappa(p)}\Delta f(p^*). \tag{25}$$

In order to keep the notation simple we will write $m^R\colon \Sigma(\frac{3}{2}R) \to \mathbb{S}^2$ for the continuation of $m^R$ whose correct name would be $(m^R)^*$. This implies that we have to write the energy of $m^R : \Sigma(R) \to \mathbb{S}^2$ as $E(m^R \mathbb{1}_{\Sigma(R)})$

The highest order term of the energy $E$ with respect to the derivatives is the exchange energy $\int_\Sigma |\nabla m|^2$. The stray field energy can be considered as a lower order perturbation. We will use this fact to show $C^{0,\alpha}$ regularity using the notion of almost-minimisers.

**Definition 18.** Let $\Omega \subset \mathbb{R}^3$, $c \geq 0$, $0 < \alpha < 1$ and $A\colon \Omega \to \mathbb{R}^{3\times 3}$. Moreover, assume that $A(p)$ is symmetric for all $p \in \Omega$ and that there exist positive constants $\underline{\lambda}$, $\overline{\lambda}$ with $|\underline{\lambda} f|^2 \leq |Af|^2 \leq |\overline{\lambda} f|^2$ for all $f\colon \Omega \to \mathbb{R}^3$.

A function $m\colon \Omega \to \mathbb{S}^2$ is called $(c,\alpha)$-*almost-minimiser for* $|A\nabla \cdot|^2$ if for every $\Omega' \subset \Omega$ with $|\Omega'| \leq 1$ and every map $g\colon \Omega \to \mathbb{S}^2$ with $g|_{\Omega\setminus\Omega'} = m|_{\Omega\setminus\Omega'}$ we have

$$\int_{\Omega'} |A\nabla m|^2 \;\leq\; \left(\int_{\Omega'} |A\nabla g|^2\right) + c|\Omega'|^{\frac{1+\alpha}{3}}.$$

*Remark.* Hardt and Kinderlehrer [6] use the notion of $(c,\alpha)$-almost-minimisers only for $(c,\alpha)$-almost-minimisers of $|\nabla \cdot|^2$.

**Theorem 19.** *(i) The function $m^R\colon \Sigma(\frac{3}{2}R) \to \mathbb{S}^2$ is a $(c,\alpha)$-almost-minimiser for $|\frac{1}{\sqrt{\kappa}}A^R\nabla\cdot|^2$ with $c = 4\left(1+\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}\right)$ and $\alpha = \frac{1}{2}$.*

*(ii) Let $u$ be a weak solution of $\Delta u = \operatorname{div}(m^R\mathbb{1}_{\Sigma(R)})$ in $\mathbb{R}^3$ and set*

$$\zeta\colon \Sigma\left(\tfrac{3}{2}R\right) \to \mathbb{R}^3, \qquad p \mapsto \frac{1}{\kappa(p)}\left(\nabla u(p^*) - \left(\nabla u(p^*)\cdot m^R(p)\right) m^R(p)\right).$$

*Then $m^R$ is a weak solution of*

$$-\operatorname{div}\left(\frac{1}{\kappa}\left(A^R\right)^2 \nabla m^R\right) \;=\; \frac{1}{\kappa}\,|A^R\nabla m^R|^2 m^R - \zeta \quad \textit{in } \Sigma(\tfrac{3}{2}R). \tag{26}$$

*Proof.* (i) First assume $\Omega \subset \Sigma$, $|\Omega| \leq 1$ and let $v\colon \Sigma \to \mathbb{S}^2$ be a map with $v|_{\Sigma\setminus\Omega} = m|_{\Sigma\setminus\Omega}$. Then $E(m^R\mathbb{1}_{\Sigma(R)}) \leq E(v)$ and with (12) we have

$$\begin{aligned}
\int_\Sigma |\nabla m^R|^2 - \int_\Sigma |\nabla v|^2 \;&\leq\; E_H(v) - E_H(m^R)\\
&\leq\; \|m^R - v\|^2_{L^2(\Omega)} + 2\|m^R - v\|_{L^2(\Omega)}\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}\\
&\leq\; 4|\Omega| + 4\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}\sqrt{|\Omega|}\\
&\leq\; 4\left(1+\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}\right)|\Omega|^{\frac{1+0.5}{3}}.
\end{aligned}$$

Now consider arbitrary $\Omega \subset \Sigma(\frac{3}{2}R)$ with $|\Omega| \leq 1$. We set

$$v_*\colon \Sigma(R)\setminus\Sigma\left(\tfrac{1}{2}R\right) \to \mathbb{S}^2, \quad (x,y) \mapsto v\left(x, \tfrac{y}{|y|}(2R - |y|)\right).$$

Then, with (23) and (24) we have

$$\int_{\Omega\setminus\Sigma} \frac{1}{\kappa}|A^R\nabla v|^2 \;=\; \int_{(\Omega\setminus\Sigma)^*} |\nabla v_*|^2,$$

and therefore

$$\begin{aligned}
&\int_\Omega \left|\frac{1}{\sqrt{\kappa}}A^R\nabla m^R\right|^2 - \left|\frac{1}{\sqrt{\kappa}}A^R\nabla v\right|^2 \\
&= \int_{\Omega\cap\Sigma} |\nabla m^R|^2 - |\nabla v|^2 + \int_{(\Omega\setminus\Sigma)^*} |\nabla m^R|^2 - |\nabla v_*|^2 \\
&\le 4\left(1+\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}\right)\left(|\Omega\cap\Sigma|^{\frac{1+0.5}{3}} + |(\Omega\setminus\Sigma)^*|^{\frac{1+0.5}{3}}\right) \\
&\le 4\left(1+\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}\right)\;|\Omega|^{\frac{1+0.5}{3}}.
\end{aligned}$$

(ii) The function $m^R\mathbb{1}_{\Sigma(R)}$ is a local minimiser of $E$ with the constraint $|m|\equiv 1$, so it satisfies the Euler-Lagrange equation

$$0 \;=\; \delta_m E(m^R\mathbb{1}_{\Sigma(R)}) - (\delta_m E(m^R\mathbb{1}_{\Sigma(R)})\cdot m^R)m^R \quad \text{in } \Sigma(R), \tag{27}$$
$$\text{where } \delta_m E(m^R\mathbb{1}_{\Sigma(R)}) \;=\; -2\Delta m^R + 2\nabla u.$$

Since $|m^R| = 1$, we have $\partial_i m^R \perp m^R$ for all $i\in\{1,2,3\}$, and therefore

$$0 \;=\; \sum_i \partial_i(\partial_i m^R\cdot m^R) = \Delta m^R\cdot m^R + |\nabla m^R|^2, \quad \text{i.e.,}\; -\Delta m^R\cdot m^R = |\nabla m^R|^2.$$

With this identity (27) becomes

$$0 \;=\; -\Delta m^R + \nabla u - |\nabla m^R|^2 m^R - (\nabla u\cdot m^R)m^R \quad \text{in } \Sigma(R). \tag{28}$$

Using (24) and (25), we have for all $p\in\Sigma(\frac{3}{2}R)$ the relation

$$\begin{aligned}
&-\operatorname{div}\left(\frac{1}{\kappa(p)}\left(A^R(p)\right)^2\;\nabla m^R(p)\right) \;=\; -\frac{1}{\kappa(p)}\Delta m(p^*) \\
&= \frac{1}{\kappa(p)}\left(-\nabla u(p^*) + |\nabla m^R(p^*)|^2 m^R(p^*) + (\nabla u(p^*)\cdot m^R(p^*))m^R(p^*)\right) \\
&= \frac{1}{\kappa(p)}\;|A^R(p)\nabla m^R(p)|^2 m^R(p) - \zeta(p).
\end{aligned}$$

Remembering that $\nabla m^R \in L^2(\Sigma(\frac{3}{2}R))$, we have for every test function $\eta\in C_c^\infty(\Sigma(\frac{3}{2}))$ the equality

$$\int_{\Sigma(\frac{3}{2}R)} \frac{1}{\kappa}\left(\left(A^R\right)^2\nabla m^R\right)\cdot\nabla\eta \;=\; \int_{\Sigma(\frac{3}{2}R)}\left(\frac{1}{\kappa}\;|A^R\nabla m^R|^2 m^R - \zeta\right)\eta,$$

that is, $m^R$ is a weak solution of (26). □

## 5.2 Preliminary lemmas regarding scaled $L^2$-estimates

All considerations regarding regularity are in the spirit of Morrey-Campanato theory. The idea is to replace $L^p$- and $C^{k,\alpha}$-estimates by scaled $L^2$-estimates. For example, there exists an integral characterisation of Hölder continuous functions [4, Thm 1.2, p. 70]. The bound on the $C^{0,\alpha}$-norm does depend on the domain of the function, but by a simple rescaling argument we get a version of the estimate, that still depends on the shape of the domain but not on the size.

**Theorem 20.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with piecewise smooth boundary. Let $0 < \eta \leq 1$ and let $f\colon \eta\Omega \to \mathbb{R}^n$, $n \in \mathbb{N}$ such that*

$$\int_{B_r(a)\cap\eta\Omega} |f - \langle f\rangle_{B_r(a)\cap\eta\Omega}|^2 \;\leq\; C_f r^{3+\alpha}$$

*for all $B_r(a)$ with $r \leq \eta\,\mathrm{diam}(\Omega)$. Then we have for all $p, p' \in \eta\Omega$ the relation*

$$|f(p) - f(p')| \;\leq\; C_{\mathrm{Camp}} C_f |p - p'|^{\frac{\alpha}{2}},$$

*where $C_{\mathrm{Camp}} = C_{\mathrm{Camp}}(\Omega, \alpha)$.*

To get the integral estimates needed in Theorem 20 we often compare the integral over an arbitrary ball $B_\rho(a)$ with the integral over the ball $B_{\eta\rho}(a)$ that is by a fixed factor $\eta$ smaller. We have the following Lemma.

**Lemma 21.** *Let $f\colon B_{\rho_0} \to \mathbb{R}$ be a map, let $\gamma > 0$ and assume that there exists $\eta \in ]0,1[$ such that for all $\rho \leq \rho_0$*

$$\frac{1}{(\eta\rho)^\gamma}\int_{B_{\eta\rho}} |f| \;\leq\; \frac{1}{2\rho^\gamma}\left(\int_{B_\rho} |f|\right) + C_0. \tag{29}$$

*Then we have for all $r \leq \rho \leq \rho_0$ the relation*

$$\frac{1}{r^\gamma}\int_{B_r} |f| \;\leq\; \frac{1}{(\eta\rho)^\gamma}\left(\int_{B_\rho} |f|\right) + \frac{2C_0}{\eta^\gamma}.$$

*Proof.* Induction yields for all $n \in \mathbb{N}$, $0 < \rho < \rho_0$ the estimate

$$\frac{1}{(\eta^n\rho)^\gamma}\int_{B_{\eta^n\rho}} |f| \;\leq\; \frac{1}{2^n\rho^\gamma}\left(\int_{B_\rho} |f|\right) + C_0\sum_{i=0}^{n-1}\frac{1}{2^i} \;\leq\; \frac{1}{\rho^\gamma}\left(\int_{B_\rho} |f|\right) + 2C_0.$$

For arbitrary $r \leq \rho$ choose $k$ such that $\eta^{k+1}\rho < r \leq \eta^k\rho$. Then we have

$$\frac{1}{r^\gamma}\int_{B_r} |f|^2 \;\leq\; \frac{1}{(\eta^{k+1}\rho)^\gamma}\int_{B_{\eta^k\rho}} |f| \;\leq\; \left(\frac{1}{(\eta\rho)^\gamma}\int_{B_\rho} |f|\right) + \frac{2}{\eta^\gamma}C_0.$$

□

In the following $\Omega \subset \mathbb{R}^3$ is a domain. To show estimates like (29), we often write a function $u$ as the sum of the weak solution $v$ of an elliptic equation and the rest $w$. In Lemma 22 below we compare the integral $\int_{B_r} |\nabla v|^2$ over balls of different sizes. The proof of Lemma 22 relies on standard estimates for elliptic operators. For details see [9]

**Lemma 22.** *Let $0 < \rho \leq 1$, $A\colon B_\rho \to \mathbb{R}^{3\times 3}$ and let $v \in H^1(B_\rho)$ be a weak solution of $Lv = \operatorname{div}(A\nabla v) = 0$ in $B_\rho$. Moreover, assume that*

$$q \cdot (Aq) \geq \underline{\lambda}^2 |q|^2, \quad |q' \cdot (Aq)| \leq \overline{\lambda}^2 |q||q'| \quad \text{for all } q, q' \in \mathbb{R}^3.$$

*Then we have for all $0 < \eta \leq 1$*

$$\frac{1}{(\eta\rho)^3} \int_{B_{\eta\rho}} |\nabla v|^2 \;\leq\; \frac{C_{\mathrm{inEst}}}{\rho^3} \int_{B_\rho} |\nabla v|^2, \tag{30}$$

$$\frac{1}{(\eta\rho)^5} \int_{B_{\eta\rho}} |\nabla v - \langle \nabla v\rangle_{B_{\eta\rho}}|^2 \;\leq\; \frac{C_{\mathrm{inEst}}}{\rho^5} \int_{B_\rho} |\nabla v - \langle \nabla v\rangle_{B_\rho}|^2, \tag{31}$$

*where $C_{\mathrm{inEst}} = C_{\mathrm{inEst}}(\underline{\lambda}, \overline{\lambda}, K)$ and*

$$K \;:=\; \max\left(\|\nabla A\|_{C^0(B_\rho)}\rho,\, \|D^2 A\|_{C^0(B_\rho)}\rho^2,\, \|D^3 A\|_{C^0(B_\rho)}\rho^3\right).$$

One simple example for the method of comparing integrals of balls of different sizes is the following estimate, which we will use in Section 7.

**Lemma 23.** *Let $u$ be the solution of $\Delta u = \operatorname{div} m^R \mathbb{1}_{\Sigma(R)}$ in $\mathbb{R}^3$. Assume that $R \leq 1$. Then for every $0 < \gamma < 3$ there exists an absolute constant $C_\gamma$ such that for all $r \leq 1$, $a \in \mathbb{R}^3$ we have $\int_{B_r(a)} |\nabla u| \leq C_\gamma r^\gamma$.*

*Proof.* For $a \in \mathbb{R}^3$, $0 < \rho \leq 1$ define $v, w\colon B_\rho(a) \to \mathbb{R}^3$ as the solutions of

$$\Delta v \;=\; 0 \quad \text{in } B_\rho(a), \qquad v \;=\; u \quad \text{on } \partial B_\rho(a), \tag{32}$$

$$\Delta w \;=\; \operatorname{div}(m^R \mathbb{1}_{\Sigma(R)}) \quad \text{in } B_\rho(a), \qquad w \;=\; 0 \quad \text{on } \partial B_\rho(a). \tag{33}$$

Then $u|_{B_\rho(a)} = v + w$. Using Lemma 22, we find an absolute constant $C_1$ such that for each $0 < \eta < 1$

$$\|\nabla v\|^2_{L^2(B_{\eta\rho})} \;\leq\; C_1 \eta^3 \|\nabla v\|^2_{L^2(B_\rho)}.$$

Testing (33) with $w$ yields

$$\|\nabla w\|^2_{L^2(B_\rho(a))} = -\int_{B_\rho(a)} \operatorname{div} m^R\, w = \int_{B_\rho(a)} m^R \cdot \nabla w \leq \sqrt{\frac{4}{3}\pi\rho^3}\, \|\nabla w\|_{L^2(B_\rho(a))},$$

so we have $\|\nabla w\|^2_{L^2(B_\rho(a))} \leq \frac{4}{3}\pi\rho^3$. Since $v$ is the minimiser of $\|\nabla f\|^2_{L^2(B_\rho(a))}$ in the set $\{f \in H^1(B_\rho(a)) : f = u \text{ on } \partial B_\rho(a)\}$, we have in particular

$$\int_{B_\rho(a)} |\nabla v|^2 \;\leq\; \int_{B_\rho(a)} |\nabla u|^2.$$

Now choose $\eta$ such that $C_1 \eta^{3-\gamma} \leq \frac{1}{4}$. Then

$$\begin{aligned} \frac{1}{(\eta\rho)^\gamma} \|\nabla u\|^2_{L^2(B_{\eta\rho}(a))} \;&\leq\; \frac{2}{(\eta\rho)^\gamma}\left(\|\nabla v\|^2_{B_{\eta\rho}(a)} + \|\nabla w\|^2_{B_{\eta\rho}(a)}\right) \\ &\leq\; \left(\frac{1}{2\rho^\gamma}\int_{B_\rho(a)} |\nabla u|^2\right) + \frac{8}{3\eta^\gamma}\pi \end{aligned}$$

and Lemma 21 yields for all $r \leq 1$

$$\frac{1}{r^\gamma}\|\nabla u\|^2_{L^2(B_r(a))} \;\leq\; \frac{1}{\eta^\gamma}\|\nabla u\|^2_{L^2(B_1(a))} + \frac{16}{3\eta^{2\gamma}}\pi \;\leq\; \underbrace{\frac{1}{\eta^\gamma}E(m^R \mathbb{1}_{\Sigma(R)}) + \frac{16}{3\eta^{2\gamma}}\pi}_{=:C}\,.$$

Thus,

$$\frac{1}{r^\gamma}\|\nabla u\|_{L^1(B_r(a))} \;\leq\; \frac{1}{r^\gamma}\|\nabla u\|_{L^2(B_r(a))}\sqrt{|B_r|} \;\leq\; C\sqrt{\frac{4}{3}\pi}.$$

□

# 6 A decay estimate for almost-minimisers and $C^{0,\frac{1}{4}}$ regularity of $m^R$

In this section we compare the integral $\frac{1}{r^{1+\alpha}}\int_{B_r}|\nabla m|^2$ for $(c,\alpha)$-minimizers $m$ over balls of different sizes. The arguments are essentially the same as the arguments in [6]. For details on how the proofs have to be adapted for $(c,\alpha)$-minimizers for $|A\nabla\cdot|^2$ instead of $|\nabla\cdot|^2$ see [9].

First, using a comparison function, we find a hybrid inequality for almost-minimisers.

**Lemma 24.** *Let $\mu \in \mathbb{R}^3$, let $B_\rho(a) \subset \Omega$, and let $m$ be an $(c,\alpha)$-minimizer for $|A\nabla\cdot|^2$ in $\Omega$ with $\underline{\lambda}|f| \leq |Af| \leq \overline{\lambda}|f|$ for all $f \in \mathbb{R}^3$. Then we have for all $0 < \tau < 1$ the relation*

$$\begin{aligned}&\frac{1}{\frac{1}{2}\rho}\int_{B_{\frac{\rho}{2}}(a)}|\nabla m|^2\\ &\leq \frac{\overline{\lambda}^2}{\underline{\lambda}^2}\frac{\tau}{\rho}\left(\int_{B_\rho(a)}|\nabla m|^2\right) + 6\cdot 10^4\frac{\overline{\lambda}^2}{\underline{\lambda}^2}\frac{1}{\tau\rho^3}\left(\int_{B_\rho(a)}|m-\mu|^2\right) + c\rho^\alpha. \qquad (34)\end{aligned}$$

The following lemma relies on the observation that, when we appropriately rescale almost-minimisers that have small energy on small balls, we get a sequence of functions that converges to the solution of an elliptic equation. For solutions of elliptic equations we have Lemma 22, so for elements of the sequence that are close to this solution we have a similar estimate.

**Lemma 25.** *For $n \in \mathbb{N}$ let $m_n$ be an $(c,\alpha)$-almost-minimiser for $|A_n\nabla\cdot|^2$ in some domain $\Omega_n$ and assume that there exist constants $C$, $\underline{\lambda}$, $\overline{\lambda}$ such that*

$$\|\nabla A_n\|_{C^1(\Omega_i)} \leq C, \qquad \underline{\lambda}|v| \leq |A_n v| \leq \overline{\lambda}|v| \text{ for all } n \in \mathbb{N},\ v \in \mathbb{R}^3.$$

*Let $B_{r_n}(a_n) \subset \Omega_n$ and set*

$$\epsilon_n^2 \;:=\; \frac{1}{r_n}\int_{B_{r_n}(a_n)}|\nabla m_n|^2.$$

*If*

$$\lim_{n\to\infty} r_n = 0, \qquad \lim_{n\to\infty}\epsilon_n = 0, \qquad \lim_{n\to\infty}\frac{r_n^\alpha}{\epsilon_n^2} = 0,$$

*then there exists $C_1(\underline{\lambda},\overline{\lambda}) > 0$ such that for each $0 < \eta < 1$ there exists a subsequence $(m_{n_k})_{k\in\mathbb{N}}$ of $(m_n)_{n\in\mathbb{N}}$ such that for all elements of the subsequence and for all $\beta \in [\eta, 1]$*

$$\frac{1}{(\beta r_{n_k})^3}\int_{B_{\beta r_{n_k}}(a_{n_k})}\left|m_{n_k} - \langle m_{n_k}\rangle_{B_{\beta r_{n_k}}(a_{n_k})}\right|^2 \;\le\; C_1\beta^2\epsilon_{n_k}^2.$$

Lemmas 24 and 25 imply the decay estimate for almost-minimisers.

**Theorem 26.** *Let $m$ be a $(c,\alpha)$-almost-minimiser for $|A\nabla\cdot|^2$ in $\Omega$ with $\underline{\lambda}|f| \le |Af| \le \overline{\lambda}|f|$ for all $f \in \mathbb{R}^3$. Then there exist positive constants $\epsilon_0, r_0, C_2, C_3$ and $\eta < 1$, all depending only on $c$, $\alpha$, $\underline{\lambda}$ and $\overline{\lambda}$, with the following properties: If $r \le \rho \le r_0$ with $B_\rho(a) \subset \Omega$ and $\frac{1}{\rho}\int_{B_\rho(a)}|\nabla m|^2 < \epsilon_0$, we have the estimates*

$$\frac{1}{(\eta\rho)^{1+\alpha}}\int_{B_{\eta\rho}(a)}|\nabla m|^2 \;\le\; \max\left(\frac{1}{2\rho^{1+\alpha}}\left(\int_{B_\rho}|\nabla m|^2\right),\, C_2\right), \tag{35}$$

$$\frac{1}{r^{1+\alpha}}\int_{B_r(a)}|\nabla m|^2 \;\le\; C_3\left(\frac{1}{\rho^{1+\alpha}}\left(\int_{B_\rho}|\nabla m|^2\right)+1\right). \tag{36}$$

Combining this decay estimate with the characterisation theorem for Hölder continuous functions (Theorem 20), we prove that the minimisers $m^R$ are Hölder continuous. For $x_0 \in \mathbb{R}$ we set

$$Z_R(x_0) \;:=\; [x_0 - R, x_0 + R] \times D_R \tag{37}$$

**Theorem 27.** *There exists $C_s > 0$ such that for all $R$ small enough and all $p, q \in \Sigma(\frac{3}{2}R)$ with $|p-q| \le R$ we have $m^R(p) - m^R(q) \le C_s|p-q|^{\frac{1}{4}}$.*

*Proof.* The functions $m^R$ are $(c,\alpha)$-almost-minimisers for $|\frac{1}{\sqrt{\kappa}}A^R\nabla\cdot|^2$ in $\Sigma(\frac{3}{2}R)$ with $c = 4+\sqrt{E(m^R\mathbb{1}_{\Sigma(R)})}$ and $\alpha = \frac{1}{2}$ (Theorem 19). Thus for $R \le 1$ we have uniform constants $\epsilon_0$, $r_0$, $C_3$ in Theorem 26. Assume that $R$ is so small that

$$R \le 1, \quad R \le r_0, \quad \frac{1}{R}E(m^R\mathbb{1}_{\Sigma(R)}) \le \frac{\epsilon_0}{4}, \quad \frac{3}{R^{1+\frac{1}{2}}}E(m^R\mathbb{1}_{\Sigma(R)}) \le 1.$$

Then we have for all $p \in \Sigma(R)$ the estimate

$$\frac{1}{\frac{1}{2}R}\int_{B_{\frac{1}{2}R}(p)}|\nabla m^R|^2 \;\le\; \frac{2}{R}\int_{\Sigma(\frac{3}{2}R)}|\nabla m^R|^2 \;\le\; \frac{4}{R}\|\nabla m^R\|^2_{L^2(\Sigma(R))} \;\le\; \epsilon_0,$$

and with Theorem 26 we get

$$\begin{aligned}\int_{B_r(p)}|\nabla m^R|^2 \;&\le\; C_3 r^{1+\frac{1}{2}}\left(1+\frac{2^{1.5}}{R^{1+\frac{1}{2}}}\int_{B_{\frac{1}{2}R(p)}}|\nabla m^R|^2\right)\\ &\le\; C_3 r^{1+\frac{1}{2}}\left(1+\frac{3}{R^{1+\frac{1}{2}}}E(m^R\mathbb{1}_{\Sigma(R)})\right) \;\le\; 2C_3 r^{1+\frac{1}{2}}.\end{aligned}$$

Now the Point-caré inequality (20) implies

$$\int_{B_r(p)}|m^R - \langle m^R\rangle_{B_r(p)}|^2 \;\le\; 128 C_3 r^{3+\frac{1}{2}}.$$

If $p, q \in \Sigma(\frac{3}{2}R)$ and $|p-q| < R$, then there exists $x_0 \in \mathbb{R}$ such that $p^*, q^* \in Z_R(x_0)$. We apply Theorem 20 for $\Omega := Z_1$ and $\eta := R$ and get

$$\begin{aligned} |m(p) - m(q)| &= |m(p^*) - m(q^*)| \leq 128 C_3 C_{\text{Camp}} \left(Z_1, \tfrac{1}{2}\right) |p^* - q^*|^{\frac{1}{4}} \\ &\leq 128 C_3 C_{\text{Camp}} \left(Z_1, \tfrac{1}{2}\right) |p - q|^{\frac{1}{4}}. \end{aligned}$$

□

# 7 Uniform $C^{1,\beta}$ regularity of the functions $m^R$ and convergence results

The aim of this section is to prove good regularity of the functions $m^R$ to get strong convergence results. In the first subsection we show that we do not have strong oscillations of $\nabla m^R$ on the balls of diameter $R$. In the second subsection we use methods as in the Morrey-Campanato approach to regularity (cf. [4]) to show that on smaller scales we have even less oscillations.

## 7.1 Convergence on balls of radius $R$

In this section we will use the fact that the functions $m^R$ are energy minimising to bound $\nabla m^R$ on the scale of $R$. We know that $m^{\text{red}}$ satisfies the differential equation $|\partial_x m^{\text{red}}| - \frac{1}{\sqrt{2}}|m_y^{\text{red}}| = 0$. The first lemma shows that the functions $m^R$ satisfy this differential equation approximately, i.e., that the difference $\left||\partial_x m^R| - \frac{1}{\sqrt{2}}|m_y^R|\right|$ is small. For $m \colon \Sigma \to \mathbb{R}^3$ let $\overline{m}$, $\tilde{m}$ be as in (6) and (7).

**Lemma 28.** *For all $\beta < 1$ we have*

$$\lim_{R \to 0} \frac{1}{R^{3+\beta}} \int_{\Sigma(R)} \left( |\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}} |\overline{m}_y^R| \right)^2 = 0.$$

*Proof.* Because of Lemma 14, the functions $\overline{m}^R(\cdot, 0)$ converges in $H^1(\mathbb{R})$ to $m^{\text{red}}$. Thus, using the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$, we can assume $|\overline{m}^R| \geq \frac{1}{2}$ for sufficiently small $R$. Moreover, after reducing $R$, we can assume $\|\tilde{m}^R\|_{L^2(\Sigma(R))} \leq CR^3$ (Lemma 15) and $\|\nabla m^R\|_{L^2(\Sigma(R))} \leq 3R$ (Theorem 7). Since

$$\begin{aligned} \left|\partial_x \overline{m}^R\right| &= \left|\partial_x \left( \frac{\overline{m}^R}{|\overline{m}^R|} |\overline{m}^R| \right)\right| = \sqrt{|\overline{m}^R|^2 \left|\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}\right|^2 + (\partial_x |\overline{m}^R|)^2} \\ &\geq |\overline{m}^R| \left|\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}\right| \end{aligned}$$

with (9) we have

$$\begin{aligned}\int_{\Sigma(R)} |\partial_x \overline{m}^R| \cdot |\overline{m}^R_y| \;&\geq\; \int_{\Sigma(R)} |\overline{m}^R|^2 \left|\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}\right| \frac{|\overline{m}^R_y|}{|\overline{m}^R|} \\ &=\; \underbrace{\int_{\Sigma(R)} \left|\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}\right| \frac{|\overline{m}^R_y|}{|\overline{m}^R|}}_{T_1} \underbrace{-|\tilde{m}^R|^2 \left|\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}\right| \frac{|\overline{m}^R_y|}{|\overline{m}^R|}}_{T_2}.\end{aligned}$$

To bound the first term $T_1$ from below, we set $\theta := \arccos\left(\frac{\overline{m}^R_x}{|\overline{m}^R|}\right)$ and calculate

$$T_1 \;=\; \int_{\Sigma(R)} |\partial_x \theta| \cdot |\sin(\theta)| \geq 2\pi R^2.$$

For the second term $T_2$ we use the assumptions on $R$ and the pointwise estimates $|\tilde{m}^R| \leq 1$, $|\overline{m}^R| \leq 1$. We have

$$\begin{aligned}T_2 \;&\geq\; -\int_{\Sigma(R)} |\tilde{m}^R|^2 \left|\frac{\partial_x \overline{m}^R}{|\overline{m}^R|}\right| \frac{|\overline{m}^R_y|}{|\overline{m}^R|} \;\geq\; -4 \int_{\Sigma(R)} |\tilde{m}^R| \left|\partial_x \overline{m}^R\right| \\ &\geq\; -4 \|\tilde{m}^R\|_{L^2(\Sigma(R))} \|\nabla m^R\|_{L^2(\Sigma(R))} \;\geq\; -12CR^4.\end{aligned}$$

Using this calculation and the equality $E^{\mathrm{red}}(m^{\mathrm{red}} 1\!\!1_{\Sigma(R)}) = \sqrt{8}\pi$, we obtain

$$\begin{aligned}&\left\| |\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}} |\overline{m}^R_y| \right\|^2_{L^2(\Sigma(R))} \\ &= \frac{1}{2} \left\|\overline{m}^R_y\right\|^2_{L^2(\Sigma(R))} + \left\|\partial_x \overline{m}^R\right\|^2_{L^2(\Sigma(R))} - \int_{\Sigma(R)} \sqrt{2}\, |\partial_x \overline{m}^R| \cdot |\overline{m}^R_y| \\ &\leq \frac{1}{2} \left\|\overline{m}^R_y\right\|^2_{L^2(\Sigma(R))} + E_{ex}(m 1\!\!1_{\Sigma(R)}) - R^2 E^{\mathrm{red}}(m^{\mathrm{red}}) + 12\sqrt{2}CR^4 \\ &= \underbrace{\frac{1}{2} \left\|\overline{m}^R_y\right\|^2_{L^2(\Sigma(R))} - E_H(\overline{m}^R 1\!\!1_{\Sigma(R)})}_{S_1} + \underbrace{E_H(\overline{m}^R 1\!\!1_{\Sigma(R)}) - E_H(m^R 1\!\!1_{\Sigma(R)})}_{S_2} \\ &\quad + \underbrace{E(m^R 1\!\!1_{\Sigma(R)}) - R^2 E^{\mathrm{red}}(m^{\mathrm{red}})}_{S_3} + 12\sqrt{2}CR^4.\end{aligned}$$

We consider each summand separately. By Lemma 12 we know $\lim_{R\to 0} \frac{1}{R^{3+\beta}} S_1 = 0$. Since $\lim_{R\to 0} \frac{1}{R^2} E(m^R 1\!\!1_{\Sigma(R)}) = \sqrt{8}\pi$ (Theorem 7), for the second summand, (12) and Lemma 15 imply

$$\begin{aligned}\lim_{R\to 0} \frac{1}{R^{3+\beta}} S_2 \;&\leq\; \lim_{R\to 0} \frac{1}{R^{3+\beta}} \left(2\sqrt{E(m^R 1\!\!1_{\Sigma(R)})} \|\tilde{m}^R\|_{L^2(\Sigma(R))} + \|\tilde{m}^R\|^2_{L^2(\Sigma(R))}\right) \\ &=\; 0.\end{aligned}$$

Finally, using Theorem 16 we get $\lim_{R\to 0} \frac{1}{R^{3+\beta}} S_3 \;=\; 0.$ ☐

Combining the bound on $\nabla \tilde{m}^R$ of Lemma 15 with the estimate for $\partial_x \overline{m}$ of Lemma 28 we get the following Lemma.

**Lemma 29.** *For $Z_R$ as in (37) and for all $\beta < 1$ we have uniformly in $x_0$*

$$\lim_{R\to 0} \frac{1}{|R|^{3+\beta}} \int_{Z_R(x_0)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{Z_R(x_0)}\right|^2 \;=\; 0.$$

*Proof.* With (21) we have

$$\begin{aligned}
&\int_{Z_R(x_0)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{Z_R(x_0)}\right|^2 \\
&\leq \int_{Z_R(x_0)} \left| |\nabla \tilde{m}^R| + \left| \partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \left\langle \overline{m}^R_y \right\rangle_{Z_R(x_0)} \right| \right|^2 \\
&\leq 3 \left( \int_{\Sigma(R)} |\nabla \tilde{m}^R|^2 + \left| \partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}^R_y \right|^2 \right) + \frac{3}{2} \int_{Z_R(x_0)} \left| \overline{m}^R_y - \left\langle \overline{m}^R_y \right\rangle_{Z_R(x_0)} \right|^2 .
\end{aligned}$$

Lemma 15 and Lemma 28 imply

$$\lim_{R\to 0} \frac{3}{R^{3+\beta}} \left( \int_{\Sigma} |\nabla \tilde{m}^R|^2 + \left| \partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}^R_y \right|^2 \right) \;=\; 0,$$

and the Poincaré inequality [5, p.164] yields

$$\begin{aligned}
\lim_{R\to 0} \frac{3}{R^{3+\beta}} \int_{Z_R(x_0)} \left| \overline{m}^R_y - \left\langle \overline{m}^R_y \right\rangle_{Z_R(x_0)} \right|^2 \;&\leq\; \lim_{R\to 0} \frac{12}{R^{1+\beta}} \|\partial_x \overline{m}^R_y\|^2_{L^2(Z_R(x_0))} \\
&\leq\; \lim_{R\to 0} \frac{12}{R^{1+\beta}} E(m^R \mathbb{1}_{\Sigma(R)}) \;=\; 0.
\end{aligned}$$

□

From Lemma 29 we can deduce the main theorem of this subsection.

**Theorem 30.** *For all $\beta < 1$ we have*

$$\lim_{R\to 0} \sup_{p\in\Sigma(R)} \frac{1}{R^{3+\beta}} \int_{B_{\frac{R}{2}}(p)} |\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_{\frac{R}{2}}(p)}|^2 \;=\; 0, \tag{38}$$

$$\lim_{R\to 0} \sup_{p\in\Sigma(R)} \frac{1}{R^{2+\beta}} \int_{B_{\frac{R}{2}}(p)} |\partial_x m^R|^2 \;=\; 0. \tag{39}$$

*Proof.* With (23) and (24) we have for all $x_0 \in \mathbb{R}$

$$\begin{aligned}
\int\limits_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}\setminus Z_R(x_0)} |\nabla m^R|^2 \;&\leq\; \int\limits_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}\setminus Z_R(x_0)} \frac{3}{\kappa} \left| A^R \nabla m^R \right|^2 \\
&=\; 3 \int\limits_{Z_R(x_0)\setminus[x_0-R,x_0+R]\times D_{\frac{R}{2}}} |\nabla m^R|^2 \\
&\leq\; 3 \int_{Z_R(x_0)} |\nabla m^R|^2 .
\end{aligned}$$

Because of the special form of $A^R$ we have the same estimate for $\nabla_y m^R$ instead of $\nabla m^R$. This implies for all $(x_0, y_0) \in \Sigma(R)$

$$\int_{B_{\frac{R}{2}}(x_0,y_0)} |\nabla_y m^R|^2 \;\le\; \int_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}} |\nabla m^R|^2 \;\le\; 4\int_{\Sigma(R)} |\nabla_y m^R|^2.$$

Therefore, using (21) and (18), we obtain

$$\begin{aligned}
&\lim_{R\to 0}\sup_{p\in\Sigma(R)} \frac{1}{R^{3+\beta}} \int_{B_{\frac{R}{2}}(p)} |\nabla_y m^R - \langle \nabla_y m\rangle_{B_{\frac{R}{2}}(p)}|^2 \\
&\le \lim_{R\to 0}\sup_{p\in\Sigma(R)} \frac{1}{R^{3+\beta}} \int_{B_{\frac{R}{2}}(p)} |\nabla_y m^R|^2 \;=\; 0. \qquad (40)
\end{aligned}$$

To show the analogous statement for $\partial_x m^R$, we note that we have $\partial_x m^R(p) = \partial_x m^R(p^*)$, where $p^*$ as in (22). As a consequence of (21), (23), and Lemma 29 we obtain

$$\begin{aligned}
&\lim_{R\to 0}\sup_{(x_0,y_0)\in\Sigma(R)} \frac{1}{R^{3+\beta}} \int_{B_{\frac{R}{2}}(x_0,y_0)} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{B_{\frac{R}{2}(x_0,y_0)}}|^2 \\
&\le \lim_{R\to 0}\sup_{(x_0,y_0)\in\Sigma(R)} \frac{1}{R^{3+\beta}} \int_{B_{\frac{R}{2}}(x_0,y_0)} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{Z_R(x_0)}|^2 \\
&\le \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{1}{R^{3+\beta}} \int_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{Z_R(x_0)}|^2 \\
&\le \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{4}{R^{3+\beta}} \int_{Z_R(x_0)} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{Z_R(x_0)}|^2 \\
&= 0.
\end{aligned}$$

We now prove (39). For all $x_0 \in \mathbb{R}$ we have

$$\begin{aligned}
\left|\int_{Z_R(x_0)} \partial_x m^R\right| \;&\le\; \int_{Z_R(x_0)} \left(\left||\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right| + \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right) \\
&\le\; \sqrt{|Z_R|}\,\left\||\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right\|_{L^2(Z_R(x_0))} + \frac{1}{\sqrt{2}}|Z_R|
\end{aligned}$$

Thus Lemma 28 implies

$$\lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{1}{R^{2+\beta}} \left|\int_{Z_R(x_0)} \partial_x m^R\right| \;=\; 0,$$

and with Lemma 29 we obtain

$$\lim_{R\to 0}\sup_{(x_0,y_0)\in\Sigma(R)} \frac{1}{R^{2+\beta}} \int_{B_{\frac{R}{2}}(x_0,y_0)} |\partial_x m^R|^2 \;\le\; \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{4}{R^{2+\beta}} \int_{Z_R(x_0)} |\partial_x m^R|^2$$

$$= \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{4}{R^{2+\beta}} \left( \int_{Z_R(x_0)} \left|\partial_x m^R - \langle \partial_x m^R\rangle_{Z_R(x_0)}\right|^2 + \left| \int_{Z_R(x_0)} \partial_x m^R \right|^2 \right)$$

$$= 0.$$

□

## 7.2 Bounds on balls with radius smaller than $R$

In this subsection, let $\zeta$ be as in Theorem 19. Moreover, choose $a \in \Sigma(R)$, $\rho \le \frac{1}{2}R$ and define $v, w\colon B_\rho(a) \to \mathbb{R}^3$ as the weak solutions of

$$\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\,\nabla v\right) = 0 \qquad \text{in } B_\rho(a), \tag{41}$$

$$v = m^R \qquad \text{on } \partial B_\rho(a), \tag{42}$$

$$\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\,\nabla w\right) = \frac{1}{\kappa}\,|A^R\nabla m^R|^2 m^R + \zeta \qquad \text{in } B_\rho(a), \tag{43}$$

$$w = 0 \qquad \text{on } \partial B_\rho(a). \tag{44}$$

These definitions are valid for the rest of this section. For $v$ we have the estimates of Lemma 22 uniformly in $R$.

**Lemma 31.** *There exists a constant $C_{\mathrm{InEst}}$ such that for all $R \le 1$, $a \in \Sigma(R)$, $\rho \le \frac{1}{2}R$, $0 < \eta < 1$,*

$$\frac{1}{(\eta\rho)^3}\int_{B_{\eta\rho}(a)} |\nabla v|^2 \;\le\; \frac{C_{\mathrm{inEst}}}{\rho^3}\int_{B_\rho(a)} |\nabla v|^2 \tag{45}$$

$$\frac{1}{(\eta\rho)^5}\int_{B_{\eta\rho}(a)} \left|\nabla v - \langle\nabla v\rangle_{B_{\eta\rho}(a)}\right|^2 \;\le\; \frac{C_{\mathrm{inEst}}}{\rho^5}\int_{B_\rho(a)} \left|\nabla v - \langle\nabla v\rangle_{B_\rho(a)}\right|^2 \tag{46}$$

*Proof.* $A^R$ is symmetric and

$$\frac{1}{3}|\nabla v| \;\le\; \frac{1}{\kappa}|\nabla v|^2 \;\le\; \left|\frac{1}{\sqrt{\kappa}}A^R\nabla v\right|^2 \;\le\; \kappa|\nabla v|^2 \;\le\; 3|\nabla v|^2.$$

For each radius $R$ we have a different function $\kappa$. Removing this ambiguity and writing $\kappa_R$ we have

$$\left\| D^n\left(\frac{1}{\sqrt{\kappa}_R}A^R\right)\right\|_{C_0(\Sigma(\frac{3}{2}R))} = \frac{1}{R^n}\left\| D^n\left(\frac{1}{\sqrt{\kappa}_1}A^1\right)\right\|_{C_0(\Sigma(\frac{3}{2}))}$$

Therefore we get, with the notation of Lemma 22, a uniform lower bound for $\underline{\lambda}$ and uniform upper bounds for $\overline{\lambda}$ and $K$. Thus Lemma 22 implies the result. □

Using the techniques described in Section 5, we will prove $C^{1,\beta}$-regularity by showing scaled $L^2$-estimates. First, in Lemma 32, we will consider $\frac{1}{r^\gamma}\int_{B_r}\left|\nabla m^R\right|^2$ for $\gamma < 3$. Then, in Lemma 33, we will use Lemma 32 to treat $\frac{1}{r^\gamma}\int_{B_r}|\nabla m^R - \langle\nabla m^R\rangle_{B_r}|^2$ for $3 \le \gamma < 3 + \frac{1}{8}$.

**Lemma 32.** *For each $\gamma < 3$ there exists $R_0 = R_0(\gamma)$ such that for all $R \le R_0$, all $a \in \Sigma(R)$ and all $r \le \frac{1}{2}R$ we have*

$$\frac{1}{r^\gamma}\int_{B_r(a)}\left|\nabla m^R\right|^2 \;\le\; 1.$$

*Proof.* We assume that $R \le 1$ is so small that with Theorem 27

$$|m^R(p) - m^R(p')| \le C_s|p-p'|^{\frac{1}{4}} \qquad \text{for all } p, p' \in \Sigma(\tfrac{3}{2}R) \text{ with } |p-p'| < R.$$

First, we show that $|w|$ is small. Since $\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\nabla v_i\right) = 0$ in $B_\rho(a)$ for $i \in \{1,2,3\}$ the maximum principle yields

$$\sup_{p\in B_\rho(a)} v_i(p) \;=\; \sup_{p\in\partial B_\rho(a)} m_i^R(p), \qquad \inf_{p\in B_\rho(a)} v_i(p) \;=\; \inf_{p\in\partial B_\rho(a)} m_i^R(p).$$

Thus

$$|w| \;=\; |m^R - v| \;\le\; \sum_{i=1}^{3}\sup_{p\in\partial B_\rho(a)} m_i^R(p) - \inf_{p\in\partial B_\rho(a)} m_i^R(p) \;\le\; 6C_s\rho^{\frac{1}{4}}. \tag{47}$$

Since $|\zeta(p)| \le |\nabla u(p)|$ for all $p \in \mathbb{R}^3$, Lemma 23 implies

$$\int_{B_\rho(a)}|\zeta| \;\le\; C_\gamma\rho^\gamma. \tag{48}$$

Testing (43) with $w$ yields

$$\begin{aligned}
\int_{B_\rho(a)}|\nabla w|^2 \;&\le\; 3\int_{B_\rho(a)}\frac{1}{\kappa}|A^R\nabla w|^2 \;=\; 3\int_{B_\rho(a)}\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\nabla w\right)w\\
&=\; 3\int_{B_\rho(a)}\frac{1}{\kappa}|A^R\nabla m^R|^2 m w + \zeta w\\
&\le\; 3\,\|w\|_{C_0(B_\rho(a))}\int_{B_\rho(a)}3|\nabla m^R|^2 + |\zeta|\\
&\le\; 54C_s\rho^{\frac{1}{4}}\left(\int_{B_\rho(a)}\left|\nabla m^R\right|^2\right) + 18C_sC_\gamma\rho^{\frac{1}{4}+\gamma}.
\end{aligned}$$

Moreover, using Lemma 31, for all $\eta \le 1$ we have

$$\begin{aligned}
\int_{B_{\eta\rho}(a)}|\nabla v|^2 \;&\le\; C_{\text{inEst}}\eta^3\int_{B_\rho(a)}|\nabla v|^2 \;\le\; 3C_{\text{inEst}}\eta^3\int_{B_\rho(a)}\frac{1}{\kappa}|A^R\nabla v|^2\\
&\overset{*}{\le}\; 3C_{\text{inEst}}\eta^3\int_{B_\rho(a)}\frac{1}{\kappa}|A^R\nabla m^R|^2 \;\le\; 9C_{\text{inEst}}\eta^3\int_{B_\rho(a)}|\nabla m^R|^2.
\end{aligned}$$

For inequality $(*)$ we have used that $v$ is the minimiser of

$$g \mapsto \int_{B_\rho(a)} \frac{1}{\kappa} |A^R \nabla g|^2 \quad \text{in } \left\{ g \in H^1(B_\rho(a)) \,:\, g|_{\partial B_\rho(a)} = m^R|_{\partial B_\rho(a)} \right\}.$$

Combining the estimates, we get for all $0 < \eta \le 1$

$$\begin{aligned} \frac{1}{(\eta\rho)^\gamma} \int_{B_{\eta\rho}(a)} |\nabla m^R|^2 \;&\le\; \frac{2}{(\eta\rho)^\gamma} \left( \int_{B_{\eta\rho}(a)} |\nabla v|^2 \right) + \frac{2}{(\eta\rho)^\gamma} \left( \int_{B_\rho(a)} |\nabla w|^2 \right) \\ &\le \left( 18 C_{\mathrm{inEst}} \eta^{3-\gamma} + \frac{108 C_s \rho^{\frac14}}{\eta^\gamma} \right) \left( \frac{1}{\rho^\gamma} \int_{B_\rho(a)} |\nabla m^R|^2 \right) + \frac{36 C_s C_\gamma}{\eta^\gamma} \rho^{\frac14}. \end{aligned}$$

Now let $\eta$ be the largest number such that

$$18 C_{\mathrm{inEst}} \eta^{3-\gamma} \;\le\; \frac14,$$

and let $r_0$ be the largest number such that

$$\frac{108 C_s}{\eta^\gamma} r_0^{\frac14} \;\le\; \frac14, \qquad \frac{36 C_s C_\gamma}{\eta^\gamma} r_0^{\frac14} \;\le\; \frac14 \eta^\gamma.$$

Then, if $\rho \le r_0$, we have

$$\frac{1}{(\eta\rho)^\gamma} \int_{B_{\eta\rho}(a)} |\nabla m^R|^2 \;\le\; \frac{1}{2\rho^\gamma} \int_{B_{\eta\rho}(a)} |\nabla m^R|^2 + \frac14 \eta^\gamma,$$

so Lemma 21 yields for all $r \le \rho \le r_0$

$$\frac{1}{r^\gamma} \int_{B_r(a)} |\nabla m^R|^2 \;\le\; \frac{1}{(\eta\rho)^\gamma} \left( \int_{B_\rho(a)} |\nabla m^R|^2 \right) + \frac12.$$

Using Theorem 30, we can find $R_0 \le r_0$ such that for all $R \le R_0$ and all $a \in \Sigma(R)$ we have $\frac{1}{R^\gamma} \int_{B_{\frac{R}{2}}(a)} |\nabla m^R|^2 \le \frac12 \eta^\gamma$. Then we have for all $r \le \frac12 R$ the estimate

$$\frac{1}{r^\gamma} \int_{B_r(a)} |\nabla m^R|^2 \;\le\; 1.$$

□

**Lemma 33.** *For all $\beta < \frac18$ there exist positive constants $R_1 = R_1(\beta)$, $C = C(\beta)$ such that for all $R \le R_1$, all $r \le \frac12 R$ and all $a \in \Sigma(R)$ we have*

$$\int_{B_r(a)} |\nabla m^R - \langle \nabla m^R \rangle_{B_r(a)}|^2 \;\le\; C r^{3+2\beta}.$$

*Proof.* Set $\gamma := 3 + 2\beta - \frac14$, let $R_0(\gamma)$ as in Lemma 32 and assume that $R \le R_0(\gamma)$. Then, using (47), (48), and Lemma 32, we have for all $a \in \Sigma(R)$, $\rho \le \frac12 R$

$$\begin{aligned} \int_{B_\rho(a)} |\nabla w|^2 \;&\le\; 3 \int_{B_\rho(a)} \frac{1}{\kappa} |A^R \nabla w|^2 \;=\; 3 \int_{B_\rho(a)} \left( \frac{1}{\kappa} |A^R \nabla m^R|^2 m^R \cdot w + \zeta \cdot w \right) \\ &\le\; 6 C_s \rho^{\frac14} \int_{B_\rho} 3 |\nabla m^R|^2 + |\zeta| \;\le\; 6 C_s (3 + C_\gamma) \rho^{\frac14 + \gamma} \\ &=\; 6 C_s (3 + C_\gamma) \rho^{3+2\beta}. \end{aligned}$$

The function $s : p \mapsto v(p) - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \cdot (p-a)$ is in

$$S := \left\{ g : B_\rho(a) \to \mathbb{R}^3 \mid g_i(p) = m_i^R(p) - \left\langle \nabla m_i^R \right\rangle_{B_\rho(a)} \cdot (p-a) \text{ on } \partial B_\rho(a) \right\}$$

and satisfies

$$\operatorname{div}\left( \frac{1}{\kappa} (A^R)^2 \nabla s \right) = 0 \quad \text{in } B_\rho(a).$$

Thus $s$ is a minimiser of $m \mapsto \int_{B_\rho(a)} \left| \frac{1}{\sqrt{\kappa}} A^R \nabla m \right|^2$ in $S$. We have in particular

$$\int\limits_{B_\rho(a)} \left| \frac{1}{\sqrt{\kappa}} A^R \left( \nabla v - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right) \right|^2 \le \int\limits_{B_\rho(a)} \left| \frac{1}{\sqrt{\kappa}} A^R \left( \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right) \right|^2 .$$

Therefore, with Lemma 31 and (21), we obtain

$$\begin{aligned}
\int_{B_{\eta\rho}(a)} \left| \nabla v - \langle \nabla v \rangle_{B_{\eta\rho}(a)} \right|^2 &\le C_{\text{inEst}} \eta^5 \int_{B_\rho(a)} \left| \nabla v - \langle \nabla v \rangle_{B_\rho(a)} \right|^2 \\
&\le C_{\text{inEst}} \eta^5 \int_{B_\rho(a)} \left| \nabla v - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right|^2 \\
&\le 3C_{\text{inEst}} \eta^5 \int_{B_\rho(a)} \left| \frac{1}{\sqrt{\kappa}} A^R \left( \nabla v - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right) \right|^2 \\
&\le 3C_{\text{inEst}} \eta^5 \int_{B_\rho(a)} \left| \frac{1}{\sqrt{\kappa}} A^R \left( \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right) \right|^2 \\
&\le 9C_{\text{inEst}} \eta^5 \int_{B_\rho(a)} \left| \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right|^2 .
\end{aligned}$$

Thus, using again (21),

$$\begin{aligned}
&\frac{1}{(\eta\rho)^{3+2\beta}} \int_{B_{\eta\rho}(a)} \left| \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_{\eta\rho}(a)} \right|^2 \\
&\le \frac{1}{(\eta\rho)^{3+2\beta}} \int_{B_{\eta\rho}(a)} \left| \nabla m^R - \langle \nabla v \rangle_{B_{\eta\rho}(a)} \right|^2 \\
&\le \frac{2}{(\eta\rho)^{3+2\beta}} \int_{B_{\eta\rho}(a)} \left| \nabla v - \langle \nabla v \rangle_{B_{\eta\rho}(a)} \right|^2 + \frac{2}{(\eta\rho)^{3+2\beta}} \int_{B_\rho(a)} |\nabla w|^2 \\
&\le \frac{18 C_{\text{inEst}} \eta^{2-2\beta}}{\rho^{3+2\beta}} \left( \int_{B_\rho(a)} \left| \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right|^2 \right) + \frac{12 C_s (3 + C_\gamma)}{\eta^{3+2\beta}} .
\end{aligned}$$

Now let $\eta$ be the largest numbers such that $18 C_{\text{inEst}} \eta^{2-2\beta} \le \frac{1}{2}$. Then, with Lemma 21, for all $0 < r < \rho$, we get

$$\begin{aligned}
&\frac{1}{r^{3+2\beta}} \int_{B_r(a)} \left| \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_r(a)} \right|^2 \\
&\le \frac{1}{\eta^{3+2\beta}} \frac{1}{\rho^{3+2\beta}} \int_{B_\rho(a)} \left| \nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)} \right|^2 + \frac{24 C_s (3 + C_\gamma)}{\eta^{6+4\beta}} .
\end{aligned}$$

In particular, setting $\rho := R$ and using Theorem 30, we see that there exists a constant $C$ such that for all $r \le R \le R_0(\gamma)$

$$\int_{B_r(a)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_r(a)}\right|^2 \le C r^{3+2\beta}.$$

□

Now we come to the main regularity theorem. We prove uniformly good regularity for the functions $m^R$. These bounds of strong norms then imply convergence in slightly weaker norms.

**Theorem 34.** *For each $\beta < \frac{1}{8}$ there exist positive constants $R_{C^{1,\beta}}$, $C_{C^{1,\beta}}$, such that for all $R \le R_{C^{1,\beta}}$*

*Proof.* Let $R_1 = R_1(\beta)$ as in Lemma 33, and let $R \le \min\{R_1, 1\}$ be so small that Lemma 15 and Lemma 28 yield

$$\|\partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}_y^R\|^2_{L^2(\Sigma)} \le R^{3+\frac{1}{2}}, \qquad \|\nabla \tilde{m}^R\|^2_{L^2(\Sigma)} \le R^{3+\frac{1}{2}}. \tag{49}$$

Using the integral characterisation of Hölder continuous functions (Theorem 20) we get locally, on the scale of $R$, a uniform bound of $\nabla m^R$ in $C^{0,\beta}$, i.e., there exists $C$ depending only on $\beta$, such that

$$|\nabla m^R(p) - \nabla m^R(p')| \le C|p - p'|^\beta \qquad \text{if } |p - p'| \le R. \tag{50}$$

In particular, we have $\left(\nabla m^R(p) - \left\langle \nabla m^R \right\rangle_{Z_R(x)}\right) \le 2CR^\beta$ for all $p \in Z_R(x)$ where $Z_R(x)$ as in (37). Therefore

$$\begin{aligned}
&\left|\nabla m^R(p) - \frac{1}{\sqrt{2}} \left\langle \overline{m}_y^R \right\rangle_{Z_R(x)}\right| \\
&\le \left| \left(\nabla m^R(p) - \left\langle \nabla m^R \right\rangle_{Z_R(x)}\right) + \left\langle \nabla \tilde{m}^R \right\rangle_{Z_R(x)} \right. \\
&\quad \left. + \left( \left\langle \partial_x \overline{m}^R \right\rangle_{Z_R(x)} - \frac{1}{\sqrt{2}} \left\langle \overline{m}_y^R \right\rangle_{Z_R(x)} \right) \right| \\
&\le 2CR^\beta + \frac{\sqrt{|Z_R|}}{|Z_R|} \left\|\nabla \tilde{m}^R\right\|_{L^2(Z_R)} + \frac{\sqrt{|Z_R|}}{|Z_R|} \left\| \partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}_y^R \right\|_{L^2(\Sigma)} \\
&\overset{*}{\le} 2CR^\beta + \sqrt{\frac{R^{3+\frac{1}{2}}}{|Z_R|}} + \sqrt{\frac{R^{3+\frac{1}{2}}}{|Z_R|}} \le (2C+1)R^\beta.
\end{aligned}$$

For the estimate $(*)$ we have used (49).
Since $\left\langle \overline{m}_y^R \right\rangle_{Z_R(x)} \le 1$, this calculation shows that $|\nabla m^R|$ is bounded by some constant $\tilde{C}$, which then yields

$$\begin{aligned}
&|\nabla m^R(p) - \nabla m^R(p')| \\
&\le \left|\nabla m^R(p) - \left\langle \overline{m}_y^R \right\rangle_{Z_R(x)}\right| + \left|\left\langle \overline{m}_y^R \right\rangle_{Z_R(x)} - \left\langle \overline{m}_y^R \right\rangle_{Z_R(x')}\right| + \left|\nabla m^R(p') - \left\langle \overline{m}_y^R \right\rangle_{Z_R(x')}\right| \\
&\le 2(2C+1)R^\beta + \tilde{C}|x - x'|.
\end{aligned}$$

Thus we have for all $p, p' \in \Sigma(R)$ the estimate

$$\left|\nabla m^R(p) - \nabla m^R(p')\right| \le \begin{cases} C|p-p'|^\beta & \text{if } |p-p'| \le R \\ (4C+2+\tilde{C})|p-p'|^\beta & \text{if } R < |p-p'| \le 1 \\ 2\tilde{C}|p-p'|^\beta, & \text{if } 1 < |p-p'|. \end{cases}$$

□

**Theorem 35.** *(i) For $R$ small enough, $m^R \in H^2(\Sigma(R)) + \chi \cap C^1(\Sigma(R))$.*
*(ii) We have*

$$\lim_{R\to 0} \frac{1}{R}\|m^R - m^{\text{red}}\|_{H^1(\Sigma(R))} = 0, \tag{51}$$

$$\lim_{R\to 0} \|m^R - m^{\text{red}}\|_{C^1(\Sigma(R))} = 0, \tag{52}$$

*Proof.* (i) Because of Theorem 34 we have $m^R \in C^1(\Sigma(R))$. Moreover $m^R$ in $H^1(\Sigma, \mathbb{R}^3) + \chi$ so the right hand side of (26) is in $L^2(\Sigma)$ and by standard elliptic theory $m^R \in H^2(\Sigma(R)) + \chi$.

(ii) Lemma 14 implies (51) and Theorem 34 implies (52). □

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