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Moving domain walls in magnetic nanowires

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Abstract

This paper investigates the reversal of magnetic nanowires via a perturbation argument from the static case. We consider the gradient flow equation of the micromagnetic energy including the nonlocal stray field energy. For thin wires and weak external magnetic fields we show the existence of travelling wave solutions. These travelling waves are almost constant on the cross section and can thus be seen as moving domain walls of a type called transverse wall.

1 Introduction

Because of possible technical applications [1, 10] in the recent years there has been a growing interest in magnetic nanowires and especially in their reversal modes. It is known that the reversal of the magnetisation starts at one end of the wire and then a domain wall separating the already reversed part from the not yet reversed part is propagating through the wire.

In the micromagnetic model, the evolution of the magnetisation is described by the Landau-Lifshitz-Gilbert (LLG) equation. We simplify this equation taking the overdamped limit, that is, we consider the gradient flow equation of the micromagnetic energy. Viewing static domain walls as travelling waves with speed 0, we show the existence of travelling wave solutions for thin wires and weak external magnetic fields via a perturbation argument. This argument relies crucially on the fact that the wires are thin, since we need strong regularity of the static domain wall. We have proved strong regularity in the case of thin wires [7], and we cannot expect it for thick wires where the examples of low energy configurations are vortex walls which have a singularity and are not even continuous [6].

For thin wires, static domain walls are almost constant on the cross section [6]. Thus, after perturbing the equation with a weak external field, the moving domain walls are still almost constant on the cross section. Such a reversal mode has been observed in numerical simulations [4, 5, 11] and is called transverse mode.

Various models for the transverse mode have been analysed previously. Thiaville and Nakatani [10] study a one dimensional model for the transverse mode and compare it with numerical simulations. Carbou and Labbé [3] consider a similar model. They prove that one dimensional domain walls are asymptotically stable. Sanchez [9] considers the limit of the Landau-Lifshitz equation when the diameter of the domain and the exchange coefficient in the equation simultaneously tend to zero and performs an asymptotic expansion.

The final goal in understanding the transverse mode is to find solutions to the full Landau-Lifshitz-Gilbert equation, to describe their properties, and to rigorously derive a reduced theory. This paper is a step towards that goal that, contrary to the other approaches, takes into account the full three dimensional structure of the problem. We expect that the methods developed in this paper can be applied to find solutions for the full Landau-Lifshitz-Gilbert equation.

1.1 Static domain walls

We work in the framework of micromagnetism. This is a mesoscopic continuum theory that assigns a nonlocal nonconvex energy to each magnetisation m from the domain $\Sigma \subset \mathbb{R}^3$ to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Experimentally observed ground states correspond to minimisers of the micromagnetic energy functional. When appropriately rescaled, for a soft magnetic material with an external field of strength h in direction of $\vec{e}_x$ this energy is

$$E_h(m) = \int_\Sigma \left(|\nabla m|^2 + h\vec{e}_x \cdot m\right) + \int_{\mathbb{R}^3} |H(m)|^2. \tag{1}$$

Here $H(m) : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection of m on gradient fields, i.e.,

$$H(m) = \nabla u \qquad \text{with } \Delta u = \operatorname{div} m \text{ in } \mathbb{R}^3. \tag{2}$$

We consider magnetisations where the domain $\Sigma(R) = \mathbb{R} \times D_R$ is an infinite cylinder with radius R and set

$$\mathcal{M}(R) := \left\{m : \Sigma(R) \to \mathbb{S}^2 \,|\, E_0(m) < \infty\right\}. \tag{3}$$

To specify the conditions at $\pm\infty$ we define a smooth function $\chi : \mathbb{R} \to \mathbb{R}^3$ with $\lim_{x\to\pm\infty} \chi(x) = \pm\vec{e}_x$ and set

$$\chi : \mathbb{R} \to \mathbb{R}^3, \qquad x \mapsto \tanh(x)\vec{e}_x. \tag{4}$$

In [6] we have shown that for $m : \Sigma(R) \to \mathbb{S}^2$ the condition $E_0(m) < \infty$ is equivalent to the statement that one of the maps four maps $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma(R))$. Thus, to single out the magnetisations that correspond to a 180 degree domain wall we define

$$\mathcal{M}_l(R) := \left\{m : \Sigma(R) \to \mathbb{S}^2 \,|\, m - \chi\vec{e}_x \in H^1(\Sigma(R))\right\}. \tag{5}$$

For every $R > 0$ there exist energy minimising 180 degree domain walls, i.e., minimisers of E_0 in $\mathcal{M}_l(R)$ [6]. For $R \to 0$ the energy minimisation problem $\Gamma-$converges to a reduced, one dimensional problem whose minimiser can be calculated explicitly to be

$$m^{\text{red}} \colon \mathbb{R} \to \mathbb{S}^2, \qquad x \mapsto \left(\tanh(\tfrac{x}{\sqrt{2}}), \frac{1}{\cosh(\frac{x}{\sqrt{2}})}, 0\right). \tag{6}$$

In [7] we have shown that the minimiser converge to m^{red} not only in a topology implied by the energy estimates but also in stronger norms.

Theorem 1. *Let m^R be a minimiser of E_0 in $\mathcal{M}_l(R)$.*
(i) For R small enough, $m^R \in H^2(\Sigma(R)) + \chi \cap C^1(\Sigma(R))$.
(ii) We have

$$\begin{aligned} \lim_{R\to 0} \frac{1}{R}\|m^R - m^{\mathrm{red}}\|_{H^1(\Sigma(R))} &= 0, \\ \lim_{R\to 0} \|m^R - m^{\mathrm{red}}\|_{C^1(\Sigma(R))} &= 0. \end{aligned}$$

1.2 The dynamic model

We assume that the evolution of the magnetisation can be described by gradient flow of the energy under the condition $|m| \equiv 1$ with Neumann boundary conditions, that is,

$$\partial_t m = -\delta_m E_h(m) + (\delta_m E_h(m)\cdot m)m \quad \text{in } \Sigma(R), \qquad \partial_\nu m = 0 \quad \text{on } \partial\Sigma(R), \tag{7}$$

where

$$\delta_m E_h(m) = -2\Delta m + 2H(m) - h\vec{e}_x. \tag{8}$$

This equation is the overdamped limit of the Landau-Lifshitz-Gilbert equation. We are interested in travelling wave solutions. Because of the rotational symmetry of the cylinder we have to take into account that the solutions may rotate around the axis of the cylinder. We set

$$Q_\phi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}, \qquad \tilde{Q}_\phi := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{9}$$

and note that $\partial_t Q_{\omega t} = \omega \tilde{Q}_{\omega t + \frac{\pi}{2}}$. Rotating travelling waves with speed c and angular velocity ω satisfy

$$m(t,x,y) = Q_{\omega t}\, m(0, Q_{-\omega t}(x - ct, y)).$$

Defining

$$\Phi(m) := \begin{pmatrix} m_x \\ m_{y_2} \\ -m_{y_1} \end{pmatrix} + \begin{pmatrix} 0 & & \\ & \partial_{y_1} m_{y_1} & \partial_{y_2} m_{y_1} \\ & \partial_{y_1} m_{y_2} & \partial_{y_2} m_{y_2} \end{pmatrix} \begin{pmatrix} 0 \\ y_2 \\ -y_1 \end{pmatrix}, \tag{10}$$

we have

$$\begin{aligned} &\partial_t m(t,x,y) \\ &\quad= \omega \tilde{Q}_{\omega t + \frac{\pi}{2}} m\,(0, Q_{-\omega t}(x-ct,y)) - c Q_{\omega t}\, \partial_x m\,(0, Q_{-\omega t}(x-ct,y)) \\ &\qquad - Q_{\omega t} \nabla_y m(0, Q_{-\omega t}(x-ct,y))\, \omega \tilde{Q}_{-\omega t + \frac{\pi}{2}} \vec{y} \\ &\quad= -c\partial_x m(t,x,y) + \omega \tilde{Q}_{\frac{\pi}{2}}\, m(t,x,y) - \omega \nabla_y m(t,x,y) \tilde{Q}_{\frac{\pi}{2}} \vec{y} \\ &\quad= -c\partial_x m(t,x,y) - \omega \Phi(m(t,x,y)). \end{aligned}$$

In particular, rotating travelling waves that are a solution of (7) satisfy the stationary equation

$$\begin{aligned} -\delta_m E_h(m) + (\delta_m E_h(m)\cdot m)m + c\partial_x m + \omega\Phi(m) &= 0 \quad \text{in } \Sigma(R), \\ \partial_\nu m &= 0 \quad \text{on } \partial\Sigma(R). \end{aligned} \tag{11}$$

To find solutions of (11) we consider first the case $h = 0$ and then use a perturbation argument. For this we have to work in a function space that is large enough to contain the solutions and small enough that the left hand side of (11) is differentiable in this function space. As we will see, $H^2(\Sigma(R), \mathbb{R}^3) + \chi$ is a good choice. In this space we have to restrict the search to solutions with $|m| \equiv 1$. We have to include further conditions in the set of admissible solutions to break the translation invariance and the rotation invariance of the problem. For $c = 0$, $\omega = 0$, $h = 0$ equation (11) simplifies to

$$0 = -\delta_m E_0(m) + (\delta_m E_0(m) \cdot m)m \quad \text{in } \Sigma(R), \qquad \partial_\nu m = 0 \quad \text{on } \partial\Sigma(R). \quad (12)$$

This is the Euler Lagrange equation for the energy E_0 under the condition $|m| = 1$. Thus, Theorem 1 implies that, for $R > 0$ small enough, minimisers m^R of the energy E_0 are solutions of (12) in $H^2(\Sigma(R), \mathbb{R}^3) + \chi$.

We proceed a follows.

(1.) Depending on m^R we define the set of admissible functions $\mathcal{S}$ and show that $\mathcal{S}$ is a Banach submanifold of $H^2(\Sigma(R), \mathbb{R}^3) + \chi$.

(2.) We find a continuously differentiable function

$$N \colon \mathcal{S} \times L^2(\Sigma(R), \mathbb{R}) \times \mathbb{R}^3 \to L^2(\Sigma(R), \mathbb{R}^3) \times \mathbb{R}$$

such that (m, c, ω, h) is a solution of (11) if and only if there exists $\alpha \in L^2(\Sigma(R), \mathbb{R})$ that satisfies $N(m, \alpha, c, \omega, h) = (0, h)$.

(3.) We show that the derivative DN of N in $(m^R, 0, 0, 0, 0)$ is invertible.

(4.) Then, according to the inverse function theorem [12, Theorem 73.B, p.552], there exists a neighbourhood U of $(m^R, 0, 0, 0, 0)$ and a neighbourhood V of $(0, 0)$ such that $N|_U \to V$ is bijective. In particular, there exists $h_0 > 0$, such that for all $|h| < h_0$ there is $m_h, \alpha_h, c_h, \omega_h$ with $N(m_h, \alpha_h, c_h, \omega_h, h) = 0$. In other words, for all $|h| < h_0$ there exists a solution of (11).

In Section 2 we go through the steps (1.)–(4.) to show the existence of travelling wave solutions for small radii and small external magnetic field. The arguments of Section 2 use the invertibility of an operator representing the "interesting" part of $DN(m^R, 0, 0, 0, 0)$. This invertibility is shown in Section 3 and relies on the fact that m^R is close to m^{red}.

1.3 Definitions and Notation

The letter p denotes a point in $\mathbb{R}^3$ and has the components $p = (x, y_1, y_2) = (x, y)$. A map f with values in $\mathbb{R}^3$ has the components $f = (f_x, f_{y_1}, f_{y_2})$. We write f_y for $(0, f_{y_1}, f_{y_2})$, i.e., we view f_y as a map to $\{0\} \times \mathbb{R}^2$. For a set $A \subset L^2(\mathbb{R}^n)$, we denote the closure of A in $L^2(\mathbb{R}^n)$ by A_{L^2} and the characteristic function by $\mathbb{1}_A$. For a, $b \in \mathbb{R}^n$, $n \in \mathbb{N}$ we denote the scalar product by $a \cdot b$. For $\Omega \subset \mathbb{R}^3$ and $f, g \colon \Omega \to \mathbb{R}^n$, $n \in \mathbb{N}$, we set

$$\langle f, g \rangle_\Omega \;:=\; \int_\Omega f(p) \cdot g(p)\, dp,$$

whenever the integral on the right hand side is defined. Moreover we set

$$D_R(p) := \{q \in \mathbb{R}^2 \ : \ |p-q| < R\}, \quad D_R := D_R(0), \quad \Sigma := \Sigma(R) := \mathbb{R} \times D_R.$$

Note that we write Σ instead of $\Sigma(R)$ when the radius R is clear from the context. The definitions of χ in (4), of $\mathcal{M}_l$ in (5), and of Φ in (10) remain valid. With m^{red} as in (6) we define

$$m_R^{\text{red}} \colon \Sigma(R) \to \mathbb{S}^2, \quad (x,y) \mapsto m^{\text{red}}(x)\mathbb{1}_{D_R}(y), \tag{13}$$

For $m \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ let $H(m) : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection of m on gradient fields as in (2). For $m : \Sigma(R) \to \mathbb{R}^3$ the micromagnetic energy without external magnetic field is denoted by $E(m) = E(m,R)$ and the micormagnetic energy incuding the external magnetic field is denoted by $E_h(m) = E_h(m,R)$.

Finally, let $m^R \colon \Sigma(R) \to \mathbb{S}^2$ always be a minimiser of E in $\mathcal{M}_l(R)$. To break the translation and rotation invariance we additionally require

$$\|m^R - m^{\text{red}}\|_{L^2(\Sigma)} \le \|v - m^{\text{red}}\|_{L^2(\Sigma)} \quad \text{for all other minimizers } v \in \mathcal{M}_l(R).$$

2 The perturbation argument

As described above, the first step in the perturbation argument is to show that we are working on a sufficiently smooth manifold. Set

$$\mathcal{S}^R := \left\{ f \in H^2(\Sigma(R), \mathbb{R}^3) + \chi \;\middle|\; \begin{array}{ll} |f| \equiv 1, & \partial_\nu f = 0 \text{ on } \partial\Sigma, \\ \left\langle \partial_x m^R, f \right\rangle_\Sigma = 0, & \left\langle \Phi(m^R), f \right\rangle_\Sigma = 0. \end{array} \right\},$$

$$T\mathcal{S}^R := \left\{ f \in H^2(\Sigma(R), \mathbb{R}^3) \;\middle|\; \begin{array}{ll} f \cdot m^R \equiv 0, & \partial_\nu f = 0 \text{ on } \partial\Sigma, \\ \left\langle \partial_x m^R, f \right\rangle_\Sigma = 0, & \left\langle \Phi(m^R), f \right\rangle_\Sigma = 0. \end{array} \right\}.$$

Lemma 2. *There exists $R_0 > 0$, such that for all $R \le R_0$ the set $\mathcal{S}^R$ is a submanifold of $H^2(\Sigma, \mathbb{R}^3) + \chi$. The tangent space of $\mathcal{S}^R$ in m^R is $T\mathcal{S}^R$.*

Proof. We show the Lemma in two steps. We define

$$\mathcal{W}^R := \left\{ m \in H^2(\Sigma, \mathbb{R}^3) \,\middle|\, \partial_\nu m|_{\partial\Sigma} = 0, \ \left\langle m, \partial_x m^R \right\rangle_\Sigma = 0, \ \left\langle m, \Phi(m^R) \right\rangle_\Sigma = 0 \right\}.$$

First, since $\partial_x m^R, \Phi(m^R) \in L^2(\Sigma)$ and since the trace of a function in $H^2(\Sigma)$ is in $H^1(\partial\Sigma)$, the set $\mathcal{W} + \chi$ is a closed affine subspace of $H^2(\Sigma, \mathbb{R}^3) + \chi$.

Second, we show that $\mathcal{S}^R$ is a submanifold of $\mathcal{W}^R + \chi$. Set

$$\phi \colon \mathcal{W}^R + \chi \to \left\{ f \in H^2(\Sigma, \mathbb{R}) \;\middle|\; \partial_\nu f|_{\partial\Sigma} = 0 \right\}, \qquad m \mapsto |m| - 1,$$

then $\mathcal{S}^R = \phi^{-1}(0)$. On $\{m \in \mathcal{W}^R : |\phi(m)| < 1\}$ the function ϕ is continuously differentiable and the derivative in m is

$$D\phi(m) \colon \mathcal{W}^R \to \left\{ f \in H^2(\Sigma, \mathbb{R}) \;\middle|\; \partial_\nu f|_{\partial\Sigma} = 0 \right\}, \qquad g \mapsto \frac{g \cdot m}{|m|}. \tag{14}$$

If R is small enough, for every $m \in \mathcal{S}^R$ the differential $D\phi(m)$ is surjective: Since the equality $\partial_x m_R^{\mathrm{red}} \cdot \Phi(m_R^{\mathrm{red}}) = 0$ implies

$$\begin{aligned} &\det \begin{pmatrix} \langle \partial_x m_R^{\mathrm{red}}, \partial_x m_R^{\mathrm{red}} \rangle_\Sigma & \langle \partial_x m^R, \Phi(m_R^{\mathrm{red}}) \rangle_\Sigma \\ \langle \partial_x m_R^{\mathrm{red}}, \Phi(m_R^{\mathrm{red}}) \rangle_\Sigma & \langle \Phi(m_R^{\mathrm{red}}), \Phi(m_R^{\mathrm{red}}) \rangle_\Sigma \end{pmatrix} \\ &= \pi R^2 \left(\|\partial_x m_R^{\mathrm{red}}\|^2_{L^2(\mathbb{R})} + \|\Phi(m_R^{\mathrm{red}})\|_{L^2(\mathbb{R})} \right), \end{aligned}$$

so with Theorem 1 (ii) there exists R_0 such that for all $R \le R_0$ we have

$$\det \begin{pmatrix} \langle \partial_x m^R, \partial_x m^R \rangle_\Sigma & \langle \partial_x m^R, \Phi(m^R) \rangle_\Sigma \\ \langle \partial_x m^R, \Phi(m^R) \rangle_\Sigma & \langle \Phi(m^R), \Phi(m^R) \rangle_\Sigma \end{pmatrix} > 0.$$

Therefore, for every $f \in H^2(\Sigma, \mathbb{R})$ with $\partial_\nu f|_{\partial\Sigma} = 0$ we can find unique numbers b_1, b_2 such that

$$\begin{aligned} \langle fm + b_1 \partial_x m^R + b_2 \Phi(m^R),\ \partial_x m^R \rangle_\Sigma &= 0, \\ \langle fm + b_1 \partial_x m^R + b_2 \Phi(m^R),\ \Phi(m^R) \rangle_\Sigma &= 0, \end{aligned}$$

and $fm + b_1 \partial_x m^R + b_2 \Phi(m^R)$ is a pre-image of f in $\mathcal{W}^R$. Moreover, since in a Hilbert space every subspace splits, in particular $D\phi^{-1}(0)$ splits. Thus 0 is a regular value of ϕ and we can apply [12, Thm. 73C, p.556] to conclude that $\mathcal{S}^R$ is a submanifold of $\mathcal{W}^R + \chi$. Because of (14) the space $T\mathcal{S}^R$ is the tangent space of $\mathcal{S}^R$ in m^R. □

We consider the map

$$s\colon \mathcal{S}^R \to L^2(\Sigma, \mathbb{R}^3), \quad m \mapsto -\delta_m E_h(m) + (\delta_m E_h(m) \cdot m)m,$$

that is, with (8),

$$s(m) = \underbrace{2(\Delta m - (\Delta m \cdot m)m - H(m) + (H(m) \cdot m)m)}_{s_1} + \underbrace{h\vec{e}_x - (h\vec{e}_x \cdot m)m}_{s_2}.$$

The space $H^2(\Sigma, \mathbb{R}) + \chi$ embeds into $C^0(\Sigma, \mathbb{R})$, and functions $m \mapsto \Delta m$, and $m \mapsto H(m)$ are continuous linear maps from $\mathcal{S}^R$ to $L^2(\Sigma, \mathbb{R}^3)$. For the last statement see [8, Lemma 2.6]. Thus $s_1\colon \mathcal{S}^R \to L^2(\Sigma, \mathbb{R}^3)$ is well defined and continuously differentiable.

Moreover, we have

$$|h\vec{e}_x - (h\vec{e}_x \cdot m)m| = h\left|(1 - m_x^2)\vec{e}_x + m_x m_y\right| \le 2h|m_y|,$$

so $s_2\colon \mathcal{S}^R \to L^2(\Sigma, \mathbb{R}^3)$ is well defined and continuously differentiable, too.

Thus we can define the continuously differentable map

$$\begin{aligned} &N^R\colon \mathcal{S}^R \times L^2(\Sigma, \mathbb{R}) \times \mathbb{R}^3 \to L^2(\Sigma, \mathbb{R}^3) \times \mathbb{R}, \\ &(m, \alpha, c, \omega, h) \mapsto \\ &\quad (-\delta_m E_h(m, R) + (\delta_m E_h(m, R) \cdot m)m + c\partial_x m + \omega\Phi(m) + \alpha m,\ h). \end{aligned}$$

Since $(-\delta_m E_h(m, R) + (\delta_m E_h(m, R) \cdot m)m + c\partial_x m + \omega\Phi(m)) \perp m$ for all $m \in \mathcal{S}^R$ we have $N^R(m, \alpha, c, \omega, h) = (0, h)$ if and only if m is a solution of (11) and $\alpha = 0$.

The differential of N^R in $(m^R, 0_{L^2(\Sigma,\mathbb{R})}, 0_{\mathbb{R}^3})$ is

$$\begin{aligned} DN^R(m^R,0,0)\colon T\mathcal{S}^R \times L^2(\Sigma, R^3)\times\mathbb{R}^3 \;&\to\; L^2(\Sigma,\mathbb{R}^3)\times\mathbb{R} \\ (g,\alpha,c,\omega,h) \;&\mapsto\; (-L^R(g) + c\partial_x m^R + \omega\Phi(m_0) + \alpha m^R,\ h), \end{aligned}$$

where

$$\begin{aligned} L^R\colon H^2(\Sigma,\mathbb{R}^3) &\to L^2(\Sigma,\mathbb{R}^3) \\ g \;\mapsto\; & \delta_m E(g,R) - (\delta_m E(g,R)\cdot m^R)m^R \\ & - (\delta_m E(m^R,R)\cdot g)m^R - (\delta_m E(m^R,R)\cdot m^R)g. \end{aligned} \tag{15}$$

We will consider the restrictions of L^R to different subspaces of $H^2(\Sigma,\mathbb{R}^3)$. We will call this restrictions L^R as well, but name always the domain and the range.

Lemma 3. *For all $R>0$ and all $g, f\in T\mathcal{S}^R$ we have*

$$L^R(g) \;=\; \delta_m E(g,R) - (\delta_m E(g,R)\cdot m^R)m^R - (\delta_m E(m^R,R)\cdot m^R)g, \tag{16}$$

$$L^R(g)\cdot f \;=\; \delta_m E(g,R)\cdot f - (\delta_m E(m^R,R)\cdot m^R)g\cdot f. \tag{17}$$

Moreover $L^R(T\mathcal{S}^R)\subseteq (T\mathcal{S}^R)_{L^2}$ and the operator $L^R\colon T\mathcal{S}^R\to(T\mathcal{S}^R)_{L^2}$ is symmetric.

Proof. Since m^R is a solution of (12), $\delta_m E(m^R,R)$ is pointwise parallel to m^R. The elements of $T\mathcal{S}^R$ are pointwise orthogonal to m^R. This implies (16) and (17). Since the elements of $T\mathcal{S}^R$ satisfy Neumann boundary conditions, for all $g,f\in T\mathcal{S}^R$ we have $\langle L^R f, g\rangle_\Sigma = \langle f, L^R g\rangle_\Sigma$.

It remains to show that $L^R(T\mathcal{S}^R)\subseteq(T\mathcal{S}^R)_{L^2}$. We have

$$(T\mathcal{S}^R)_{L^2} \;:=\; \left\{ f\in L^2(\Sigma(R),\mathbb{R}^3) \;\middle|\; \begin{array}{l} f\cdot m^R\equiv 0, \\ \langle\partial_x m^R, f\rangle_\Sigma = 0, \quad \langle\Phi(m^R), f\rangle_\Sigma = 0 \end{array} \right\}.$$

Looking at (16), we see that $L^R(g)\perp m^R$. Set $v(t,x,y):=m^R(x+t,y)$. Then $v(t,\cdot)$ satisfies for all $t\in\mathbb{R}$ the equation

$$0 \;=\; \delta_m E(v(t,\cdot),R) - (\delta_m E(v(t,\cdot),R)\cdot v(t,\cdot))\,v(t,\cdot),$$

therefore we have for all $g\in T\mathcal{S}^R$

$$\begin{aligned} 0 \;&=\; \partial_t \left.\langle \delta_m E(v(t,\cdot),R) - (\delta_m E(v(t,\cdot),R)\cdot v(t,\cdot))\,v(t,\cdot),\ g\rangle_\Sigma\right|_{t=0} \\ &=\; \langle L(\partial_x m^R), g\rangle_\Sigma \;=\; \langle L(g), \partial_x m^R\rangle_\Sigma\,. \end{aligned}$$

Analogously, with Q_ϕ as in (9) we have for $w(\phi,x,y):=Q_\phi(m^R(Q_{-\phi}(x,y))$ the equation

$$0 \;=\; \delta_m E(w(\phi,\cdot),R) - (\delta_m E(w(\phi,\cdot),R)\cdot v(\phi,\cdot))\,w(\phi,\cdot)$$

and thus for all $g\in T\mathcal{S}^R$

$$\begin{aligned} 0 \;&=\; \partial_\phi\, \langle \delta_m E(w(\phi,\cdot),R) - (\delta_m E(w(\phi,\cdot),R)\cdot v(\phi,\cdot))\,w(\phi,\cdot),\ g\rangle_\Sigma|_{\phi=0} \\ &=\; \langle L(\Phi(m^R)), g\rangle_\Sigma \;=\; \langle L(g), \Phi(m^R)\rangle_\Sigma\,. \end{aligned}$$

□

Note that $DN^R(m^R, 0, 0)$ is bijective if and only if

(a) $\partial_x m^R$ and $\Phi(m^R)$ are linearly independent,

(b) $L^R\colon T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ bijective.

Since $\lim_{R\to 0} \|m^R - m^{\text{red}}\|_{C^1(\Sigma(R))} = 0$ and since $\partial_x m^{\text{red}}$ and $\Phi(m^{\text{red}})$ are linearly independent, (a) is satisfied if R is small enough. In Section 3 we will show that (b) is satisfied for small R, too. Altogether, we have the following theorem.

Theorem 4. *(m, c, ω) is a solution of (11) if and only if there exists $\alpha \in L^2(\Sigma, \mathbb{R})$ such that $N^R(m, \alpha, c, \omega, h) = (0, h)$.*

The function N^R is continuously differentiable and, if R is small enough, $DN^R(m^R, 0, 0)$ is bijective.

If N^R is continuously differentiable and $DN^R(m^R)$ is invertible, according to the inverse function theorem [12, Theorem 73.B, p. 552] there exists a neighbourhood U of $(m^R, 0_{L^2(\Sigma,\mathbb{R})}, 0_{\mathbb{R}^3})$ and a neighbourhood V of $(0_{L^2(\Sigma,\mathbb{R}^3)}, 0_{\mathbb{R}})$ such that $N^R|_U \to V$ is bijective. So for every h small enough, we can find $m_h, \alpha_h, c_h, \omega_h$ such that $N^R(m_h, \alpha_h, c_h, \omega_h, h) = 0$. That is, we have proved our main theorem.

Theorem 5. *For all $R > 0$ small enough there exists $h_R > 0$ such that for all h with $h < h_R$ there is exists a solution (m_h, c_h, ω_h) of (11).*

3 Invertibility of L^R

The goal of this section is to prove the following theorem.

Theorem 6. *For R small enough, the operator $L^R\colon T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$, as defined in (15), is invertible, and its inverse is continuous.*

We proceed in two steps. First, we define a map L_0^R and show that for functions m in a certain space $T\mathcal{S}_0^R$ we have $\langle L_0^R(m), m\rangle_\Sigma \geq \frac{1}{4}\|m\|^2_{L^2(\Sigma)}$. Then we prove that, for small R, the operator L^R is similar to L_0^R and the space $T\mathcal{S}^R$ is similar to $T\mathcal{S}_0^R$.

The map L_0^R is related to the energy functional

$$E^0(\cdot, R)\colon \mathcal{M}(R) \to \mathbb{R}, \qquad m \mapsto \int_\Sigma |\partial_x m|^2 + \frac{1}{2}|m_y|^2 + 20R^2|\nabla_y m|^2.$$

Then we have

$$\delta_m E^0(m, R) \;=\; -2\partial_{xx} m + (0, m_{y_1}, m_{y_2}) - 40R^2\Delta_y m.$$

Lemma 7. *The minimiser of E^0 in $\mathcal{M}_l(R)$ is unique up to translation and rotation. It is given by*

$$m_R^{\text{red}}\colon \quad (x, y) \mapsto m^{\text{red}}(x) = \left(\tanh\left(\tfrac{x}{\sqrt{2}}\right),\ \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)},\ 0\right),$$

and we have

$$
\begin{aligned}
|\partial_x m_R^{\mathrm{red}}(x,y)| &= \frac{1}{\sqrt{2}}|m_y(x,y)| \\
\frac{\partial_x m_R^{\mathrm{red}}(x,y)}{|\partial_x m_R^{\mathrm{red}}(x,y)|} &= \left(\frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)}, -\tanh\left(\tfrac{x}{\sqrt{2}}\right), 0 \right), \\
\Phi(m_R^{\mathrm{red}}(x,y)) &= \left(0, 0, \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)} \right).
\end{aligned}
$$

Proof. For every function $m \in \mathcal{M}_l(R)$ we have

$$
E^0(m) = \int_{D_R} E_{\mathrm{red}}(m(\cdot,y))\, dy + 20R^2 \|\nabla_y m\|^2_{L^2(\Sigma)}.
$$

Since m^{red} is the only minimiser of E_{red} in $\{m \in H^1(\mathbb{R}) + \chi : |m| = 1\}$, up to translation and rotation [8, Lemma 2.26], the function m_R^{red} is the only minimiser of E^0 in $\mathcal{M}_l(R)$, up to translation and rotation.

A direct calculation yields the results for $\partial_x m_R^{\mathrm{red}}$ and $\Phi(m_R^{\mathrm{red}})$. □

Remark. Since the domain of definition will not always be clear from the context, for the rest of this section we will distinguish the functions m^{red} and m_R^{red}.

We now set, in analogy to (15),

$$
\begin{aligned}
&L_0^R \colon H^2(\Sigma, \mathbb{R}^3) \to L^2(\Sigma, \mathbb{R}^3), \\
&\quad g \mapsto \delta_m E^0(g,R) - (\delta_m E^0(g,R) \cdot m_R^{\mathrm{red}}) m_R^{\mathrm{red}} \\
&\qquad - (\delta_m E^0(m_R^{\mathrm{red}},R) \cdot g) m_R^{\mathrm{red}} - (\delta_m E^0(m_R^{\mathrm{red}},R) \cdot m_R^{\mathrm{red}}) g, \qquad (18)
\end{aligned}
$$

and define

$$
T\mathcal{S}_0^R := \left\{ f \in H^2(\Sigma(R), \mathbb{R}^3) \;\middle|\; \begin{array}{ll} f \cdot m_R^{\mathrm{red}} \equiv 0, & \partial_\nu f = 0 \text{ on } \partial\Sigma, \\ \left\langle \partial_x m_R^{\mathrm{red}}, f \right\rangle = 0, & \left\langle \Phi(m_R^{\mathrm{red}}), f \right\rangle = 0. \end{array} \right\}.
$$

Lemma 8. *For all $R > 0$ and all $g, f \in T\mathcal{S}_0^R$ we have*

$$
L_0^R(g) = \delta_m E^0(g,R) - (\delta_m E^0(g,R) \cdot m_R^{\mathrm{red}}) m_R^{\mathrm{red}} - (\delta_m E^0(m_R^{\mathrm{red}},R) \cdot m_R^{\mathrm{red}}) g, \qquad (19)
$$

$$
L_0^R(g) \cdot f = \delta_m E^0(g,R) \cdot f - (\delta_m E^0(m_R^{\mathrm{red}},R) \cdot m_R^{\mathrm{red}}) g \cdot f. \qquad (20)
$$

Moreover, $L_0^R(T\mathcal{S}_0^R) \subseteq (T\mathcal{S}_0^R)_{L^2}$, and the operator $L_0^R \colon T\mathcal{S}_0^R \to (T\mathcal{S}_0^R)_{L^2}$ is symmetric.

Proof. We can argue exactly as in Lemma 3. □

Theorem 9. *For all $R > 0$ and all $m \in T\mathcal{S}_0^R$ we have*

$$
\left\langle L_0^R(m), m \right\rangle_\Sigma \geq \frac{1}{4} \|m\|^2_{L^2(\Sigma)}
$$

Proof. The relations $|\partial_x m^{\rm red}| = \frac{|m_y^{\rm red}|}{\sqrt{2}}$ and

$$\partial_{xx} m^{\rm red} \cdot m^{\rm red} + |\partial_x m^{\rm red}|^2 \;=\; \partial_x(\partial_x m^{\rm red} \cdot m^{\rm red}) \;=\; 0$$

imply

$$\delta_m E^0(m_R^{\rm red}, R) \cdot m_R^{\rm red} \;=\; -2\partial_{xx} m_R^{\rm red} \cdot m_R^{\rm red} + |(m_R^{\rm red})_y|^2 \;=\; 2|(m_R^{\rm red})_y|^2.$$

Thus, with Lemma 8, for all $g, h \in T\mathcal{S}_0^R$ we have

$$\begin{aligned} L_0^R(g) \cdot h \;&=\; \delta_m E^0(g, R) \cdot h - (\delta_m E^0(m_R^{\rm red}, R) \cdot m_R^{\rm red}) g \cdot h \\ &=\; \big(\delta_m E_0(g, R) - 2|(m_R^{\rm red})_y|^2 g\big) \cdot h. \end{aligned}$$

We define the vector $\vec{e}_s$ to be the unit vector in direction of $\partial_x m^{\rm red}$, i.e.,

$$\vec{e}_s(x) \;:=\; \frac{\partial_x m^{\rm red}(x)}{|\partial_x m^{\rm red}(x)|} \;=\; \big(m_{y_1}^{\rm red}(x), m_x^{\rm red}(x),\ 0\big)\,,$$

and the sets

$$\begin{aligned} \mathcal{W}_1 \;&:=\; \left\{ m \in T\mathcal{S}_0^R \;:\; \int_{D_R} m(x,y)\, dy \equiv 0 \right\}, \\ \mathcal{W}_2 \;&:=\; \big\{ m \in T\mathcal{S}_0^R \;:\; m(x,y) \;=\; \alpha(x)\, \vec{e}_{y_2} \text{ for some } \alpha \in H^2(\mathbb{R}, \mathbb{R}) \big\}\,, \\ \mathcal{W}_3 \;&:=\; \big\{ m \in T\mathcal{S}_0^R \;:\; m(x,y) \;=\; \alpha(x)\, \vec{e}_s(x) \;\text{ for some } \alpha \in H^2(\mathbb{R}, \mathbb{R}) \big\}\,. \end{aligned}$$

Then $T\mathcal{S}_0^R$ is the direct sum of $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$, and we have $L_0^R(\mathcal{W}_i) \subset (\mathcal{W}_i)_{L^2}$ for $i \in \{1,2,3\}$.

<u>Assume $m \in \mathcal{W}_1$.</u> Using the Poincaré inequalitywe have

$$\begin{aligned} &\big\langle L_0^R m, m \big\rangle_\Sigma \\ &\;=\; 40R^2 \|\nabla_y m\|_{L^2(\Sigma)}^2 + 2\|\partial_x m\|_{L^2(\Sigma)}^2 + \|m_y\|_{L^2(\Sigma)}^2 - 2\|(m_R^{\rm red})_y m\|_{L^2(\Sigma)}^2 \\ &\;\geq\; \frac{40}{16} \|m\|_{L^2(\Sigma)}^2 - 2\|m\|_{L^2(\Sigma)}^2 \;=\; \frac{1}{2}\|m\|_{L^2(\Sigma)}^2. \end{aligned}$$

<u>Assume $m \in \mathcal{W}_2$.</u> Then $m(x,y) = \alpha(x) \mathbb{1}_{D_R}(y) \vec{e}_{y_2}$ for some $\alpha \in H^2(\mathbb{R}, \mathbb{R})$, we have

$$L_0^R(m)|_{(x,y)} \;=\; \big(-2\,\partial_{xx}\alpha(x) + \alpha(x) - 2|m_y^{\rm red}(x)|^2 \alpha(x)\big)\, \mathbb{1}_{D_R}(y)\, \vec{e}_{y_2}. \qquad (21)$$

$$\big\langle L_0^R(m), m \big\rangle_\Sigma \;=\; \pi R^2 \left(2\|\partial_x \alpha\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} (1 - 2|m_y^{\rm red}|^2)\alpha^2 \right) \qquad (22)$$

and

$$1 - 2|m_y^{\rm red}(x)|^2 \;\geq\; \frac{1}{4} \qquad \text{for } |x| \geq 1.6. \qquad (23)$$

Since $\Phi(m_R^{\rm red}) \cdot \vec{e}_{y_2}$ is positive (Lemma 7), and since $\big\langle \Phi(m_R^{\rm red}), m \big\rangle = 0$, the function α has to change sign.

First, assume that α changes sign in $[-1.6, 1.6]$. We have

$$\inf_{\{f:[-1.6,1.6]\to\mathbb{R}, f \text{ changes sign}\}} \left(\frac{2\|\partial_x f\|_{L^2([-1.6,1.6])}^2}{\|f\|_{L^2([-1.6,1.6])}^2} \right) \;=\; \frac{2\pi^2}{3.2^2},$$

the infimum is attained and the minimizers are multiples of $x \mapsto \sin\left(\frac{\pi}{3.2}x\right)$. Thus we have

$$\begin{aligned} 2\|\partial_x \alpha\|^2_{L^2([-1.6,1.6])} &+ \int_{-1.6}^{1.6} (1-2|m_y^{\mathrm{red}}|^2)\alpha^2 \\ &\geq 2\|\partial_x \alpha\|^2_{L^2([-1.6,1.6])} - \|\alpha\|^2_{L^2([-1.6,1.6])} \\ &\geq \left(\frac{2\pi^2}{3.2^2} - 1\right) \|\alpha\|^2_{L^2([-1.6,1.6])}. \end{aligned}$$

and therefore, with (23) and (22)

$$\left\langle L_0^R(m), m\right\rangle_\Sigma \geq \pi R^2 \left(\left(\frac{2\pi^2}{3.2^2} - 1\right)\|\alpha\|^2_{[-1.6,1.6]} + \frac{1}{4}\|\alpha\|^2_{L^2([-1.6,1.6])}\right) \geq \frac{1}{4}\|m\|^2_\Sigma.$$

Now assume that α does not change sign in $[-1.6, 1.6]$ and let S_- be the set where α has the opposite sign as in $[-1.6, 1.6]$. With Lemma 7 we see that $\Phi(m^{\mathrm{red}_R}(x,y)) \cdot \vec{e}_{y_2} \geq 0.5$ for $|x| < 1.6$, and since $\left\langle \Phi(m_R^{\mathrm{red}}) \cdot \vec{e}_{y_2}, \alpha\right\rangle_{\mathbb{R}} = 0$ we have

$$\begin{aligned} \sqrt{1.6}\|\alpha\|_{L^2[-1.6,1.6]} &\leq \left|\left\langle \Phi(m_R^{\mathrm{red}}) \cdot \vec{e}_{y_2}, |\alpha|\right\rangle_{[-1.6,1.6]}\right| \leq \left|\left\langle \Phi(m_R^{\mathrm{red}}) \cdot \vec{e}_{y_2}, |\alpha|\right\rangle_{S_-}\right| \\ &\leq \int_{S_-} 2e^{-\frac{|x|}{\sqrt{2}}}|\alpha| \leq \int_{S_-} \sqrt{8}e^{-\frac{|x|}{\sqrt{2}}}|\partial_x\alpha| \\ &\leq \left\|\sqrt{8}e^{-\frac{|x|}{\sqrt{2}}}\right\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])} \|\partial_x\alpha\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])} \\ &\leq 1.1\|\partial_x\alpha\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])}. \end{aligned}$$

Thus (23) implies

$$\begin{aligned} \left\langle L_0^R(m), m\right\rangle_\Sigma &\geq \pi R^2 \left(\frac{1}{2}\|\alpha\|^2_{\mathbb{R}\setminus[-1.6,1.6]} + 2\|\partial_x\alpha\|^2_{L^2(\mathbb{R}\setminus[-1.6,1.6])} - \|\alpha\|^2_{L^2[-1.6,1.6]}\right) \\ &\geq \pi R^2 \left(\frac{1}{2}\|\alpha\|^2_{\mathbb{R}\setminus[-1.6,1.6]} + \left(\frac{2\sqrt{1.6}}{1.1} - 1\right)\|\alpha\|^2_{L^2[-1.6,1.6]}\right) \\ &\geq \frac{1}{2}\|m\|^2_\Sigma. \end{aligned}$$

<u>Assume $m \in \mathcal{W}_3$.</u> Then $m(x,y) = \alpha(x)\mathbb{1}_{D_R}(y)\vec{e}_s(x)$ for some $\alpha \in H^2(\mathbb{R},\mathbb{R})$. The function $L_0^R(m)$ is pointwise parallel to $\vec{e}_s$, we have $\partial_x\vec{e}_s \cdot \vec{e}_s = 0$ and

$$\begin{aligned} 0 &= \partial_x(\partial_x\vec{e}_s \cdot \vec{e}_s) = |\partial_x\vec{e}_s|^2 + \partial_{xx}\vec{e}_s \cdot \vec{e}_s \\ &= |\partial_x m^{\mathrm{red}}|^2 + \partial_{xx}\vec{e}_s \cdot \vec{e}_s = \frac{1}{2}|m_y^{\mathrm{red}}|^2 + \partial_{xx}\vec{e}_s \cdot \vec{e}_s. \end{aligned}$$

So $\partial_{xx}(\alpha\vec{e}_s) \cdot \vec{e}_s = \partial_{xx}\alpha + \frac{1}{2}|m_y^{\mathrm{red}}|^2\alpha$. Moreover, we have $\vec{e}_s \cdot \vec{e}_y = m_x^{\mathrm{red}}$ and therefore

$$\begin{aligned} L_0^R(m) \cdot \vec{e}_s &= -2\partial_{xx}\alpha + |m_y^{\mathrm{red}}|^2\alpha + |m_x^{\mathrm{red}}|^2\alpha - 2|m_y^{\mathrm{red}}|^2\alpha \\ &= -2\partial_{xx}\alpha + \left(1 - 2|m_y^{\mathrm{red}}|^2\right)\alpha \qquad (24) \end{aligned}$$

Comparing (24) and (21), we can conclude like in the case $m \in \mathcal{W}_2$ that $\left\langle L^R(m), m\right\rangle \geq \frac{1}{4}\|m\|^2_{L^2(\Sigma)}$. □

The next lemma compares the operators L_0^R and L^R on the space $H^2(\Sigma)$.

Lemma 10. *For each $\epsilon > 0$ there exists a radius $R_\epsilon > 0$ such that*

$$\left\langle L_0^R(m), m \right\rangle_\Sigma - \left\langle L^R(m), m \right\rangle_\Sigma \;\le\; \epsilon \|m\|^2_{H^1(\Sigma)}$$

for all $R < R_\epsilon$ and all $m \in H^2(\Sigma)$.

Proof. For $\epsilon \in]0,1]$ we can find $\tilde{R}_\epsilon \le \min\left(\frac{1}{\sqrt{20}}, \epsilon\right)$ such that for all $R < \tilde{R}_\epsilon$ the following inequalities hold (Theorem 1):

$$\|m_R^{\text{red}} - m^R\|_{C^1(\Sigma)} \le \epsilon, \quad \|m_R^{\text{red}} - m^R\|_{L^2(\Sigma)} \le \epsilon R, \quad \|\nabla_y m^R\|_{L^\infty(\Sigma)} \le \epsilon.$$

Define

$$\mathcal{A}(R) := \{f \in H^1_{\text{loc}}(\Sigma(R), \mathbb{R}^3) : f \text{ is constant on each cross section}\}.$$

Because of [8, Lemma 2.24], after reducing $\tilde{R}_\epsilon$ we can assume that

$$\begin{aligned}
\|H((m_R^{\text{red}})_x \vec{e}_x, R)\|^2_{L^2(\mathbb{R}^3)} &\;<\; \epsilon^2 R^2 && \text{for all } R \le \tilde{R}_\epsilon, \\
\frac{1}{2}\|g_y\|^2_{L^2(\Sigma)} - E_{\sigma\sigma}(g, R) &\;<\; \epsilon^2 \|g_y\|^2_{H^1(\Sigma)} && \text{for all } R \le \tilde{R}_\epsilon,\ g \in \mathcal{A}(R).
\end{aligned}$$

Moreover by [8, Lemma 2.10] we know that

$$\|H(g)\|^2_{L^2(\mathbb{R}^2)} = \|H(g_x \vec{e}_x)\|^2_{L^2(\mathbb{R}^2)} + \|H(g_y)\|^2_{L^2(\mathbb{R}^2)} \quad \text{for all } R \le \tilde{R}_\epsilon,\ g \in \mathcal{A}(R). \tag{25}$$

For $R < \tilde{R}_\epsilon$ and $m \in T\mathcal{S}^R$ we have

$$\begin{aligned}
&\left\langle L_0^R m, m \right\rangle_\Sigma - \left\langle L^R m, m \right\rangle_\Sigma \\
&\quad = \underbrace{\left\langle \delta_m E^0(m,R), m \right\rangle_\Sigma - \left\langle \delta_m E(m,R), m \right\rangle_\Sigma}_{A} \\
&\quad\quad \underbrace{- \left\langle |(m_R^{\text{red}})_y|^2, |m|^2 \right\rangle_\Sigma + 2\left\langle H(m^R) \cdot m^R, |m|^2 \right\rangle_\Sigma}_{B} \\
&\quad\quad + \underbrace{\int_\Sigma \left(-2|\partial_x m_R^{\text{red}}|^2 + 2|\partial_x m^R|^2 + 2|\nabla_y m^R|^2\right) m^2}_{C}
\end{aligned}$$

We estimate the summands separately. We decompose m in $\overline{m}$ and $\tilde{m}$.

$$\overline{m}(x,y) := \int_{D_R} m(x,y)\, dy, \qquad \tilde{m}(x,y) := m(x,y) - \overline{m}(x)$$

Since $40R^2 \le 2$ and since $\|f\|_{L^2(\Sigma)} \ge \|H(f)\|_{L^2(\mathbb{R}^3)}$ for every $f \in L^2(\Sigma, \mathbb{R}^3)$, we get for the first summand

$$\begin{aligned}
A \;&=\; \|m_y\|^2_{L^2(\Sigma)} + \left(\frac{40}{R^2} - 2\right)\|\nabla_y m\|_{L^2(\Sigma)} - 2\|H(m)\|_{L^2(\mathbb{R}^3)} \\
&\le\; \|m_y\|^2_{L^2(\Sigma)} - 2\|H(m)\|_{L^2(\mathbb{R}^3)} \\
&=\; \|\overline{m}_y\|^2_{L^2(\Sigma)} - 2\|H(\overline{m})\|_{L^2(\mathbb{R}^3)} + \|\tilde{m}_y\|^2_{L^2(\Sigma)} - 2\|H(\tilde{m})\|_{L^2(\mathbb{R}^3)} + 4\int_\Sigma H(m)\,\tilde{m} \\
&\le\; \|\overline{m}_y\|^2_{L^2(\Sigma)} - 2\|H(\overline{m}_y)\|^2_{L^2(\mathbb{R}^3)} + \|\tilde{m}_y\|^2_{L^2(\Sigma)} + 4\|\overline{m}\|_{L^2(\Sigma)}\|\tilde{m}\|_{L^2(\Sigma)} \\
&\le\; 2\epsilon\|\overline{m}_y\|^2_{H^1(\Sigma)} + \|\tilde{m}_y\|^2_{L^2(\Sigma)} + 4\|m\|_{L^2(\Sigma)}\|\tilde{m}\|_{L^2(\Sigma)}.
\end{aligned}$$

Using the Poincaré inequality and the assumption $R \le \epsilon$ we obtain

$$A \;\le\; \epsilon\|\overline{m}_y\|^2_{H^1(\Sigma)} + 16R^2\|\nabla\tilde{m}\|^2_{L^2(\Sigma)} + 16R\|\nabla\tilde{m}\|_{L^2(\Sigma)}\|\overline{m}\|_{L^2(\Sigma)} \;\le\; 33\epsilon\|m\|^2_{H^1(\Sigma)}.$$

For the second summand we calculate

$$\begin{aligned} B \;=\; & \underbrace{\int_\Sigma \big((m_R^{\text{red}})_y - 2H(m_R^{\text{red}})\big)\cdot m_R^{\text{red}}\,|m|^2}_{B_1} + \underbrace{2\int_\Sigma H(m_R^{\text{red}})\cdot(m_R^{\text{red}} - m^R)\,|m|^2}_{B_2} \\ & + \underbrace{2\int_\Sigma H(m_R^{\text{red}} - m^R)\cdot m^R\,|m|^2}_{B_3}, \end{aligned}$$

$$\begin{aligned} |B_1| \;&\le\; \|(m_R^{\text{red}})_y - 2H(m_R^{\text{red}})\|_{L^2(\Sigma)}\|m_R^{\text{red}}\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \\ &\le\; \big(\|2H((m_R^{\text{red}})_x\vec{e}_x)\|_{L^2(\Sigma)} + \|(m_R^{\text{red}})_y - 2H((m_R^{\text{red}})_y)\|_{L^2(\Sigma)}\big)\,\|m\|^2_{L^4(\Sigma)} \\ &\le\; 2\sqrt{\epsilon^2R^2} + 2\sqrt{\epsilon^2\pi R^2\|m_y^{\text{red}}\|^2_{H^1(\mathbb{R})}}\;\|m\|^2_{L^4(\Sigma)} \;\le\; 6\epsilon R\|m\|^2_{L^4(\Sigma)}, \\ |B_2| \;&\le\; 2\|H(m_R^{\text{red}})\|_{L^2(\Sigma)}\|m_R^{\text{red}} - m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \\ &\overset{(*)}{=}\; 2\sqrt{\|H((m_R^{\text{red}})_y)\|^2_{L^2(\mathbb{R}^3)} + \|H((m_R^{\text{red}})_x\vec{e}_x)\|^2_{L^2(\mathbb{R}^3)}}\;\|m_R^{\text{red}} - m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \\ &\le\; 2\sqrt{\frac{\pi}{2}R^2\|(m_R^{\text{red}})_y\|^2_{L^2(\mathbb{R})} + \|H((m_R^{\text{red}})_x\vec{e}_x)\|^2_{L^2(\mathbb{R}^3)}}\;\epsilon\|m\|^2_{L^4(\Sigma)} \\ &\le\; 2\sqrt{\sqrt{2}\pi R^2 + \epsilon^2R^2}\;\epsilon\|m\|^2_{L^4(\Sigma)} \;\le\; 6\epsilon R\|m\|^2_{L^4(\Sigma)}, \\ |B_3| \;&\le\; 2\|m_R^{\text{red}} - m^R\|_{L^2(\Sigma)}\|m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \;\le\; 2R\epsilon\|m\|^2_{L^4(\Sigma)}. \end{aligned}$$

For $(*)$ we have used (25). Because of the Sobolev embedding $H^1(\Sigma(1)) \hookrightarrow L^4(\Sigma(1))$ there exists a constant C_{Sobolev} such that

$$\|u\|_{L^4(\Sigma(1))} \;\le\; C_{\text{Sobolev}}\|u\|_{H^1(\Sigma(1))} \quad \text{for all } u\colon \Sigma(1) \to \mathbb{R}^n.$$

Rescaling implies for all $R \le 1$

$$\|u\|_{L^4(\Sigma(R))} \;\le\; \frac{1}{\sqrt{R}}C_{\text{Sobolev}}\|u\|_{H^1(\Sigma(R))} \quad \text{for all } u\colon \Sigma(R) \to \mathbb{R}^n.$$

Thus,

$$|B| \;\le\; 14C^2_{\text{Sobolev}}\epsilon\|m\|^2_{H^1(\Sigma)}.$$

Since $\partial_x m^{\text{red}} = \frac{1}{\sqrt{2}}|m_y^{\text{red}}| \le \frac{1}{\sqrt{2}}$ (Lemma 7) the third summand C can be estimated by

$$\begin{aligned} C &\le 2\|\partial_x m_R^{\text{red}} - \partial_x m^R\|_{L^\infty(\Sigma)}\|\partial_x m_R^{\text{red}} + \partial_x m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^2(\Sigma)} + 2\epsilon\|m\|^2_{L^2(\Sigma)} \\ &\le 2\epsilon\left(\frac{2}{\sqrt{2}} + \epsilon\right)\|m\|^2_{L^2(\Sigma)} + 2\epsilon\|m\|^2_{L^2(\Sigma)} \;\le\; 7\epsilon\|m\|_{L^2(\Sigma)}, \end{aligned}$$

and therefore we have for all $R \le \tilde{R}_\epsilon$

$$\big\langle L_0^R m, m\big\rangle_\Sigma - \big\langle L^R m, m\big\rangle_\Sigma \;\le\; (40 + 14C^2_{\text{Sobolev}})\epsilon\|m\|^2_{H^1(\Sigma)}.$$

□

Using Lemma 10, we transfer the result of Lemma 8 to the operator L^R.

Lemma 11. *For each $0 < \epsilon < \frac{1}{4}$ there exists R_ϵ such that*

$$\left\langle L^R(m), m\right\rangle_\Sigma \;\geq\; \left(\frac{1}{4} - \epsilon\right) \|m\|^2_{L^2(\Sigma)}$$

for all $R < R_\epsilon$ and all $m \in T\mathcal{S}^R$.

Proof. Let $P_0 \colon H^2(\Sigma) \to T\mathcal{S}_0^R$ be the L^2-orthogonal projection. Since

$$m_R^{\text{red}} \perp \partial_x m_R^{\text{red}}, \quad m_R^{\text{red}} \perp \Phi(m_R^{\text{red}}), \quad \left\langle \partial_x m_R^{\text{red}}, \Phi(m_R^{\text{red}})\right\rangle_\Sigma = 0,$$

we have for all $m \in T\mathcal{S}^R$

$$\begin{aligned} &P_0(m)\\ &= m - (m \cdot (m_R^{\text{red}} - m^R))m_R^{\text{red}} + \left\langle m, \partial_x m^R - \partial_x m_R^{\text{red}}\right\rangle_\Sigma \frac{\partial_x m_R^{\text{red}}}{\|\partial_x m_R^{\text{red}}\|^2_{L^2(\Sigma)}}\\ &\quad + \left\langle m, \Phi(m^R) - \Phi(m_R^{\text{red}})\right\rangle_\Sigma \frac{\Phi(m_R^{\text{red}})}{\|\Phi(m_R^{\text{red}})\|^2_{L^2(\Sigma)}}, \end{aligned}$$

that is,

$$\begin{aligned} &\|m\|_{L^2(\Sigma)} - \|P_0(m)\|_{L^2(\Sigma)}\\ &\qquad \leq \|m\|_{L^2(\Sigma)}\|m^R - m_R^{\text{red}}\|_{L^\infty(\Sigma)} + \|m\|_{L^2(\Sigma)}\frac{\|\partial_x m^R - \partial_x m_R^{\text{red}}\|_{L^2(\Sigma)}}{\|\partial_x m_R^{\text{red}}\|_{L^2(\Sigma)}}\\ &\qquad\quad + \|m\|_{L^2(\Sigma)}\frac{\|\Phi(m^R) - \Phi(m_R^{\text{red}})\|_{L^2(\Sigma)}}{\|\Phi(m_R^{\text{red}})\|_{L^2(\Sigma)}} \end{aligned}$$

Thus, with Theorem 1, we can find $\tilde{R}_\epsilon$ such that

$$\|m\|_{L^2(\Sigma)} - \|P_0(m)\|_{L^2(\Sigma)} \;\leq\; \epsilon\|m\|_{L^2(\Sigma)} \qquad \text{for all } R \leq \tilde{R}_\epsilon.\ m \in T\mathcal{S}^R.$$

After reducing R_ϵ we can assume that Lemma 10 implies

$$\left\langle L^R m, m\right\rangle_\Sigma \;\geq\; \left\langle L_0^R m, m\right\rangle_\Sigma - \epsilon\|m\|^2_{H^1(\Sigma)} \qquad \text{for all } R \leq \tilde{R}_\epsilon,\ m \in T\mathcal{S}^R.$$

Then we have

$$\begin{aligned} \left\langle L^R m, m\right\rangle_\Sigma &\geq (1-\epsilon)\left\langle L_0^R m, m\right\rangle_\Sigma + \epsilon\|\nabla m\|^2_{L^2(\Sigma)}\\ &\quad - \epsilon\left(\int_\Sigma \underbrace{\left(2|\nabla m_R^{\text{red}}|^2 + |(m_R^{\text{red}})_y|^2\right)}_{\leq 2} m^2\right) - \epsilon\|m\|^2_{H^1(\Sigma)}\\ &\geq (1-\epsilon)\left\langle L_0^R m, m\right\rangle_\Sigma - 3\epsilon\|m\|^2_{L^2(\Sigma)}. \end{aligned}$$

Since the operator L_0^R is the second variation of the energy E^0 and since m_0^R is a minimiser of the energy, the operator L_0^R is positive semidefinite. Moreover, it is symmetric on the set $\{m \in H^2(\Sigma,\mathbb{R}^3) : \partial_\nu m|_{\partial\Sigma} = 0\}$, so $L_0^R(T\mathcal{S}_0^R) \subset (T\mathcal{S}_0^R)_{L^2}$ implies

$$\begin{aligned}
\left\langle L_0^R m, m\right\rangle_\Sigma &= \left\langle L_0^R(P_0(m)), P_0(m)\right\rangle_\Sigma + \left\langle L_0^R(m - P_0(m)), m - P_0(m)\right\rangle_\Sigma \\
&\geq \left\langle L_0^R(P_0(m)), P_0(m)\right\rangle_\Sigma \geq \frac{1}{4}\|P_0(m)\|^2_{L^2(\Sigma)} \\
&\geq \frac{1-\epsilon}{4}\|m\|^2_{L^2(\Sigma)}.
\end{aligned}$$

Thus, $\left\langle L^R m, m\right\rangle_\Sigma \geq \frac{1}{4}\|m\|^2_{L^2(\Sigma)} - 4\epsilon\|m\|^2_{L^2(\Sigma)}$. □

Lemma 12. *There exists $\lambda, C > 0$, $\tilde{R} > 0$ such that*

$$\begin{aligned}
\|L^R(g) + \lambda g\|_{L^2(\Sigma)} &\geq \|g\|_{H^2(\Sigma)} \\
\|L^R(g)\|_{L^2(\Sigma)} &\leq C\|g\|_{H^2(\Sigma)}
\end{aligned}$$

for all $R \leq \tilde{R}$ and all $g \in T\mathcal{S}^R$.

Proof. We split the operator in two parts and set

$$\begin{aligned}
L_H^R(g) &:= 2\left(H(g) - (H(g)\cdot m^R)m^R - (H(m^R)\cdot g)m^R - (H(m^R)\cdot m^R)g\right), \\
L_\Delta^R(g) &:= 2\left(-\Delta g + (\Delta g\cdot m^R)m^R + (\Delta m^R\cdot g)m^R + (\Delta m^R\cdot m^R)g\right).
\end{aligned}$$

First, we show $H(m^R) \in L^\infty(\Sigma)$ using a result by Carbou and Fabrie [2] for bounded domains. Let $\eta : \Sigma \to [0,1]$ be a smooth function with

$$\eta(p) = 1 \ \text{ for } p \in [-1,1]\times D_R, \qquad \eta(p) = 0 \ \text{ for } p' \in \Sigma\setminus([-2,2]\times D_R),$$

and set $\eta_x : (x', y') \mapsto \eta(x' - x, y)$.

Then [2, Lemma 2.3] and the Sobolev embedding $W^{1,4}(\Sigma) \hookrightarrow L^\infty(\Sigma)$ imply that there exist constants C, $\tilde{C}$ independent of x such that

$$\begin{aligned}
\|H(m\cdot\eta_x)\|_{L^\infty(\Sigma)} &\leq \tilde{C}\|H(m\cdot\eta_x)\|_{W^{1,4}(\Sigma)} \\
&\leq \|m\cdot\eta_x\|_{W^{1,4}(\Sigma)} \leq (2\pi)^{\frac{1}{4}}\|m\|_{C^1(\Sigma)} \qquad (26)
\end{aligned}$$

Moreover, using the representation of H in terms of K_i of [8, Lemma 2.6] we obtain for all $p = (x, y) \in \Sigma$ the estimate

$$\begin{aligned}
&H(m\cdot(1-\eta_x))(p) \\
&\quad \leq 3\left(\|K_i\|_{L^1(\Sigma\setminus([-1,1]\times D_R))} + \|K_i\|_{L^1(\partial\Sigma\setminus([-1,1]\times\partial D_R))}\right)\|m\|_{C^1(\Sigma)}. \qquad (27)
\end{aligned}$$

The combination of (26) and (27) implies that $\|H(m^R)\|_{L^\infty(\Sigma)}$ is finite.

Since $\|H(g)\|_{L^2(\Sigma)} \leq \|g\|_{L^2(\Sigma)}$ and since $\|m^R\|_{L^\infty(\Sigma)} = 1$, we have for all $g \in (T\mathcal{S})_{L^2}$

$$\|L_H^R(g)\|_{L^2(\Sigma)} \leq \left(4 + 4\|H(m^R)\|_{L^\infty(\Sigma)}\right)\|g\|_{L^2(\Sigma)}. \qquad (28)$$

Thus the operator $L_H^R : (T\mathcal{S})_{L^2} \to (T\mathcal{S})_{L^2}$ is continuous.

Since $\partial_i m^R \perp m^R$ ($i \in \{1,2,3\}$) and since $g \perp m^R$ for all $g \in T\mathcal{S}^R$, we have

$$\begin{aligned}
0 &= \Delta(m^R\cdot g) = \Delta m^R\cdot g + 2\nabla m^R\cdot\nabla g + m^R\cdot\Delta g, \\
0 &= \operatorname{div}(\nabla m^R\cdot m^R) = \Delta m^R\cdot m^R + |\nabla m^R|^2,
\end{aligned}$$

and therefore

$$L^R_\Delta(g) \;=\; -2\Delta g - 4(\nabla m^R \cdot \nabla g)m^R - 2|\nabla m^R|^2 g. \tag{29}$$

Moreover, we have the estimate

$$\begin{aligned}
\|-2\Delta g + \lambda g\|^2_{L^2(\Sigma)} &= \int_\Sigma \left(4|\Delta g|^2 - 4\Delta g \cdot \lambda g + \lambda^2 g^2\right) \\
&= \int_\Sigma \left(4|D^2 g|^2 + 4\lambda|\nabla g|^2 + \lambda^2 |g|^2\right) \\
&\geq \frac{1}{3}\left(2\|D^2 g\|_{L^2(\Sigma)} + 2\sqrt{\lambda}\|\nabla g\|_{L^2(\Sigma)} + \lambda\|g\|_{L^2(\Sigma)}\right)^2.
\end{aligned}$$

This yields

$$\begin{aligned}
&\|L^R_\Delta(g) + \lambda g\|_{L^2(\Sigma)} \\
&\geq \|-2\Delta g + \lambda g\|_{L^2(\Sigma)} - 4\|m^R\|_{C^1}\|\nabla g\|_{L^2(\Sigma)} - 2\|m^R\|^2_{C^1(\Sigma)}\|g\|_{L^2(\Sigma)} \\
&\geq \frac{2}{\sqrt{3}}\|D^2 g\|_{L^2(\Sigma)} + \left(\frac{2\sqrt{\lambda}}{\sqrt{3}} - 4\|m^R\|_{C^1(\Sigma)}\right)\|\nabla g\|_{L^2(\Sigma)} \\
&\quad + \left(\frac{\lambda}{\sqrt{3}} - 2\|m^R\|^2_{C^1(\Sigma)}\right)\|g\|_{L^2(\Sigma)},
\end{aligned}$$

and we can choose λ such that

$$\|L^R_\Delta(g) + \lambda g\|_{L^2(\Sigma)} \geq \|g\|_{H^2(\Sigma)}. \tag{30}$$

Combining (28) and (29), we obtain the second estimate. □

Using the above estimates, we show that the operator L^R is bijective and has an continuous inverse.

Lemma 13. *There exists $\tilde{R} > 0$ such that for all $R < \tilde{R}$*

(i) the operator $L^R : T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ is injective,

(ii) $L^R(T\mathcal{S}^R)$ is dense in $(T\mathcal{S}^R)_{L^2}$,

(iii) $L^R(T\mathcal{S}^R)$ is closed in $(T\mathcal{S}^R)_{L^2}$,

(iv) the operator $L^R : T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ is bijective,

(v) the operator $(L^R)^{-1} : (T\mathcal{S}^R)_{L^2} \to T\mathcal{S}^R$ is bounded.

Proof. Let $\tilde{R}$ be so small and λ, C so large that for all $R \leq \tilde{R}$ and all $g \in T\mathcal{S}^R$ Lemma 11 and Lemma 12 imply

$$\left\langle L^R(m), m\right\rangle_\Sigma \;\geq\; \frac{1}{8}\|m\|^2_{L^2(\Sigma)}, \tag{31}$$

$$\|L^R(g) + \lambda g\|_{L^2(\Sigma)} \;\geq\; \|g\|_{H^2(\Sigma)}, \tag{32}$$

$$\|L^R(g)\|_{L^2(\Sigma)} \;\leq\; C\|g\|_{H^2(\Sigma)}. \tag{33}$$

(i) This is a direct implication of (31).

(ii) We show the statement by contradiction and assume that $L^R(T\mathcal{S}^R)$ is not dense in $(T\mathcal{S}^R)_{L^2}$. Then there exists $v \in (T\mathcal{S}^R)_{L^2}$ that is orthogonal on $L^R(T\mathcal{S}^R)$, and there exists w in $T\mathcal{S}^R$ such that

$$\langle w, g\rangle_\Sigma < \frac{1}{10C}\|w\|_{L^2(\Sigma)}\|g\|_{L^2(\Sigma)} \qquad \text{for all } \ g \in L^R(T\mathcal{S}^R),\ g \not\equiv 0.$$

Thus with (33) we have

$$\left\langle L^R(w), w\right\rangle_\Sigma \ < \ \frac{1}{10C}\|L^R(w)\|_{L^2(\Sigma)}\|w\|_{L^2(\Sigma)} \ \leq \ \frac{1}{10}\|w\|^2_{L^2(\Sigma)}.$$

This is in contradiction to (31).

(iii) Let $(L^R(g_n))_{n\in\mathbb{N}}$, $g_n \in T\mathcal{S}^R$ be a sequence that converges to some $h_0 \in (T\mathcal{S}^R)_{L^2}$ in $L^2(\Sigma, \mathbb{R}^3)$. We have to show $h_0 \in L^R(T\mathcal{S}^R)$.

Because of (31), the sequence $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Sigma, \mathbb{R}^3)$, so the sequence $(L^R(g_n) + \lambda g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Sigma, \mathbb{R}^3)$, too. Thus the estimate (32) implies that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Sigma, \mathbb{R}^3)$ converging to some $g_0 \in T\mathcal{S}^R$. Using (33), we obtain

$$L^R(g_0) \ = \ \lim_{n\to\infty} L^R(g_n) \ = \ h_0.$$

(iv) Since $L^R(T\mathcal{S}^R)$ is dense and closed in $(T\mathcal{S}^R)_{L^2}$, we have $L^R(T\mathcal{S}^R) = (T\mathcal{S}^R)_{L^2}$. Thus, with (i), $L^R : T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ is bijective.

(v) The arguments for (iii) imply: If $L^R(g_0) = \lim_{n\to\infty} L^R(g_n)$ in $L^2(\Sigma)$ then g_n converges to g_0 in $H^2(\Sigma)$. □

Theorem 6 summarises the statements of Lemma 13.

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