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Travelling waves with a singularity in magnetic nanowires

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Abstract

We model the evolution of the magnetisation in an infinite cylinder by harmonic map heat flow with an additional external field. Using variational methods, we prove the existence of corotationally symmetric travelling wave solutions with a moving vortex. We moreover show that for weak and strong fields the travelling waves connect the original state antiparallel to the external magnetic field with the totally reversed state in direction of the external field. Our results match numeric simulations. For thicker wires several groups have observed a reversal mode where a domain wall with a corotational symmetry and a vortex is propagating through the wire.

1 Introduction

Because of the possible use of arrays of magnetic nanowires as future storage devices [1], in the recent years there has been a growing interest in the magnetic reversal process of such wires. It is known that in a long magnetic wire the magnetisation starts to reverse at one end of the wire and then a domain wall is propagating through the wire.

In numeric simulations, several groups, e.g. [5, 8, 20], have observed two different reversal modes. In the *transversal mode* observed in thin wires, the magnetisation is almost constant on each cross section of the wire. In the *vortex mode* observed in thick wires, in first approximation the magnetisation is corotaionally symmetric and tangential to the boundary. Thus at one point the magnetisation forms a vortex. This configuration is moving along the wire. When looking closer, in some simulations [8] one can observe a more complicated behaviour. The vortex seems not to perform a steady movement.It lacks behind, and periodically the old vortex is annihilated and a new vortex is created. The vortex wall moves much faster than the transversal wall.

Since the field of research is quite new, there is not much mathematical literature about magnetic nanowires. Static domain walls have been considered in [10, 12, 4]. We have analysed the energy scaling of optimal domain walls [10] and proved uniform regularity for domain walls in thin wires [12]. Carbou and Labbé [4] have investigated the stability of transverse walls in the limit $R \to 0$.

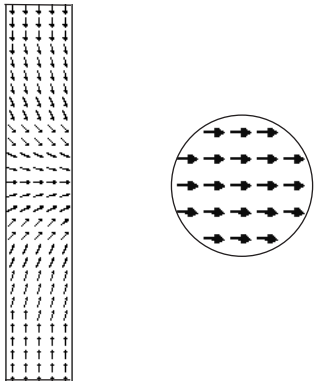

Figure 1: Transverse Mode: longitudinal section and cross section

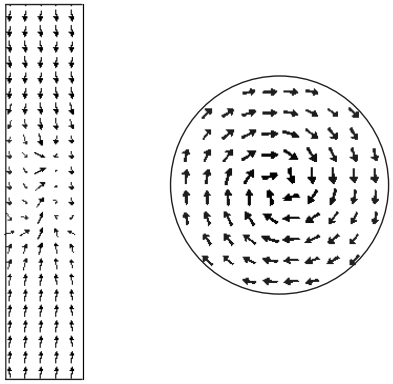

Figure 2: Vortex Mode: longitudinal section and cross section

The transverse mode has been studied in [18, 11]. Sanchez [18] has considered the limit of the Landau-Lifshitz equation when the diameter of the domain and the exchange coefficient in the equation simultaneously tend to zero and performs an asymptotic expansion. We have shown the existence of travelling wave solutions to the overdamped limit of the Landau-Lifshitz-Gilbert equation [11] using a perturbation argument from the static case.

This is the first article considering a dynamic model for the vortex mode. We show that, for a simplified equation that captures the highest order terms with respect to the derivatives, there exist travelling wave solutions with corotational symmetry and thus a moving singularity.

1.1 The model

We model the evolution of the magnetisation by harmonic map heat flow with an external magnetic field in direction of the wire. Harmonic map heat flow follows from the Landau-Lifshitz-Gilbert equation when we consider the overdamped limit and keep only the highest order terms with respect to the derivatives

$$\partial_t m = \Delta m - (\Delta m \cdot m)m, \qquad |m| \equiv 1. \tag{1}$$

This equation has been extensively studied, but most of the time on bounded domains (cf. [19] and references therein). Bertsch, Muratov and Primi [3] consider it in an infinite cylinder and investigate travelling wave solutions. Since the equation itself does not contain any driving force, the travelling waves have to be "pulled" by the boundary conditions. Our methods to prove existence of travelling waves rely on the same variational principles as the methods used in that article.

In order to have travelling wave solutions without imposing any special boundary conditions, we have to include the external field $h\vec{e}_x$

$$\partial_t m = \Delta m + h\vec{e}_x - (\Delta m \cdot m)m) - hm_x\vec{e}_x, \qquad |m| \equiv 1. \tag{2}$$

This equation corresponds to gradient flow of the energy

$$E(m) \;=\; \frac{1}{2}|\nabla m|^2 + h m_x \tag{3}$$

under the condition $|m| \equiv 1$. At first glance this model may seem inappropriate because the vortex mode only appears in thick wires, where the stray field energy is important [10]. However, we have the following picture in mind: The *existence* of the singularity in the vortex mode is due to the strong influence of the stray field energy that prevents surface charges, but the properties of the *evolution* of a magnetisation with a singularity is mainly determined by the highest order terms with respect to the derivatives.

We change to a coordinate system that is better adapted our problem. Let $\Sigma := \mathbb{R} \times D_R := \mathbb{R} \times \{y \in \mathbb{R}^2 \,:\, |y| \leq R\}$ be the infinite cylinder with radius R. We describe the magnetisation $m\colon \Sigma \to \mathbb{S}^2$ by spherical coordinates $(\gamma, \theta)\colon \Sigma \to [0, 2\pi[\times[0, \pi]$ (cf. Figure 3). Then

$$m = \begin{pmatrix} -\cos\theta \\ \sin\theta\,\cos\gamma \\ \sin\theta\,\sin\gamma \end{pmatrix} \quad \text{and} \quad \nabla m = \begin{pmatrix} \sin\theta\nabla\theta \\ \cos\theta\,\cos\gamma\,\nabla\theta - \sin\theta\,\sin\gamma\,\nabla\gamma \\ \cos\theta\,\sin\gamma\,\nabla\theta + \sin\theta\cos\gamma\,\nabla\gamma \end{pmatrix}.$$

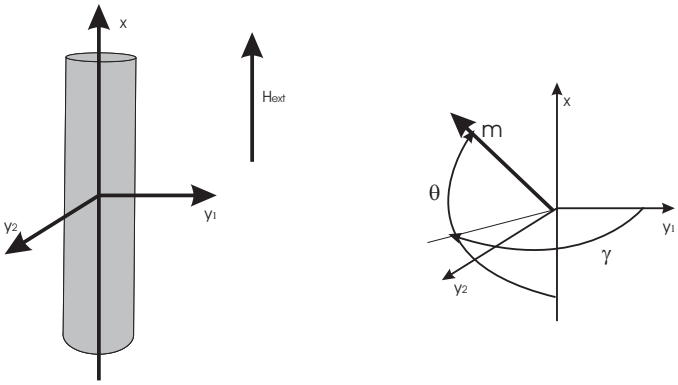

Figure 3: The coordinate system in the domain and in the range

In spherical coordinates, the assumption that the magnetisation is tangential to the closest boundary in each point (x, y) with $y \neq 0$ is equivalent to the equality $\gamma = \arctan\left(\frac{-y_1}{y_3}\right)$. In this case we have $|\nabla\gamma|^2 = \frac{1}{|y|^2}$. Transforming (3) and (2) we get

$$E(\theta) \;=\; \int_\Sigma \frac{1}{2}|\nabla\theta|^2 + \frac{1}{2|y^2|}\sin^2(\theta) + h(\cos\theta) \tag{4}$$

$$\partial_t\theta \;=\; -\delta_\theta E \;=\; \Delta\theta - \frac{1}{2|y^2|}\sin(2\theta) + h\sin\theta. \tag{5}$$

Since we are interested in travelling wave solutions of this equation, we replace $\partial_t\theta$ by $-c\partial_x\theta$, where c is the speed of the travelling wave. To have θ in the interval $[0, 1]$ and to normalise R to 1 we rescale. That is, we measure θ in

multiples of π, length in multiples of R, the magnetic field in multiples of $\frac{1}{R^2}$, and time in multiples of R^2. Since there is no influence from the rest of the space on the magnetisation of the wire, the appropriate boundary conditions are Neumann boundary conditions, which are also the natural boundary conditions for the energy E. Thus we get the equation

$$\Delta\theta + c\partial_x\theta + f^0(y,\theta) = 0 \ \text{ in } \Sigma, \qquad \partial_\nu\theta = 0 \ \text{ on } \partial\Sigma, \tag{6}$$

where

$$f^0(\theta, y) \ := \ -\frac{1}{2\pi y^2}\sin(2\pi\theta) + \frac{h}{\pi}\sin(\pi\theta).$$

1.2 The main results

In Section 2 and Section 3 we treat the problem of existence and speed of travelling waves. The following theorem summarises our results.

Theorem 1. *For all $h > 0$ there exists a monotone solution (c, θ) of (6) such that $u(x, 0) \in \{0, 1\}$ almost everywhere and $\underline{c} \le c \le \overline{c}$ with*

$$\overline{c} := 2\sqrt{h} \quad \text{and} \quad \underline{c} := \begin{cases} \frac{2h}{5\pi} & \text{for } 0 < h < h_c \\ 2\sqrt{h - k_0^2} & \text{for } h_c \le h \end{cases}$$

where $k_0 \approx 1.8$ is the first root of the Bessel function J_1 and $h_c \approx 0.05 + k_0^2$ is the smaller one of the two solutions of the equation $\frac{2h}{5\pi} = 2\sqrt{h - k_0^2}$.

There are two possibilities:

- *In the variational case there exists a solution $(c^\dagger, \theta^\dagger)$ such that $\theta^\dagger$ is a minimiser of the functional $\Phi^0_{c^\dagger}$ (See (12)) and $(c^\dagger)^2 > 4(h - k_0^2)$.*
- *In the non-variational case there is a solution (c^*, θ^*) such that $c^* = 2\sqrt{h - k_0^2}$.*

There exists some $h_0 \in [h_c, \infty]$ such that for all $h < h_0$ we have the variational case and for all $h \ge h_0$ we have the non-variational case.

In Section 4 and Section 5 we consider possible end states and find the following theorem.

Theorem 2. *We have*

$$\lim_{x\to-\infty} \theta(x, \cdot) = \theta_- = 0 \qquad \text{in } C^\infty_{\text{loc}}(D_1 \setminus \{0\}),$$
$$\lim_{x\to\infty} \theta(x, \cdot) = \theta_+ \qquad \text{in } C^\infty_{\text{loc}}(D_1 \setminus \{0\})$$

for some semistable stationary θ_+ state (see Definition 31). For $h \le 2$ and $h \ge k_0^2 + 1$ we know $\theta_+ \equiv 1$. Here k_0 is as in Theorem 1. In the variational case, $\theta(x, \cdot)$ converges also in $L^\infty(D_1)$ to the endstates $\theta_\pm$.

Remark 3. In the variational case with $\theta_+ \equiv 1$, because of convergence in $L^\infty(D_1)$, there has to exist some point x_0 such that $\theta^\dagger$ jumps in x_0 form zero to one . Thus in this case we have a discontinuous travelling wave. In the other cases we may have this jump as well, but it is also possible that on the whole x-axis we have $\theta(\cdot, 0) \equiv 0$ or $\theta(\cdot, 0) \equiv 1$.

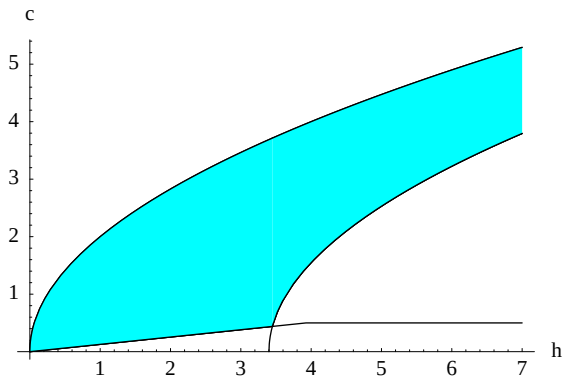

Figure 4: Upper bound $\overline{c}$ and lower bound $\underline{c}$ for the speed

Remark 4. Theorem 1 is a combination of Theorem 25 and Theorem 28 in Section 3.2, Theorem 2 is a short version of Theorem 52 in Section 5.2.

1.3 Outline

In Section 2 we recall results of [17] concerning variational methods for travelling wave problems.

In Section 3 we apply these results to our problem and show that for each $h > 0$ there exist solutions of (6). Moreover we derive some properties of the travelling waves, including bounds on the speed.

Clearly, the possible end states are stationary states, i.e., solutions of (6) that do not depend on x. In Section 4, we analyse properties of stationary states.

In Section 5 we first show that the stationary state at $-\infty$ is semistable. Then we combine the results about stationary states with the results of Section 3. We conclude that for $h < 2$ and $h \geq 4.38$ the solutions of (6) found in Section 3 converge to zero at $+\infty$ and converge to one at $-\infty$.

Notation. The letter p denotes a point in $\mathbb{R}^3$, with the component $x \in \mathbb{R}$ in direction of the wire and $y \in \mathbb{R}^2$ orthogonal to the wire. $a \cdot b$ is the $\mathbb{R}^3$ scalar product. The characteristic function of a set A is denoted by $\mathbb{1}_A$.

2 Variational methods for travelling wave problems

The difficulty of (6) is the singularity of the function f at $y = 0$. When we replace f by a continuous function, the problem becomes much simpler. We set

$$\begin{aligned} \Delta\theta + c\partial_x\theta + f^\epsilon(y,\theta) &= 0 \quad \text{in } \Sigma, \\ \partial_\nu\theta &= 0 \quad \text{on } \partial\Sigma, \end{aligned} \tag{7}$$

where

$$f^\epsilon(\theta, y) := -\frac{1}{2\pi y^2 + \epsilon^2} \sin(2\pi\theta) + \frac{h}{\pi} \sin(\pi\theta).$$

Now we could use standard methods to find solutions of (7), applying techniques relying on the maximum principle, as described in [2]. However, these methods do not give uniform bounds on the speed c^ϵ, so it is impossible to prove the convergence of a subsequence of solutions of (7) for $\epsilon \to 0$. Therefore we use a variational principle for travelling wave equations. Such a principle was developed by Heinze [7] and by Lucia, Muratov and Novaga [16, 15, 14, 17].

In this section we present some of the results of [17]. In that article the authors give a comprehensive treatment of the properties of the travelling waves that correspond to minimisers of a certain functional. In addition they use variational methods to find travelling wave solutions in the case where minimisers of the functional do not exist.

We use a different orientation than [17]. In [17], waves have a positive speed and connect an energetically favourable stable state v_- with $0 < v_- \leq 1$ at $-\infty$ with the state $v_+ = 0$ at $+\infty$. In this paper, waves have a negative speed and connect the state $w_- = 0$ at $-\infty$ with an energetically favourable stable state w_+ with $0 < w_+ \leq 1$ at $+\infty$.

Muratov and Novaga [17] study the solutions (c, u) of the equations

$$\left.\begin{aligned} \Delta u + c\,\partial_x u + \nabla_y \phi \cdot \nabla_y u + f(u, y) &= 0 \quad \text{in } \Sigma_\Omega \\ u &= 0 \quad \text{on } \partial\Sigma_\Omega^\pm, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Sigma_\Omega^0. \end{aligned}\right\} \tag{8}$$

Here, $\Sigma_\Omega = \mathbb{R} \times \Omega$, the boundary $\partial\Sigma_\Omega$ is of class C^2 and the disjoint union of $\partial\Sigma_\Omega^0$ and $\partial\Sigma_\Omega^\pm$. The term $\nabla\phi$ is a convection term that is important for the application in combustion theory that is studied by the authors. We need the results of [17] only in the case $\phi = 0$ and $\Sigma_\Omega^\pm = \emptyset$, thus we consider the equation

$$\begin{aligned} \Delta u + c\,\partial_x u + f(u, y) &= 0 \quad \text{in } \Sigma_\Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Sigma_\Omega. \end{aligned} \tag{9}$$

In order to guarantee that there is a solution of (9) such that u has values in $]0, 1[$, we have to make the following assumptions:

(H1) The function $f\colon [0, 1] \times \overline{\Omega} \to \mathbb{R}$ satisfies for all $y \in \Omega$ the equations

$$f(0, y) = 0, \qquad f(1, y) \leq 0.$$

(H2) For some $\gamma \in]0, 1[$,

$$f \in C^{0,\gamma}([0, 1] \times \Omega), \quad \partial_u f \in C^{0,\gamma}([0, 1] \times \Omega).$$

To formulate a third hypothesis, we need some definitions. Let $L^2_c(\Sigma_\Omega)$ be the Hilbert space with the weighted norm

$$\|u\|_{L^2_c(\Sigma_\Omega)} := \sqrt{\int_{\Sigma_\Omega} e^{cx} u^2 \, dp}$$

and let $H^1_c(\Sigma_\Omega)$ be the Hilbert space of all functions for which

$$\|u\|_{H^1_c(\Sigma_\Omega)} := \sqrt{\|u\|^2_{L^2_c(\Sigma_\Omega)} + \|\nabla u\|^2_{L^2_c(\Sigma_\Omega)}}$$

is finite. We define the functional

$$\Phi_c : H^1_c(\Sigma_\Omega) \to \mathbb{R}, \qquad u \mapsto \int_{\Sigma_\Omega} e^{cx} \left(\frac{1}{2}|\nabla u|^2 + V(u,y) \right) dp,$$

where

$$V(u,y) = \begin{cases} 0 & \text{for } u < 0, \\ -\int_0^u f(s,y)ds & \text{for } 0 \le u \le 1, \\ -\int_0^1 f(s,y)ds & \text{for } 1 < u. \end{cases}$$

so that $\partial_u V = -f(u)\mathbb{1}_{[0,1]}$. Note that (9) is the Euler-Lagrange equation of the functional Φ_c, thus every critical point u of Φ_c that satisfies $0 \le u \le 1$ is a solution of (9). Finally, define the auxiliary functional

$$I : H^1(\Omega) \to \mathbb{R}, \qquad v \mapsto \int_\Omega \frac{1}{2}|\nabla_y v|^2 + V(v,y)\, dy.$$

In order to investigate (9), it is important to consider the linearisation of (9) at the end states at $\pm\infty$.

Let μ be the smallest eigenvalue of $-\Delta u - \partial_u f(0,y)$ and let ψ be the corresponding eigenfunction. Analogously, let $\tilde{\mu}$ be the smallest eigenvalue of $-\Delta\tilde{\psi} - \partial_u f(u_+,y)\tilde{\psi}$, where u_+ is the end state at $+\infty$, and let $\tilde{\psi}$ be the corresponding eigenfunction.

Using these definitions, we can formulate the hypothesis that is crucial for the existence of a number $c^\dagger$ such that $\Phi_{c^\dagger}$ has a nontrivial minimiser.

(H3) There exists $c < 0$ satisfying $c^2 + 4\mu > 0$ and $u \in H^1_c(\Sigma_\Omega)$, such that $\Phi_c(u) \le 0$ and $u \not\equiv 0$.

Definition 5. The number c in (H3) is called *admissible trial velocity.*

We now present the two main results about the existence of travelling waves and their properties. Theorem 6 corresponds to [17, Theorem 3.3] and considers the case when (H3) is satisfied, Theorem 7 corresponds to [17, Theorem 4.2] and considers the case when (H3) is not satisfied.

Theorem 6. *Under hypotheses (H1)-(H3) there exists a unique $c^\dagger < 0$ such that there exists a minimiser $u \not\equiv 0$ of $\Phi_{c^\dagger}$ in $H^1_{c^\dagger}(\Sigma_\Omega)$. This minimiser is unique up to translation. Moreover we have:*

(1.) $c^\dagger \le c < 0$, where c is the admissible trial velocity given by assumption (H3). We have $u \in C^2(\Sigma_\Omega) \cap W^{1,\infty}(\overline{\Sigma}_\Omega)$, and u solves (9) with $c = c^\dagger$.

(2.) $u(x,y)$ is strictly increasing in x for each $y \in \Omega$. We have

$$\lim_{x\to-\infty} u(x,\cdot) = 0 \quad \text{in } C^1(\overline{\Omega}), \qquad \lim_{x\to+\infty} u(x,\cdot) = u_+ \quad \text{in } C^1(\overline{\Omega}),$$

where $u_+ \colon \Omega \to]0,1]$ is a critical point of I with $I(u_+) < 0$.

(3.) Set $\lambda_- := \frac{1}{2}\left(c^\dagger + \sqrt{(c^\dagger)^2 + 4\mu}\right)$. *There exists* $a > 0$ *and* $\lambda > \lambda_-$ *such that*

$$\lim_{t\to-\infty} \left\| \left(u(x,y) - a\psi(y)e^{\lambda_- x}\right) e^{-\lambda x} \right\|_{C^1(]-\infty,t]\times\overline{\Omega})} = 0.$$

(4.) We have $\tilde{\mu} \geq 0$. *Set* $\lambda_+ := \frac{1}{2}\left(c^\dagger - \sqrt{(c^\dagger)^2 + 4\tilde{\mu}}\right)$. *If* $\tilde{\mu} > 0$, *then there exists* $a > 0$ *and* $\lambda > -\lambda_+$ *such that*

$$\lim_{t\to+\infty} \left\| \left(u_+ - u(x,y) - a\psi(y)e^{\lambda_+ x}\right) e^{\lambda x} \right\|_{C^1([t,\infty[\times\overline{\Omega})} = 0.$$

Theorem 7. *Assume that hypotheses (H1) and (H2) hold, whereas hypothesis (H3) is not satisfied. Assume in addition that there exists a function* $v \in H^1(\Omega)$, *such that* $I(v) < 0$. *Then there exists* $u \in C^2(\Sigma_\Omega) \cap W^{1,\infty}(\Sigma_\Omega)$ *which solves* (9) *with* $c = c_0 := -2\sqrt{|\mu|}$. *Furthermore,* u *has the asymptotic behaviour*

$$u(x,y) = (a - bx)\psi e^{-\frac{1}{2}c_0 x} + O(e^{\lambda x}) \quad \text{as } x \to -\infty$$

for some $\lambda > \frac{c_0}{2}$, *and either* $b > 0$ *or* $b = 0$ *and* $a > 0$. *Assertions (2) and (4) of Theorem 6 still hold.*

Since we will work only with solutions provided by the two theorems above, we make the following definition.

Definition 8. A solution (c, u) of (9) is called an *MN-solution* if it is a solution provided by Theorem 6 or in Theorem 7. In the first case it is called a *variational MN-solution*, in the latter a *non-variational MN-solution*.

There is no easy criterion to decide whether or not (H3) is satisfied. However, there are necessary and sufficient conditions for (H3). On the one hand, [17, Remark 3.8] and [17, Theorem 3.9] yield the following result.

Proposition 9. *If* $\mu \geq 0$, *and if there exists a function* $v \in H^1(\Omega)$ *such that* $I(v) < 0$, *then hypothesis* $(H3)$ *is satisfied.*

On the other hand, in a certain case we know that (H3) is not satisfied ([17, Proposition 4.1]).

Proposition 10. *Under hypotheses (H1) and (H2) assume that* $\mu < 0$ *and*

$$\frac{2}{u_0^2}\int_0^{u_0} f(s,y)\, ds \;\leq\; \partial_u f(0,y) \quad \text{for all } y \in \Omega,\ u_0 \in [0,1]. \tag{10}$$

Then the hypothesis $(H3)$ *is not satisfied.*

Remark 11. If $f(s,y) \leq s\partial_u f(0,y)$ for all $s \in [0,1]$, then (10) is satisfied.

Both in the variational and in the non-variational case we have the existence of a solution of (9), and a statement about the speed c of the travelling wave. For variational MN-solutions we know $c^2 \geq -4\mu$, for non-variational MN-solutions we have $c = -2\sqrt{-\mu}$. It is easy to see that the speed depends monotonously on the potential V.

Corollary 12. *For* $f = f_1$, $f = f_2$, *let* (u_1, c_1) *and* (u_2, c_2) *be MN-solutions of* (9). *If* $V_1 \leq V_2$ *then* $c_1 \leq c_2$, *i.e.,* $|c_1| \geq |c_2|$.

Proof. First, assume that (u_2, c_2) is a non-variational MN-solution. Then

$$c_2 \;=\; -2\sqrt{-\mu_2} \;\geq\; -2\sqrt{-\mu_1} \;\geq\; c_1.$$

Now assume that (u_2, c_2) is a variational MN-solution. If $c_2^2 + 4\mu_1 \leq 0$ then surely $c_1 \leq -2\sqrt{-\mu_1} \leq c_2$. Otherwise the combination

$$c_1 \leq -2\sqrt{-\mu_1} < c_2, \qquad \Phi^1_{c_2}(u_2) \leq \Phi^2_{c_2}(u_2) \leq 0$$

implies that c_2 is an admissible trial velocity for f_1 and we again have $c_1 \leq c_2$. □

2.1 A sketch of the proof of Theorem 6

We now give a short sketch of the proof of Theorem 6. This gives us the opportunity to present some lemmas, used in the proof and necessary for us later.

To show Theorem 6, Muratov and Novaga use an auxiliary constrained variational problem. They minimise the functional Φ_c over the set

$$B_c := \left\{ u \in H^1_c(\Sigma_\Omega) \;\middle|\; \frac{1}{2}\|\partial_x u\|^2_{L^2_c(\Sigma_\Omega)} = 1 \right\}. \tag{11}$$

First they show that there is a minimiser of Φ_c over the set B_c, and then how the minimiser of the constrained problem is related to the global minimiser of $\Phi_{c^\dagger}$. To show that the minimiser of the constrained problem exists, they use the direct method. They prove that for an arbitrary function $w \in H^1_c(\Sigma_\Omega)$, the $H^1_c(\Sigma_\Omega)$ norm of w is bounded by $\Phi_c(w)$ and $\|\partial_x w\|^2_{L^1_c(\Sigma_\Omega)}$ (Lemma 13). and that the functional Φ_c is weakly lower semi-continuous (Lemma 14).

Lemma 13. *Assume that f satisfies (H1) and (H2), and define*

$$C_1 := \min_{(u,p)\in[0,1]\times\Sigma_\Omega} \left(\frac{V(u(p), y)}{x^2} \right).$$

Then for each $u \in H^1_c(\Sigma_\Omega)$ we have

$$\begin{aligned} \|u\|^2_{L^2_c(\Sigma_\Omega)} \;&\leq\; \frac{4}{c^2}\|\partial_x u\|^2_{L^2_c(\Sigma_\Omega)}, \\ \|\nabla_y u\|^2_{L^2_c(\Sigma_\Omega)} \;&\leq\; 2\Phi_c(u) - \frac{8C_1}{c^2}\|\partial_x u\|^2_{L^2_c(\Sigma_\Omega)}. \end{aligned}$$

Proof. This lemma corresponds to [14, Lemma 5.1 and Lemma 5.2]. □

Lemma 14. *Let f satisfy hypotheses (H1) and (H2), and let $c^2 + 4\mu > 0$. Then the functional Φ_c is sequentially weakly lower semi-continuous on $H^1_c(\Sigma_\Omega)$.*

Proof. See [14, Prop. 5.5] and the proof of [17, Thm 3.3]. □

To transfer the results for the constrained problem to the full problem, they use the following relation between the functionals Φ_{c_1} and Φ_{c_2}.

Lemma 15. *Define* $v(x) := u\left(\frac{c_1}{c_2}x\right)$. *Then*

$$\Phi_{c_1}(v) \;=\; \frac{c_2}{c_1}\left(\Phi_{c_2}(u) + \frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\int_{\Sigma_\Omega} e^{c_2 x}|\partial_x u(x)|^2\, dx\right).$$

Proof. This Lemma can be verified by simple calculation. □

Remark 16. Note that Lemma 15 implies that (H3) is equivalent to:

(H3') There exists $c < 0$ satisfying $c^2 + 4\mu > 0$ and $u \in H^1_c(\Sigma_\Omega)$ such that $\Phi_c(u) < 0$ and $u \not\equiv 0$.

Remark 17. Let $u \not\equiv 0$ be a minimiser of $\Phi_{c^\dagger}$. Then $\Phi_{c^\dagger}(u) = 0$, so Lemma 15 implies $c^\dagger = \sup\{c > 0 : c \text{ is admissible trial velocity}\}$.

Summarising their result about the constrained minimiser we have the following lemma.

Lemma 18. *Assume that (H1)-(H3) are satisfied, let* $c^\dagger$ *be as in Theorem 6 and let* c *be an admissible trial velocity for (H3). Then there exists a minimiser* u *of* Φ_c *over the set* B_c*, we have* $c^\dagger = c\sqrt{1 - \Phi_c(u_c)}$*, and* $v(x, y) := u(\frac{c^\dagger}{c}x, y)$ *is a minimiser of* $\Phi_{c^\dagger}$.

Proof. The proof is part of the proof of [17, Theorem 3.3]. □

3 Existence and properties of travelling wave solutions modelling the vortex mode

In this section we show the existence of solutions of (6). In the first subsection we use the theorems of the preceding section to find and investigate solutions of (7) and in particular to find good bounds on the speed. In the second subsection we pass to the limit $\epsilon \to 0$.

3.1 Preliminary lemmas and properties of travelling wave solutions for a regularised equation

In view of the variational problem, we define for all $\epsilon \geq 0$

$$\begin{aligned}
V_h^\epsilon &\colon [0,1] \times D_1 \to \mathbb{R}, \quad (u, r) \mapsto \frac{1}{2\pi^2(r^2 + \epsilon^2)}\sin^2(\pi u) + \frac{h}{\pi^2}(\cos(\pi u) - 1),\\
I_h^\epsilon &\colon H^1(D_1) \to \overline{\mathbb{R}}, \qquad u \mapsto \int_{D_1} \frac{1}{2}|\nabla_y u|^2 + V_h^\epsilon(u, y)\, dy, \quad I_h := I_h^0,\\
\Phi_{h,c}^\epsilon &\colon H^1_c(\Sigma) \to \overline{\mathbb{R}}, \qquad u \mapsto \int_\Sigma \left(\frac{1}{2}|\nabla u|^2 + V_h^\epsilon(u, y)\right) e^{cx}\, dp,
\end{aligned}$$

$$V_h := V_h^0, \quad I_h := I_h^0, \quad \Phi_{h,c} := \Phi_{h,c}^0. \tag{12}$$

V_h^ϵ can be split in two parts: a positive part V_+^ϵ which is monotonously decreasing in ϵ and independent of h, and a negative part V_{h-} that is independent of ϵ:

$$V_+^\epsilon\colon [0,1]\times[0,1]\to\mathbb{R}, \qquad (u,r)\mapsto \frac{1}{2\pi^2(r^2+\epsilon^2)}\sin^2(\pi u),$$
$$V_{h-}\colon [0,1]\times[0,1]\to\mathbb{R}, \qquad (u,r)\mapsto \frac{h}{\pi^2}(\cos(\pi u)-1).$$

Definition 19. For $\epsilon\geq 0$, let μ_ϵ be the smallest eigenvalue of

$$\Delta u - \frac{1}{|y|^2+\epsilon^2}u + \mu u = 0 \ \text{ in } D_1, \quad \partial_\nu u = 0 \ \text{ on } \partial D_1,$$

and let ψ_ϵ be the corresponding positive eigenfunction with $\|\psi_\epsilon\|_{L^2(D_1)}=1$.

Remark 20. The smallest eigenvalue of

$$\Delta u - \frac{1}{|y|^2+\epsilon^2}u + hu + \mu u = 0 \ \text{ in } D_1, \quad \partial_\nu u = 0 \ \text{ on } \partial D_1, \tag{13}$$

is $\mu_\epsilon - h$. A simple inspection of the defining equation yields $\mu_0 = k_0^2$ and $\psi(r) = J_1(k_0 r)$. Here and in the following, J_1 is the first Bessel function of first kind (cf.[9]), and $k_0\approx 1.84$ is the first root of the derivative of J_1.

Lemma 21. *For $\epsilon\geq 0$, the eigenvalue μ_ϵ depends continuously on ϵ.*

Proof. For $\epsilon>0$ the statement follows directly from the representation

$$\mu_\epsilon \;=\; \inf_{\{v:\|v\|_{H^1(\Omega)}=1\}} \int_{D_1} \frac{1}{2}|\nabla_y v|^2 + \frac{v^2}{|y|^2+\epsilon^2}$$

Now assume $\epsilon=0$. Since $\epsilon_1\geq\epsilon_2$ implies $\mu_{\epsilon_1}\leq\mu_{\epsilon_2}$, it suffices to show that there is a positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, such that $\lim_{n\to\infty}\mu_{\epsilon_n}\geq\mu_0$. Since $\|\nabla\psi^\epsilon\|^2_{L^2(D_1)}\leq\mu_\epsilon$ is bounded, there is a positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, such that $(\psi^{\epsilon_n})_{n\in\mathbb{N}}$ converges weakly in $H^1(D_1)$ and strongly in $L^2(D_1)$ to some function $v_{\lim}$. We have

$$\begin{aligned}\mu_0 \;&\leq\; \lim_{\delta\to 0}\int_{D_1\setminus D_\delta}\frac{1}{2}|\nabla_y v_{\lim}|^2 + \frac{v_{\lim}^2}{|y|^2}\\ &=\; \lim_{\delta\to 0}\lim_{n\to\infty}\int_{D_1\setminus D_\delta}\frac{1}{2}|\nabla_y\psi_{\epsilon_n}|^2 + \frac{\psi_{\epsilon_n}^2}{|y|^2+\epsilon_n^2}\,dp \;\leq\; \lim_{n\to\infty}\mu_{\epsilon_n}.\end{aligned}$$

□

Using the monotonicity properties of the speed (Corollary 12) and special trial functions, we find upper and lower bounds on the speed of the solutions of (7).

Lemma 22. *For all $\epsilon>0$ there exists an MN-solution (c^ϵ,u^ϵ) of (7). Moreover there exist constants $\overline{c}$, $\underline{c}>0$, such that $\underline{c}<|c^\epsilon|\leq\overline{c}$ if ϵ small enough. We have*

$$\overline{c} := 2\sqrt{h} \quad \textit{and} \quad \underline{c} := \begin{cases}\frac{2h}{5\pi} & \textit{for } 0<h<h_c\\ 2\sqrt{h-k_0^2} & \textit{for } h_c\leq h,\end{cases}$$

where $h_c\approx 0.05+k_0^2$ is the smaller one of the two solutions of the equation $\frac{2h}{5\pi}=2\sqrt{h-k_0^2}$.

Proof. The existence of MN-solutions follows from Theorem 6 and Theorem 7. Corollary 12 implies that the MN-solution (c, u) of (7) is slower than the MN-solution $(\overline{c}, \overline{u})$ of

$$\begin{aligned} \Delta\overline{u} + \overline{c}\partial_x\overline{u} + \frac{h}{\pi}\sin(\pi\overline{u}) &= 0 \quad \text{in } \Sigma, \\ \partial_\nu\overline{u}(x,y) &= 0 \quad \text{on } \partial\Sigma. \end{aligned} \tag{14}$$

The term $\frac{h}{\pi}\sin(\pi\overline{u})$ satisfies equation (10), so Proposition 10, Theorem 7, and Definition 8 yield that $(\overline{c}, \overline{u})$ is a non-variational MN-solution. The smallest eigenvalue of $-\Delta v - hv$ in D_1 with Neumann boundary conditions is $-h$, thus we have $\overline{c} = 2\sqrt{h}$.

If $h \geq k_0^2$, using Remark 20 and the statements about the speed in Theorem 6 and Theorem 7, we find the lower bound

$$|c| \;\geq\; 2\sqrt{h-\mu_\epsilon} \;\geq\; 2\sqrt{h-\mu_0} \;=\; 2\sqrt{h-k_0^2} \;=:\; c_b.$$

To find a lower bound for $h < k_0^2$, we use a trial function v and find some c_s such that (H3) is satisfied for f^ϵ when ϵ is small enough. We define

$$v(x,r) = \begin{cases} e^{\frac{\lambda x}{\sqrt{r}}} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

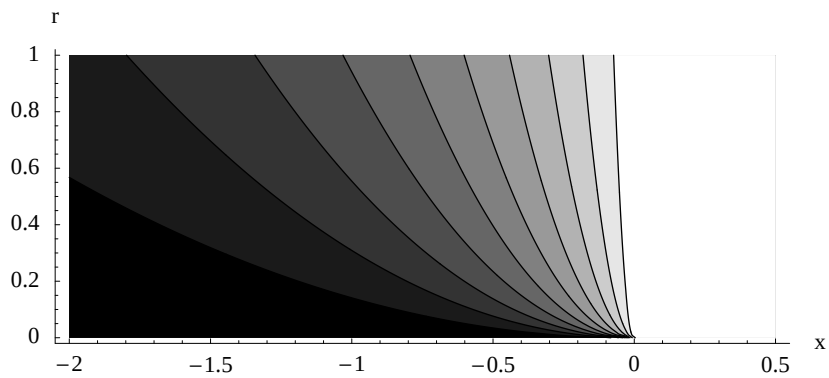

Figure 5: contour plot of v

Then we can estimate the terms of $\Phi_c(v)$ by

$$\begin{aligned} \int_\Sigma \frac{1}{2}(\partial_x v(p))^2 e^{cx}\, dp &\leq \frac{2\pi\lambda^2}{3(2\lambda+c)}, \\ \int_\Sigma \frac{1}{2}|\nabla_y v|^2 e^{cx}\, dp &\leq \frac{8\pi\lambda^2}{(2\lambda+c)^3}, \\ \int_\Sigma V_h^0(v,y)\, e^{cx}\, dp &\leq \frac{2h}{\pi c} + \frac{2}{\pi(2\lambda+c)}. \end{aligned}$$

Now choose $\lambda := 2$, and assume $c \geq -0.5$. Then for all $\epsilon \geq 0$ we have

$$\Phi^{\epsilon}_{h,c} \;\leq\; \Phi^{0}_{h,c}(v) \;<\; 5 + \frac{2h}{\pi c}.$$

Thus, for $c < 0$ and $|c| \leq \underline{c}_s := \min\left(\frac{2}{5\pi}h, \frac{1}{2}\right)$, the functional $\Phi^{\epsilon}_{h,c}(v)$ is negative, and for $h < k_0^2 + 1$ we have $c_s^2 > 4(h - k_0^2)$. Now continuity of μ_ϵ yields that, for $h < k_0^2 + 1$ and ϵ small enough, hypothesis (H3) is satisfied and we have $|c| \geq c_s$. □

Lemma 23. *For $\epsilon > 0$, let (c^ϵ, u^ϵ) be a monotone solution of (7) and set $u^\epsilon_+ = \lim_{x\to+\infty} u(x,\cdot)$. Then u^ϵ is rotationally symmetric and*

$$\begin{aligned} c^\epsilon \left\|\partial_x u^\epsilon\right\|^2_{L^2([a,b]\times D_1)} &= \int_{D_1} V^\epsilon_h(u^\epsilon, y) + \frac{1}{2}(\nabla_y u^\epsilon)^2 - \frac{1}{2}(\partial_x u^\epsilon)^2 \, dy \Big|_{x=a}^{x=b}, \\ c^\epsilon \left\|\partial_x u^\epsilon\right\|^2_{L^2(\Sigma)} &= I^\epsilon_h(u^\epsilon_+) \end{aligned}$$

Proof. To show rotational symmetry, we parametrise D_1 by polar coordinates (r, ϕ), and differentiate (7) by x and by ϕ:

$$\begin{aligned} \Delta\partial_x u^\epsilon + c\partial_x\partial_x u^\epsilon \; \partial_x u^\epsilon + \partial_u f^\epsilon &= 0 \quad \text{in } \Sigma, \qquad & \partial_\nu\partial_x u^\epsilon &= 0 \quad \text{on } \partial\Sigma \\ \Delta\partial_\phi u^\epsilon + c\partial_x\partial_\phi u^\epsilon + \partial_x u^\epsilon \partial_u f^\epsilon &= 0 \quad \text{in } \Sigma, \qquad & \partial_\nu\partial_\phi u^\epsilon &= 0 \quad \text{on } \partial\Sigma. \end{aligned}$$

Thus, both $\partial_x u^\epsilon$ and $\partial_\phi u^\epsilon$ are eigenfunctions of the operator

$$u \mapsto -\Delta u + c\partial_x u + u\partial_u f$$

with Neumann boundary conditions. Since u^ϵ is monotone in x, the function $\partial_x u$ is nonnegative, so 0 is the smallest eigenvalue and all eigenfunctions for the eigenvalue zero are multiples of $\partial_x u$. This has to hold in particular for $\partial_\phi u$. Since $u^\epsilon(x, r, \phi) = u^\epsilon(x, r, \phi + 2\pi)$, the function $\partial_\phi u^\epsilon$ can not be entirely positive or entirely negative, thus $\partial_\phi u^\epsilon \equiv 0$.

The first equation in the statement of Lemma 23 can be verified by testing (7) with $\partial_x u^\epsilon$ and using partial integration. The second equation follows from the first as the limit for $a \to -\infty$ and $b \to +\infty$. □

3.2 Travelling wave solutions for the original equation

In this subsection we will construct solutions of (6) as a limit of solutions of (7). We could do this purely on the level of differential equations, but if the travelling waves in the sequence are variational MN-solutions, we have more information, and we would like transfer this information to the limit. In particular, we want to have a limit that is the minimiser of the limit functional. So we will first show a lower semi-continuity result.

Lemma 24. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions converging weakly in $H^1_c(\Sigma)$ to some function $v \in H^1_c(\Sigma)$. If $\delta := c^2 + 4(k_0^2 - h) > 0$, then*

$$\Phi_{h,c}(v) \;\leq\; \liminf_{n\to\infty} \Phi^{\epsilon_n}_{h,c}(v_n) \;\leq\; \liminf_{n\to\infty} \Phi_{h,c}(v_n).$$

Proof. Since μ_ϵ depends continuously on ϵ (Lemma 21), and since $\mu_0 = k_0^2$ (Remark 20), there is some ϵ_0 such that $c^2 + 4(\mu_\epsilon - h) \geq \frac{\delta}{2} > 0$ for all $\epsilon < \epsilon_0$. So for all $\epsilon < \epsilon_0$ the functionals $\Phi_{h,c}^{\epsilon_n}$ are weakly lower semi-continuous on $H_c^1(\Sigma)$ (Lemma 14), and thus

$$\begin{aligned}
&\limsup_{n\to\infty} \left(\Phi_{h,c}(v) - \Phi_{h,c}^{\epsilon_n}(v_n)\right) \\
&\quad \leq \limsup_{n\to\infty} (\Phi_{h,c}(v) - \Phi_{h,c}^{\epsilon_n}(v)) + \underbrace{\limsup_{n\to\infty} (\Phi_{h,c}^{\epsilon_n}(v) - \Phi_{h,c}^{\epsilon_n}(v_n))}_{\leq 0} \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{2\pi^2} \int_\Sigma e^{cx} \left(\frac{1}{|y|^2} - \frac{1}{|y|^2 + \epsilon_n^2}\right) \sin^2(\pi v)\, dp \qquad (15) \\
&\quad \overset{(*)}{\leq} \lim_{\rho\to 0} \limsup_{n\to\infty} \frac{1}{2\pi^2} \int_{\Sigma\setminus(\mathbb{R}\times D_\rho)} e^{cx} \left(\frac{1}{|y|^2} - \frac{1}{|y|^2 + \epsilon_n^2}\right) \sin^2(\pi v)\, dp \qquad (16) \\
&\quad \leq \lim_{\rho\to 0} \limsup_{n\to\infty} \frac{1}{2\pi^2} \left(\frac{1}{\rho^2} - \frac{1}{(\rho^2 + \epsilon_n^2)}\right) \|v\|^2_{L_c^2(\Sigma)} \quad = \quad 0.
\end{aligned}$$

The estimate $(*)$ needs some explanation. The positive parts of (15) and (16) are independent of n and thus equal. For the negative part, we have for all $\delta > 0$ the estimate

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{2\pi^2} \int_\Sigma -e^{cx} \frac{1}{|y|^2 + \epsilon_n^2} \sin^2(\pi v)\, dp \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{2\pi^2} \int_{\Sigma\setminus(\mathbb{R}\times D_\rho)} -e^{cx} \frac{1}{|y|^2 + \epsilon_n^2} \sin^2(\pi v).
\end{aligned}$$

Thus we can pass to the limit $\delta \to 0$ on the right hand side. Since $\Phi_{h,c}^{\epsilon_n}(v_n) \leq \Phi_{h,c}(v_n)$ for all $\epsilon \geq 0$, we have proved the statement. □

Theorem 25. *Assume that there exists $c < 0$ satisfying $c^2 + 4(k_0^2 - h) \geq \delta > 0$, and $\tilde{u} \in H_c^1(\Sigma)$ such that $\Phi_{h,c}(\tilde{u}) \leq 0$. Then there exists $(c^\dagger, u)$ such that $u \in H_{c^\dagger}^1(\Sigma)$ is a minimiser of $\Phi_{h,c^\dagger}$. Moreover, $(c^\dagger, u^\dagger)$ satisfies (6), and we have:*

(i) $\Phi_{h,c^\dagger}(u) = 0$ and $e^{\frac{c^\dagger x}{2}} u(x,y)$ is bounded.

(ii) u is monotonously increasing in x and rotationally symmetric and $\underline{c} \leq |c^\dagger| \leq \overline{c}$ with $\underline{c}, \overline{c}$ as in Lemma 22.

(iii) There is at most one point $x \in \mathbb{R}$ where $u(x,0)$ is neither zero nor one.

(iv) The minimiser u is unique up to translation.

(v) $|c^\dagger| = |c^\dagger(h)|$ depends monotonously increasing and continuously on h. Up to translation, the minimiser u depends continuously in $H^1_{\mathrm{loc}}(\overline{\Sigma})$ on h.

Proof. Let $c < c^\dagger$ be an admissible trial velocity in (H3) for $\epsilon = 0$. Since μ_ϵ depends continuously on ϵ (Lemma 21) and since $\Phi_{h,c}^\epsilon(\tilde{u}) \leq \Phi_{h,c}(\tilde{u})$ (Lemma 24), there exists $\epsilon_0 > 0$ such that c is an admissible trial velocity for all $\epsilon \leq \epsilon_0$. Thus for all $\epsilon \leq \epsilon_0$ there exists $(c_\epsilon^\dagger, u_\epsilon)$ such that $u_\epsilon \not\equiv 0$ is a minimiser of $\Phi_{h,c_\epsilon^\dagger}^\epsilon$ (Theorem 6).

Since the functions u_ϵ are in different spaces $H^1_{c_\epsilon^\dagger}$, we rescale. We set $v_\epsilon(x,y) := u_\epsilon(\frac{c}{c_\epsilon^\dagger}(x - a_\epsilon), y)$, where a_ϵ is chosen such that $v_\epsilon(x,y) \in B_c$ with B_c as in (11).

Since the minimisers u_ϵ of $\Phi^\epsilon_{h,c^\dagger_\epsilon}$ are unique up to translation (Theorem 6), Lemma 18 implies that for $\epsilon \le \epsilon_0$ the functions v_ϵ are minimisers of $\Phi^\epsilon_{h,c}$ in B_c. We have $\Phi^\epsilon_{h,c}(v_\epsilon) = \frac{c}{c^\dagger_\epsilon} - \frac{c^\dagger_\epsilon}{c} \le \frac{c}{c^\dagger} - \frac{c^\dagger}{c} < 0$ (Lemma 15), so $\|v_\epsilon\|_{H^1_c(\Sigma)}$ is uniformly bounded and there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero such that $(v_{\epsilon_n})_{n\in\mathbb{N}}$ converges weakly in $H^1_c(\Sigma)$ to some $v \in H^1_c(\Sigma)$. Lemma 24 implies $0 > \lim_{n\to\infty} \Phi_{h,c}(v_{\epsilon_n}) \ge \Phi_{h,c}(v)$.

We show that v is a minimiser of $\Phi_{h,c}$ in B_c. Because of weak convergence in $H^1_c(\Sigma)$, we have $\int_\Sigma \frac{1}{2}|\partial_x v|^2 \le 1$. If $\int_\Sigma \frac{1}{2}|\partial_x v|^2 < 1$ there exists $a > 0$ such that $\tilde{v}\colon (x,y) \mapsto v(x+a,y)$ is in B_c and we have

$$\Phi_{h,c}(\tilde{v}) \;=\; \frac{\int_\Sigma \frac{1}{2}|\partial_x \tilde{v}|^2}{\int_\Sigma \frac{1}{2}|\partial_x v|^2}\Phi_{h,c}(v) \;<\; \Phi_{h,c}(v) \le \lim_{n\to\infty} \Phi_{h,c}(v_{\epsilon_n}).$$

Thus there exists $n_0 \in \mathbb{N}$ such that $\Phi_{h,c}(\tilde{v}) < \Phi_{h,c}(v_{\epsilon_n})$ for all $n \ge n_0$. This is a contradiction to the fact that the functions v_{ϵ_n} were chosen as minimisers of $\Phi^{\epsilon_n}_{h,c}$ in B_c.

If $\Phi_{h,c}(v) < \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c}(v_{\epsilon_n})$ or if v is not a minimiser of $\Phi_{h,c}$ in B_c, then, again, there is a function $\tilde{v} \in B_c$ such that $\Phi^{\epsilon_n}_{h,c}(\tilde{v}) \le \Phi_{h,c}(\tilde{v}) < \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c}(v_{\epsilon_n})$ for all $n \in \mathbb{N}$ and, again, this is impossible. Thus v is a minimiser of $\Phi_{h,c}$ in B_c.

Now define $c^\dagger := c\sqrt{1-\Phi_{h,c}(v)}$ and set $u(x,y) := v(\frac{c^\dagger}{c}x, y)$. Arguing as in the proof of [17, Theorem 3.3], we show that u is a minimiser of $\Phi_{h,c^\dagger}$: For any $w \in H^1_c(\Sigma)$, $w \not\equiv 0$ define $\tilde{w}(x,y) := w\left(\frac{c(x-a)}{c^\dagger}, y\right)$, where a is chosen such that $\tilde{w} \in B_c$. Using Lemma 15 we have

$$\begin{aligned} e^{c^\dagger a}\Phi_{h,c^\dagger}(w) \;&=\; \frac{c}{c^\dagger}\left(\Phi_{h,c}(\tilde{w}) + \left(\frac{(c^\dagger)^2}{c^2} - 1\right)\frac{1}{2}\int_\Sigma e^{cx}|\partial_x \tilde{w}(x)|^2\,dx\right) \\ &=\; \frac{c}{c^\dagger}\left(\Phi_{h,c}(\tilde{w}) - \Phi_{h,c}(v)\right). \end{aligned}$$

Since v was a minimiser of $\Phi_{h,c}(v)$ in B_c, the right hand side of the equation is non-negative and the minimum is attained for $\tilde{w} = v$, i.e. $w = u$. Thus u is a minimiser of $\Phi_{h,c^\dagger}$, which immediately implies that u satisfies (6). Moreover, $v \in H^1_c(\Sigma)$ implies $u \in H^1_{c^\dagger}(\Sigma)$

(i) If $\Phi_{h,c^\dagger}(u) \ne 0$ we could decrease $\Phi_{h,c^\dagger}$ by translating u. This contradicts the assumption that u is a minimiser of $\Phi_{h,c^\dagger}$, thus $\Phi_{h,c^\dagger}(u) = 0$. Lemma 32 below implies $I_0(w) \ge \frac{1}{\pi}\left(1 - \cos(\pi\|w\|_{L^\infty(D_1)})\right)$ for all $w\colon D_1 \to \mathbb{R}$. Since, in addition, $2t^2 \le 1 - \cos(\pi t) \le \frac{\pi^2}{2}t^2$ for all $t \in [0,1]$, we have

$$\begin{aligned} 0 \;&=\; \Phi_{h,c^\dagger}(u) \\ &\ge\; \int_{\mathbb{R}} e^{c^\dagger x}\left(I_0(u(x,\cdot)) + \frac{1}{2}\|\partial_x - \int_{D_1}\frac{hr}{\pi^2}(1-\cos(\pi u(x,y))\,dy\right)\,dx \\ &\ge\; \int_{\mathbb{R}} e^{c^\dagger x}\frac{1}{\pi}\left(1 - \cos(\|\pi u(x,\cdot)\|_{L^\infty(D_1)})\right)\,dx \;-\; \frac{\pi}{2}h\|u\|^2_{L^2_{c^\dagger}(\Sigma)} \\ &\ge\; \int_{\mathbb{R}} \frac{2}{\pi}e^{c^\dagger x}\|u(x,\cdot)\|^2_{L^\infty(D_1)}\,dx \;-\; h\pi\|u\|^2_{L^2_{c^\dagger}(\Sigma)}. \end{aligned}$$

Since the second summand is finite, the first summand has to be finite as well,

and since u is monotone we have

$$\int_{\mathbb{R}} \frac{2}{\pi} e^{c^\dagger x} \|u(x,\cdot)\|^2_{L^\infty(D_1)}\, dx \;\geq\; \sup_{x_0\in\mathbb{R}} \int_{-\infty}^{x_0} e^{c^\dagger x} \|u(x_0,\cdot)\|^2_{L^\infty(D_1)}\, dx$$
$$\geq\; \sup_{x_0\in\mathbb{R}} \frac{2}{\pi c^\dagger} e^{c^\dagger x_0} \|u(x_0,\cdot)\|^2_{L^\infty(D_1)}.$$

This can only be finite if $\|u(x,\cdot)\|_{L^\infty(D_1)} e^{\frac{c^\dagger}{2}x}$ is bounded.

(ii) The statements follow immediately from the fact that for $\epsilon > 0$ the functions v_ϵ are monotonously increasing in x and rotationally symmetric (Lemma 23) and from the bounds on $c^\dagger_\epsilon$ of Lemma 22.

(iii) Since $\Phi_{h,c^\dagger}(u)$ is finite and u is monotone, there is at most one point where u is neither zero or one.

(iv) The proof is due to S. Heinze [7]. Let u_1, u_2 be nontrivial minimisers of $\Phi_{h,c^\dagger}$. After a translation we can assume that there is some point (x^*, y^*), $y^* \neq 0$ such that $u_1(x^*, y^*) = u_2(x^*, y^*)$. Set $\overline{u} = \max(u_1, u_2)$ and $\underline{u} = \min(u_1, u_2)$. Then $\Phi_{h,c^\dagger}(\overline{u}) + \Phi_{h,c^\dagger}(\underline{u}) = 0$, thus $\overline{u}$ and $\underline{u}$ are minimisers as well. Set $w := \overline{u} - \underline{u}$ and set

$$g(x,y) := \begin{cases} \frac{f^0(\overline{u}(x,y),y) - f^0(\underline{u}(x,y),y)}{\overline{u}(x,y) - \underline{u}(x,y)} & \text{if } \overline{u}(x,y) \neq \underline{u}(x,y), \\ 0 & \text{otherwise.} \end{cases}$$

Then $|g(x,y)| \leq \frac{1}{|y|^2} + h$, and the function w satisfies

$$w \geq 0, \quad w(x^*, y^*) = 0, \quad \Delta w + c\partial_x w + \min(0, g(x,y))w \leq 0.$$

So the strong maximum principle [6, Theorem 3.5] implies $w \equiv 0$.

(v) Corollary 12 implies that $|c^\dagger|$ is monotonously increasing in h. Let h_0 be such that (H3) is satisfied. Since, for any $v \in L^2_c$ such that $\Phi_{c,h}(v)$ is finite, $\Phi_{c,h}(v)$ depends continuously on h, for any admissible trial velocity $c < c^\dagger(h_0)$ there exists a neighborhood $U_c(h_0)$ such that c is an admissible trial velocity for all $h \in U_c(h_0)$. Thus Remark 17 implies $c^\dagger(h_0) \leq \lim_{h\to h_0} c^\dagger(h)$. Now fix some admissible trial velocity $c_0 < c^\dagger(h_0)$ and let $(h_n)_{n\in\mathbb{N}}$, $h_n \in U(h_0)$ be a sequence converging to h_0. As we have seen at the beginning of this proof, there exist minimisers v_n of $\Phi_{h_n,c}$ in B_c. These minimisers of the constrained problem are uniformly bounded in $H^1_c(\Sigma)$ (Lemma 13), thus there exists a subsequence converging weakly in $H^1_c(\Sigma)$ to some $v \in H^1_c(\Sigma)$. Since the functional $\Phi_{h_0,c}$ is lower semi-continuous (Lemma 24), we have

$$\lim_{n\to\infty} \Phi_{h_n,c}[v_n] \;\geq\; \lim_{n\to\infty} \Big(\Phi_{h_0,c}[v_n] - (h_n - h_0)\|v_n\|^2_{L^2_c(\Sigma)}\Big)$$
$$\geq\; \Phi_{c,h_0}[v] \;\geq\; \Phi_{h_0,c}(\tilde{v}),$$

where $\tilde{v}$ is the function v translated in a way that $\frac{1}{2}\|v\|^2_{L^2_c(\Sigma)} = 1$. Thus

$$c_0^\dagger \;=\; c\sqrt{1 - \Phi_{h_0,c}[v_0]} \;\geq\; c\sqrt{1 - \Phi_{h_0,c}[\tilde{v}]}$$
$$\geq\; \lim_{n\to\infty} c\sqrt{1 - \Phi_{h_n,c}[v_n]} \;=\; \lim_{n\to\infty} c_n^\dagger \;\geq\; c_0^\dagger,$$

and we conclude that all inequalities have to be equalities. In particular, $v = \tilde{v}$ has to be a constrained minimiser of $\Phi_{h_0,c}$. Since minimisers are unique up to translation, constrained minimisers are unique, thus $v = v_0$. So all convergent subsequences of $(v_n)_{n\in\mathbb{N}}$ have to converge weakly in $H^1_c(\Sigma)$ to v_0, which implies that $(v_n)_{n\in\mathbb{N}}$ itself converges to v_0. The minimisers u_n of $\Phi_{h_n,c_n^\dagger}$ are the functions v_n, rescaled by a factor that depends continuously on $c_n^\dagger$. We can conclude that, up to translation, the functions u_n depend continuously in $H^1_{\mathrm{loc}}(\overline{\Sigma})$ on h. □

Remark 26. For $h \leq h_c$, where h_c as in Lemma 22, we can use the trial function described in the proof of Lemma 22, so the conditions of Theorem 25 are certainly satisfied. Note that $h_c > k_0^2$.

If the conditions of Theorem 25 are not satisfied, we have to work with the differential equation and use the following lemma.

Lemma 27. *Let u^ϵ satisfy (7). Then there is a constant $C = C(l, \rho, |c|)$ independent of ϵ and monotonously increasing in $|c|$ such that*

$$\|u^\epsilon\|_{H^2([-l,l]\times D_1\setminus D_\rho)} \;\leq\; C(l,\rho,|c|)\left(\|f^\epsilon\|_{L^\infty([-2l,2l]\times D_1\setminus D_{\frac{\rho}{2}})} + 1\right).$$

Proof. By standart ellipitc estimates. □

Theorem 28. *Assume that there exists no pair $(c, \tilde{u}) \in \mathbb{R} \times H^1_c(\Sigma)$ such that $c^2 + 4(k_0^2 - h) > 0$ and $\Phi_{h,c}(\tilde{u}) \leq 0$. Then there exists a solution (c^*, u) of (6) such that:*

(i) u is monotonously increasing in x and rotationally symmetric. Moreover,

$$\inf\left\{x \in \mathbb{R} \mid \exists\, y \in D_1 \setminus D_{\frac{1}{2}} \text{ s.th. } u(x,y) \geq 0.33\right\} = 0.$$

(ii) $c^* = -2\sqrt{h - k_0^2}$.

(iii) $\|\partial_x u\|_{L^2(\mathbb{R}\times D_1)} \;\leq\; \sqrt{\frac{2h}{\pi|c^*|}}$ and

$$\int_{[a,b]\times D_1} \frac{1}{2}(\nabla u)^2 + V_+^0(u,y) \;\leq\; \frac{h}{\pi}\left(4(2-a) + \frac{1}{|c^*|}\right) \quad \textit{for all } a,b \in \mathbb{R}.$$

(iv) There is at most one point $x \in \mathbb{R}$ where $u(x, 0)$ is neither zero nor one.

Proof. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let $(u_n, c_n)_{n\in\mathbb{N}}$ be a sequence of MN-solutions of (7) with $\epsilon = \epsilon_n$ such that

$$0 \;=\; \inf\left\{x \in \mathbb{R} \mid \exists\, y \in D_1 \setminus D_{\frac{1}{2}} \text{ s.th. } u_n(x,y) \geq 0.33\right\}. \tag{17}$$

With Lemma 27, the bound on c^n (Lemma 22) implies for each $l, \rho > 0$ a bound on $\|u_n\|_{H^2([-l,l]\times D_1\setminus D_\rho)}$.

Thus we have weak convergence of a subsequence of $(u_n)_{n\in\mathbb{N}}$ in $H^2_{\mathrm{loc}}(\overline{\Sigma}\setminus\mathbb{R}\times\{0\})$. A bootstrap argument gives convergence in $H^k_{\mathrm{loc}}(\overline{\Sigma}\setminus\mathbb{R}\times\{0\})$ for all $k \in \mathbb{N}$ and thus convergence in $C^\infty_{\mathrm{loc}}(\overline{\Sigma}\setminus\mathbb{R}\times\{0\})$.

After passing to a subsequence, we can assume that $(u_n, c_n)_{n\in\mathbb{N}}$ converges to (u, c).

(i) Since all MN-solutions are monotonously increasing, the functions u_n are monotonously increasing. They are rotationally symmetric (Lemma 23), and, by assumption, (17) is satisfied. Thus the statement follows from convergence in $C^\infty_{\mathrm{loc}}(\Sigma \setminus \mathbb{R} \times \{0\})$.

(ii) Either

$$c^* \;=\; \lim_{n\to\infty} c_n \;=\; \lim_{n\to\infty} -2\sqrt{h-\mu_{\epsilon_n}} \;=\; -2\sqrt{h-k_0^2}$$

or

$$c^* \;=\; \lim_{n\to\infty} c_n \;<\; \lim_{n\to\infty} -2\sqrt{h-\mu_{\epsilon_n}}.$$

In the latter case we can assume that for all $n \in \mathbb{N}$ the functions u_n are minimisers of $\Phi^{\epsilon_n}_{h,c_n}$. Set $v_n(x) := u_n(\frac{c^*}{c_n}x - a_n)$ where a_n is chosen such that $v_n \in B^c$. Then $\|v_n\|_{H^1_c(\Sigma)}$ is uniformly bounded, so the functions v_n converge, up to a subsequence, weakly in H^1_c to some function v. Since the velocity c^ϵ is montonously decreasing in ϵ (Corollary 12), we have $c^* \geq c_n$ for all $n \in \mathbb{N}$. Thus, using Lemma 15 and Lemma 24, we have

$$0 \;=\; \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c_k}(u_n) \;\geq\; \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c^*}(v_n) \;\geq\; \Phi_{h,c^*}(v).$$

This contradicts the assumptions.

(iii) For all $n \in \mathbb{N}$, Lemma 23 implies

$$c_n \, \|\partial_x u_n\|^2_{L^2(\mathbb{R}\times D_1)} \;\geq\; \int_{D_1} V^{\epsilon_n}_h(1,y)\, dy \;=\; -\frac{2}{\pi}h.$$

Thus

$$\|\partial_x u\|^2_{L^2(\Sigma)} \;=\; \lim_{\rho\to 0}\lim_{n\to\infty} \|\partial_x u_n\|^2_{L^2(\Sigma\setminus(\mathbb{R}\times D_\rho))} \;\leq\; \lim_{n\to\infty} \|\partial_x u_n\|^2_{L^2(\Sigma)} \;\leq\; \frac{2h}{\pi|c^*|}. \tag{18}$$

For the second equation we use Lemma 23 again. Letting b tend to ∞ in the first equation of Lemma 23, for all $a \in \mathbb{R}$ we have

$$\begin{aligned}
&\int_{D_1} V^{\epsilon_n}_+(u_n(a,y),y) + \frac{1}{2}|\nabla_y u_n(a,y)|^2\, dy \\
&= -\|\partial_x u_n\|_{L^2([a,\infty[\times D_1} + \frac{1}{2}\|\partial_x u_n(a,\cdot)\|^2_{L^2(D_1)} - \int_{D_1} V_{h-}(u_n(a,y),y)\, dy \\
&\leq \frac{1}{2}\|\partial_x u_n(a,\cdot)\|^2_{L^2(D_1)} + \frac{2h}{\pi}.
\end{aligned}$$

Integration from a_1 to a_2 and passing to the limit $n \to \infty$ yields

$$\begin{aligned}
&\int_{[a_1,a_2]\times D_1} V^0_+(u,y) + \frac{1}{2}|\nabla_y u|^2 \\
&= \lim_{\rho\to 0}\lim_{n\to\infty} \int_{[a_1,a_2]\times(D_1\setminus D_\rho)} V^{\epsilon_n}_+(u_n,y) + \frac{1}{2}|\nabla_y u_n|^2 \\
&\leq \lim_{n\to\infty} \int_{[a_1,a_2]\times D_1} V^{\epsilon_n}_+(u_n,y) + \frac{1}{2}|\nabla_y u_n|^2 \\
&\leq \lim_{n\to\infty} \frac{2h(a_2-a_1)}{\pi} + \frac{1}{2}\|\partial_x u_n\|^2_{L^2([a_1,a_2]\times D_1)} \\
&\leq \frac{2h(a_2-a_1)}{\pi} + \frac{h}{\pi|c^*|}.
\end{aligned}$$

(iv) Set $r = |y|$. Since u is rotationally symmetric, we can write $u(x,r)$ instead of $u(x,y)$ in the calculation below. We know from (iii) that for almost every $a \in \mathbb{R}$

$$\begin{aligned}
\infty \;&>\; \int_{D_1} 2V_+^0(u(a,y),y) + |\nabla_y u(a,y)|^2 \, dy \\
&=\; \int_0^1 \frac{1}{\pi r}\sin^2(\pi u(a,r)) + \pi r(\partial_r u(a,r))^2 \, dr \\
&\geq\; \int_0^1 2|\sin(\pi u(x,r))\partial_r u(a,r)| \, dr \;\;\geq\; \int_0^1 \frac{2}{\pi}|\partial_r \cos(\pi u(a,r))| \, dr
\end{aligned}$$

and see that $u(a,\cdot)$ is continuous for almost all $a \in \mathbb{R}$. If $u(a,\cdot)$ is continuous and $\int_{D_1} V_+(u,y)$ is finite, either $u(a,0) = 0$ or $u(a,0) = 1$. Since $u(x,0)$ is monotonously increasing in x, there is at most one point a^* such that $u(a,0)$ is neither zero nor one. □

4 Stationary states

In this section we investigate stationary states, i.e. solutions of (6) that do not depend on x and therefore solve

$$\Delta u + f^0(y,u) = 0 \;\text{ in } D_1, \qquad \partial_\nu u = 0 \;\text{ on } \partial D_1. \tag{19}$$

We restrict our attention to radially symmetric solutions u with values in $[0,1]$ for which $I_h(u)$ as defined in (12) is finite. Such functions u depend only on the scalar variable $r = |y|$. For $a, b \in \mathbb{R}$, $v : [a,b] \to \mathbb{R}$ we define

$$\tilde{I}_h(v,[a,b]) \;:=\; \int_a^b \frac{\pi r}{2}(v')^2 + \frac{1}{2\pi r}\sin^2(\pi v) + \frac{h\,r}{\pi}(\cos(\pi u) + 1) \, dr,$$

and for $v : [0,1] \to \mathbb{R}$ we set

$$\tilde{I}_h(v) := \tilde{I}_h(v,[0,1]).$$

Then, as functions of r, the maps u are the critical points of $\tilde{I}_h$. Moreover, they are exactly the solutions of the ordinary differential equation

$$-u'' - \frac{1}{r}u' + \frac{1}{2\pi r^2}\sin(2\pi u) - \frac{h}{\pi}\sin(\pi u) = 0, \quad u'(1) = 0 \tag{20}$$

that are contained in

$$\mathcal{A} \;:=\; \left\{ u \in H^1_{\text{loc}}(]0,1],[0,1]) \;:\; \tilde{I}_h(u) < \infty \right\}.$$

Remark 29. If $u \in \mathcal{A}$ then $\lim_{r\to 0} u(r) = 0$ or $\lim_{r\to 0} u(r) = 1$.

We define

$$\mathcal{A}_0 \;:=\; \{u \in \mathcal{A} : \lim_{r\to 0} u(r) = 0\}, \qquad \mathcal{A}_1 \;:=\; \{u \in \mathcal{A} : \lim_{r\to 0} u(r) = 1\}.$$

If u is a local minimiser of $\tilde{I}_h$ in $\mathcal{A}_i$ ($i \in \{0,1\}$) then u is a local minimiser of $\tilde{I}_h$ in $H^1_{\text{loc}}(]0,1])$.

For $\rho, k \in]0,1]$ we set

$$\begin{aligned} \mathcal{W}(k,\rho) &:= \{v \in \mathcal{A}_0 : v[0,\rho] \subset [0,k],\ v(\rho) = k\}, \\ E_h(k,\rho) &:= \inf\{\tilde{I}_h(u,[0,\rho]) : u \in \mathcal{W}(k,\rho)\}. \end{aligned}$$

Remark 30. For $r \in \left]0, \sqrt{\frac{1}{2h}}\right]$, $t \in \left[0, \frac{1}{8}\right]$ we have

$$\begin{aligned} &\partial_{tt}\left(\frac{1}{2\pi r}\sin^2(\pi t) + \frac{hr}{\pi}(\cos(\pi t)+1)\right) \\ &\quad = \frac{\pi}{r}\cos(2\pi t) - h\pi r\cos(\pi t) \ \geq\ \left(\frac{\pi}{r} - \frac{h\pi r}{2}\right)\cos(2\pi t) \ >\ 0. \end{aligned}$$

Thus for $\rho \in \left]0, \sqrt{\frac{1}{2h}}\right]$, $k \in \left[0, \frac{1}{8}\right]$ the functional $\tilde{I}(\cdot,[0,\rho])$ is convex on $\mathcal{W}(k,\rho)$.

Definition 31. A function u is called a *semistable stationary state* or a *semistable solution* of (20) if u is a solution of (20) whose second variation is nonnegative, i.e., where for all $v \in \mathcal{A}_0$

$$\pi\int_0^1 r(v')^2 + \left(\frac{1}{r}\cos(2\pi u) - hr\cos(\pi u)\right)v^2\ dr \ \geq\ 0. \tag{21}$$

This is equivalent to all eigenvalues of L_u being nonnegative, where

$$L_u(\phi) \ :=\ -\phi'' - \frac{1}{r}\phi' + \left(\frac{1}{r^2}\cos(2\pi u) - h\cos(\pi u)\right)\phi. \tag{22}$$

4.1 Stationary states without external magnetic field

We consider the functional $\tilde{I}_0$ for fixed boundary value $u(1) = k$. Since $\tilde{I}_0(u) = \tilde{I}_0(1-u)$ for all $u \in \mathcal{A}$, we can assume $u \in \mathcal{A}_0$ without loss of generality. Using the Modica-Mortola trick, we can determine the value of $E_0(k,\rho)$ as well as the minimisers. In a second lemma we show that these minimisers are the only solutions of (20) for $h = 0$.

Lemma 32. *For all $k \in [0,1]$ we have $E_0(k,\rho) = \frac{1}{\pi}(1-\cos(\pi k))$. The minimum is attained, and the minimiser is*

$$\xi_a\colon [0,\rho] \to [0,1], \quad r \mapsto \frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{a^2r^2+1}}\right) \qquad \textit{where } a = \frac{1}{\rho}\tan\left(\frac{\pi}{2}k\right).$$

It satisfies the differential equation

$$r\xi_a' = \frac{1}{\pi}\sin(\pi\xi_a),$$

and we have

$$\xi_a(r) \sim \frac{2}{\pi}ar \quad \textit{for } r \to 0, \qquad \xi_a(r) \sim 1 - \frac{2}{\pi}\cdot\frac{1}{ar} \quad \textit{for } r \to \infty.$$

For a sketch of ξ_a for different values of a see Figure 6.

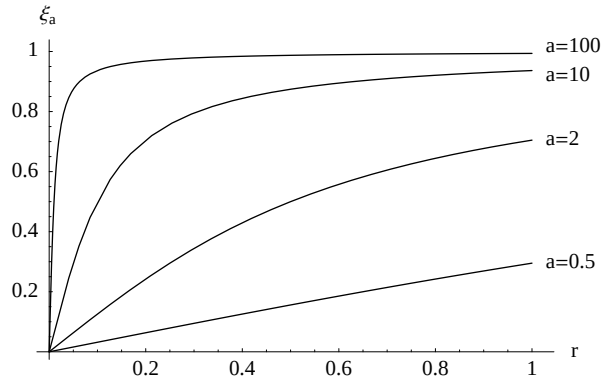

Figure 6: The functions ξ_a for different values of a

Proof. For all functions $\xi \in \mathcal{A}_0$ we have

$$\begin{aligned} \tilde{I}_0(\xi, [0,\rho]) &= \int_0^\rho \frac{\pi r}{2}|\xi'|^2 + \frac{1}{2\pi r}\sin^2(\pi\xi)\, dr \;\geq\; \int_0^\rho |\sin(\pi\xi)\,\xi'|\, dr \\ &= \frac{1}{\pi}\int_0^\rho |(\cos(\pi\xi))'|\, dr \;\geq\; \frac{1}{\pi}(1-\cos(\pi\xi(\rho))). \end{aligned} \tag{23}$$

Assume that ξ is a minimiser with $\xi(\rho) = k$. Then ξ is a monotonously increasing function that satisfies (23) with equality. The latter is the case if and only if

$$r|\xi'(r)|^2 \;=\; \frac{1}{\pi^2 r}\sin^2(\pi\xi(r)) \qquad \text{for all } r \in \mathbb{R},$$

that is, if ξ satisfies the differential equation

$$r\partial_r\xi \;=\; \frac{1}{\pi}\sin(\pi\xi), \qquad \xi(0) = 0, \qquad \xi(\rho) \;=\; k. \tag{24}$$

A solution of this equation is

$$\xi_a \colon \mathbb{R}_0^+ \to [0,1], \qquad r \mapsto \frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{a^2r^2+1}}\right),$$

where a can be calculated from k via

$$\frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{a^2\rho^2+1}}\right) \;=\; k, \quad \text{i.e.,}\; a \;=\; \frac{1}{\rho}\tan\left(\frac{\pi}{2}k\right).$$

Applying the uniqueness theorem for differential equations in $r = k$, we see that the function ξ_a is the only solution. The statements about the asymptotic behaviour of ξ_a can be found by direct inspection. □

In the following, ξ_a will always refer to the function described above.

Lemma 33. *The only functions $u \in \mathcal{A}$ that satisfy*

$$-u'' - \frac{1}{r}u' + \frac{1}{2\pi r^2}\sin(2\pi u) = 0 \tag{25}$$

are the functions ξ_a and $1-\xi_a$ where $a \in \mathbb{R}_0^+$. In particular, besides $u \equiv 0$ and $u \equiv 1$, there is no solution $u \in \mathcal{A}$ of (25) with $u'(1) = 0$.

Proof. Let u be a solution of (25), and without loss of generality assume $u \in \mathcal{A}_0$. Moreover, let $\rho > 0$ be such that $u([0,\rho]) \subset [0, \frac{1}{4}[$. Then the functional $\tilde{I}_0$ is convex on $\mathcal{W}(\rho, u(\rho))$. Lemma 32 implies

$$u|_{[0,\rho]} = \xi_a|_{[0,\rho]} \qquad \text{where } a = \frac{1}{\rho}\tan\left(\frac{\pi}{2}u(\rho)\right),$$

and since u and ξ_a solve the same differential equation, we have $u = \xi_a$ on the whole interval $[0,1]$. Since $\xi_a'(r) \neq 0$ for all $a, r \in \mathbb{R}^+$, the second statement follows immediately. □

4.2 Monotonicity properties of stationary states

In this subsection we use the functions ξ_a as comparison functions to find properties of solutions of (20). As a result of this subsection we will obtain the following theorem.

Theorem 34. *Let u be a solution of* (20)*. Then either $u \equiv 1$ or $u \in \mathcal{A}_0$ and for all $r_0 \in]0,1]$ we have $u(r) \geq \frac{u(r_0)}{r_0} r$. If u is semistable, then u is monotonously increasing.*

Lemma 35. *The function $u \equiv 1$ is the only solution of* (20) *in $\mathcal{A}_1$.*

Proof. Assume $u \in \mathcal{A}_1$, $u \not\equiv 1$ is a solution of (20). Define

$$\tilde{\mathcal{W}}(\rho) \;:=\; \left\{ v \in \mathcal{A}_1 : \, v(r) \geq \frac{3}{4} \text{ for all } r < \rho, \; v(\rho) = u(\rho) \right\}.$$

Since u is continuous there exists a $\rho_0 > 0$ such that $u(r) > \frac{3}{4}$ for all $r < \rho_0$. For all $\rho < \rho_0$ the functional $\tilde{I}_h(\cdot, [0,\rho])$ is convex on $\tilde{\mathcal{W}}(\rho)$ and u is a minimiser of $\tilde{I}_h(\cdot, [0,\rho])$ in $\tilde{\mathcal{W}}(\rho)$. For all $a \in \mathbb{R}^+$ for which u and $1 - \xi_a$ coincide in some point $\rho \in [0, \rho_0]$, we have $u|_{[0,\rho]} \geq 1 - \xi_a|_{[0,\rho]}$, since otherwise the map $\tilde{u}(r) := \max\{u(r), 1 - \xi_a(r)\}$ satisfies the inequality $\tilde{I}_h(\tilde{u}, [0,\rho]) < \tilde{I}_h(u, [0,\rho])$, which contradicts the minimality of u. We define

$$a_0 \;:=\; \sup\{a : \text{there exists } r > 0 \text{ such that } 1 - \xi_a(r) > u(r)\}.$$

The number a_0 is finite: The map u attains some minimum $w > 0$ and setting $a_1 := \frac{1}{\rho_0}\tan(\frac{\pi}{2}(1-w))$ we have $1 - \xi_{a_1}|_{[\rho_0,1]} \leq w \leq u$. With the above considerations we have $1 - \xi_{a_1}|_{[0,\rho_0]} \leq u|_{[0,\rho_0]}$ as well, thus $a_0 \leq a_1$. Figure 7 shows a sketch of the situation.

For all $r \in [0,1]$ we have $1 - \xi_{a_0}(r) \leq u(r)$. There are three possibilities:

(1.) $1 - \xi_{a_0}(r) < u(r)$ for all $0 < r \leq 1$. This is the case if and only if the intersection point s_a of $1 - \xi_a$ and u goes to zero for a to a_0 from below. But since we have shown that s_a cannot be smaller than ρ_0, this is impossible.

(2.) $1 - \xi_{a_0}(r) < u(r)$ for all $0 < r < 1$, but $1 - \xi_{a_0}(1) = u(1)$. Since $1 - \xi_{a_0}'(1) < 0 = u'(1)$, there is $\rho_1 < 1$ with $1 - \xi_{a_0}(r) > u(r)$ for all $\rho_1 < r < 1$. This is a contradiction to $1 - \xi_{a_0} \leq u$.

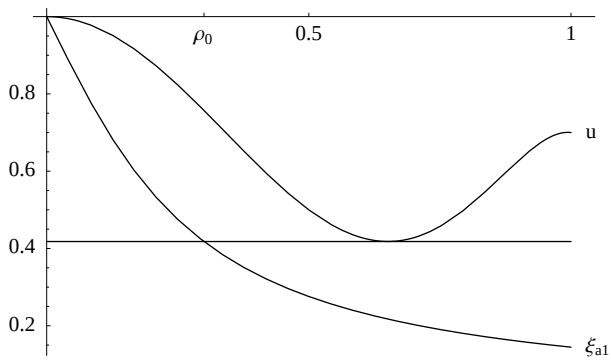

Figure 7: The function ξ_{a_1}

(3.) There is some $r_0 \in]0,1[$ such that $1 - \xi_{a_0}(r_0) = u(r_0)$. Then $u'(r_0) = \xi'_{a_0}(r_0)$ and the equations (20) and (25) imply

$$u''(r_0) + \frac{h}{\pi}\sin(\pi r_0) = -\xi''_{a_0}(r_0), \quad \text{i.e., } u''(r_0) < -\xi''_{a_0}.$$

Thus there is some neighbourhood U of r_0 such that the inequality $(1 - \xi_{a_0})(r) > u(r)$ holds for all $r \in U \setminus \{r_0\}$. This is a contradiction to $1 - \xi_a \leq u$.

So all three possibilities lead to a contradiction, and we have shown that there is no nontrivial solution of (20) in $\mathcal{A}_1$. □

Lemma 36. *Let u be a solution of* (20) *in $\mathcal{A}_0$, $r_0 \in]0,1]$ and $a_0 \in \mathbb{R}^+$ such that $u(r_0) = \xi_{a_0}(r_0)$. Then we have $u(r) \geq \xi_a(r)$ for all $r < r_0$.*

Proof. The proof of this lemma is similar to the proof of Lemma 35. There is some $\rho_0 > 0$ such that for all $\rho < \rho_0$ the functional $\tilde{I}_h(\cdot, [0,\rho])$ is convex on the set $\mathcal{W}(\rho, u(\rho))$. Therefore the statement of the lemma is obvious for all $r_0 \leq \rho_0$. We define

$$a_1 \;:=\; \inf\{a : \text{there exists } r \in [0, r_0] \text{ such that } \xi_a(r) > u(r)\}\,.$$

Then $\xi_{a_1}(r) \leq u(r)$ for all $r \in [0, r_0]$.

If $a_1 = a_0$ the statement of the lemma holds. Otherwise there are two possibilities:

(1.) ξ_{a_1} touches u in some point $r_1 \in]0, r_0[$.

(2.) $\xi_{a_1}(r) < u(r)$ for all $0 < r \leq r_0$.

We can exclude both possibilities like in Lemma 35. □

Since for all $a \in \mathbb{R}^+$ the second derivative of ξ_a is negative, Lemma 36 implies:

Corollary 37. *Let u be a solution of* (20) *in $\mathcal{A}_0$. Then we have for all $r_0 > 0$ and all $r \in [0, r_0]$ the inequality $u(r) \geq \frac{u(r_0)}{r_0} r$.*

If we assume additionally that u is semistable, we obtain the following monotonicity result.

Lemma 38. *Let $u \not\equiv 1$ be a semistable solution of* (20)*. Then u is in $\mathcal{A}_0$ and monotonously increasing.*

Proof. Since u is not in $\mathcal{A}_1$ (Lemma 35), u is in $\mathcal{A}_0$. We assume $u \not\equiv 0$ and define

$$e^t := r, \qquad \tilde{u}(t) := u(e^t), \qquad \phi := \partial_t \tilde{u} = r\partial_r u.$$

Then $\partial_t \phi = r^2 \partial_{rr} u + r\partial_r u$. On the one hand, equation (20) can be transformed to

$$-\partial_{tt}\tilde{u} + \frac{1}{2\pi}\sin(2\pi\tilde{u}) - \frac{he^{2t}}{\pi}\sin(\pi\tilde{u}) \;=\; 0,$$

and differentiation with respect to t yields

$$-\partial_{tt}\phi + \cos(2\pi\tilde{u})\phi - he^{2t}\cos(\pi\tilde{u})\phi \;=\; \frac{2he^{2t}}{\pi}\sin(\pi\tilde{u}). \tag{26}$$

On the other hand, (21) implies for all $v \in \mathcal{A}_0$, $\tilde{v}(t) := v(e^t)$

$$\begin{aligned} 0 \;&\le\; \int_0^1 r(\partial_r v(r))^2 + \left(\frac{1}{r}\cos(2\pi u(r)) - hr\cos(\pi u(r))\right) v(r)^2 \, dr \\ &= \int_{-\infty}^0 \left(e^t(e^{-t}\partial_t v(e^t))^2 + \left(e^{-t}\cos(2\pi u(e^t)) - he^t\cos(\pi u(e^t))\right) v(e^t)^2\right) e^t \, dt \\ &= \int_{-\infty}^0 (\partial_t \tilde{v}(t))^2 + \cos(2\pi\tilde{u}(t))\tilde{v}(t)^2 - he^{2t}\cos(\pi\tilde{u}(t))\tilde{v}(t)^2 \, dt. \end{aligned} \tag{27}$$

Assume that u is not monotonously increasing. Then there exists $t_0 < t_1 \le 0$ such that $\phi(t_0) = \phi(t_1) = 0$ and $\phi(t) < 0$ for $t_0 < t < t_1$. Set

$$\tilde{\phi}(t) := \begin{cases} \phi(t) & \text{if } t_0 \le t \le t_1 \\ 0 & \text{otherwise.} \end{cases}$$

We test equation (26) with $\tilde{\phi}$.

$$\int_{-\infty}^0 (\partial_t\tilde{\phi})^2 + \cos(2\pi\tilde{u})\tilde{\phi}^2 - he^{2t}\cos(\pi\tilde{u})\tilde{\phi}^2 \, dt = \int_{-\infty}^0 \underbrace{\frac{2he^{2t}}{\pi}\sin(\pi\tilde{u})}_{>0} \tilde{\phi} \, dt < 0.$$

This is a contradiction to (27). □

4.3 Nonexistence of minimisers in $\mathcal{A}_0$

Theorem 39. *Set*

$$\begin{aligned} b_h \;&:=\; \inf\{\tilde{I}_h(u) \,:\, u \in \mathcal{A}_0\}, \\ b_\infty \;&:=\; \inf\{\tilde{I}_0(u, \mathbb{R}_0^+) \,:\, u(0) = 0, \lim_{r\to\infty} u(r) = 1\}. \end{aligned}$$

If $h \le 2$ then the constant zero function is the only minimiser of $\tilde{I}_h$ in $\mathcal{A}_0$. If $h > 2$ there exists no minimiser of $\tilde{I}_h$ in $\mathcal{A}_0$ and $b_h = b_\infty = \frac{2}{\pi}$.

Proof. In view of Lemma 32 we see that $b_\infty = \frac{2}{\pi}$. We show the inequality $b_h \le b_\infty$ by calculating $\tilde{I}_h(\xi_a)$ for large a. We have

$$\tilde{I}_h(\xi_a) \le b_\infty + \int_0^1 \frac{h\,r}{\pi}(\cos(\pi\xi_a)+1).$$

Since $\lim_{a\to\infty} \xi_a(r) = \lim_{a\to\infty} \xi_1(ar) = 1$, for $a \to \infty$ the functions ξ_a converge locally uniformly to 1 and $\int_0^1 hr(\cos(\pi\xi_a)+1)$ becomes arbitrarily small. Thus

$$b_h \;\le\; \lim_{a\to\infty} \tilde{I}_h(\xi_a) \;=\; b_\infty.$$

According to Lemma 38, we can limit our search for minimisers to monotonously increasing functions. So let $u \in \mathcal{A}_0$, $u \not\equiv 0$ be a monotonously increasing function. We have

$$\tilde{I}_h(u) \;=\; \tilde{I}_0(u) + \int_0^1 \frac{h\,r}{\pi}(\cos(\pi u(r))+1)\,dr.$$

To bound the first summand from below, we calculate as in the proof of Lemma 32

$$\tilde{I}_0(u) \;\ge\; \frac{1}{\pi}\int_0^1 |\cos(\pi u)'| \;=\; \frac{1-\cos(\pi u(1))}{\pi}.$$

To bound the second summand from below, we use the monotonicity of u

$$\begin{aligned}\int_0^1 \frac{h\,r}{\pi}(\cos(\pi u(r))+1)\,dr \;&>\; \int_0^1 \frac{h\,r}{\pi}(\cos(\pi u(1))+1)\,dr\\ &=\; \frac{h}{2\pi}(\cos(\pi u(1))+1).\end{aligned}$$

Combining theses estimates, we obtain

$$\begin{aligned}\tilde{I}_h(u) \;&>\; \frac{-\cos(\pi u(1))+1}{\pi} + \frac{h}{2\pi}(\cos(\pi u(1))+1)\\ &=\; \frac{2}{\pi} + \left(\frac{h}{2\pi} - \frac{1}{\pi}\right)(\cos(\pi u(1))+1).\end{aligned}$$

We can conclude that if $h > 2$ then $\tilde{I}_h(u) > \frac{2}{\pi} = b_\infty$, i.e., $b_h = b_\infty$ but the infimum is not attained. Otherwise, if $h \le 2$, $\tilde{I}_h(u) > \frac{2h}{\pi} = \tilde{I}_h(0)$. Thus a function $u \in \mathcal{A}_0$, $u \not\equiv 0$ is never a minimiser of $\tilde{I}_h$ in $\mathcal{A}_0$: Either the constant zero function has a smaller energy, or the infimum of the energy b_h is not attained at all. □

Corollary 40. *If $h \le 2$ then $u \equiv 1$ is the only solution u of* (20) *such that $\tilde{I}_h(u) \le \tilde{I}_h(0)$.*

4.4 Nonexistence of semistable stationary states for a large external magnetic field

Recall the definiton of k_0 from Remark 20. The main result in this subsection is:

Theorem 41. *(i) If $h > k_0^2$, the only solution $u \in \mathcal{A}$ with $u|_{[\frac{1}{k_0},1]} \leq 0.33$ is $u \equiv 0$.*

(ii) If $h \geq k_0^2 + 1$, the only semistable solution $u \in \mathcal{A}$ of (20) is $u \equiv 1$.

Here, as introduced in Remark 20, $k_0 \approx 1.84$ is the first root of J_1, the first Bessel function of first kind. We prove the result in two steps and distinguish the regimes "$u(1)$ is small" and "$u(1)$ is large".

Lemma 42. *(i) If $h > k_0^2$ the only solution $u \in \mathcal{A}$ of (20) such that $u|_{[\frac{1}{k_0},1]} \leq 0.33$ is $u \equiv 0$.*

(ii) If $h \geq k_0^2 + 1$ there exists no semistable solution $u \in \mathcal{A}$ of (20) with $u(1) \leq 0.62$.

Proof. Let $u \in \mathcal{A}_0$, $u \not\equiv 0$ be a solution of (20) and set $v(r) := J_1(\frac{r}{k_0})$. Then

$$-v'' - \frac{1}{r}v' + \left(-k_0^2 + \frac{1}{r^2}\right) v \;=\; 0, \qquad v'(1) \;=\; 0. \tag{28}$$

Since $\int_0^1 r u'(r)^2 \, dr$ is finite, there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero such that $\epsilon_n u'(\epsilon_n)$ converges to zero. Thus

$$\begin{aligned}\int_0^1 r u'' v + u' v \;&=\; \lim_{n\to\infty} \left(r u' v|_{\epsilon_n}^1 - \int_{\epsilon_n}^1 r u' v' \right) \\ &=\; \lim_{n\to\infty} \left(-r u v'|_{\epsilon_n}^1 + \int_{\epsilon_n}^1 r u v'' + u v' \right) \;=\; \int_0^1 r u v'' + u v'.\end{aligned}$$

We test equation (20) with $rv(r)$

$$\begin{aligned}0 \;&=\; \int_0^1 -(u'' + \frac{1}{r}u')rv + \left(-\frac{h}{\pi}\sin(\pi u) + \frac{1}{2\pi r^2}\sin(2\pi u)\right) rv \\ &=\; \underbrace{\int_0^1 -ru(v'' + \frac{1}{r}v') + \left(-k_0^2 + \frac{1}{r^2}\right) ruv}_{T_1} + \underbrace{\int_0^1 (k_0^2 - h)\frac{\sin(\pi u)}{\pi} rv}_{T_2} \\ &\quad + \underbrace{\int_0^1 \underbrace{\left(k_0^2\left(u - \frac{\sin(\pi u)}{\pi}\right) + \frac{1}{r^2}\left(-u + \frac{\sin(2\pi u)}{2\pi}\right)\right)}_{t_3(u,r)} rv}_{T_3}\end{aligned} \tag{29}$$

Because of (28) the term T_1 is zero and since $h > k_0^2$ the term T_2 is negative.

(i) For $r < \frac{1}{k_0}$ and arbitrary $0 < s < 1$ the term $t_3(s,r)$ is negative. A numerical calculation shows that $t_3(s,1) < 0$ for all $0 < s < 0.33$. Since the term $t_3(s,r)$ is monotonously increasing in r, this implies $t_3(s,r) < 0$ for all $0 < s \leq 0.33$, $0 < r \leq 1$. Thus if $u|_{[\frac{1}{k_0},1]} \leq 0.33$ then $T_3 < 0$ and $T_1 + T_2 + T_3 < 0$. This is a contradiction to (29), so there is no solution of (20) with $u|_{[\frac{1}{k_0},1]} \leq 0.33$.

(ii) Assume that u is a semistable solution of (20) with $u(1) \leq 0.62$. Since u is monotonously increasing (Theorem 34), with $u(r) \geq r\,u(1)$ (Corollary 37) we have

$$\sin(\pi u(r)) \;\geq\; 2u(r) \;\geq\; 2r\,u(1).$$

Moreover, a numerical calculation gives $t_3(u(1),1) \;\le\; 0.4u(1)$, so

$$t_3(u(r),r) \;\le\; t_3(u(r),1) \;\le\; t_3(u(1),1) \;\le\; 0.4\,u(1).$$

Since $t_3(r,s) \le 0$ for $r \le \frac{1}{k_0}$, and since $v(1)r \le v(r) \le v(1)$, we can calculate

$$\begin{aligned} T_2 + T_3 \;&\le\; \int_0^1 (k_0^2 - h)\frac{\sin(\pi u(r))}{\pi}\, rv(r)\; dr + \int_{\frac{1}{k_0}}^1 t_3(u(r),r)\, rv(r)\; dr \\ &\le -\int_0^1 \frac{2}{\pi} u(1)v(1)\, r^3\; dr + \int_{\frac{1}{k_0}}^1 0.4u(1)v(1)\, r\; dr \\ &\le \left(-\frac{1}{2\pi} + 0.2\left(1 - \frac{1}{k_0^2}\right)\right) u(1)v(1) < 0. \end{aligned}$$

Again we have a contradiction to (29). □

To show the nonexistence of semistable stationary solutions of (20) for large values of $u(1)$ we need an additional lemma.

Lemma 43. *Let u be a solution of* (20)*. Then*

$$\int_0^1 hr\; (\cos(\pi u(r)) - \cos(\pi u(1)))\; dr \;=\; \frac{1}{4}\sin^2(\pi u(1)).$$

Proof. For $\Omega \subset [0,1]$ and $v\colon [0,1] \to \mathbb{R}$ we define the functional J_h by

$$J_h(u,\Omega) \;:=\; \int_\Omega \frac{hr}{\pi}(\cos(\pi u) + 1)\; dr.$$

For $\epsilon > 0$ we moreover set

$$u_\epsilon\colon [0,1] \to \mathbb{R}, \qquad u_\epsilon(r) := \begin{cases} u(\frac{r}{1-\epsilon}) & \text{if } 0 \le r < 1-\epsilon \\ u(1) & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \tilde{I}_h(u_\epsilon) \;&=\; \tilde{I}_0(u_\epsilon,[0,1-\epsilon]) + J_h(u_\epsilon,[0,1-\epsilon]) \\ &\quad +\; \tilde{I}_0(u_\epsilon,[1-\epsilon,1]) + J_h(u_\epsilon,[1-\epsilon,1]). \end{aligned}$$

We calculate the summands

$$\tilde{I}_0(u_\epsilon,[0,1-\epsilon]) \;=\; \tilde{I}_0(u)$$

$$\begin{aligned} J_h(u_\epsilon,[0,1-\epsilon]) \;&=\; \int_0^{1-\epsilon} \frac{hr}{\pi}\cos\left(\pi u\left(\tfrac{r}{1-\epsilon}\right)\right)\; dr \\ &=\; \int_0^1 \frac{ht(1-\epsilon)}{\pi}\cos(\pi u(t))\;(1-\epsilon)\; dt \\ &=\; J_h(u,[0,1])\cdot(1 - 2\epsilon + \epsilon^2) \end{aligned}$$

$$
\begin{aligned}
\tilde{I}_0(u_\epsilon, [1-\epsilon, 1]) &= \int_{1-\epsilon}^{1} \frac{1}{2\pi r} \sin^2(\pi u(1))\, dr \\
&= \frac{1}{2\pi} \sin^2(\pi u(1)) \cdot (-\log(1-\epsilon)) \\
&= \frac{1}{2\pi} \sin^2(\pi u(1)) \cdot \left(\epsilon - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right)
\end{aligned}
$$

$$
\begin{aligned}
J_h(u_\epsilon, [1-\epsilon, 1]) &= \int_{1-\epsilon}^{1} \frac{h\, r}{\pi} (\cos(\pi u(1)) + 1)\, dr \\
&= \frac{h}{\pi} (\cos(\pi u(1)) + 1) \cdot \left(\epsilon - \frac{1}{2}\epsilon^2\right).
\end{aligned}
$$

The derivative with respect to ϵ is

$$
\partial_\epsilon \tilde{I}_h(u_\epsilon)|_{\epsilon=0} \;=\; -2J_h(u, [0,1]) + \frac{1}{2\pi} \sin^2(\pi u(1)) + \frac{h}{\pi}(\cos(\pi u(1)) + 1)).
$$

Since u is a stationary state we have $\partial_\epsilon I_h(u_\epsilon)|_{\epsilon=0} = 0$. This implies

$$
J_h(u, [0,1]) \;=\; \frac{1}{4\pi} \sin^2(\pi u(1)) + \frac{h}{2\pi}(\cos(\pi u(1)) + 1),
$$

which is equivalent to

$$
\int_0^1 hr\, (\cos(\pi u(r)) - \cos(\pi u(1)))\, dr \;=\; \frac{1}{4} \sin^2(\pi u(1)).
$$

□

Using this lemma, we now show that there are no nontrivial semistable solutions for $h \geq k_0^2 + 1$ and large $u(1)$.

Lemma 44. *For $h \geq k_0^2 + 1$ there exist no semistable solutions $u \in \mathcal{A}_0$ of* (20) *with $u(1) \geq 0.62$.*

Proof. Assume that $u(1) \geq 0.62$ and set $r_0 := 0.69$. Then for $a := \tan(\frac{\pi}{2}u(1))$ we have $\xi_a(1) = u(1)$ and for all $r \in [r_0, 1]$ we have $u(r) \geq \xi_a(r) \geq 0.5$. In particular we have the estimates

$$
\sin(\pi u(r)) \geq \sin(\pi u(1)), \quad \sin(2\pi u(r)) \leq 0 \quad \text{for all } r \in [r_0, 1]\,. \tag{30}
$$

We prove the lemma in two steps: First, using Lemma 43, we show

$$
\frac{u(1) - u(r_0)}{\sin(\pi u(1))} \;\leq\; \frac{1}{2\pi\, h r_0^2} \;\leq\; \frac{1.0502}{\pi\, h}, \tag{31}
$$

then, using equation (20), we show

$$
\frac{u(1) - u(r_0)}{\sin(\pi u(1))} \;\geq\; 0.05455 \frac{h}{\pi}. \tag{32}
$$

Combining (31) and (32) we get

$$
h \;\leq\; \sqrt{\frac{1.0502}{0.05455}} < 4.388 < k_0^2 + 1.
$$

Step 1: Lemma 43 states that

$$\underbrace{\int_0^1 hr\left(\cos(\pi u) - \cos(\pi u(1))\right) dr}_{LS} \;=\; \frac{1}{4}\sin^2(\pi u(1)). \tag{33}$$

Using (30) and the fact that u is monotone (Theorem 34), we can bound the left hand side LS from below:

$$\begin{aligned} LS \;&\geq\; \int_0^{r_0} hr(\cos(\pi u(r)) - \cos(\pi u(1))\, dr \\ &\geq\; \int_0^{r_0} hr\left(\cos\left(\pi u\left(r_0\right)\right) - \cos(\pi u(1))\right)\, dr \\ &\geq\; \frac{\pi}{2} h r_0^2 \left(u(1) - u\left(r_0\right)\right) \sin(\pi u(1)). \end{aligned}$$

Combining this estimate with equation (33) yields (31).

Step 2: We test equation (20) with $r\ln(\frac{r}{r_0}) \cdot 1_{[r_0,1]}$

$$\begin{aligned} 0 \;=\; &\int_{r_0}^1 \left(-u''(r) - \frac{1}{r}u'(r)\right) r\ln\left(\tfrac{r}{r_0}\right)\, dr \\ &+ \int_{r_0}^1 \left(\frac{1}{2\pi r^2}\sin(2\pi u(r)) - \frac{h}{\pi}\sin(\pi u(r))\right) r\ln\left(\tfrac{r}{r_0}\right)\, dr. \end{aligned} \tag{34}$$

Since $(ru')' = u' + u''r$, partial integration of the first summand yields

$$\begin{aligned} &\int_{r_0}^1 \left(-u''(r) - \frac{1}{r}u'(r)\right)\, r\ln\left(\frac{r}{r_0}\right)\, dr \\ &= \underbrace{-ru'\ln\left(\frac{r}{r_0}\right)\Big|_{r=r_0}^{r=1}}_{=0} + \int_{r_0}^1 u'(r)\, dr \;=\; u(1) - u(r_0). \end{aligned}$$

Using (30), we can bound the second summand

$$\begin{aligned} &\int_{r_0}^1 \left(\frac{1}{2\pi r^2}\sin(2\pi u(r)) - \frac{h}{\pi}\sin(\pi u(r))\right)\, r\ln\left(\frac{r}{r_0}\right)\, dr \\ &\leq\; -\frac{h}{\pi}\,\sin(\pi u(1)) \int_{r_0}^1 r\ln\left(\frac{r}{r_0}\right) \;\leq\; -0.05455\,\frac{h}{\pi}\,\sin(\pi u(1)). \end{aligned}$$

So equation (34) implies

$$u(1) - u(r_0) \;\geq\; 0.05455\,\frac{h}{\pi}\,\sin(\pi u(1)),$$

which is equivalent to (32). □

4.5 Stationary states have a bounded derivative

In this subsection we will often use the following version of the maximum principle.

Lemma 45. *Given $r_0 < r_1$, let*

$$a, c :]r_0, r_1[\to \mathbb{R}^+, \quad b :]r_0, r_1[\to \mathbb{R}, \quad u, v : [r_0, r_1] \to \mathbb{R}$$

be functions such that

$$u(r_0) = v(r_0), \quad u(r_1) = v(r_1),$$

$$au'' + bu' - cu \ > \ av'' + bv' - cv \qquad \text{on }]r_0, r_1[.$$

Then $u \le v$ on $[r_0, r_1]$.

Proof. Assume the conditions of the lemma hold but the conclusion is false. Then $u - v$ has a positive maximum in some point x. Since $u'(x) - v'(x) = 0$, the conditions imply $u''(x) - v''(x) = \frac{c(x)}{a(x)}(u(x) - v(x)) > 0$. This is in contradiction to the assumption that $u - v$ attains a maximum in x. □

Let u be a solution of (20). Using the maximum principle, we can prove bounds of the form $u(r) \le Cr^\alpha$ for any $\alpha < 1$. Unfortunately, we cannot prove $u(r) \le Cr$ directly. So we first prove the bound $u(r) \le C\sqrt{r}$ using the maximum principle and then use the fact that for small r_0 the function u is a minimiser of $I_h(\cdot, [0, r_0])$ in $\mathcal{W}(k, \rho)$.

Lemma 46. *For each solution $u \in \mathcal{A}_0$ of (20) there exists a number K_0 such that $u(r) \le K_0\sqrt{r}$.*

Proof. We choose $\rho_0 < \frac{1}{2\sqrt{h}}$ such that $u(r) \le \frac{1}{4}$ for all $r \le \rho_0$. Then, for $0 < r \le \rho_0$, we have the estimates

$$\begin{aligned} \frac{h}{\pi}\sin(\pi u(r)) \ &\le \ \frac{1}{4\pi r^2}\sin(\pi u(r)) \ \le \ \frac{1}{4\pi r^2}\sin(2\pi u(r)), \\ \sin(2\pi u(r)) \ &> \ \pi u(r), \end{aligned}$$

and therefore

$$\begin{aligned} 0 \ &= \ u''(r) + \frac{1}{r}u'(r) - \frac{1}{2\pi r^2}\sin(2\pi u(r)) + \frac{h}{\pi}\sin(\pi u(r)) \\ &\le \ u''(r) + \frac{1}{r}u'(r) - \frac{1}{4\pi r^2}\sin(2\pi u(r)) \\ &\le \ u''(r) + \frac{1}{r}u'(r) - \frac{1}{4r^2}u(r). \end{aligned}$$

Set $v : [0, \rho_0] \to \mathbb{R}, r \mapsto \frac{u(\rho_0)}{\sqrt{\rho_0}}\sqrt{r}$. Then v satisfies the differential equation $v''(r) + \frac{1}{r}v'(r) - \frac{1}{4r^2}v(r) = 0$, so the maximum principle (Lemma 45) implies

$$u(r) \ \le \ v(r) \ = \ \frac{u(\rho_0)}{\sqrt{\rho_0}}\sqrt{r} \ \le \ \frac{\sqrt{r}}{\sqrt{\rho}_0} \qquad \text{for } 0 < r \le \rho_0.$$

Since for all $r \ge \rho_0$ the estimate $u(r) \le 1 \le \frac{\sqrt{r}}{\sqrt{\rho}_0}$ is trivially true, we can set $K_0 := \frac{1}{\sqrt{\rho}_0}$. □

Lemma 47. *Let $u \in \mathcal{A}_0$ be a solution of* (20) *and set $\alpha(r) := u'(r) - \frac{1}{\pi r}\sin(\pi u)$. Then the function $r \mapsto \frac{1}{r}|\alpha(r)|$ is integrable.*

Proof. First we calculate

$$\begin{aligned}
&\tilde{I}_h(u,[0,\rho]) \\
&= \int_0^\rho \frac{r}{2}(u'(r))^2 + \frac{1}{2\pi^2 r}\sin^2(\pi u(r)) + \frac{h\,r}{\pi^2}(\cos(\pi u(r)) - 1)\,dr \\
&= \int_0^\rho \frac{1}{\pi}\sin(\pi u(r))\,u'(r) + \frac{r}{2}\left(u'(r) - \frac{1}{\pi r}\sin(\pi u(r))\right)^2 \\
&\quad + \frac{h\,r}{\pi^2}(\cos(\pi u(r)) - 1)\,dr \\
&= 1 - \cos(\pi u(\rho)) + \int_0^\rho \frac{1}{2} r\,\alpha(r)^2 + \frac{h\,r}{\pi^2}(\cos(\pi u(r)) - 1)\,dr.
\end{aligned}$$

Choose ρ_0 such that $\rho_0 \le \sqrt{\frac{1}{2h}}$ and $u|_{[0,\rho_0]} \le \frac{1}{8}$. Then, in particular, for all $\rho \le \rho_0$ the function u is a minimiser of $\tilde{I}_h(\cdot,[0,\rho])$ in $\mathcal{W}(u(\rho),\rho)$ (Remark 30). It suffices to show that $\int_0^{\rho_0} \frac{1}{r}|\alpha(r)|$ is finite.
Using Lemma 32 we have for all $\rho \le \rho_0$

$$\tilde{I}_h(u,[0,\rho]) < \inf_{v \in \mathcal{W}(u(\rho),\rho)} \tilde{I}_0(v,[0,\rho]) = 1 - \cos(\pi u(\rho)).$$

Since $u \le K_0\sqrt{r}$, we have in particular

$$\begin{aligned}
\int_0^\rho r\,\alpha(r)^2 &\le 2\int_0^\rho \frac{hr}{\pi^2}\left(1 - \cos(\pi K_0\sqrt{r})\right)\,dr \\
&\le \int_0^\rho h\,K_0^2\,r^2\,dr \;=\; \frac{h}{3}K_0^2\,\rho^3.
\end{aligned}$$

We calculate $\int \frac{1}{r}\alpha\,dr$ on the intervals $I_k := [2^{-k-1}\rho_0, 2^{-k}\rho_0]$.

$$\begin{aligned}
\int_{I_k} \frac{1}{r}|\alpha(r)|\,dr &= \int_{I_k} r^{\frac{1}{2}}\,|\alpha(r)|\,r^{-\frac{3}{2}}\,dr \\
&\le \sqrt{\int_0^{2^{-k}\rho_0} r\,\alpha(r)^2\,dr}\,\sqrt{\int_{I_k} r^{-3}\,dr} \\
&\le \sqrt{\frac{1}{3}h\,K_0^2\,2^{-3k}\,\rho_0^3}\,\sqrt{2^{(2k+1)}\rho_0^{-2}} \\
&\le \sqrt{h\rho_0}\,K_0\,2^{-\frac{k}{2}}.
\end{aligned}$$

Thus we have the estimate

$$\int_0^{\rho_0} \frac{1}{r}\,|\alpha(r)|\,dr \;\le\; \sqrt{h\rho_0}\,K_0\sum_{k=0}^{\infty} 2^{-\frac{k}{2}} \;\le\; 4\sqrt{h\rho_0}\,K_0.$$

□

Theorem 48. *For each solution u of* (20) *there exists a number K_1 such that we have $u(r) \le K_1 r$ and $u'(r) \le K_1$ for all $r \in [0,1]$.*

Proof. We define $v(r) := \frac{1}{r}u(r)$ and set $\alpha(r) := u'(r) - \frac{1}{\pi r}\sin(\pi u)$. Then

$$\begin{aligned} |v'(r)| &= \left|\frac{1}{r}u'(r) - \frac{1}{r^2}u(r)\right| = \left|\frac{1}{\pi r^2}\sin(\pi u) + \frac{1}{r}\alpha(r) - \frac{1}{r^2}u(r)\right| \\ &\le \frac{1}{r}|\alpha(r)| + \frac{\pi}{2r^2}\, u(r)^3 \le \frac{1}{r}|\alpha(r)| + \frac{\pi K_0^3}{2\sqrt{r}}. \end{aligned}$$

Since $r \mapsto \frac{1}{r}|\alpha(r)|$ is integrable (Lemma 47), v' is also integrable, and $v(r) = \frac{1}{r}u(r)$ is bounded by some number K_1. Now Corollary 37 implies $u'(r) \le \frac{u}{r} \le K_1$. □

We can use this information on solutions u of (20) to show a similar bound for the eigenfunctions of A_u.

Theorem 49. *Let $u \in \mathcal{A}_0$ be a solution of* (20)*, and let ϕ be an eigenfunction for some eigenvalue λ of A_u as defined in* (22)*. Then there is a number K_2 such that $\phi(r) \le K_2 r$ and $\phi'(r) \le K_2$ for all $r \in [0,1]$.*

Proof. To prove this lemma, we use the fact that $u(r) \le K_1 r$ and the maximum principle. For all $r \le r_0 := \frac{1}{2K_1}$ we have

$$\begin{aligned} 0 &= \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\cos(2\pi u_-) + h\cos(\pi u_-)\phi + \lambda\phi \\ &\le \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\cos(2\pi K_1 r)\phi + (h+\lambda)\phi \\ &\le \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}(1 - \frac{1}{2}(2\pi K_1 r)^2)\phi + (h+\lambda)\phi \\ &= \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\phi + c_1^2\phi, \qquad \text{where } c_1 := \sqrt{2\pi K_1^2 + h + \lambda} \end{aligned}$$

As before let J_1 denote the first Bessel function of first kind. Then $j\colon x \mapsto J_1(c_1 x)$ is a solution of $j'' + \frac{1}{r}j' - \frac{1}{r^2}j + c_1^2 j = 0$ with $j(0) = 0$ and bounded derivative on $[0,1]$. Using the maximum principle (Lemma 45) and setting $c_2 := \frac{\phi(r_0)}{J_1(r_0)}$ we have the inequality $c_2(c_1 r) \ge \phi(r)$ for all $r \le r_0$. In particular, since the derivative of J_1 is bounded, there is a number K_2 such that $\phi(r) \le K_2 r$. On the other hand, $\phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\phi < 0$. Since linear functions g are solutions of $g'' + \frac{1}{r}g' - \frac{1}{r^2}g = 0$, the maximum principle yields the estimate $\phi(r) \ge \frac{\phi(\rho)}{\rho}r$ for all $\rho \in [0,1]$ and all $r \in [0,\rho]$. Thus we get $\phi'(\rho) \le \frac{\phi(\rho)}{\rho} \le K_2$ for all $\rho \in [0,1]$. □

5 Possible end states of travelling waves modelling the vortex mode

In this section we consider possible end states of solutions of (6). In the first subsection we discuss the properties of end state $u_\pm$ at $\pm\infty$ and show that they are rotationally symmetric, semistable, stationary states with finite energy. In the second subsection we use the results about stationary states of Section 4 to show that $u_+ \equiv 1$ for small and large external magnetic field.

5.1 Properties of the end states

It is easy to see that end states are rotationally symmetric stationary states with finite energy.

Lemma 50. *Let (u, c) be a solution of* (6) *provided by Theorem 25 or Theorem 28. Then $u(x, \cdot)$ converges for $x \to \pm\infty$ in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$ to maps $u_{\pm\infty}$. These are rotationally symmetric and satisfy*

$$\begin{aligned} \Delta_y u_\pm + f^0(u_\pm, y) &= 0 \quad \text{in } D_1, \\ \partial_\nu u_\pm &= 0 \quad \text{on } \partial D_1. \end{aligned} \tag{35}$$

For the energy of the end states we have $I_h(u_\pm) < \infty$.

Proof. Monotonicity of u in x, Lemma 27 and a bootstrap argument imply that $u(x, \cdot)$ converges for $x \to \pm\infty$ in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$ to some functions $u_\pm : D_1 \to [0, 1]$. Thus in particular $\partial_x u(x, y)$ and $\partial_{xx} u(x, y)$ converge to zero for all $y \in D_1 \setminus \{0\}$, and passing to the limit in (6) yields (35). Since u is rotationally symmetric, the functions $u_\pm$ are rotationally symmetric as well.

If u is a minimiser of $\Phi^\dagger_c$ then $I_h(u_\pm)$ is obviously finite. If u is a non-variational solution, we use Theorem 28 (iii). Because of convergence in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$ we have

$$\begin{aligned} I_0(u_\pm) &= \lim_{\delta\to 0} \lim_{x\to\pm\infty} \int_{D_1\setminus D_\delta} \int_x^{x+1} \frac{1}{2}|\nabla_y u(t, y)|^2 + V(u(t, y))\, dt\, dy \\ &\le \lim_{x\to\pm\infty} \int_x^{x+1} I_0(u(t, \cdot))\, dt \;\le\; \frac{h}{\pi}\left(4 + \frac{1}{|c^*|}\right) \;<\; \infty. \end{aligned} \tag{36}$$

□

The difficulty in the proof of the following theorem lies in the singularity of $f^0(u, y)$ for $y = 0$. If the function f^0 was smooth we could use the proof of Heinze [7, Thm. 2.4]. To overcome the problem we will use that, close to $y = 0$, Theorem 48 and Theorem 49 provide good bounds for the functions we are considering.

Theorem 51. *Let u be a solution of* (6) *provided by Theorem 25 or Theorem 28. Then u_+ is a semistable stationary state.*

Proof. First, we introduce some notation for this proof. For functions $w, \tilde{w} \colon D_1 \to \mathbb{R}$ we set

$$\begin{aligned} L_{u_+}(w) &:= -\Delta_y w - \partial_u f^0(u_+, y) w \\ &= -\Delta w + \frac{1}{|y|^2}\cos(2\pi u_+) w - h\cos(\pi u_+) w, \\ N(w) &:= -\Delta_y w - (f^0(u_+, y) - f^0(u_+ - w, y)). \end{aligned}$$

Let μ be the smallest eigenvalue of L_{u_+}, and let ϕ be the corresponding eigenfunction. Without loss of generality we can assume $u_+ \in A_0$. Then we have the bounds $u_+(y) \le K_1|y|$ and $\phi(y) \le K_2|y|$ for some constants K_1, K_2 (Theorem 48, Theorem 49).

For $v(x,y) := u_+(y) - u(x,y)$, Equation (6) yields

$$\partial_{xx}v + c\partial_x v \;=\; -\partial_{xx}u - c\partial_x u \;=\; \Delta_y u \underbrace{-\Delta_y u_+ - f^0(u_+,y)}_{=0} + f^0(u,y) \;=\; N(v).$$

Our strategy is to prove that, if μ is negative, then there is $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$ we have $\langle N(v(x,\cdot),\phi\rangle_{D_1} < 0$. Since $c\partial_x v$ is positive, this will imply

$$\langle \partial_{xx}v(x,\cdot),\phi\rangle_{D_1} \;<\; -\langle \partial_x v(x,\cdot),\phi\rangle_{D_1} \;<\; 0.$$

Thus $\partial_x\langle v(x,\cdot),\phi\rangle_{D_1}$ goes exponentially to $-\infty$ as x tends to $+\infty$ and we get a contradiction to the fact that v is bounded.

The operator N can be written as

$$\begin{aligned} N(w) \;&=\; -\Delta_y w - \int_0^w \partial_u f^0(u_+ - t, y)\; dt \\ &=\; -\Delta_y w - \left(\int_0^w \partial_u f^0(u_+,y) - \int_0^t \partial_{uu} f^0(u_+ - s, y)\; ds\; dt\right) \\ &=\; L(w) + \int_0^w \int_0^t \frac{2\pi}{|y|^2}\sin(2\pi u_+ - s) - h\pi\sin(\pi u_+ - s)\; ds\; dt, \end{aligned}$$

and we have the estimate

$$\begin{aligned} N(w) \;&\leq\; L(w) + \int_0^w \int_0^t \frac{2\pi}{|y|^2}(2\pi u_+ - s)\; ds\; dt \\ &\leq\; L(w) + \frac{4\pi^2}{|y|^2}u_+ \int_0^w \int_0^t 1\; ds\; dt \\ &=\; Lw + \frac{2\pi^2}{|y|^2}u_+ w^2 \;\leq\; L(w) + \frac{2\pi^2 K_1}{|y|}w^2. \end{aligned} \tag{37}$$

<u>*Claim 1*</u>*: For each $\lambda > 0$ there exists $\delta > 0$ such that for all functions $w\colon D_1 \to \mathbb{R}$ with Neumann boundary values and $0 \leq w \leq u_+$ the inequality $\langle w,\phi\rangle_{D_1} \leq \delta\|w\|_{L^1}$ implies $\int_{D_1} N(w) \geq \lambda\|w\|_{L^1}$.*

We have

$$\begin{aligned} N(w) \;&=\; -\Delta w + \frac{1}{|y|^2}\cos(2\pi u_+)w + \int_0^w \int_0^t \frac{2\pi}{|y|^2}\sin(2\pi u_+ - s)\; ds\; dt \\ &\quad+\; \frac{h}{\pi}\underbrace{(-\cos(\pi u_+) + \cos(\pi u_+ - w))}_{\geq -\pi w}. \end{aligned}$$

If $u_+ < \frac{1}{6}$ then $\sin(2\pi u_+ - s)$ is positive for all $s < u_+$ and $\cos(2\pi u_+) \geq \frac{1}{2}$. Choose r_0 such that $r_0 \leq \frac{1}{\sqrt{4\lambda+4h}}$ and $u_+|_{D_{r_0}} \leq \frac{1}{6}$. Then

$$\begin{aligned} N(w)|_{D_{r_0}} \;&\geq\; -\Delta w(y)\; dy \;+ \frac{1}{2r_0^2}w - hw \\ \int_{D_1} N(w) \;&\geq\; \int_{D_1} -\Delta w(y)\; dy \;+ \frac{1}{2r_0^2}\|w\|_{L^1(D_{r_0})} - h\|w\|_{L^1(D_1)} \\ &\geq\; (2\lambda + 2h)\|w\|_{L^1(D_{r_0})} - h\|w\|_{L^1(D_1)}. \end{aligned} \tag{38}$$

Set $\delta := \frac{1}{2}\min_{y\in D_1\setminus D_{r_0}}\phi(y)$. Then we have $\langle\phi,w\rangle_{D_1} \geq 2\delta\|w\|_{L^1(D_1\setminus D_{r_0})}$, so the inequality $\langle\phi,w\rangle_{D_1} \leq \delta\|w\|_{L^1(D_1)}$ implies $\|w\|_{L^1(D_{r_0})} \geq \frac{1}{2}\|w\|_{L^1(D_1)}$, and in particular (38) yields $\int_{D_1} N(w) \geq \lambda\|w\|_{L^1(D_1)}$.

<u>*Claim 2*</u>*: There are numbers $x_0 \in \mathbb{R}$, $\delta > 0$ such that for all $x > x_0$ we have $\langle v(x,\cdot),\phi\rangle_{D_1} \geq \delta\|v(x,\cdot)\|_{L^1(D_1)}$.*

Choose $\tilde{\delta}$ such that Claim 1 holds for $\lambda := 2\pi^2K_1^2$ and set

$$\delta \;:=\; \min\left(\tilde{\delta}, \frac{\langle v(0,\cdot),\phi\rangle}{\|v(0,\cdot)\|_{L^1(D_1)}}\right).$$

If $\langle v(x,\cdot),\phi\rangle_{D_1} \geq \delta\|v(x.\cdot)\|_{L^1(D_1)}$ for all $x \geq 0$ there is nothing to show. Otherwise set

$$\begin{aligned} x_0 \;&:=\; \inf\left\{x \geq 0 \mid \langle v(x,\cdot),\phi\rangle_{D_1} < \delta\|v(x,\cdot)\|_{L^1(D_1)}\right\}, \\ x_1 \;&:=\; \sup\left\{x > x_0 \mid \langle v(\tilde{x},\cdot),\phi\rangle_{D_1} < \delta\|v(x,\cdot)\|_{L^1(D_1)} \text{ for all } \tilde{x} \in]x_0,x_1[\right\}. \end{aligned}$$

Then $x_0 \in [0,\infty[$, $x_1 \in]0,\infty]$. By the choice of $\tilde{\delta}$, we have the relation $\int_{D_1} N(v(x,\cdot)) \geq 2\pi^2K_1^2\|v(x,\cdot)\|_{L^1(D_1)}$ for all $x \in]x_0,x_1[$, thus

$$\begin{aligned} 0 \;&=\; \int_{D_1} \partial_{xx}v(x,y) + c\partial_x v(x,y) - N(v(x,y),y)\,dy \\ &\geq\; \partial_{xx}\|v(x,\cdot)\|_{L^1(D_1)} + c\,\partial_x\|v(x,\cdot)\|_{L^1(D_1)} - 2\pi^2K_1^2\|v(x,\cdot)\|_{L^1(D_1)}. \end{aligned}$$

On the other hand, (37) implies

$$\begin{aligned} 0 \;&=\; \langle\partial_{xx}v(x,\cdot),\phi\rangle_{D_1} + \langle c\partial_x v(x,\cdot),\phi\rangle_{D_1} - \langle N(v(x,\cdot),\cdot),\phi\rangle_{D_1} \\ &\leq\; \partial_{xx}\langle v(x,\cdot),\phi\rangle_{D_1} + c\partial_x\langle v(x,\cdot),\phi\rangle_{D_1} - 2\pi^2K_1^2\langle v(x,\cdot),\phi\rangle_{D_1}. \end{aligned}$$

If x_1 is finite, by definition $\langle v(x,\cdot),\phi\rangle_{D_1}$ and $\delta\|v(x,\cdot)\|_{L^1(D_1)}$ agree at x_0 and x_1. Otherwise $\langle v(x,\cdot),\phi\rangle_{D_1}$ and $\delta\|v(x,\cdot)\|_{L^1(D_1)}$ agree at x_0, and we have $0 = \lim_{x\to\infty}\langle v(x,\cdot),\phi\rangle_{D_1} = \lim_{x\to\infty}\delta\|v(x,\cdot)\|_{L^1(D_1)}$. In both cases the maximum principle yields $\delta\|v(x,\cdot)\|_{L^1(D_1)} \leq \langle v(x,\cdot),\phi\rangle_{D_1}$ for all $x \in]x_1,x_0]$. This is a contradiction to the definition of x_0.

<u>*Claim 3*</u>*: The eigenvalues of L_{u_+} are nonnegative.*

Assume that the smallest eigenvalue μ of L_{u_+} is negative. Using (37), we have

$$\begin{aligned} \langle N(v(x,\cdot)),\phi\rangle_{D_1} \;&\leq\; \langle L(v(x.\cdot)),\phi\rangle_{D_1} + \int_{D_1}\frac{2\pi^2K_1}{|y|}v(x,y)^2\phi(y)\,dy \\ &\leq\; \mu\langle\phi,v(x.\cdot)\rangle_{D_1} + 2\pi^2K_1K_2\|v(x,\cdot)\|_{L^\infty(D_1)}\|v(x,\cdot)\|_{L^1(D_1)}. \end{aligned}$$

So with Claim 2 we have for all $x \leq x_0$ the relation

$$\langle N(v(x,\cdot)),\phi\rangle_{D_1} \;\leq\; \left(\delta\mu + 2\pi^2K_1K_2\|v(x,\cdot)\|_{L^\infty(D_1)}\right)\;\|v(x,\cdot)\|_{L^1(D_1)}.$$

Since $v(x,y) \leq K_1|y|$ and $v(x,\cdot)$ converges to zero in $C^\infty_{\text{loc}}(D_1\setminus\{0\})$, we have that $\|v(x,\cdot)\|_{L^\infty}$ converges to zero as well, that is, the first summand inside the brackets is negative, and the second converges to 0. Thus there is some $x_2 \in \mathbb{R}$ such that $\langle N(v(x.\cdot))\phi\rangle < 0$ for all $x < x_2$. As discussed at the beginning of the proof, this implies that $\partial_x\langle v(x,\cdot),\phi\rangle$ goes exponentially to $-\infty$ as x tends to $+\infty$, which is a contradiction to the fact that $v \leq u_+$ is bounded. $\square$

5.2 Combining the facts

In this subsection we combine the results about stationary states with the results about travelling wave solutions.

Theorem 52. *Let u be a solution provided by Theorem 25 or Theorem 28, and set $u_\pm(y) := \lim_{x\to\pm\infty} u(x,y)$. Then*

(1.) $u_- \equiv 0$.

(2.) If $h \le 2$, we have $u_+ \equiv 1$.

(3.) If $h \ge k_0^2 + 1$, we have $u_+ \equiv 1$.

(4.) If $2 < h < k_0^2 + 1$, the function u_+ could be 1 or some other semistable stationary solution of (20). For such a solution we have $u_+(0) = 0$, u_+ is monotonously increasing, u_+' is bounded and $u_+'(0) > 0$.

(5.) If u is a variational solution, $u(x,\cdot)$ converges in $L^\infty(D_1)$ to u_+.

Proof. For variational solutions (1.) is clear and for non-variational solutions it is a consequence of Theorem 28 and Theorem 41.

(2.) follows from Corollary 40 and Lemma 50.

(3.) follows from Theorem 41 and Theorem 51.

(4.) follows from Theorem 34 and Theorem 51.

(5.) For $u_+ \in \mathcal{A}_0$ the statement is obvious. Otherwise $u \equiv 1$ (Theorem 34), and we show the statement by contradiction. We assume that there is a minimiser u and a number $\delta > 0$ such that $\|1 - u(x,\cdot)\|_{L^\infty(D_1)} > \delta$ for all $x \in \mathbb{R}$. Because of convergence in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$, for each $\epsilon > 0$ there is an x_ϵ, such that $u|_{D_1\setminus D_\epsilon} > 1 - \epsilon$ for all $x \ge x_\epsilon$. For ϵ small enough, by modifying u on $D_\epsilon \times \mathbb{R} \cup D_R \times]x_\epsilon, \infty[$, we can construct a function v with $v(x,r) = 1$ for $x \ge x_\epsilon + 1$ and $\Phi^0_{h,c^\dagger}(v) < \Phi^0_{h,c^\dagger}(u)$. This is a contradiction to u being a variational solution For details on how to construct v see [13]. □

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