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On a Diffuse Interface Model for Two-Phase Flows of Viscous, Incompressible Fluids with Matched Densities

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On a Diffuse Interface Model for Two-Phase Flows of Viscous, Incompressible Fluids with Matched Densities

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Abstract

We study a diffuse interface model for the flow of two viscous incompressible Newtonian fluids of the same density in a bounded domain. The fluids are assumed to be macroscopically immiscible, but a partial mixing in a small interfacial region is assumed in the model. Moreover, diffusion of both components is taken into account. This leads to a coupled Navier-Stokes/Cahn-Hilliard system, which is capable to describe the evolution of droplet formation and collision during the flow. We prove existence of weak solutions of the non-stationary system in two and three space dimensions for a class of physical relevant and singular free energy densities, which ensures – in contrast to the usual case of a smooth free energy density – that the concentration stays in the physical reasonable interval. Furthermore, we present some results on regularity and uniqueness of weak solutions. In particular, we obtain that unique "strong" solutions exist in two dimensions globally in time and in three dimensions locally in time. Moreover, we show that for any weak solution the concentration is uniformly continuous in space and time. Because of this regularity, we are able to show that any weak solution becomes regular for large times and converges as $t \to \infty$ to a solution of the stationary system. These results are based on a regularity theory for the Cahn-Hilliard equation with convection and singular potentials in spaces of fractional time regularity as well as on a result on maximal regularity of a Stokes system with variable viscosity and forces in $L^2(0,\infty;H^s(\Omega))$, $s \in [0,1]$, which are new themselves.

Key words: Two-phase flow, free boundary value problems, diffuse interface model, mixtures of viscous fluids, Cahn-Hilliard equation, Navier-Stokes equation
AMS-Classification: 76T99, 76D27, 76D03, 76D05, 76D45, 35B40, 35B65, 35Q30, 35Q35,

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1 Introduction and Main Result

In the present contribution we study a system describing the flow of viscous incompressible Newtonian fluids of the same density, but different viscosity. Although it is assumed that the fluids are macroscopically immiscible, the model takes a partial mixing on a small length scale measured by a parameter $\varepsilon > 0$ into account. Therefore the classical sharp interface between both fluids is replaced by an interfacial region and an order parameter related to the concentration difference of both fluids is introduced.

The model goes back to Hohenberg and Halperin [16] with the name "model H". Gurtin et al. [15] gave a continuum mechanical derivation based on the concept of microforces. The model is a so-called *diffuse interface model*. These have been successfully used during last years to describe flows of two or more macroscopically fluids beyond the occurrence of topological singularities of the separating interface (e.g. coalescence or formation of drops). We refer to Anderson and McFadden [5] for a review on that topic.

This model leads to a Navier-Stokes/Cahn-Hilliard system:

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c)Dv) + \nabla p = -\varepsilon \operatorname{div}(\nabla c \otimes \nabla c) \qquad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$\partial_t c + v \cdot \nabla c = m\Delta\mu \qquad \text{in } \Omega \times (0,\infty), \tag{1.3}$$

$$\mu = \varepsilon^{-1}\phi(c) - \varepsilon\Delta c \qquad \text{in } \Omega \times (0,\infty). \tag{1.4}$$

Here v is the mean velocity, $Dv = \frac{1}{2}(\nabla v + \nabla v^T)$, p is the pressure, c is an order parameter related to the concentration of the fluids (e.g. the concentration difference or the concentration of one component), and Ω is a suitable bounded domain. Moreover, $\nu(c) > 0$ is the viscosity of the mixture, $\varepsilon > 0$ is a (small) parameter, which will be related to the "thickness" of the interfacial region, and $\phi = \Phi'$ for some suitable energy density Φ specified below. It is assumed that the densities of both components as well as the density of the mixture are constant and for simplicity equal to one. We note that capillary forces due to surface tension are modeled by an extra contribution $\varepsilon\nabla c \otimes \nabla c$ in the stress tensor leading to the term on the right-hand side of (1.1). Moreover, we note that in the modeling diffusion of the fluid components is taken into account. Therefore $m\Delta\mu$ is appearing in (1.3), where $m > 0$ is the mobility coefficient, which is assumed to be constant. (The case $m = 0$ corresponds to a pure transport of the components without diffusion.)

We close the system by adding the boundary and initial conditions

$$v|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.5}$$

$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.6}$$

$$(v,c)|_{t=0} = (v_0, c_0) \qquad \text{in } \Omega. \tag{1.7}$$

Here (1.5) is the usual no-slip boundary condition for viscous fluids, n is the exterior normal on $\partial\Omega$, $\partial_n\mu|_{\partial\Omega} = 0$ means that there is no flux of the components through the boundary, and $\partial_n c|_{\partial\Omega} = 0$ describes a "contact angle" of $\pi/2$ of the diffused interface and the boundary of the domain.

The total energy of the system above is given by $E(c, v) = E_1(c) + E_2(v)$, where

$$E_1(c) = \frac{1}{2}\int_\Omega \varepsilon|\nabla c(x)|^2\,dx + \int_\Omega \varepsilon^{-1}\Phi(c(x))\,dx, \tag{1.8}$$

$$E_2(v) = \frac{1}{2}\int_\Omega |v(x)|^2\,dx. \tag{1.9}$$

Here the Ginzburg-Landau energy $E_1(c)$ describes an interfacial energy associated with the region where c is not close to the minima of $\Phi(c)$ and $E_2(v)$ is the kinetic energy of the fluid. The system is dissipative. More precisely, for sufficiently smooth solutions

$$\frac{d}{dt}E(c(t), v(t)) = -\int_\Omega \nu(c(t))|Dv(t)|^2\,dx - m\int_\Omega |\nabla\mu(t)|^2\,dx.$$

There is a large literature on the mathematical analysis of free boundary value problems related to fluids with a classical sharp interface. Most results are a priori limited to flows without singularities in the interface. There are some attempts to construct weak solutions of a two-phase flow of two viscous, incompressible, immiscible fluids with a classical sharp interface. But so far there is no satisfactory existence theory of weak solutions in the case that capillary forces are taken into account. We refer to [1, 2] for a review and some results in this direction.

There are only few results on the mathematical analysis of diffuse interface models in fluid mechanics and the system above. First results on existence of strong solutions, if $\Omega = \mathbb{R}^2$ and Φ is a suitably smooth double well potential were obtained by Starovoitov [26]. More complete results were presented by Boyer [7] in the case that $\Omega \subset \mathbb{R}^d$ is a periodical channel and f is a suitably smooth double well potential. The author showed the existence of global weak solutions, which are strong and unique if either $d = 2$ or $d = 3$ and $t \in (0, T_0)$ for a sufficiently small $T_0 > 0$. Moreover, the case of the physical relevant logarithmic potential (1.10) presented below is also considered in connection with a degenerate mobility $m = m(c) \to 0$ suitably as $c \to \pm 1$. In this case existence of weak solutions with $c(t, x) \in [-1, 1]$ is shown. The system (1.1)-(1.4) was also briefly discussed by Liu and Shen [19].

It is the scope of the present contribution to present a more complete mathematical theory of existence, uniqueness, regularity of solutions to (1.1)-(1.7) and asymptotic behavior as $t \to \infty$. Qualitatively, our results are similar to the known result on the uncoupled Navier-Stokes system, cf. e.g. Sohr [25]. Of course the results are limited by the fact that the regularity and uniqueness of weak solution of the non-stationary Navier-Stokes system in three space dimensions is an unsolved problem.

Moreover, it is the purpose of this work to present a theory for a class of physically relevant free energy densities Φ. More precisely, we assume throughout the article:

Assumption 1.1 *Let $\Phi \in C([a,b]) \cap C^2((a,b))$ such that $\phi = \Phi'$ satisfies*

$$\lim_{s\to a}\phi(s) = -\infty, \qquad \lim_{s\to b}\phi(s) = \infty, \qquad \phi'(s) \geq -\alpha$$

for some $\alpha \in \mathbb{R}$. Furthermore, we assume that $\nu \in C^2([a,b])$ is a positive function.

We extend $\Phi(x)$ by $+\infty$ if $x \notin [a,b]$. Hence $E_1(c) < \infty$ implies $c(x) \in [a,b]$ for almost every $x \in \Omega$.

Often c is a just the concentration difference of both components and $[a,b] = [-1,1]$. But it is mathematically useful to consider a general interval.

Remark 1.2 The latter assumptions are motivated by the so-called regular solution model free energy suggested by Cahn and Hilliard [9]:

$$\Phi(c) = \frac{\theta}{2}\left((1+c)\ln(1+c) + (1-c)\ln(1-c)\right) - \frac{\theta_c}{2}c^2, \tag{1.10}$$

where $\theta, \theta_c > 0$, $a = -1$, $b = 1$. Here the logarithmic terms are related to the entropy of the system. In the theory of the Cahn-Hilliard equation, this free energy is usually approximated by a suitable smooth free energy density. But then one cannot ensure that the concentration difference stays in the physical reasonable interval $[-1,1]$ due to the lack of a comparison principle for fourth order diffusion equation. As was first shown by Elliott and Luckhaus [12], using the latter free energy density, the associated Cahn-Hilliard equation admits a unique solution with $c(t,x) \in (-1,1)$ almost everywhere. For further references and results in that direction we refer to Abels and Wilke [3].

We note that (1.1) can be replaced by

$$\partial_t v + v\cdot\nabla v - \operatorname{div}(\nu(c)Dv) + \nabla g = \mu\nabla c \tag{1.11}$$

with $g = p + \frac{\varepsilon}{2}|\nabla c|^2 + \varepsilon^{-1}\Phi(c)$ since

$$\mu\nabla c = \nabla\left(\frac{\varepsilon}{2}|\nabla c|^2 + \varepsilon^{-1}\Phi(c)\right) - \varepsilon\operatorname{div}(\nabla c\otimes\nabla c). \tag{1.12}$$

In the following we will for simplicity assume that $\varepsilon = 1$ and $m = 1$. But all result are valid for general $\varepsilon > 0, m > 0$. Moreover, $\mu\nabla c$ in (1.11) can be replaced by $P_0\mu\nabla c$ if g is replaced by $g - m(\mu)c$, where $m(\mu)$ is the mean value of μ in Ω and $P_0\mu = \mu - m(\mu)$, cf. Section 2 below.

Furthermore, let $Q_{(s,t)} = \Omega\times(s,t)$, $Q_t = Q_{(0,t)}$, and $Q = Q_{(0,\infty)}$. We refer to Section 2 for the definition of the function spaces in the following.

Definition 1.3 (Weak Solution)
Let $0 < T \leq \infty$. A triple (v,c,μ) such that

$$\begin{aligned}
&v \in BC_w(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^d),\\
&c \in BC_w(0,T;H^1(\Omega)),\ \phi(c) \in L^2_{\mathrm{loc}}([0,T);L^2(\Omega)), \nabla\mu \in L^2(Q_T)
\end{aligned}$$

is called a weak solution of (1.1)-(1.7) on $(0,T)$ *if*

$$-(v,\partial_t\psi)_{Q_T}-(v_0,\psi|_{t=0})_\Omega+(v\cdot\nabla v,\psi)_{Q_T}+(\nu(c)Dv,D\psi)_{Q_T}=(\mu\nabla c,\psi)_{Q_T} \quad (1.13)$$

for all $\psi\in C^\infty_{(0)}([0,T)\times\Omega)^d$ *with* $\operatorname{div}\psi=0$,

$$\begin{aligned} -(c,\partial_t\varphi)_{Q_T}-(c_0,\varphi|_{t=0})_\Omega+(v\cdot\nabla c,\varphi)_{Q_T} &= -(\nabla\mu,\nabla\varphi)_{Q_T} && (1.14)\\ (\mu,\varphi)_{Q_T} &= (\phi(c),\varphi)_{Q_T}+(\nabla c,\nabla\varphi)_{Q_T} && (1.15)\end{aligned}$$

for all $\varphi\in C^\infty_{(0)}([0,T)\times\overline{\Omega})$*, and if the (strong) energy inequality*

$$E(v(t),c(t))+\int_{Q_{(t_0,t)}}\nu(c)|Dv|^2\,d(x,\tau)+\int_{Q_{(t_0,t)}}|\nabla\mu|^2\,d(x,\tau)\le E(v(t_0),c(t_0)) \quad (1.16)$$

holds for almost all $0\le t_0<T$ *including* $t_0=0$ *and all* $t\in[t_0,T)$.

Throughout the article $\Omega\subset\mathbb{R}^d$, $d=2,3$, will denote a bounded domain with C^3-boundary if no other assumption are made. Our main results are as follows:

THEOREM 1.4 (Global Existence of Weak Solutions)
For every $v_0\in L^2_\sigma(\Omega)$, $c_0\in H^1(\Omega)$ *with* $c_0(x)\in[a,b]$ *almost everywhere there is a weak solution* (v,c,μ) *of (1.1)-(1.7) on* $(0,\infty)$*. Moreover, if* $d=2$*, then (1.16) holds with equality for all* $0\le t_0\le t<\infty$*. Finally, every weak solution on* $(0,\infty)$ *satisfies*

$$\nabla^2 c,\phi(c)\in L^2_{loc}([0,\infty);L^r(\Omega)),\frac{t^{\frac12}}{1+t^{\frac12}}c\in BUC(0,\infty;W^1_q(\Omega)) \quad (1.17)$$

where $r=6$ *if* $d=3$ *and* $1<r<\infty$ *is arbitrary if* $d=2$ *and* $q>3$ *is independent of the solution and initial data. If additionally* $c_0\in H^2_N(\Omega):=\{c\in H^2(\Omega):\partial_n c|_{\partial\Omega}=0\}$ *and* $-\Delta c_0+\phi_0(c_0)\in H^1(\Omega)$*, then* $c\in BUC(0,\infty;W^1_q(\Omega))$.

We note that the regularity statement $t^{\frac12}/(1+t^{\frac12})c\in BUC(0,\infty;W^1_q(\Omega))$ with $q>d$ for *any* weak solution in the latter theorem is a crucial ingredient for obtaining higher regularity of weak solutions. This is one of the most difficult steps in the analysis. It is essentially based on the regularity result of the Cahn-Hilliard equation with convection and singular potential in spaces of fractional time regularity presented in Lemma 3.2 below. This result and a careful interpolation argument using the density of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ for $|s|<\frac12$ leads to the latter statement, cf. proof of Theorem 1.4 in Section 6 as well as Remark 3.3 below.

Because of $c\in BUC(\delta,\infty;W^1_q(\Omega))$, $q>d$, for all $\delta>0$ and $\delta=0$ for suitable inital data, one is able to use a result on maximal regularity for an associated Stokes system with variable viscosity, cf. Proposition 4.5 below, to conclude higher regularity for the velocity v in the case of small or large times and in the case $d=2$, which is enough to obtain a (locally) unique solution. More precisely, the results are as follows:

Proposition 1.5 (Uniqueness)
Let $0 < T \leq \infty$, $q = 3$ if $d = 3$ and let $q > 2$ if $d = 2$. Moreover, assume that $v_0 \in W^1_{q,0}(\Omega) \cap L^2_\sigma(\Omega)$ and let $c_0 \in H^1_{(0)}(\Omega) \cap C^{0,1}(\overline{\Omega})$ with $c_0(x) \in [a,b]$ for all $x \in \Omega$. If there is a weak solution (v, c, μ) of (1.1)-(1.7) on $(0,T)$ with $v \in L^\infty(0,T; W^1_q(\Omega))$ and $\nabla c \in L^\infty(Q_T)$, then any weak solution (v', c', μ') of (1.1)-(1.7) on $(0,T)$ with the same initial values and $\nabla c' \in L^\infty(Q_T)$ coincides with (v, c, μ).

THEOREM 1.6 (Regularity of Weak Solutions)
Let $c_0 \in H^2_N(\Omega)$ such that $E_1(c_0) < \infty$ and $-\Delta c_0 + \phi(c_0) \in H^1(\Omega)$.

1. *Let $d = 2$ and let $v_0 \in V_2^{1+s}(\Omega)$ with $s \in (0,1]$. Then every weak solution (v, c) of (1.1)-(1.7) on $(0, \infty)$ satisfies*

$$v \in L^2(0,\infty; H^{2+s'}(\Omega)) \cap H^1(0,\infty; H^{s'}(\Omega)) \cap BUC([0,\infty); H^{1+s-\varepsilon}(\Omega))$$

   for all $s' \in [0, \frac{1}{2}) \cap [0, s]$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(0,\infty; L^r(\Omega))$ for every $1 < r < \infty$. In particular, the weak solution is unique.

2. *Let $d = 2, 3$. Then for every weak solution (v, c, μ) of (1.1)-(1.7) on $(0,\infty)$ there is some $T > 0$ such that*

$$v \in L^2(T,\infty; H^{2+s}(\Omega)) \cap H^1(T,\infty; H^s(\Omega)) \cap BUC([T,\infty); H^{2-\varepsilon}(\Omega))$$

   for all $s \in [0, \frac{1}{2})$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(T,\infty; L^r(\Omega))$ with $r = 6$ if $d = 3$ and $1 < r < \infty$ if $d = 2$.

3. *If $d = 3$ and $v_0 \in V_\gamma^{s+1}(\Omega)$, $s \in (\frac{1}{2}, 1]$, then there is some $T_0 > 0$ such that every weak solution (v, c) of (1.1)-(1.7) on $(0, T_0)$ satisfies*

$$v \in L^2(0,T_0; H^{2+s'}(\Omega)) \cap H^1(0,T_0; H^{s'}(\Omega)) \cap BUC([0,T_0]; H^{1+s-\varepsilon}(\Omega))$$

   for all $s' \in [0, \frac{1}{2})$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(0,T_0; L^6(\Omega))$. In particular, the weak solution is unique on $(0, T_0)$.

Finally, because of the regularity of any weak solution for large times, we are able to modify the proof in [3], based on the Lojasiewicz-Simon inequality, to show convergence to stationary solutions as $t \to \infty$.

THEOREM 1.7 (Convergence to Stationary Solution)
Assume that $\Phi\colon (a,b) \to \mathbb{R}$ is analytic and let (v, c, μ) be a weak solution of (1.1)-(1.7). Then $(v(t), c(t)) \rightharpoonup_{t\to\infty} (0, c_\infty)$ in $H^{2-\varepsilon}(\Omega)^d \times H^2(\Omega)$ for all $\varepsilon > 0$ and for some $c_\infty \in H^2(\Omega)$ with $\phi(c_\infty) \in L^2(\Omega)$ solving the stationary Cahn-Hilliard equation

$$-\Delta c_\infty + \phi(c_\infty) = const. \qquad \text{in } \Omega, \tag{1.18}$$

$$\partial_n c_\infty|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega, \tag{1.19}$$

$$\int_\Omega c_\infty(x)\,dx = \int_\Omega c_0(x)\,dx. \tag{1.20}$$

Remark 1.8 We note that the latter convergence result shows that asymptotically solutions of the model H show the "right behaviour" in the sense that the velocity goes to zero and the diffuse interface tends to a diffuse interface with constant "mean curvature $-\Delta c_\infty + \phi(c_\infty)$" for large times might as one observes in real world two-phase flows. But it also indicates that the model H and the Cahn-Hilliard equation might share some common (unwanted) effects for large times. In particular we note that it was shown by Sternberg and Zumbrun [28] that in a strictly convex domain and for sufficiently small $\varepsilon > 0$ the diffuse interface of a *stable* stationary solution of the Cahn-Hilliard equation is connected. This suggests that there are no stable stationary solutions of the model H that represent two or more droplets, which is clearly observed in real world two-phase flows. This is related to a *coarsening effect*, known for the Cahn-Hilliard equation and which might be present also in the model H, where mass from smaller droplets diffuses to larges droplets until finally (in a stricly convex domain) almost all mass is contained in one droplet. More precise studies of the dynamics of the model H for large times might be the topic of future works.

Let us comment on the novelties: In comparision with the few known results for the system (1.1)-(1.4) we present the first results on existence, uniqueness and regularity for a class of singular free energies, including the physically important logarithmic free energy (1.10), which assures that $c(t,x) \in (a,b)$ almost everywhere. – We note that in [7] the free energy (1.10) is consider together with a degenerating mobility $m(c) \to_{c\to\pm 1} 0$; but only existence of solutions with $c(t,x) \in [-1,1]$ is shown. No higher regularity or uniqueness was obtained. - In order to deal with the singular free energies, we extend the results of [3], based on perturbation results for monotone operator, to include the convective term in (1.3). In order to show that $c \in BUC(\delta,\infty; W^1_q(\Omega))$ for some $q > d$ and any weak solution, it is essential to work in spaces with *fractional regularity in time* since v has only very limited regularity in time and space, cf. Lemma 3.2 and Remark 3.3. This is one of the most crucial steps in the analysis. The latter regularity for c is essential in order to get higher regularity from the linear Stokes system with variable viscosity in various situations. The necessary results on maximal regularity are obtained by perturbation arguments from the case with constant viscosity. Even these result seem to be original since the Stokes with variable viscosity is little studied in the literature. We note that the results above hold true for a suitable smooth free energy density $\Phi(c)$ as e.g. $\Phi(c) = (c^2-1)^2$ with even simpler proofs since the regularity of c for solutions of (1.3)-(1.4) is easily obtained by standard results on parabolic partial differential equations. But even in that case the regularity for large times and in particular the convergence as $t \to \infty$, which is based on that, are new results.

By the assumption on Φ we have the decomposition

$$\Phi(s) = \Phi_0(s) - \frac{\alpha}{2}c^2, \qquad \phi(s) = \phi_0(s) - \alpha c \tag{1.21}$$

where $\Phi_0 \in C([a,b]) \cap C^2((a,b))$ is convex. This will be the key point in the following analysis of the Cahn-Hilliard equation (1.3)-(1.4). The condition $\lim_{c\to a}\phi_0(c) = -\infty, \lim_{c\to b}\phi_0(c) = \infty$ for $\phi_0 = \Phi_0'$ will keep the concentration difference c in the (physical reasonable) interval $[a,b]$ and ensures that the subgradient of the associated functional is a single valued function with a suitable domain, cf. Theorem 2.3 below.

The structure of the article is as follows: In Section 2 we fix the notation and summarize some basic results on the used function spaces, monotone operators and subgradients. Then we start with an existence and regularity theory of the separated systems. In Section 3 we derive the needed results for a Cahn-Hilliard equation with convection, i.e., (1.3)-(1.4) for given v. In particular, we prove that $\left(\frac{t}{1+t}\right)^{\frac{1}{2}} c \in BUC(0,\infty;W^1_q(\Omega))$ for some $q > 3$ under regularity assumptions on v, which are satisfied by any weak solution. This is done with aid of suitable estimates in vector-valued Besov spaces. In order to prove existence of solutions we use the method of [3], which is based on a decomposition of the associated operators in a monotone operator plus a Lipschitz perturbation. Then in Section 4 we study the Stokes and Navier-Stokes equation with variable viscosity $\nu(c)$ for a fixed $c \in BUC(0,\infty;W^1_q(\Omega))$, $q > d$, in fractional L^2-Sobolev spaces. Section 5 is devoted to the proof of Theorem 1.4. In Section 6 the uniqueness and regularity results are shown. Finally, in Section 7 we prove the convergence to stationary solutions as $t \to \infty$ with the aid of the regularity results and the Lojasiewicz-Simon inequality.

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2 Preliminaries

For a set M the power set will be denoted by $\mathcal{P}(M)$ and χ_M denotes its characteristic function. Moreover, we denote $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$, $a \otimes b = (a_i b_j)_{i,j=1}^n$ for $a, b \in \mathbb{R}^d$, $A_{\text{sym}} = \frac{1}{2}(A + A^T)$, and $[A,B] = AB - BA$ for two operators A, B. If X is a Banach space and X' is its dual, then

$$\langle f, g\rangle \equiv \langle f, g\rangle_{X',X} = f(g), \qquad f \in X', g \in X,$$

denotes the duality product. We write $X \hookrightarrow\hookrightarrow Y$ if X is compactly embedded into Y. Moreover, if H is a Hilbert space, $(\cdot,\cdot)_H$ denotes its inner product. In the following all Hilbert spaces will be real-valued and separable.

2.1 Function Spaces

Spaces of continuous functions: The usual spaces of bounded continuous, Hölder continuous, Lipschitz continuous, k-times differentiable and smooth functions on a closed set A are denoted by $C(A)$, $C^\alpha(A)$ for $0 < \alpha < 1$, $C^{0,1}(A)$, $C^k(A)$, and $C^\infty(A)$, respectively. Here all derivatives and moduli of continuity (as long as they exist) are assumed to be bounded as well as. Furthermore, $C_0^\infty(\Omega) \equiv \mathcal{D}(\Omega)$ denotes the space of smooth and compactly supported functions $f\colon \Omega \to \mathbb{R}$. If $A \subset \mathbb{R}^d$, then

$$C_{(0)}^\infty(A) = \left\{ f\colon A \to \mathbb{R} : f = F|_A, F \in C_0^\infty(\mathbb{R}^d), \operatorname{supp} f \subseteq A \right\}.$$

Finally, let $0 < T \leq \infty$ and let X be a Banach space. Then $BC(0,T;X)$ is the Banach space of all bounded and continuous $f\colon [0,T) \to X$ equipped with the supremum norm and $BUC(0,T;X)$ is the subspace of all bounded and uniformly continuous functions. Moreover, we define $BC_w(0,T;X)$ as the topological vector space of all bounded and weakly continuous functions $f\colon [0,T) \to X$.

Spaces of integrable functions: If $M \subseteq \mathbb{R}^d$ is measurable, $L^q(M)$, $1 \leq q \leq \infty$ denotes the usual Lebesgue-space, $\|.\|_q$ its norm, and $(.,.)_M \equiv (.,.)_{L^2(M)}$. Moreover, $L^q(M;X)$ denotes its vector-valued variant of strongly measurable q-integrable functions/essentially bounded functions, where X is a Banach space. If $M = (a,b)$, we write for simplicity $L^q(a,b;X)$ and $L^q(a,b)$. Furthermore, $f \in L^q_{\mathrm{loc}}([0,\infty);X)$ if and only if $f \in L^q(0,T;X)$ for every $T > 0$. Moreover, $L^q_{\mathrm{uloc}}([0,\infty);X)$ denotes the *uniformly local* variant of $L^q(0,\infty;X)$ consisting of all measurable $f\colon [0,\infty) \to X$ such that

$$\|f\|_{L^q_{\mathrm{uloc}}([0,\infty);X)} = \sup_{t \geq 0} \|f\|_{L^q(t,t+1;X)} < \infty.$$

Recall that, if X is a Banach space with the Radon-Nikodym property, then

$$L^q(M;X)' = L^{q'}(M;X') \qquad \text{for every } 1 \leq q < \infty$$

by means of the duality product $\langle f, g\rangle = \int_M \langle f(x), g(x)\rangle_{X',X}\, dx$ for $f \in L^{q'}(M;X')$, $g \in L^q(M;X)$. If X is reflexive or X' is separable, then X has the Radon-Nikodym property, cf. Diestel and Uhl [11].

Moreover, recall the Lemma of Aubin-Lions: If $X_0 \hookrightarrow\hookrightarrow X_1 \hookrightarrow X_2$ are Banach spaces, $1 < p < \infty$, $1 \leq q < \infty$, and $I \subset \mathbb{R}$ is a bounded interval, then

$$\left\{ v \in L^p(I;X_0) : \frac{dv}{dt} \in L^q(I;X_2) \right\} \hookrightarrow\hookrightarrow L^p(I;X_1). \tag{2.1}$$

See J.-L. Lions [18] for the case $q > 1$ and Simon [23] or Roubíček [20] for $q = 1$.

Finally, let (X_0, X_1) be a compatible couple of Banach spaces, i.e., there is a Hausdorff topological vector space Z such that $X_0, X_1 \hookrightarrow Z$, cf. Bergh and Löfström [6], and let $(.,.)_{[\theta]}$ and $(.,.)_{\theta,r}$, $\theta \in [0,1], 1 \leq r \leq \infty$, denote the complex and real interpolation functor, respectively. Then for all $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$, and $\theta \in (0,1)$

$$(L^{p_0}(M;X_0), L^{p_1}(M;X_1))_{[\theta]} = L^p(M;(X_0,X_1)_{[\theta]}), \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{2.2}$$

cf. [6, Theorem 5.1.2.]. Moreover, we will use that, if $X_1 \hookrightarrow X_0$, then

$$(X_0, X_1)_{\theta_1, q_1} \hookrightarrow (X_0, X_1)_{\theta_0, q_0} \qquad \text{if } 0 \leq \theta_0 < \theta_1 \leq 1, 1 \leq q_0, q_1 \leq \infty, \tag{2.3}$$

which follows from [6, Theorem 3.4.1].

Sobolev, Bessel potential, and Besov spaces: $W^m_q(\Omega)$, $m \in \mathbb{N}_0$, $1 \leq q \leq \infty$, denotes the usual L^q-Sobolev space, $W^m_{q,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^m_q(\Omega)$, and $W^{-m}_q(\Omega) = (W^m_{q',0}(\Omega))'$. The L^2-Bessel potential spaces are denoted by $H^s(\Omega)$, $s \in \mathbb{R}$, which are defined by restriction of distributions in $H^s(\mathbb{R}^d)$ to Ω, cf. Triebel [29, Section 4.2.1]. It is well-known that

$$(H^{s_0}(\mathbb{R}^d), H^{s_1}(\mathbb{R}^d))_{[\theta]} = (H^{s_0}(\mathbb{R}^d), H^{s_1}(\mathbb{R}^d))_{\theta,2} = H^s(\mathbb{R}^d), \quad s = (1-\theta)s_0 + \theta s_1, \tag{2.4}$$

for all $\theta \in (0,1)$ $s_0, s_1 \in \mathbb{R}$, cf. [6, Theorem 6.4.5]. Furthermore, if $\Omega \subset \mathbb{R}^d$ has a continuous extension operator $E\colon W^k_2(\Omega) \to W^k_2(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ (in particular if Ω is a bounded domain with Lipschitz boundary), then $H^k(\Omega) = W^k_2(\Omega)$ by Plancharel's theorem and (2.4) holds with $\mathbb{R}^d$ replaced by Ω and all $s_0, s_1 \geq 0$ by [29, Section 1.2.4]. Moreover, we note that, if Ω is a bounded domain with C^1-boundary, then

$$H^s(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)} \quad \text{and} \quad H^s(\Omega)' = H^{-s}(\Omega) \qquad \text{for all } |s| < \frac{1}{2}, \tag{2.5}$$

cf. e.g. [29, Section 4.3.2. and 4.8.2].

Moreover, if X is a Banach space and $0 < T \leq \infty$, then $f \in W^1_p(0,T;X)$, $1 \leq p < \infty$ if and only if $f, \frac{d}{dt}f \in L^p(0,T;X)$, where $\frac{d}{dt}f$ denotes the vector-valued distributional derivative of f. Furthermore, $W^1_{p,\mathrm{uloc}}([0,\infty);X)$ is defined by replacing $L^p(0,T;X)$ by $L^p_{\mathrm{uloc}}([0,\infty);X)$ and we set $H^1(0,T;X) = W^1_2(0,T;X)$. Now let X_0, X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ densely. Then

$$W^1_p(0,T;X_0) \cap L^p(0,T;X_1) \hookrightarrow BUC(0,T;(X_0,X_1)_{1-\frac{1}{p},p}), \qquad 1 \leq p < \infty, \tag{2.6}$$

continuously, cf. Amann [4, Chapter III, Theorem 4.10.2]. Moreover, there is a continuous extension operator

$$E\colon (X_0,X_1)_{1-\frac{1}{p},p} \to W^1_p(0,\infty;X_0) \cap L^p(0,\infty;X_1) \quad \text{such that } Eu_0|_{t=0} = u_0, \tag{2.7}$$

cf. [4, Chapter III, Theorem 4.10.2]. Actually (2.6) and (2.7) follow directly from the trace method for the real interpolation, cf. [6, Corollary 3.12.3]. If additionally $X_0 = H$ is a Hilbert space and H is identified with its dual, then $X_1 \hookrightarrow H \hookrightarrow X_1'$ and

$$\frac{1}{2}\frac{d}{dt}\|f\|_H^2 = \langle \frac{d}{dt}f(t), f(t)\rangle_{X_1',X_1} \qquad \text{for almost all } t \in [0,T] \tag{2.8}$$

provided that $f \in L^p(0,T;X_1)$ and $\frac{d}{dt}f \in L^{p'}(0,T;X_1')$, $1 < p < \infty$, cf. Zeidler [30, Proposition 23.23]. In particular, (2.8) implies

$$\sup_{t\in[0,T]} \|f(t)\|_H^2 \leq 2\left(\|\partial_t f\|_{L^2(0,T;X_1')}\|f\|_{L^2(0,T;X_1)} + \|f(0)\|_H^2\right). \tag{2.9}$$

Furthermore, we define $H^{1,2}(Q_T) := L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ for $0 < T \leq \infty$.

The usual Besov spaces are denoted by $B^s_{pq}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$, cf. e.g. [6, 29]. If $\Omega \subseteq \mathbb{R}^n$ is a domain, $B^s_{pq}(\Omega)$ is defined by restriction of the elements of $B^s_{pq}(\mathbb{R}^n)$ to Ω, equipped with the quotient norm. We refer to [6, 29] for the standard results on interpolation of Besov spaces and Sobolev embeddings. We only note that $H^s(\Omega) = B^s_{22}(\Omega)$ and that, if Ω has a continuous extension operator as above, then

$$(W^k_{p_0}(\Omega), W^{k+1}_{p_1}(\Omega))_{\theta,p} = B^{k+\theta}_{pp}(\Omega) \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, k \in \mathbb{N}_0, \tag{2.10}$$

for all $\theta \in (0,1)$, cf. [29, Section 2.4.2 Theorem 1].

In order to derive some suitable estimates we will use vector-valued Besov spaces $B^s_{q\infty}(I;X)$, where $s \in (0,1)$, $1 \le q \le \infty$, I is an interval, and X is a Banach space. They are defined as

$$\begin{aligned} B^s_{q\infty}(I;X) &= \left\{ f \in L^q(I;X) : \|f\|_{B^s_{q\infty}(I;X)} < \infty \right\}, \\ \|f\|_{B^s_{q\infty}(I;X)} &= \|f\|_{L^q(I;X)} + \sup_{0<h\le 1} \|\Delta_h f(t)\|_{L^q(I_h;X)}, \end{aligned}$$

where $\Delta_h f(t) = f(t+h) - f(t)$ and $I_h = \{t \in I : t+h \in I\}$. Moreover, we set $C^s(I;X) = B^s_{\infty\infty}(I;X)$, $s \in (0,1)$. Let X_0, X_1 be two Banach spaces. Using $f(t) - f(s) = \int_s^t \frac{d}{dt} f(\tau)\, d\tau$ it is easy to show that for $1 \le q_0 < q_1 \le \infty$

$$W^1_{q_1}(I;X_1) \cap L^{q_0}(I;X_0) \hookrightarrow B^\theta_{q\infty}(I;X_\theta), \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{2.11}$$

where $\theta \in (0,1)$ and $X_\theta = (X_0,X_1)_{[\theta]}$ or $X_\theta = (X_0,X_1)_{\theta,r}$, $1 \le r \le \infty$. Finally, $B^s_{q\infty,\mathrm{uloc}}([0,\infty);X)$ is defined in the obvious way replacing $L^q(0,\infty;X)$-norms by $L^q_{\mathrm{uloc}}([0,\infty);X)$-norms.

Weak Neumann Laplace equation: Given $f \in L^1(\Omega)$, we denote by $m(f) = \frac{1}{|\Omega|}\int_\Omega f(x)\, dx$ its mean value. Moreover, for $m \in \mathbb{R}$ we set

$$L^q_{(m)}(\Omega) := \{f \in L^q(\Omega) : m(f) = m\}, \qquad 1 \le q \le \infty,$$

and $P_0 f := f - m(f)$ is the orthogonal projection onto $L^2_{(0)}(\Omega)$. Furthermore, we define

$$H^1_{(0)} \equiv H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega), \qquad (c,d)_{H^1_{(0)}(\Omega)} := (\nabla c, \nabla d)_{L^2(\Omega)}.$$

Then $H^1_{(0)}(\Omega)$ is a Hilbert space due to Poincaré's inequality. Moreover, let $H^{-1}_{(0)} \equiv H^{-1}_{(0)}(\Omega) = H^1_{(0)}(\Omega)'$. Then the Riesz isomorphism $\mathcal{R}\colon H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ is given by

$$\langle \mathcal{R}c, d\rangle_{H^{-1}_{(0)},H^1_{(0)}} = (c,d)_{H^1_{(0)}} = (\nabla c, \nabla d)_{L^2}, \qquad c,d \in H^1_{(0)}(\Omega),$$

i.e., $\mathcal{R} = -\Delta_N$ is the negative (weak) Laplace operator with Neumann boundary conditions. In particular, this means that

$$(f,g)_{H^{-1}_{(0)}} = (\nabla \Delta_N^{-1} f, \nabla \Delta_N^{-1} g)_{L^2} = (\Delta_N^{-1} f, \Delta_N^{-1} g)_{H^1_{(0)}}. \tag{2.12}$$

This implies the useful interpolation inequality

$$\|f\|_{L^2}^2 = -(\nabla\Delta_N^{-1}f, \nabla f)_{L^2} \leq \|f\|_{H^{-1}_{(0)}}\|f\|_{H^1_{(0)}} \quad \text{for all } f \in H^1_{(0)}(\Omega). \tag{2.13}$$

Moreover, we embed $H^1_{(0)}(\Omega)$ and $L^2_{(0)}(\Omega)$ into $H^{-1}_{(0)}(\Omega)$ in the standard way:

$$\langle c, \varphi\rangle_{H^{-1}_{(0)},H^1_{(0)}} = \int_\Omega c(x)\varphi(x)\,dx, \qquad \varphi \in H^1_{(0)}(\Omega).$$

Finally, we note that, if $u \in H^1_{(0)}(\Omega)$ solves $\Delta_N u = f$ for some $f \in L^q_{(0)}(\Omega)$, $1 < q < \infty$, and $\partial\Omega$ is C^2, then it follows from standard elliptic theory that $u \in W^2_q(\Omega)$ and $\Delta u = f$ a.e. in Ω and $\partial_n u|_{\partial\Omega} = 0$ in the sense of traces. If additionally $f \in W^1_q(\Omega)$ and $\partial\Omega \in C^3$, then $u \in W^3_q(\Omega)$. Moreover,

$$\|u\|_{W^{k+2}_q(\Omega)} \leq C_q\|f\|_{W^k_q(\Omega)} \qquad \text{for all } f \in W^k_q(\Omega) \cap L^q_{(0)}(\Omega), k = 0, 1, \tag{2.14}$$

with a constant C_q depending only on $1 < q < \infty$, d, and Ω.

Spaces of solenoidal vector-fields: In the following $C^\infty_{0,\sigma}(\Omega)$ denotes the space of all divergence free vector fields in $C^\infty_0(\Omega)^d$ and $L^q_\sigma(\Omega)$ is its closure in the L^q-norm. The corresponding Helmholtz projection is denoted by P_q, cf. e.g. Simader and Sohr [22]. We note that $P_q f = f - \nabla p$, where $p \in L^q_{\mathrm{loc}}(\overline{\Omega})$ with $p \in W^1_q(\Omega) \cap L^q_{(0)}(\Omega)$ is the solution of the weak Neumann problem

$$(\nabla p, \nabla\varphi)_\Omega = (f, \nabla\varphi) \quad \text{for all } \varphi \in C^\infty_{(0)}(\overline{\Omega}). \tag{2.15}$$

In particular, this implies that $P_2 f \in H^k(\Omega)^d \cap L^2_\sigma(\Omega)$ if $f \in H^k(\Omega)^d$, $k = 0, 1, 2$, and Ω is a bounded domain with C^3-boundary by the regularity of the weak Neumann problem discussed above.

Moreover, we denote $H^s_\sigma(\Omega) = H^s(\Omega)^d \cap L^2_\sigma(\Omega)$ for $s \geq 0$, $V^s_2(\Omega) = H^s_\sigma(\Omega) \cap H^1_0(\Omega)^d$ for $s \geq 1$, and $V_2(\Omega) = V^1_2(\Omega)$. Because of Korn's inequality $V_2(\Omega)$ can be normed by $\|Dv\|_2$.

Some useful estimates: Firstly, if $f \in H^2(\Omega)$ and $\Omega \subseteq \mathbb{R}^d$, $d \leq 3$ is a bounded domain with Lipschitz boundary, then

$$\|f\|_\infty \leq C\|f\|_{L^2}^{1-\frac{d}{4}}\|f\|_{H^2}^{\frac{d}{4}}. \tag{2.16}$$

The latter estimate follows from the fact that

$$(L^2(\mathbb{R}^d), H^2(\mathbb{R}^d))_{\frac{d}{4},1} = B^{\frac{d}{2}}_{21}(\mathbb{R}^d) \hookrightarrow B^0_{\infty 1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$$

and that Ω has a continuous extension operator $E\colon H^k(\Omega) \to H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$, cf. Stein [27, Chapter VI, Section 3.2]. Here $B^s_{pq}(\mathbb{R}^d)$ denote the usual Besov spaces and we have used [6, Theorems 6.4.5 and 6.5.1]. Moreover, we note that

$$\|fg\|_{W^1_p} \leq C\|f\|_{W^1_r}\|g\|_{W^1_p} \quad \text{for all } 1 \leq p \leq r, r > d, \tag{2.17}$$

which can be easily proved using the Sobolev embedding theorem. Finally, if $d \leq 3$,

$$\|fg\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s+\frac{1}{2}}(\Omega)} \|f\|_{H^{s+1}(\Omega)} \tag{2.18}$$

for all $s \geq 0$ and bounded Lipschitz domains since $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^3(\Omega)$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$. This can be proved by first showing the case $s \in \mathbb{N}_0$ and then using bilinear complex interpolation, cf. [6, Theorem 4.4.1].

We will frequently use the following simple lemma:

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open set and let $c_k \in L^\infty(\Omega)$ be a bounded sequence such that $c_k \to_\infty c$ in $L^1_{\mathrm{loc}}(\Omega)$. Then for every $f \in L^r(\Omega)$, $1 \leq r < \infty$,*

$$c_k f \to_{k\to\infty} cf \qquad \text{in } L^r(\Omega).$$

Proof: Let $\varepsilon > 0$. Since $C_0^\infty(\Omega)$ is dense in $L^r(\Omega)$, $1 \leq r < \infty$, there is some $\varphi \in C_0^\infty(\Omega)$ such that $\|f-\varphi\|_r \leq \varepsilon$. Moreover, there is some N_ε such that $\|\varphi(c_k-c)\|_r \leq \varepsilon$ for every $k \geq N_\varepsilon$, where we use that $c_k \to_{k\to\infty} c$ in $L^1_{\mathrm{loc}}(\Omega)$ implies the convergence in $L^r_{\mathrm{loc}}(\Omega)$ since c_k are uniformly bounded. Hence

$$\begin{aligned} \|f(c_k-c)\|_r &\leq \|\varphi(c_k-c)\|_r + C\|(f-\varphi)(c_k-c)\|_r \\ &\leq \|\varphi(c_k-c)\|_r + 2\sup_{k\in\mathbb{N}}\|c_k\|_\infty \|f-\varphi\|_r \leq (1+2\sup_{k\in\mathbb{N}}\|c_k\|_\infty)\varepsilon \end{aligned}$$

for every $k \geq N_\varepsilon$. This proves the lemma. ■

2.2 Evolution Equations for Monotone Operators

We refer to Brézis [8] and Showalter [21] for basic results in the theory of monotone operators. In the following we just summarize some basic facts and definitions. Let H be a real-valued and separable Hilbert space. Recall that $\mathcal{A}\colon H \to \mathcal{P}(H)$ is a monotone operator if

$$(w-z, x-y)_H \geq 0 \qquad \text{for all } w \in \mathcal{A}(x), z \in \mathcal{A}(y).$$

Moreover, $\mathcal{D}(A) = \{x \in H : \mathcal{A}(x) \neq \emptyset\}$. Now let $\varphi\colon H \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then $\mathrm{dom}(\varphi) = \{x \in H : \varphi(x) < \infty\}$ and φ is called proper if $\mathrm{dom}(\varphi) \neq \emptyset$. The subgradient $\partial\varphi\colon H \to \mathcal{P}(H)$ is defined by $w \in \partial\varphi(x)$ if and only if

$$\varphi(\xi) \geq \varphi(x) + (w, \xi - x)_H \qquad \text{for all } \xi \in H.$$

Then $\partial\varphi$ is a monotone operator and, if additionally φ is lower semi-continuous, then $\partial\varphi$ is maximal monotone, cf. [8, Exemple 2.3.4].

THEOREM 2.2 *Let H_0, H_1 be real-valued, separable Hilbert spaces such that $H_1 \hookrightarrow H_0$ densely. Moreover, let $\varphi\colon H_0 \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous functional such that $\varphi = \varphi_1 + \varphi_2$, where $\varphi_2 \geq 0$ is convex and lower semi-continuous,* $\operatorname{dom}\varphi_1 = H_1$, *and $\varphi_1|_{H_1}$ is a bounded, coercive, quadratic form on H_1 and set $\mathcal{A} = \partial\varphi$. Furthermore, assume that $\mathcal{B}\colon [0,T] \times H_1 \to H_0$ is measurable in $t \in [0,T]$ and Lipschitz continuous in $v \in H_1$ satisfying*

$$\|B(t,v_1) - B(t,v_2)\|_{H_0} \leq M(t)\|v_1 - v_2\|_{H_1} \qquad \text{for all } v_1, v_2 \in H_0, \text{ a.e. } t \in [0,T]$$

for some $M \in L^2(0,T)$. Then for every $u_0 \in \operatorname{dom}(\varphi)$ and $f \in L^2(0,T;H_0)$ there is a unique $u \in W_2^1(0,T;H_0) \cap L^\infty(0,T;H_1)$ with $u(t) \in \mathcal{D}(\mathcal{A})$ for a.e. $t > 0$ solving

$$\frac{du}{dt}(t) + \mathcal{A}(u(t)) \ni \mathcal{B}(t,u(t)) + f(t) \quad \text{for a.a. } t \in (0,T), \tag{2.19}$$

$$u(0) = u_0. \tag{2.20}$$

Moreover, $\varphi(u) \in L^\infty(0,T)$.

Proof: In the case that $\mathcal{B}$ is independent of t, the theorem is proved in [3]. – We note that the assumption $u_0 \in \mathcal{D}(A)$ in [3, Theorem 3.1] is a typo and $u_0 \in \operatorname{dom}\varphi$ is sufficient. – The latter proof directly carries over to the present case by using the estimate

$$\int_0^t |(\mathcal{B}(v_1) - \mathcal{B}(v_2), u_1 - u_2)_{H_0}|\,ds \leq \|M\|_{L^2(0,t)}\|v_1 - v_2\|_{L^2(0,t;H_1)}\|u_1 - u_2\|_{L^\infty(0,T;H_0)}$$

and using the fact that $\|M\|_{L^2(0,t)} \to_{t\to 0} 0$. ■

2.3 Subgradients

In this section we study the "convex part" of E_1 namely

$$E_0(c) = \frac{1}{2}\int_\Omega |\nabla c(x)|^2\,dx + \int_\Omega \Phi_0(c(x))\,dx, \tag{2.21}$$

where Φ_0 is the same as in (1.21). Firstly, E_0 is defined on $L^2_{(m)}(\Omega)$, $m \in (a,b)$, with

$$\operatorname{dom} E_0 = \left\{c \in H^1(\Omega) \cap L^2_{(m)}(\Omega) : c(x) \in [a,b] \text{ a.e.}\right\}.$$

We denote by $\partial E_0(c)\colon L^2_{(m)}(\Omega) \to \mathcal{P}(L^2_{(0)}(\Omega))$ the subgradient of E_0 at $c \in \operatorname{dom} E_0$ in the sense that $w \in \partial E_0(c)$ if and only if

$$(w, c' - c)_{L^2} \leq E_0(c') - E_0(c) \qquad \text{for all } c' \in L^2_{(m)}(\Omega).$$

Note that $L^2_{(m)}(\Omega)$ is an *affine* subspace of $L^2(\Omega)$ with tangent space $L^2_{(0)}(\Omega)$. This definition is the obvious generalization of the standard definition for Hilbert spaces to affine subspaces of Hilbert spaces.

The following result was proved in [3][1]:

THEOREM 2.3 *Let Φ_0, ϕ_0 be as in (1.21). Moreover, we set $\phi_0(x) = +\infty$ for $x \notin (a,b)$ and let E_0 be defined as in (2.21). Then*

$$\mathcal{D}(\partial E_0) = \left\{c \in H^2(\Omega) \cap L^2_{(m)}(\Omega) : \phi_0(c) \in L^2, \phi_0'(c)|\nabla c|^2 \in L^1, \partial_n c|_{\partial\Omega} = 0\right\}$$

and

$$\partial E_0(c) = -\Delta c + P_0 \phi_0(c). \tag{2.22}$$

Moreover, there is some $C > 0$ independent of $c \in \mathcal{D}(\partial E_0)$ such that

$$\|c\|^2_{H^2(\Omega)} + \|\phi_0(c)\|_2^2 + \int_\Omega \phi_0'(c(x))|\nabla c(x)|^2\,dx \leq C\left(\|\partial E_0(c)\|_2^2 + \|c\|_2^2 + 1\right). \tag{2.23}$$

For the following analysis it is important that (2.23) can be improved as follows:

Lemma 2.4 *Let Φ_0, ϕ_0, E_0 be as above and let $2 \leq r < \infty$. Then there is a constant C_r such that for every $c \in \mathcal{D}(\partial E_0)$ satisfying $\partial E_0(c) \in L^r(\Omega)$ we have*

$$\|c\|_{W^2_r} + \|\phi_0(c)\|_r \leq C_r\left(\|\partial E_0(c)\|_r + \|c\|_2 + 1\right). \tag{2.24}$$

Proof: Let $\psi \in C^1(\mathbb{R})$ be a non-negative function with $s\psi'(s) \geq 0$. Then multiplying (2.22) by $\psi(\phi_0(c))\phi_0(c)$ we obtain

$$\begin{aligned}(\nabla c, \nabla(\psi(\phi_0(c))\phi_0(c)))_\Omega &+ \int_\Omega \psi(\phi_0(c))|\phi_0(c)|^2\,dx \\ \leq\ & C\left(\|\partial E_0(c)\|_r + m(\phi_0(c))\right)\|\psi(\phi_0(c))\phi_0(c)\|_{r'},\end{aligned}$$

where

$$(\nabla c, \nabla(\psi(\phi_0(c))\phi_0(c)))_\Omega = \int_\Omega \phi_0'(c)|\nabla c|^2(\psi'(\phi_0(c))\phi_0(c) + \psi(\phi_0(c))\,dx \geq 0.$$

After a simple approximation we can replace ψ by $\psi_k(s) = \min(k, |s|^{r-2})$ to conclude

$$\begin{aligned}& C\left(\|\partial E_0(c)\|_r + m(\phi_0(c))\right\|\psi_k(\phi_0(c))\phi(c)\|_{r'} \\ \geq\ & \int_\Omega \psi_k(\phi_0(c))|\phi_0(c)|^2\,dx \geq \int_\Omega \psi_k(\phi_0(c))^{r'}|\phi_0(c)|^{r'}\,dx\end{aligned}$$

since $|\phi_0(c)|^{2-r'} = (|\phi_0(c)|^{r-2})^{\frac{1}{r-1}} \geq \psi_k(\phi_0(c))^{\frac{1}{r-1}}$. Therefore

$$\|\psi_k(\phi_0(c))\phi_0(c)\|_{r'}^{\frac{r'}{r}} \leq C\left(\|\partial E_0(c)\|_r + |m(\phi_0(c))|\right) \leq C\left(\|\partial E_0(c)\|_r + \|c\|_2 + 1\right)$$

because of (2.23). Finally, passing $k \to \infty$ the estimate of the second term in (2.24) follows. The estimate of the first term then follows by using (2.22) and (2.14). ■

[1]We note that the estimate (2.23) is stated in [3, Theorem 4.3] without "+1" on the right-hand side, which holds true if $m = 0$ and $\phi'(0) = 0$, which is assumed in the proof of [3, Theorem 4.3]. By a simple transformation one can reduce to that case; but then one has to add +1 on the right-hand side or modify the estimate in another suitable way.

Corollary 2.5 *Let E_0 be defined as above and extend E_0 to a functional $\widetilde{E}_0\colon H^{-1}_{(0)}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ by setting $\widetilde{E}_0(c) = E_0(c)$ if $c \in \operatorname{dom} E_0$ and $\widetilde{E}_0(c) = +\infty$ else. Then $\widetilde{E}_0$ is a proper, convex, and lower semi-continuous functional, $\partial\widetilde{E}_0$ is a maximal monotone operator with $\partial\widetilde{E}_0(c) = -\Delta_N \partial E_0(c)$, and*

$$\mathcal{D}(\partial\widetilde{E}_0) = \{c \in \mathcal{D}(\partial E_0) : \partial E_0(c) = -\Delta c + P_0\phi_0(c) \in H^1_{(0)}(\Omega)\}. \tag{2.25}$$

Moreover, for every $c \in \mathcal{D}(\partial\widetilde{E}_0)$

$$\|c\|_{W^2_r} + \|\phi_0(c)\|_r \leq C_r \left(\|\partial\widetilde{E}_0(c)\|_{H^1_{(0)}} + \|c\|_2 + 1\right), \tag{2.26}$$

where $r = 6$ if $d = 3$ and $2 \leq r < \infty$ is arbitrary if $d = 2$.

Proof: The first part is the same as [3, Corollary 4.4.]. The last statement follows from Lemma 2.4 and the Sobolev embedding theorem. ■

3 Cahn-Hilliard Equation with Convection

3.1 Existence and Regularity Theory

In this section we consider

$$\begin{aligned} \partial_t c + v \cdot \nabla c &= \Delta\mu && \text{in } \Omega \times (0,\infty), && (3.1)\\ \mu &= \phi(c) - \Delta c && \text{in } \Omega \times (0,\infty), && (3.2)\\ \partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,\infty), && (3.3)\\ c|_{t=0} &= c_0 && \text{in } \Omega && (3.4) \end{aligned}$$

for given c_0 with $E_1(c_0) < \infty$ and $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; V_2(\Omega))$. Here $\phi = \Phi'$ and Φ is as in Assumption 1.1 and E_1 is as in (1.8). We assume with loss of generality that $m(c_0) = 0$. By a simply shift of c, Φ, and $[a,b]$ by a constant we can always reduce to that case. Moreover, we can also assume that $0 \in (a,b)$ since $a = 0$ or $b = 0$ and $E_1(c_0) < \infty$ and $m(c_0) = 0$ implies that $c_0 \equiv 0$.

We consider (3.1)-(3.4) as an evolution equation on $H^{-1}_{(0)}(\Omega)$ in the following way:

$$\begin{aligned} \partial_t c(t) + \mathcal{A}(c(t)) + \mathcal{B}(v(t))c(t) &= 0, \qquad t > 0, && (3.5)\\ c|_{t=0} &= c_0 && (3.6) \end{aligned}$$

where $\mathcal{A}(c) = \partial\widetilde{E}_0(c)$ and

$$\langle \mathcal{B}(v)c, \varphi\rangle_{H^{-1}_{(0)}, H^1_{(0)}} = (v \cdot \nabla c, \varphi)_{L^2} - \alpha(\nabla c, \nabla\varphi)_{L^2}, \quad \varphi \in \mathcal{D}(\mathcal{B}(v)) = H^1_{(0)}(\Omega).$$

I.e., $\mathcal{A}(c) = \Delta_N(\Delta c - P_0\phi_0'(c)), \mathcal{B}(v)c = v \cdot \nabla c + \alpha\Delta_N c$, where $\Delta_N\colon H^1_{(0)}(\Omega) \subset H^{-1}_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ is the Laplace operator with Neumann boundary conditions as above, which is considered as an unbounded operator on $H^{-1}_{(0)}(Omega)$. Finally, we note that $\mathcal{A}$ is a strictly monotone operator since

$$\begin{aligned}&(\mathcal{A}(c_1)-\mathcal{A}(c_2),c_1-c_2)_{H^{-1}_{(0)}}\\ =\;&(-\Delta(c_1-c_2)+\phi_0(c_1)-\phi_0(c_2),c_1-c_2)_{L^2}\geq\|\nabla(c_1-c_2)\|_2^2\end{aligned}\tag{3.7}$$

for all $c_1,c_2\in\mathcal{D}(\mathcal{A})$.

In order to apply Theorem 2.2 we use that by Corollary 2.5, $\mathcal{A}=\partial\widetilde{E}_0$ is a maximal monotone operator with $\widetilde{E}_0=\varphi_1+\varphi_2$,

$$\varphi_1(c)=\frac{1}{2}\int_\Omega|\nabla c(x)|^2\,dx,\quad \varphi_2(c)=\int_\Omega\Phi_0(c(x))\,dx,$$

$\operatorname{dom}\varphi_1=H^1_{(0)}(\Omega)$, and $\operatorname{dom}\varphi_2=\operatorname{dom}\varphi=\{c\in H^1_{(0)}:c(x)\in[a,b]\text{ a.e.}\}$. Obviously, $\varphi_1|_{H^1_{(0)}(\Omega)}$ is a bounded, coercive quadratic form on $H^1_{(0)}(\Omega)$. Furthermore,

$$\begin{aligned}|(v(t)\cdot\nabla c,\varphi)_\Omega| &= |(vP_0c,\nabla\varphi)_{L^2}|\\ &\leq C\|v(t)P_0c\|_{L^2}\|\nabla\varphi\|_{L^2}\leq\begin{cases}C'\|v(t)\|_{L^2}\|\nabla\varphi\|_{L^2}\\ C'\|v(t)\|_{L^3}\|\nabla c\|_{L^2}\|\nabla\varphi\|_{L^2}\end{cases}\end{aligned}\tag{3.8}$$

for all $c,\varphi\in H^1_{(0)}(\Omega)$ with $E_1(c)<\infty$ since $\operatorname{div}v=0$, $n\cdot v|_{\partial\Omega}=0$, and $|c(x)|\leq\max(|a|,|b|)$ almost everywhere. Hence

$$\|\mathcal{B}(v(t))c\|_{H_0}\leq C\,(1+\|v(t)\|_{H^1})\,\|c\|_{H^1_{(0)}}$$

for almost every $t\in[0,\infty)$ and all $c\in H^1_{(0)}(\Omega)$, where $M(t):=C(1+\|v(t)\|_{H^1})\in L^2(0,T)$ for every $T>0$. Hence Theorem 2.2 is applicable to the operators $\mathcal{A}$, $\mathcal{B}(t)$ defined above.

The main result of this section is the following:

THEOREM 3.1 *Let $v\in L^2(0,\infty;V_2(\Omega))\cap L^\infty(0,\infty;L^2_\sigma(\Omega))$. Then for every $c_0\in H^1_{(0)}(\Omega)$ with $E_1(c_0)<\infty$ there is a unique solution $c\in BC(0,\infty;H^1_{(0)}(\Omega))$ of (3.1)-(3.4) with $\partial_tc\in L^2(0,\infty;H^{-1}_{(0)}(\Omega))$, $\mu\in L^2_{\text{uloc}}([0,\infty);H^1(\Omega))$. This solution satisfies*

$$E_1(c(t))+\int_{Q_t}|\nabla\mu|^2\,d(x,\tau)=E_1(c_0)-\int_{Q_t}v\cdot\mu\nabla c\,d(x,\tau)\tag{3.9}$$

for all $t\in[0,\infty)$ and

$$\|c\|^2_{L^\infty(0,\infty;H^1)}+\|\partial_tc\|^2_{L^2(0,\infty;H^{-1}_{(0)})}+\|\nabla\mu\|^2_{L^2(Q)}\leq C\left(E_1(c_0)+\|v\|^2_{L^2(Q)}\right)\tag{3.10}$$

$$\|c\|^2_{L^2_{\text{uloc}}([0,\infty);W^2_r)}+\|\phi(c)\|^2_{L^2_{\text{uloc}}([0,\infty);L^r)}\leq C_r\left(E_1(c_0)+\|v\|^2_{L^2(Q)}\right)\tag{3.11}$$

where $r = 6$ if $d = 3$ and $1 < r < \infty$ is arbitrary if $d = 2$. Here C, C_r are independent of v, c_0. Moreover, for every $R > 0$ the solution

$$c \in Y := L^2_{\rm loc}([0,\infty); W^2_r(\Omega)) \cap H^1_{\rm loc}([0,\infty); H^{-1}_{(0)}(\Omega))$$

depends continuously on

$$(c_0, v) \in X := H^1(\Omega) \times L^2_{\rm loc}([0,\infty); L^2_\sigma(\Omega)) \quad \text{with } E_1(c_0) + \|v\|_{L^2(0,\infty;H^1)} \le R$$

with respect to the weak topology on Y and the strong topology on X.

Proof: We apply Theorem 2.2 to the choice $H_1 = H^1_{(0)}(\Omega)$, $H_0 = H^{-1}_{(0)}(\Omega)$, $f = 0$, and φ_1, φ_2 as above, where we assume w.l.o.g. that $\Phi(c) \ge 0$. This gives the existence of a unique solution $c\colon [0,\infty) \to H_0$ of (3.5)-(3.6) such that $c \in W^1_2(0,T,H_0) \cap L^\infty(0,T;H_1)$, $\varphi(c) \in L^\infty(0,T)$ for every $T > 0$ and $c(t) \in \mathcal{D}(\mathcal{A})$ for almost all $t > 0$.

Now we define μ according to (3.2). Then

$$\Delta_N \mu(t) = \Delta_N(-\Delta c(t) + \phi_0(c(t)) - \alpha c(t)) = -\partial \widetilde{E}_0(c(t)) - \alpha \Delta_N c(t). \tag{3.12}$$

due to Corollary 2.5 and therefore

$$\Delta_N \mu(t) = \partial_t c(t) + v(t) \cdot \nabla c(t) \qquad \text{in } H^{-1}_{(0)}(\Omega). \tag{3.13}$$

In particular, this implies

$$\|\nabla \mu\|_{L^2(Q_T)} \le C \left(\|\partial_t c(t)\|_{L^2(0,T;H^{-1}_{(0)})} + \|v\|_{L^2(Q)} \right) < \infty$$

for every $T > 0$ due to (3.8). Now using (2.26) and (3.12) we obtain

$$\|c(t)\|_{W^2_r} + \|\phi_0(c(t))\|_r \le C \left(\|\partial \widetilde{E}_0(c(t))\|_{H^1_{(0)}} + \|c(t)\|_2 \right) \le C' \left(\|\nabla \mu(t)\|_2 + \|\nabla c(t)\|_2 \right)$$

for all $t > 0$. This implies (3.11) once (3.10) is proved. In order to prove (3.9), we use that $E_1(c(t)) = \widetilde{E}_0(c(t)) - \frac{\alpha}{2}\|c(t)\|^2_{L^2}$. Because of [21, Lemma 4.3, Chapter IV], (2.8), (3.12), and (3.13), we have

$$\begin{aligned}
\frac{d}{dt} E_1(c(t)) &= (\partial \widetilde{E}_0(c(t)), \partial_t c(t))_{H^{-1}_{(0)}} - \alpha \langle \partial_t c(t), c(t) \rangle_{H^{-1}_{(0)}, H^1_{(0)}} \\
&= (\partial \widetilde{E}_0(c(t)), \partial_t c(t))_{H^{-1}_{(0)}} + \alpha (\Delta_N c(t), \partial_t c(t))_{H^{-1}_{(0)}} \\
&= -(\Delta_N \mu(t), \Delta_N \mu(t))_{H^{-1}_{(0)}} - (\Delta_N \mu(t), v(t) \cdot \nabla c(t))_{H^{-1}_{(0)}} \\
&= -(\nabla \mu(t), \nabla \mu(t))_{L^2} - (\mu(t), v(t) \cdot \nabla c(t))_{L^2},
\end{aligned}$$

Integration on $(0,t)$ shows (3.9). In order to obtain (3.10), we use that $|(\mu, v \cdot \nabla c)_{Q_T}| \le C\|v\|_{L^2(Q)} \|\nabla \mu\|_{L^2(Q_T)}$ due to (3.8). Thus (3.9) and Young's inequality imply

$$\|c\|^2_{L^\infty(0,\infty;H^1(\Omega))} + \|\mu\|^2_{L^2(0,\infty;H^1)} \le C \left(E_1(c_0) + \|v\|^2_{L^2(Q)} \right).$$

The estimate of $\|\partial_t c\|_{L^2(0,\infty;H^{-1}_{(0)})}$ follows from (3.13) and $\|v\cdot\nabla c\|_{L^2(0,\infty;H^{-1}_{(0)})} \leq C\|v\|_{L^2(Q)}$.

In order to prove continuous dependence on (c_0, v), let c_j, $j = 1,2$, be two solutions of (3.1)-(3.4) with $c_j|_{t=0} = c_j^0 \in \operatorname{dom} E_0$ and v replaced by v_j. We set $\tilde{c} = c_1 - c_2$ and $w = v_1 - v_2$. Then

$$\partial_t \tilde{c}(t) + \mathcal{A}(c_1(t)) - \mathcal{A}(c_2(t)) + \mathcal{B}(v_1(t))c_1(t) - \mathcal{B}(v_2(t))c_2(t) = 0$$

for a.e. $t > 0$ and $\tilde{c}(0) = c_1^0 - c_2^0$. Hence taking the inner product of the equation above with $\tilde{c}(t)$ in $L^2(s,t;H^{-1}_{(0)}(\Omega))$, $0 \leq s < t < \infty$, and using (3.7) we conclude

$$\begin{aligned}
&\frac{1}{2}\|\tilde{c}(t)\|^2_{H^{-1}_{(0)}} + \int_s^t \|\nabla\tilde{c}\|_2^2\,d\tau \\
&\leq \alpha\int_s^t \|\tilde{c}\|_2^2\,d\tau + \frac{1}{2}\|\tilde{c}(s)\|^2_{H^{-1}_{(0)}} - \int_s^t (v_1\cdot\nabla c_1 - v_2\cdot\nabla c_2, \tilde{c})_{H^{-1}_{(0)}}\,d\tau.
\end{aligned}$$

Due to (2.13) and Young's inequality

$$\begin{aligned}
&\|\tilde{c}(t)\|^2_{H^{-1}_{(0)}} + \int_s^t \|\nabla\tilde{c}\|_2^2\,d\tau \\
&\leq C\left(\int_s^t \|\tilde{c}\|^2_{H^{-1}_{(0)}}\,d\tau + \|\tilde{c}(s)\|^2_{H^{-1}_{(0)}} + \left|\int_s^t (v_1\cdot\nabla c_1 - v_2\cdot\nabla c_2, \tilde{c})_{H^{-1}_{(0)}}\,d\tau\right|\right) \quad (3.14)
\end{aligned}$$

Now

$$(v_1\cdot\nabla c_1 - v_2\cdot\nabla c_2, \tilde{c})_{H^{-1}_{(0)}} = (v_1\cdot\nabla\tilde{c}, (-\Delta_N)^{-1}\tilde{c})_\Omega - (wc_2, \nabla(-\Delta_N)^{-1}\tilde{c})_\Omega$$

and

$$\begin{aligned}
\left|\int_s^t (v_1\cdot\nabla\tilde{c}, (-\Delta_N)^{-1}\tilde{c})_\Omega\,d\tau\right| &\leq \|\nabla\tilde{c}\|_{L^2(Q_{(t,s)})}\left(\int_s^t \|v_1\|_6^2\|(-\Delta_N)^{-1}\tilde{c}\|_6^2\,d\tau\right)^{\frac{1}{2}} \\
\left|\int_s^t (wc_2, \nabla(-\Delta_N)^{-1}\tilde{c})_\Omega\,d\tau\right| &\leq C\|w\|_{L^1(s,t;L^2)}\|\nabla(-\Delta_N)^{-1}\tilde{c}\|_{L^\infty(s,t;L^2)}. \quad (3.15)
\end{aligned}$$

Hence by Young's inequality and $\|(-\Delta_N)^{-1}\tilde{c}\|_6 \leq C\|\nabla(-\Delta_N)^{-1}\tilde{c}\|_2 \leq C'\|\tilde{c}\|_{H^{-1}_{(0)}}$

$$\begin{aligned}
&\sup_{s\leq\tau\leq t}\|\tilde{c}(\tau)\|^2_{H^{-1}_{(0)}} + \int_s^t \|\nabla\tilde{c}(\tau)\|_2^2\,d\tau \\
&\leq C\left(\int_s^t (1 + \|v_1(\tau)\|^2_{H^1})\|\tilde{c}(\tau)\|^2_{H^{-1}_{(0)}}\,d\tau + \|\tilde{c}(s)\|^2_{H^{-1}_{(0)}} + \|v_1 - v_2\|^2_{L^1(s,t;L^2)}\right).
\end{aligned}$$

Thus the lemma of Gronwall yields

$$\begin{aligned}
&\sup_{s\leq\tau\leq t}\|\tilde{c}(\tau)\|^2_{H^{-1}_{(0)}} + \int_s^t \|\nabla\tilde{c}(\tau)\|_2^2\,d\tau \\
&\leq C\exp\left(\int_s^t \left(1 + \|v_1(\tau)\|^2_{H^1}\right)\,d\tau\right)\left(\|\tilde{c}(s)\|^2_{H^{-1}_{(0)}} + \|v_1 - v_2\|^2_{L^1(s,t;L^2)}\right). \quad (3.16)
\end{aligned}$$

This implies the continuous dependence of $c \in L^\infty(0,T;H^{-1}_{(0)}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ on $(c_0,v) \in X$ for every $T>0$ with respect to the strong topologies. Because of (3.10)-(3.11) the continuous dependence of $c \in Y$ with respect to the weak topology on $(c_0,v) \in X$ with $E_1(c_0) + \|v\|_{L^2(0,\infty;H^1)} \leq R$ follows.

Finally, since $\partial_t c \in L^2(0,\infty;H^{-1}_{(0)}(\Omega))$ and $c \in L^\infty(0,\infty;H^1_{(0)}(\Omega))$, (2.8) implies $c \in BUC(0,\infty;L^2(\Omega))$. Since $H^1_{(0)}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, $c \in BC_w(0,\infty;H^1_{(0)}(\Omega))$ necessarily. Moreover, because of (3.9), $E_1(c(t))$ is continuous. Hence

$$\limsup_{s\to t} \frac{1}{2}\|\nabla c(s)\|_2^2 \ \geq \ E_1(c(t)) - \int_\Omega \Phi(c(t))\,dx = \frac{1}{2}\|\nabla c(t)\|_2^2$$

because of (1.21) with $\Phi_0(c)$ convex and since $c \in BUC(0,\infty;L^2(\Omega))$. On the other hand $\frac{1}{2}\|\nabla c(t)\|_2^2 \leq \liminf_{s\to t} \frac{1}{2}\|\nabla c(s)\|_2^2$ by the weak continuity in $H^1_{(0)}(\Omega)$. Thus $t \mapsto \frac{1}{2}\|\nabla c(t)\|_2^2$ is continuous and therefore $c \in BC(0,\infty;H^1_{(0)}(\Omega))$. ■

The following improved regularity statement will be important to get higher regularity of solutions to the Navier-Stokes-Cahn-Hilliard system.

Lemma 3.2 *Let the assumption of Theorem 3.1 be satisfied and let (c,μ) be the corresponding solution of (3.1)-(3.4). Moreover, let $\kappa \equiv 1$ if $c_0 \in \mathcal{D}(\partial\widetilde{E}_0)$ and let $\kappa(t) = \left(\frac{t}{1+t}\right)^{\frac{1}{2}}$ else.*

1. *If $\partial_t v \in L^1_{\mathrm{uloc}}([0,\infty);L^2(\Omega))$ and r is as in Theorem 3.1, then (c,μ) satisfy*

$$\kappa\partial_t c \in L^\infty(0,\infty;H^{-1}_{(0)}(\Omega)) \cap L^2_{\mathrm{uloc}}(0,\infty;H^1(\Omega))$$
$$\kappa c \in L^\infty(0,\infty;W^2_r(\Omega)),\ \kappa\phi(c) \in L^\infty(0,\infty;L^r(\Omega)),\ \kappa\mu \in L^\infty(0,\infty;H^1(\Omega)).$$

2. *If $v \in B^\alpha_{\frac{4}{3}\infty,\mathrm{uloc}}([0,\infty);H^s(\Omega)) \cap BC_w(0,\infty;L^2_\sigma(\Omega))$ for some $-\frac{1}{2} < s \leq 0$ and $\alpha \in (0,1)$, then*

$$\kappa c \in C^\alpha([0,\infty);H^{-1}_{(0)}(\Omega)) \cap B^\alpha_{2\infty,\mathrm{uloc}}([0,\infty);H^1(\Omega)). \tag{3.17}$$

Finally, the same statements hold true if $[0,\infty)$ is replaced by $[0,T)$, $T<\infty$,

Remark 3.3 We note that the second part of the latter lemma is essential to obtain that any weak solution satisfies $c \in BUC([\delta,\infty);W^1_q(\Omega))$ for some $q>d$ and all $\delta>0$ in the proof of Theorem 1.4 below. Moreover, it is essential that $v(t)$ has some suitable positive regularity in time with values in $H^s(\Omega)$, $-\frac{1}{2} < s \leq 0$. In the latter case $C_0^\infty(\Omega)$ is dense in $H^s(\Omega)$ and $H^{-s}(\Omega)$ and therefore $v(t)\cdot\nabla c(t)$ is well defined for $v \in H^{-s}(\Omega)$. By (1.1) and the energy estimate one obtains directly $\partial_t v \in L^{\frac{4}{3}}_{\mathrm{uloc}}(0,\infty;H^{-1}(\Omega))$; but $\partial_t v \cdot \nabla c(t)$ is not well-defined if only $\partial_t v(t) \in H^{-1}(\Omega)$ since $\nabla c(t) \notin H^1_0(\Omega)$ in general.

Proof: Let $\partial_t^h f(t) = \frac{1}{h}(f(t+h) - f(t))$, $h > 0$. First of all, we note that, if $c_0 \in \mathcal{D}(\partial \widetilde{E}_0)$, then [8, Lemme 3.1, Chapter III] with $u(t) = c(t)$, $v(t) \equiv c_0$, $f(t) = -B(v(t))c(t)$, $g(t) \equiv \partial \widetilde{E}_0(c_0)$ implies

$$\begin{aligned} \|\partial_t^h c(0)\|_{H^{-1}_{(0)}} &\leq \frac{1}{h}\int_0^h \|\mathcal{B}(v)c - \partial\widetilde{E}_0(c_0)\|_{H^{-1}_{(0)}}\,d\tau \leq \|\mathcal{B}(v)c - \partial\widetilde{E}_0(c_0)\|_{L^\infty(0,\infty;H^{-1}_{(0)})} \\ &\leq C\left(\|(v,\nabla c)\|_{L^\infty(0,\infty;L^2)} + \|\partial\widetilde{E}_0(c_0)\|_{H^{-1}_{(0)}}\right) \qquad (3.18) \end{aligned}$$

Next let $\omega_t(\tau) \equiv 1$ or $\omega_t(\tau) = \tau - t$ and let $\partial_t v \in L^1_{\text{uloc}}(0,\infty;L^2(\Omega))$. In the case $\omega_t \equiv 1$ we use (3.16) with $c_1(t) = \frac{1}{h}c(t+h)$, $c_2 = \frac{1}{h}c(t)$, $h > 0$, $v_1(t) = \frac{1}{h}v(t+h)$, and $v_2(t) = \frac{1}{h}v(t)$. Hence $\tilde{c}(t) = \frac{1}{h}(c(t+h) - c(t)) = \partial_t^h c(t)$, $w = \partial_t^h v$, and

$$\begin{aligned} &\sup_{t\leq\tau\leq t+1} \omega_t(\tau)\|\partial_t^h c(\tau)\|^2_{H^{-1}_{(0)}} + \int_t^{t+1} \omega_t(\tau)\|\nabla\partial_t^h c(\tau)\|_2^2\,d\tau \\ &\quad\leq C(c_0,v)\left(\omega_t(\tau)\|\partial_t^h c(t)\|^2_{H^{-1}_{(0)}} + \|\partial_t^h v\|^2_{L^1(t,t+1;L^2)}\right) \qquad (3.19) \end{aligned}$$

for all $t \geq 0$ and $\omega_t(\tau) \equiv 1$. In the case $\omega_t(\tau) = \tau - t$ the proof of (3.16) can be easily modified to derive the latter inequality in this case. More precisely, one gets an additional term $\int_s^t |\partial_t \omega_t|\|c\|^2_{H^{-1}_{(0)}}\,d\tau = \int_s^t \|c\|^2_{H^{-1}_{(0)}}\,d\tau$, which can be estimated by the same quantities as in the case $\omega \equiv 1$.

Since $\partial_t^h c \to_{h\to 0} \partial_t c$ in $L^2(0,\infty;H^{-1}_{(0)}(\Omega))$, $\|\partial_t^h v\|_{L^1_{\text{uloc}}(0,T;L^2)} \leq \|\partial_t v\|_{L^1_{\text{uloc}}(0,T;L^2)}$, (3.19) with $\omega_t(t) = t - \tau$ yields $\kappa\partial_t c \in L^\infty(0,\infty;H^1_{(0)}(\Omega)) \cap L^2_{\text{uloc}}([0,\infty);H^1(\Omega))$ for $\kappa = \left(\frac{t}{1+t}\right)^{\frac{1}{2}}$. If $c_0 \in \mathcal{D}(\widetilde{E}_0)$, we can use (3.19) with $t = 0$ and $\omega_t(\tau) \equiv 1$ to conclude $\partial_t c \in L^\infty(0,\infty;H^{-1}_{(0)}) \cap L^2_{\text{uloc}}([0,\infty);H^1(\Omega))$ in this case. In both cases we can conclude further that

$$\kappa\Delta_N\mu = \kappa\partial_t c + \kappa v\cdot\nabla c \in L^\infty(0,\infty;H^{-1}_{(0)}(\Omega))$$

due to (3.8), which implies the $\kappa\nabla\mu \in L^\infty(0,\infty;L^2(\Omega))$. Because of (3.12), this shows $\kappa\partial\widetilde{E}_0 \in L^\infty(0,\infty;H^{-1}_{(0)}(\Omega))$. Using Corollary 2.5 we conclude $\kappa\phi_0(c), \kappa\nabla^2 c \in L^\infty(0,\infty;L^r(\Omega))$. Finally, $\kappa\mu \in L^\infty(0,\infty;L^2(\Omega))$ because of (3.2).

Finally, let $v \in B^\alpha_{\frac{4}{3}\infty,\text{uloc}}([0,\infty);H^{-s}(\Omega))$ for some $-\frac{1}{2} < s \leq 0$ and $\alpha \in (0,1)$. Then one can derive similarly

$$\begin{aligned} &\sup_{t\leq\tau\leq t+1, 0<h\leq 1} h^{-2\alpha}\omega_t(\tau)\|\Delta_h c(\tau)\|^2_{H^{-1}_{(0)}} + \sup_{0<h\leq 1}\int_t^{t+1} \omega_t(\tau)\|\nabla\Delta_h c(\tau)\|_2^2\,d\tau \\ &\quad\leq C(c_0,v)\left(\sup_{0<h\leq 1}\omega_t(t)h^{-2\alpha}\|\Delta_h c(t)\|^2_{H^{-1}_{(0)}} + 1\right) \qquad (3.20) \end{aligned}$$

for all $t \geq 0$. More precisely, one chooses $c_1(t) = h^{-\alpha}c(t+h)$, $c_2 = h^{-\alpha}c(t)$, $h > 0$, $v_1(t) = h^{-\alpha}v(t+h)$, and $v_2(t) = h^{-\alpha}v(t)$ in the proof of (3.16). Then $\tilde{c} = h^{-\alpha}\Delta_h c$

and $w = h^{-\alpha}\Delta_h v$ and the proof is done in the same way as for (3.16) (with ω_t added) except that one uses instead of (3.15) the estimate

$$\begin{aligned}
&\sup_{0<h\leq 1} h^{-2\alpha}\left|\int_t^{t+1}\omega_t(\tau)(\Delta_h v, \nabla c(-\Delta_N)^{-1}\Delta_h c)_\Omega\, d\tau\right| \\
&\leq C\|v\|_{B^\alpha_{\frac{4}{3}\infty,\mathrm{uloc}}([0,\infty);H^{-s})} \sup_{0<h\leq 1} h^{-\alpha}\|\omega_t^{\frac12}\nabla c(-\Delta_N)^{-1}\Delta_h c\|_{L^4(t,t+1;H^s)} \\
&\leq C(c_0,v)\|\nabla c\|_{L^4_{\mathrm{uloc}}([0,\infty);B^{\frac12}_{33})} \sup_{0<h\leq 1} h^{-\alpha}\|\omega_t^{\frac12}(-\Delta_N)^{-1}\Delta_h c\|_{L^\infty(t,t+1;H^1)} \\
&\leq C(c_0,v) \sup_{t\leq\tau\leq t+1, 0<h\leq 1} h^{-\alpha}\omega_t(\tau)^{\frac12}\|\Delta_h c(\tau)\|_{H^{-1}_{(0)}}
\end{aligned}$$

for all $t \geq 0$. Here we have used (2.5), $\|fg\|_{H^s} \leq C\|f\|_{B^{\frac12}_{33}}\|g\|_{H^1}$ for $0 < s < \frac12$, cf. e.g. [17, Theorems 6.6 and 7.2], and $\|\nabla c\|_{L^4(t,t+1;B^{\frac12}_{33})} \leq C\|\nabla c\|^{\frac12}_{L^\infty(0,\infty;L^2)}\|c\|^{\frac12}_{L^2(t,t+1;W^2_6)} \leq C(c_0,v)$ due to (2.10) and (3.11).

Choosing $\omega_t(\tau) = t-\tau$ in (3.20) yields (3.17) if $\kappa(t) = \left(\frac{t}{1+t}\right)^{\frac12}$. If $c_0 \in \mathcal{D}(\partial\widetilde{E}_0)$, then (3.18) implies

$$\sup_{0<h\leq 1} h^{-2\alpha}\|\Delta_h c(t)\|^2_{H^{-1}_{(0)}} \leq \sup_{0<h\leq 1} h^{2-2\alpha}\|\partial_t^h c(t)\|^2_{H^{-1}_{(0)}} < \infty.$$

Hence we use (3.20) with $t = 0$ and $\omega_t(\tau) \equiv 1$ to conclude that (3.17) holds also with $\kappa \equiv 1$.

Finally, if $[0,\infty)$ is replaced by $[0,T)$, $T < \infty$, one simply extends v for $t \geq T$ suitably (e.g. $v(t) = \psi(t)v(T)$ with $\psi(T) = 1$ and $\psi \in C_0^\infty(\mathbb{R})$)) and applies the first part. ∎

3.2 Lojasiewicz-Simon Inequality

First we consider solutions $c_\infty \in \mathcal{D}(\partial E_0)$ of the stationary Cahn-Hilliard equation (1.18)-(1.20), which are the critical points of the functional $E_1(c)$ on $H^{-1}_{(0)}(\Omega)$. Here E_0 denotes the convex part of E_1 as defined in (2.21).

Proposition 3.4 *Let $c_\infty \in \mathcal{D}(\partial E_0)$ be a solution of (1.18)-(1.20). Then there are constants M_j, $j = 1, 2$, such that*

$$a < M_1 \leq c_\infty(x) \leq M_2 < b \qquad \textit{for all } x \in \overline{\Omega}. \tag{3.21}$$

A proof of the proposition can be found in [3, Proposition 6.1].

Because of (3.21), one can replace the singular Φ in $E_1(c)$ by a smooth and bounded $\widetilde{\Phi}$ such that $\widetilde{\Phi}|_{[M_1,M_2]} = \Phi|_{[M_1,M_2]}$ for all c. Let $\widetilde{E}_1$ denote the corresponding functional. Therefore one can prove the following Lojasiewicz-Simon gradient inequality, which is the main tool to prove convergence to stationary solutions.

Proposition 3.5 (Lojasiewicz-Simon inequality) *Let $c' \in \mathcal{D}(\partial E_0)$ be a solution of (1.18)-(1.20) and let $\widetilde{E}_1$ be defined as above for some $a < M_1 < M_2 < b$. Then there exist constants $\theta \in (0, \frac{1}{2}], C, \delta > 0$ such that*

$$|\widetilde{E}_1(c) - \widetilde{E}_1(c')|^{1-\theta} \leq C\|D\widetilde{E}_1(c)\|_{H^{-1}_{(0)}} \tag{3.22}$$

for all $\|c-c'\|_{H^1_{(0)}} \leq \delta$, where $D\widetilde{E}_1\colon H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ denotes the Frechét derivative of $\widetilde{E}_1\colon H^1_{(0)}(\Omega) \to \mathbb{R}$.

The proposition follows from [3, Proposition 6.1].

Finally, we note that the only critical point of the quadratic energy E_2 is $v = 0$ and obviously,

$$|E_2(v) - E_2(0)|^{\frac{1}{2}} \leq C\|DE_2(v)\|_{L^2} \qquad \text{for all } v \in L^2_\sigma(\Omega)$$

since the Frechét derivative of $E_2\colon L^2_\sigma(\Omega) \to \mathbb{R}$ is $DE_2(v) = v$. I.e., the Lojasiewicz-Simon gradient inequality holds for E_2 as well. Hence under the same assumptions as in Proposition 3.5 there are constants $\theta \in (0, \frac{1}{2}], C, \delta > 0$ such that $\widetilde{E}(v,c) := \widetilde{E}_1(c) + E_2(v)$ satisfies

$$|\widetilde{E}(v,c) - \widetilde{E}(0,c')|^{1-\theta} \leq C\left(\|D\widetilde{E}_1(c)\|_{H^{-1}_{(0)}} + \|DE_2(v)\|_{L^2}\right) \tag{3.23}$$

for all $\|c - c'\|_{H^1_{(0)}} \leq \delta$, $v \in L^2_\sigma(\Omega)$.

4 Navier-Stokes System with Variable Viscosity

4.1 Stokes System with Variable Viscosity

We first consider the Stokes system

$$\begin{aligned} \partial_t v - \operatorname{div}(\nu(c)Dv) + \nabla p &= f && \text{in } \Omega \times (0,T), && (4.1)\\ \operatorname{div} v &= 0 && \text{in } \Omega \times (0,T), && (4.2)\\ v|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,T), && (4.3)\\ v|_{t=0} &= v_0 && \text{in } \Omega. && (4.4) \end{aligned}$$

for given $v_0 \in L^2_\sigma(\Omega), f \in L^2(0,T;V_2(\Omega)')$, $0 < T \leq \infty$ and measurable $c\colon Q_T \to \mathbb{R}$. Here $\nu \in C^2(\mathbb{R})$ such that $\nu(c) \geq c_0 > 0$ for all $c \in \mathbb{R}$.

As usual we call $v \in L^2(0,T;V_2(\Omega))$ a *weak solution* of (4.1)-(4.4) if

$$-(v, \partial_t\varphi)_{Q_T} + (v_0, \varphi|_{t=0})_\Omega + (\nu(c)Dv, D\varphi)_{Q_T} = \int_0^T \langle f(t), \varphi(t)\rangle_{V_2',V_2}\, dt \tag{4.5}$$

for all $\varphi \in C^\infty_{(0)}([0,T)\times\Omega)^d$ with $\operatorname{div}\varphi = 0$. Note that (4.5) implies $\partial_t v \in L^2(0,T;V_2(\Omega)')$ and therefore $c \in BUC(0,T;L^2_\sigma(\Omega))$ due to (2.8).

THEOREM 4.1 *Let* $v_0 \in L^2_\sigma(\Omega)$, $f \in L^2(0,T;V_2(\Omega)')$ *and let* $c\colon Q_T \to \mathbb{R}$ *be a measurable function, where* $0 < T \leq \infty$. *Then there is a unique solution* $v \in L^2(0,T;V_2(\Omega))$ *of (4.1)-(4.4) satisfying* $\partial_t v \in L^2(0,T;V_2(\Omega)')$ *and*

$$\|\partial_t v\|_{L^2(0,T;V_2')} + \|v\|_{L^2(0,T;V_2)} \leq C\left(\|v_0\|_2 + \|f\|_{L^2(0,T;V_2')}\right) \tag{4.6}$$

where $C = C(\nu,\Omega)$ *is independent of* f, v_0 *and* c. *Moreover, the mapping of* $(c,f,v_0) \in L^1_{\mathrm{loc}}(Q_T) \times L^2(0,T;V_2') \times L^2_\sigma(\Omega)$ *to* $v \in L^2(0,T;V_2) \cap H^1(0,T;V_2')$, $T < \infty$, *is strongly continuous.*

Proof: By the assumptions the operator

$$\langle A(c)v, \varphi\rangle_{V_2,V_2} = (\nu(c)Dv, D\varphi)_{L^2(\Omega)}, \qquad \varphi \in V_2(\Omega),$$

is a monotone mapping/positive linear operator $A(c)\colon V_2(\Omega) \to V_2(\Omega)'$ which satisfies all assumptions of [21, Proposition 4.1]. Hence the existence of a unique solution $v \in L^2(0,T;V_2(\Omega))$ with $\partial_t v \in L^2(0,T;V_2(\Omega)')$ follows directly form the latter proposition.

In order to derive (4.6), we choose $\varphi = v\chi_{[0,t]}$ in (4.5), use (2.8), and obtain

$$\sup_{0\leq t\leq T} \|v(t)\|_2^2 + \int_{Q_T} |Dv|^2 \, d(x,t) \leq C\left(\|f\|_{L^2(0,T;V_2')}\|Dv\|_{L^2(Q_T)} + \|v_0\|_2^2\right).$$

By Young's inequality we obtain the estimate of $v \in L^2(0,T;V_2(\Omega))$. The estimate of $\partial_t v \in L^2(0,T;V_2(\Omega)')$ follows from (4.5).

In order to prove the stated continuity, let $(c_j, f_j, v_0^j) \in L^1_{\mathrm{loc}}(Q_T) \times L^2(0,T;V_2(\Omega)') \times L^2_\sigma(\Omega)$, $j = 1,2$, let v_j be the corresponding weak solution of (4.1)-(4.4), and set $v := v_1 - v_2$, $v_0 = v_1^0 - v_2^0$. Then

$$\partial_t v(t) + A(c_1(t))v(t) = f_1(t) - f_2(t) - (A(c_1(t)) - A(c_2(t)))v_2(t) \quad \text{in } V_2(\Omega)'.$$

Hence taking the duality product with $v(t)$ and using (2.8) yields

$$\begin{aligned}
&\frac{1}{2}\|v(T)\|_2^2 + \int_{Q_T} \nu(c_1)|Dv|^2 \, d(x,t) \\
&= \int_0^T \langle f_1(t) - f_2(t), v(t)\rangle + \int_{Q_T} (\nu(c_1) - \nu(c_2))Dv_2 : Dv \, d(x,t) + \frac{1}{2}\|v_0\|_2^2 \\
&\leq C(R)\left(\|f_1 - f_2\|_{L^2(0,T;V_2')} + \|(\nu(c_1) - \nu(c_2))Dv_2\|_{L^2(Q_T)} + \frac{1}{2}\|v_0\|_2^2\right),
\end{aligned}$$

where $R = \max_{j=1,2}\|v_j\|_{L^2(0,T;V_2)}$. The first term on the right-hand side gets arbitrarily small if $f_1 - f_2$ is sufficiently small in $L^2(0,T;V_2(\Omega)')$. By Lemma 2.1 the second term converges to zero as c_1 converges to c_2 in $L^1_{\mathrm{loc}}(Q_T)$. Hence v_1 converges to v_2 in $L^2(0,T;V_2(\Omega))$ if (c_1, f_1, v_0^1) converge to (c_2, f_2, v_0^2) in $L^1_{\mathrm{loc}}(Q_T) \times L^2(0,T;V_2(\Omega)') \times L^2_\sigma(\Omega)$. The convergence of $\partial_t v_1$ in $L^2(0,T;V_2')$ follows from (4.5) and Lemma 2.1. ∎

Next we consider some results on higher regularity for the Stokes system (4.1)-(4.4), which are needed for the proof of Theorem 2.3. We start with the stationary system.

Lemma 4.2 *Let $\nu \in C^2(\mathbb{R})$ be as above, $c \in W_r^{1+j}(\Omega)$, $j = 0,1$, $r > d \geq 2$, with $\|c\|_{W_r^{1+j}} \leq R$, and let $v \in V_2(\Omega)$ be a solution of*

$$(\nu(c)Dv, D\varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \quad \text{for all } \varphi \in C^\infty_{0,\sigma}(\Omega), \tag{4.7}$$

where $f \in H^s(\Omega)^d$, $s \in [0, j]$. Then $v \in H^{2+s}(\Omega)^d$ and

$$\|v\|_{H^{2+s}(\Omega)} \leq C(R)\|f\|_{H^s(\Omega)}, \tag{4.8}$$

where $C(R)$ depends only on Ω, ν, $r > d$, and $R > 0$.

Proof: First of all, if $\nu(c) \equiv 1$, the statement follows from the well-known regularity theory for the stationary Stokes system with constant viscosity. More precisely, it is known that in this case for every $1 < p < \infty$, $f \in W_p^j(\Omega)$, and $j = 0,1$

$$\|v\|_{W_p^{2+j}} \leq C_p\|f\|_{W_p^j}, \qquad f \in W_p^j(\Omega), \tag{4.9}$$

cf. Galdi [14, Chapter IV, Lemma 6.1]. Moreover, there is a pressure $\pi \in W_p^{j+1}(\Omega) \cap L^p_{(0)}(\Omega)$ depending continuously on $f \in W_p^j(\Omega)$ such that

$$-\Delta v + \nabla\pi = f \qquad \text{in } \Omega. \tag{4.10}$$

Next let $s = j = 0$ and ν be as in the assumptions. Let $\varphi = \nu(c)^{-1}\psi - B[\operatorname{div}(\nu(c)^{-1}\psi)]$, where $\psi \in C^\infty_{0,\sigma}(\Omega)$ and B is the Bogovskii operator, cf. [14, Chapter III, Theorem 3.2]. – Note that

$$B\colon W^k_{p,0}(\Omega) \cap L^p_{(0)}(\Omega) \to W^{k+1}_{p,0}(\Omega), \qquad \operatorname{div} Bf = f, \tag{4.11}$$

for every $k \in \mathbb{N}_0$, $1 < p < \infty$. Hence $\|B[(\nabla\nu(c)^{-1})\psi]\|_{W_p^1} \leq C\|(\nabla\nu(c)^{-1})\psi\|_p$, $\varphi \in W^1_{p,0}(\Omega)$,$\operatorname{div}\varphi = 0$, and

$$\|\varphi\|_{W_p^1} \leq C(R)\|\psi\|_{W_p^1} \quad \text{for all } 1 < p \leq r,$$

where we have used (2.17). Then

$$\begin{aligned}(Dv, \nabla\psi)_\Omega &= (\nu(c)Dv, \nabla(\nu(c)^{-1}\psi))_\Omega - (\nu(c)Dv, (\nabla\nu(c)^{-1}) \otimes \psi)_\Omega \\ &= (\nu(c)^{-1}f, \psi)_\Omega - (\nu(c)Dv, (\nabla\nu(c)^{-1})\psi)_\Omega + (\nu(c)Dv, \nabla B[(\nabla\nu(c)^{-1}) \cdot \psi])_\Omega \\ &\quad -(f, B[(\nabla\nu(c)^{-1}) \cdot \psi])_\Omega \equiv (g, \psi)_\Omega,\end{aligned}$$

where $\|g\|_{s_0} \leq C\left(\|f\|_2 + \|Dv\|_2\right)(1 + \|\nabla c\|_r)$ with $\frac{1}{s_0} = \frac{1}{r} + \frac{1}{2}$. Thus $v \in W^2_{s_0}(\Omega)^d$ by (4.9) and therefore $v \in W^1_p(\Omega)^d$ with $\frac{1}{p} = \frac{1}{r} - \frac{1}{d} + \frac{1}{2}$. Since $r > d$, we conclude $p > 2$. Using this in the estimates above, we see that $g \in L^{\min(2,s_1)}(\Omega)$ with $\frac{1}{s_1} =$

$\frac{1}{r}+\frac{1}{p}$. If $s_1<2$, then we repeat this argument finitely many times to conclude that $g\in L^{\min(2,s_k)}(\Omega)$ with $\frac{1}{s_k}=\frac{1}{2}+\frac{1}{r}-k(\frac{1}{r}-\frac{1}{d})$ until $s_k\geq 2$. Hence $v\in H^2(\Omega)$ since $f\in L^2(\Omega)$. Moreover, (4.8) simply follows from the boundedness and linearity of the mapping $f\mapsto v$.

If $s=j=1$, then $v\in H^2(\Omega)$ by the first part. Hence there is some $\pi\in L^2(\Omega)$ such that (4.10) holds with f replaced by $g:=\nu^{-1}(c)f+(\nabla\nu(c))\cdot Dv\in W^1_{s_0}(\Omega)$. Using the same boot trapping argument it is easy to show that $g\in H^1(\Omega)$ by first showing that $g\in W^1_{s_j}(\Omega)$ for some increasing sequence s_j. Finally, the general case $s\in[0,1], j=1$, follows by interpolation. ∎

In particular, the latter lemma shows that the operator

$$A(c)\colon V_2^{2+j}\subset V_2^j(\Omega)\to V_2^j(\Omega)\colon v\mapsto A(c)v:=-P_2\operatorname{div}(2\nu(c)Dv),$$

where $j=0,1$,

$$V_2^{2+j}(\Omega)=\left\{u\in V_2^2(\Omega):A(c)u\in V_2^j(\Omega)\right\}$$

and $V_2^1(\Omega)=V_2(\Omega), V_2^0(\Omega)=L^2_\sigma(\Omega)$, is an invertible operator provided that $c\in W^{1+j}_r(\Omega)$, $r>d$. By interpolation one gets the same results for intermediate spaces $V_2^{2+s}(\Omega), V_2^s(\Omega)$, resp., where we define

$$V_2^{s+k}(\Omega)=(V_2^k(\Omega),V_2^{k+1}(\Omega))_{s,2}$$

for $s\in(0,1)$, $k=-1,0,1,2$ and where $V_2^{-1}(\Omega)=V_2(\Omega)'$. We will characterize the interpolation spaces above later.

For the following we denote

$$\Sigma_\delta=\{z\in\mathbb{C}\setminus\{0\}:|\arg z|<\delta\},\delta\in(0,\pi).$$

Lemma 4.3 *Let $c\in W^{1+j}_r(\Omega)$, $j=0,1$, $r>d=2,3$ and $s\in[0,j]$. Then $-A(c)\colon V_2^{s+2}(\Omega)\subset V_2^s(\Omega)\to V_2^s(\Omega)$ defined as above is a sectorial operator such that $\Sigma_\delta\subset\rho(-A(c))$ for every $0<\delta<\pi$. Moreover, there is a constant C_δ such that*

$$\|(\lambda+A(c))^{-1}\|_{\mathcal{L}(V_2^s(\Omega))}\leq\frac{C_\delta}{|\lambda|}\qquad\text{for every }\lambda\in\Sigma_\delta. \tag{4.12}$$

Here C_δ depends only on $\|c\|_{W^{1+j}_r}$, Ω, and ν.

Proof: By the definition of the spaces $V_2^s(\Omega), V_2^{s+2}(\Omega)$, it is sufficient to consider only the cases $s=0,1$ since the general case follows by interpolation. If $s=0$, then obviously A is a symmetric, positive operator on $L^2_\sigma(\Omega)$ due to

$$(A(c)u,v)_{L^2(\Omega)}=(2\nu(c)Du,Dv)_{L^2(\Omega)}=(u,A(c)v)_{L^2(\Omega)}$$

for all $u,v\in\mathcal{D}(A(c))=V_2^2(\Omega)$. Moreover, $A(c)$ is invertible because of Lemma 4.2. Hence $(A(c)u,u)\in(0,\infty)$ and the statement of the lemma follows e.g. from [13, Corollary 4.8], see also the remark after Corollary 4.8.

If $s = 1$, then $A(c)$ is again a symmetric, positive operator on $V_2^1(\Omega)$ if we equip $V_2^1(\Omega)$ with the inner product $(u,v)_{V_2(\Omega)} = (2\nu(c)Du, Dv)_{L^2(\Omega)}$ for $u, v \in V_2(\Omega)$. Then

$$\begin{aligned}(A(c)u,v)_{V_2(\Omega)} &= -(2\nu(c)\nabla A(c)u, Dv)_{L^2(\Omega)}\\ &= (P_2 \operatorname{div}(2\nu(c)Du), \operatorname{div}(2\nu(c)Dv))_{L^2(\Omega)} = (u, A(c)v)_{V_2(\Omega)}\end{aligned}$$

for all $u, v \in V_2^3(\Omega)$ since $P_2 \operatorname{div}(2\nu(c)Du)|_{\partial\Omega} = 0$. Moreover, $A(c)\colon V_2^3(\Omega) \to V_2^1(\Omega)$ is invertible because of Lemma 4.2. Therefore the statement follows again from [13, Corollary 4.8]. ∎

We need the following characterization of the interpolation spaces:

Lemma 4.4 *Let $\theta \in (0,1)$ with $\theta \neq \frac{3}{4}$. Then*

$$(V_2^{-1}(\Omega), V_2^1(\Omega))_{\theta,2} = \begin{cases} H_\sigma^{2\theta-1}(\Omega) & \text{if } 2\theta - 1 < \frac{1}{2}\\ H_\sigma^{2\theta-1}(\Omega) \cap H_0^{2\theta-1}(\Omega)^d & \text{if } 2\theta-1 > \frac{1}{2}\end{cases} \tag{4.13}$$

$$(V_2^1(\Omega), V_2^3(\Omega))_{\theta,2} = \begin{cases} H_\sigma^{1+2\theta}(\Omega) \cap H_0^1(\Omega)^d & \text{if } 1+2\theta < \frac{5}{2}\\ H_\sigma^{1+2\theta}(\Omega) \cap \{u|_{\partial\Omega} = Au|_{\partial\Omega} = 0\} & \text{if } 1+2\theta > \frac{5}{2}\end{cases} \tag{4.14}$$

Finally, if $0 \le \theta \le 1$ with $2\theta \neq \frac{1}{2}$, then

$$(L_\sigma^2(\Omega), V_2^2(\Omega))_{\theta,2} = V_2^{2\theta}(\Omega). \tag{4.15}$$

Proof: First we show that

$$(L_\sigma^2(\Omega), V_2(\Omega))_{\theta,2} = \begin{cases} H_\sigma^\theta(\Omega) \cap H_0^\theta(\Omega)^d & \text{if } \theta > \frac{1}{2},\\ H_\sigma^\theta(\Omega) & \text{if } \theta < \frac{1}{2}.\end{cases} \tag{4.16}$$

Then (4.13) follows by duality and reiteration, cf. [6, Theorem 3.7.1 and Theorems 3.5.3/3.5.4], where we note that

$$(V_2^{-1}(\Omega), V_2^1(\Omega))_{\frac{1}{2},2} = L_\sigma^2(\Omega) = V_2^0(\Omega)$$

follows from Theorem 4.1 together with (2.6) and (2.7). In order to prove (4.16), we define a projection $P\colon L^2(\Omega)^d \to L_\sigma^2(\Omega)$ by $v = Pu$ if and only if

$$(v, \Delta\varphi)_{L^2(\Omega)} = (u, \Delta\varphi)_{L^2(\Omega)} \qquad \text{for all } \varphi \in H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega) = V_2^2(\Omega).$$

Since $-P_2\Delta\colon V_2^2(\Omega) \to L_\sigma^2(\Omega)$ is invertible, the latter condition defines a unique $v = Pu \in L_\sigma^2(\Omega)$ and $v = u$ if $u \in L_\sigma^2(\Omega)$. Moreover, if $u \in H^1(\Omega)^d$, then there is a unique solution $v \in V_2^1(\Omega)$ of the weak Stokes equation

$$(\nabla v, \nabla\varphi) = (\nabla u, \nabla\varphi) = -(u, \Delta\varphi) \qquad \text{for all } \varphi \in C_{0,\sigma}^\infty(\Omega),$$

which coincides with Pu. Thus $P\colon H^1(\Omega)^d \to V_2^1(\Omega)$. Therefore a general theorem on interpolation spaces and retracts, cf. [29, Section 1.2.4, Theorem], yields

$$(L^2_\sigma(\Omega), V_2^1(\Omega))_{\theta,2} = P(L^2(\Omega)^d, H^1(\Omega)^d)_{\theta,2} = PH^\theta(\Omega)^d.$$

By interpolation $P\colon H^\theta(\Omega) \to H_0^\theta(\Omega)^d \cap H_\sigma^\theta(\Omega)$ for all $\theta \neq \frac{1}{2}$, where $H_0^\theta(\Omega) = H^\theta(\Omega)$ if $0 \leq \theta < \frac{1}{2}$. Hence $PH^\theta(\Omega)^d \subseteq H_\sigma^\theta(\Omega) \cap H_0^\theta(\Omega)^d$. Conversely, P is the identity on $H_\sigma^\theta(\Omega) \cap H_0^\theta(\Omega)^d \subseteq L^2_\sigma(\Omega)$ and therefore $PH^\theta(\Omega)^d = H_\sigma^\theta(\Omega) \cap H_0^\theta(\Omega)^d$ where $H_0^\theta(\Omega) = H^\theta(\Omega)$ if $0 \leq \theta < \frac{1}{2}$. This shows (4.16).

In order to prove (4.14), we use as before that $A\colon V_2^{1+2\theta}(\Omega) \to V_2^{2\theta-1}(\Omega)$ is an isomorphism for $\theta = 0, 1$. Hence

$$(V_2^1(\Omega), V_2^3(\Omega))_{\theta,2} = A^{-1}(V_2^{-1}(\Omega), V_2^1(\Omega))_{\theta,2} = A^{-1}V_2^{1-2\theta}(\Omega).$$

Moreover, because of (4.13) and Lemma 4.2, $u \in A^{-1}V_2^{1-2\theta}(\Omega)$ if and only if $u \in H_\sigma^{1+2\theta}(\Omega) \cap H_0^1(\Omega)$ and $Au = 0$ if $1 + 2\theta > \frac{1}{2}$.

Finally, by (4.13)-(4.14) and the reiteration theorems, cf. [6, Theorems 3.5.3/3.5.4], it only remains to prove $(L^2_\sigma(\Omega), V_2^2(\Omega))_{\frac{1}{2},2} = V_2^1(\Omega)$ to conclude (4.15). To this end, we use that A is an invertible and symmetric operator on $L^2_\sigma(\Omega)$. In particular, A is a monotone operator on $L^2_\sigma(\Omega)$ and

$$(Au, u)_{L^2(\Omega)} = (2\nu(c)Du, Du)_\Omega.$$

Hence A is the L^2_σ-subgradient of $\varphi(u) = (\nu(c)Du, Du)_\Omega$ and we can e.g. use [8, Theoreme 3.6, Chapter II] to conclude that for every $u_0 \in V_2(\Omega)$ there is some $u \in H^1(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; V_2^2(\Omega))$ such that $u|_{t=0} = u_0$. More precisely, u is determined as solution of the evolution equation

$$\frac{du}{dt} + Au = 0 \qquad \text{for } t > 0, \qquad u|_{t=0} = u_0.$$

Thus $(L^2_\sigma(\Omega), V_2^2(\Omega))_{\frac{1}{2},2} \supseteq V_2^1(\Omega)$. But the converse inclusion holds since for every $u \in H^1(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; V_2^2(\Omega))$ we obviously have $u_0 = u|_{t=0} \in H^1_\sigma(\Omega)$ and $u_0|_{\partial\Omega} = 0$ because of $(L^2(\Omega), H^2(\Omega))_{\frac{1}{2},2} = H^1(\Omega)$ and (2.6). ∎

Proposition 4.5 *Let $c \in BUC([0,\infty); W_r^1(\Omega))$ with $r > d \geq 2$, let $0 < T \leq \infty$, and let $2 \leq q < \infty$. If $f \in L^q(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; V_2(\Omega)')$, $v_0 \in (L^2_\sigma(\Omega), V_2^2(\Omega))_{1-\frac{1}{q},q}$ and v is the weak solution of (4.1)-(4.4), then $v \in W_q^1(0,T; L^2_\sigma(\Omega)) \cap L^q(0,T; V_2^2(\Omega))$ and*

$$\|(\partial_t v, \nabla^2 v)\|_{L^q(0,T;L^2)} \leq C_q \left(\|f\|_{L^q(0,T;L^2)\cap L^2(0,T;V_2')} + \|v_0\|_{(L^2_\sigma(\Omega), V_2^2(\Omega))_{1-\frac{1}{q},q}} \right) \tag{4.17}$$

where C_q is independent of $0 < T \leq \infty$.

If additionally, $c \in BUC(0,\infty;W_r^2(\Omega))$*,* $f \in L^2(0,T;V_2^s(\Omega))$*,* $v_0 \in (V_2^s(\Omega),V_2^{2+s}(\Omega))_{\frac12,2}$ *for some* $s \in [0,\frac12)$*, then the weak solution* v *of (4.1)-(4.4) satisfies* $v \in H^1(0,T;V_2^s(\Omega)) \cap L^2(0,T;V_2^{s+2}(\Omega))$ *and*

$$\|(\partial_t v, A(c)v)\|_{L^2(0,T;V_2^s)} \leq C\left(\|f\|_{L^2(0,T;V_2^s)} + \|v_0\|_{(V_2^s(\Omega),V_2^{2+s}(\Omega))_{\frac12,2}}\right) \tag{4.18}$$

where C *is independent of* $0 < T \leq \infty$*.*

Proof: First of all, it is sufficient to consider the case $T=\infty$ since the case $T<\infty$ can be reduced to that case by extending $f(t)$ by 0 for $t \geq T$.

Firstly, we consider the case that $c(t) = c_0 \in W_r^{1+j}(\Omega)$ with $r > d$, is constant in time. We can easily reduce to the case $v_0 = 0$ since for every $v_0 \in (X_0,X_1)_{1-\frac1q,q}$ there is some $w \in W_q^1(0,\infty;X_0) \cap L^q(0,\infty;X_1)$ with $w|_{t=0} = v_0$ and the norm of w is bounded by a constant times the norm of v_0 in $(X_0,X_1)_{1-\frac1q,q}$ because of (2.7). Then the statement of the theorem follows from Lemma 4.3 and [10, Theorem 4.4], part "(ii) implies (i)", where we note that $\mathcal{R}$-boundedness of an operator family on a Hilbert space coincides with uniform boundedness, cf. [10, Section 3.1], where the constant in (4.17)-(4.18) can be chosen to depend only on some $R > 0$ with $\|c_0\|_{W_r^{1+j}(\Omega)} \leq R$ since the constant in (4.12) depends only on $\|c_0\|_{W_r^{1+j}(\Omega)}$. Here we note that $V_2^2(\Omega) \hookrightarrow L_\sigma^2(\Omega)$ densely since $C_{0,\sigma}^\infty(\Omega) \subset V_2^2(\Omega)$ is dense in $L_\sigma^2(\Omega)$. Moreover, $V_2^3(\Omega) \hookrightarrow V_2^1(\Omega)$ densely since $A(c_0)\colon V_2^{2j+1}(\Omega) \to V_2^{2j-1}(\Omega)$, $j=0,1$, is an isomorphism and $V_2^1(\Omega) \hookrightarrow L_\sigma^2(\Omega) \hookrightarrow V_2^{-1}(\Omega)$ densely. In particular, this shows that for every $c_0 \in W_r^{1+j}(\Omega)$ the linear operator $\mathcal{L}$ associated to (4.1)-(4.4) is a bijection

$$\begin{aligned}\mathcal{L}\colon L^q(0,\infty;V_2^{s+2}) \cap W_q^1(0,\infty;V_2^s) &\to L^2(0,\infty;V_2^s) \times (V_2^s(\Omega),V_2^{s+2}(\Omega))_{1-\frac1q,q}.\\ : v &\mapsto (\partial_t v - P_2(\operatorname{div}(2\nu(c)Dv), v|_{t=0}),\end{aligned}$$

where $q=2$ if $j=s=1$ and $1<q<\infty$ if $s=j=0$. Next we note that

$$\begin{aligned}\|(A(c)-A(c'))v\|_{L^2} &\leq C\left(\|(\nu(c)-\nu(c'))\nabla^2 v\|_{L^2} + \|\nabla(\nu(c)-\nu(c'))\nabla v\|_{L^2}\right)\\ &\leq C\left(\|c-c'\|_\infty\|v\|_{H^2} + \|c-c'\|_{W_r^1}\|v\|_{W_p^1}\right)\\ &\leq C\|c-c'\|_{W_r^1}\|v\|_{H^2} \qquad (4.19)\end{aligned}$$

for all $c,c' \in W_r^1(\Omega)$, where $\frac1p = \frac12 - \frac1r$. Moreover, if $j=1$, then

$$\begin{aligned}&\|(A(c)-A(c'))v\|_{H^1} \leq C\left(\|(\nu(c)-\nu(c'))\nabla^2 v\|_{H^1} + \|\nabla(\nu(c)-\nu(c'))\nabla v\|_{H^1}\right)\\ &\leq C\left(\|c-c'\|_\infty\|v\|_{H^3} + \|c-c'\|_{W_r^2}\|v\|_{W_p^2}\right) \leq C\|c-c'\|_{W_r^2}\|v\|_{H^3}\end{aligned}$$

for all $c,c' \in W_r^2(\Omega)$, where $\frac1p = \frac12 - \frac1r$. By interpolation we obtain

$$\|(A(c)-A(c'))v\|_{H^{1+s}(\Omega)} \leq C\|c-c'\|_{W_r^2(\Omega)}\|v\|_{H^{2+s}(\Omega)} \tag{4.20}$$

for all $0 \leq s \leq 1$ if $j = 1$. Because of (4.19), (4.20), Lemma 4.4, and a simple Neumann series argument, there is some $\varepsilon > 0$ such that for every $c \in L^\infty(0,\infty; W_r^{1+j}(\Omega))$ with $\sup_{t\geq 0} \|c(t) - c_0\|_{W_r^{1+j}(\Omega)} \leq \varepsilon$ the statement of the theorem holds true with a constant C in (4.17) depending only on $\|c_0\|_{W_r^{1+j}}$.

Now let $c \in BUC([0,\infty); W_r^{1+j}(\Omega))$ and let $\varepsilon > 0$ such that the latter statement holds for all c_0 with $\|c_0\|_{W_r^{1+j}} \leq \sup_{0\leq t<\infty} \|c(t)\|_{W_r^{1+j}}$. Since $c(t)$ is uniformly continuous, there is some $\delta > 0$ such that $\|c(t) - c(t')\|_{W_r^{1+j}} \leq \varepsilon$ for all $t, t' \geq 0$ with $|t - t'| \leq \delta$. In order to localize in time, let $t_k := \delta k$, $k \in \mathbb{N}_0 \cup \{-1\}$ and set $I_k = (t_{k-1}, t_{k+1})$, $k \in \mathbb{N}_0$. Then it is easy to construct a partition of unity of $[0,\infty)$ subordinated to I_k with the following properties: Let $\varphi_k(t) = \varphi_0(t - k\tau)$, $k \in \mathbb{Z}$, be such that $\operatorname{supp} \varphi_0 \subseteq [-\tau, \tau]$ and $\sum_{k=-\infty}^{\infty} \varphi_k(t) = 1$ for all $t \in \mathbb{R}$. Then $\sum_{k=-\infty}^{\infty} \varphi_k(t) = \sum_{k=0}^{\infty} \varphi_k(t) = 1$ for $t \geq 0$ and $\|\varphi_k\|_{C^1([0,\infty))} \leq \|\varphi_0\|_{C^1([0,\infty))}$ for all $k \in \mathbb{Z}$.

Now let $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; V_2(\Omega))$ be a weak solution of (4.1)-(4.4). Then $w_k := \varphi_k v$, $k \in \mathbb{N}_0$, is a solution of (4.1)-(4.4) with f replaced by $f_k = \varphi_k f + (\partial_t \varphi_k) v$ and v_0 replaced by 0 if $k \geq 1$. Since $f_k(t) = \partial_t v_k(t) = A(c(t)) v_k(t) = 0$ for $t \notin I_k$, $A(c(t))$ can be replaced by $A(c_k(t))$ with

$$c_k(t) = \begin{cases} c(t_{k-1}) & \text{if } 0 \leq t \leq t_{k-1}, \\ c(t) & \text{if } t_{k-1} < t < t_{k+1}, \\ c(t_{k+1}) & \text{if } t \geq t_{k+1}. \end{cases}$$

Since by construction $\sup_{0\leq t<\infty} \|c_k(t) - c(t_k)\|_{W_r^{1+j}} \leq \varepsilon$ for all $k \in \mathbb{N}_0$, we can apply the result proved so far to conclude that

$$\begin{aligned} \|w_k\|_{L^q(0,\infty;V_2^{s+2})} + \|\partial_t w_k\|_{L^q(0,\infty;V_2^s)} &\leq C_q \|f_k\|_{L^q(I_k;V_2^s)}, \quad k \geq 1, \\ \|w_0\|_{L^q(0,\infty;V_2^{s+2})} + \|\partial_t w_0\|_{L^q(0,\infty;V_2^s)} &\leq C_q \left(\|f_0\|_{L^q(0,\delta;V_2^s)} + \|v_0\|_{(V_2^s, V_2^{s+2})_{1-\frac{1}{q},q}} \right) \end{aligned}$$

where $2 \leq q < \infty$ if $s = j = 0$ and $q = 2$ if $0 < s < \frac{1}{2}$. Hence

$$\begin{aligned} &\|v\|_{L^q(0,\infty;V_2^{s+2})} + \|\partial_t v\|_{L^q(0,\infty;V_2^s)} \\ &\leq \sum_{k=0}^{\infty} \left(\|w_k\|_{L^q(0,\infty;V_2^{s+2})} + \|\partial_t w_k\|_{L^q(0,\infty;V_2^s)} \right) \\ &\leq C \left(\sum_{k=1}^{\infty} \|f_k\|_{L^2(I_k;V_2^s)} + \|f_0\|_{L^2(0,\delta;V_2^s)} + \|v_0\|_{(V_2^s,V_2^{s+2})_{1-\frac{1}{q},q}} \right) \\ &\leq C \left(\|f\|_{L^q(0,\infty;V_2^s)} + \|v\|_{L^q(0,\infty;V_2^s)} + \|v_0\|_{(V_2^s,V_2^{s+2})_{1-\frac{1}{q},q}} \right) \\ &\leq C \left(\|f\|_{L^q(0,\infty;V_2^s)} + \|f\|_{L^2(0,\infty;V_2')} + \|v_0\|_{(V_2^s,V_2^{s+2})_{1-\frac{1}{q},q}} \right) \end{aligned}$$

by (4.6) where $2 \leq q < \infty$ if $s = j = 0$ and $q = 2$ if $0 < s < \frac{1}{2}$. Here we have used that $v \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; H^1(\Omega)) \hookrightarrow L^q(0,\infty; L^2(\Omega))$ in the case

$s = 0, 2 \le q < \infty$ and $v \in L^2(0,\infty;V_2(\Omega)) \hookrightarrow L^2(0,\infty;V_2^s(\Omega))$ if $0 < s < \frac{1}{2}$. This proves the proposition. ■

Next we construct strong solutions to the associated Navier-Stokes system:

$$\partial_t v + v\cdot\nabla v - \operatorname{div}(\nu(c)Dv) + \nabla p = f \quad \text{in } \Omega\times(0,T), \tag{4.21}$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega\times(0,T), \tag{4.22}$$
$$v|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{4.23}$$
$$v|_{t=0} = v_0 \quad \text{in } \Omega. \tag{4.24}$$

for given c, v_0, f and suitable $0 < T \le \infty$.

THEOREM 4.6 *Let $c \in BUC(0,\infty;W_q^{1+j}(\Omega))$, $q > d$, $d = 2,3$, $j = 0,1$. Moreover, let $v_0 \in V_2^{s+1}(\Omega)$ with $s = 0$ if $j = 0$ and $0 \le s < \frac{1}{2}$ if $j = 1$ and let $f \in L^2(0,\infty;H^s_\sigma(\Omega)))$. Then there is some $T > 0$ and a unique solution $v \in L^2(0,T;V_2^{s+2}(\Omega))\cap H^1(0,T;H^s_\sigma(\Omega))$ of (4.21)-(4.24). Furthermore, there is some $\varepsilon_0 > 0$ such that, if $\|v_0\|_{V_2^{s+1}} + \|f\|_{L^2(0,\infty;H^s_\sigma)} \le \varepsilon_0$, then there is a unique solution $v \in L^2(0,T;V_2^{s+2}(\Omega))\cap H^1(0,T;H^s_\sigma(\Omega))$ of (4.21)-(4.24) with $T = \infty$.*

Proof: We prove the theorem with the aid of the contraction mapping principle. To this end we define a mapping

$$F\colon X_T := \left\{w \in L^2(0,T;V_2^{s+2}(\Omega))\cap H^1(0,T;V_2^s(\Omega)) : w|_{t=0} = v_0\right\} \to X_T$$

as follows: Given $u \in X_T$ let $v = F(u) \in X_T$ be the solution of (4.1)-(4.4) with f replaced by $f_u := f - u\cdot\nabla u$. Then

$$\|F(u_1) - F(u_2)\|_{X_T} \le C_0 \min(T^{\frac{1}{4}},1)\max_{j=1,2}\{\|u_j\|_{X_T}\}\|u_1-u_2\|_{X_T}$$

since

$$\begin{aligned}
&\|f_{u_1} - f_{u_2}\|_{L^2(0,T;H^s)} \\
&\le \|u_1-u_2\|_{L^\infty(0,T;H^{s+1})}\|u_1\|_{L^2(0,T;H^{s+\frac{3}{2}})} + \|u_2\|_{L^\infty(0,T;H^{s+1})}\|u_1-u_2\|_{L^2(0,T;H^{s+\frac{3}{2}})} \\
&\le C_1\min(T^{\frac{1}{4}},1)\max_{j=1,2}\{\|u_j\|_{X_T}\}\|u_1-u_2\|_{X_T},
\end{aligned}$$

if $d \le 3$. Here we have used (2.6), (2.18), and that $L^\infty(0,T;H^{s+1})\cap L^2(0,T;H^{s+2}) \hookrightarrow L^4(0,T;H^{s+\frac{3}{2}}))$. This implies

$$\|F(u)\|_{X_T} \le C_2\min(T^{\frac{1}{4}},1)\|u\|^2_{X_T} + C_3\left(\|f\|_{L^2(0,\infty;H^s)} + \|v_0\|_{V_2^{s+1}(\Omega)}\right),$$

where C_j, $j = 2,3$, are independent of $0 < T \le \infty$.

In order to prove the first part, let $R := 2C_3\left(\|f\|_{L^2(0,\infty;H^s_\sigma)} + \|v_0\|_{V_2^{s+1}(\Omega)}\right)$ and choose $0 < T \leq 1$ so small that $\max\{C_0, C_2\}T^{\frac{1}{4}}R \leq \frac{1}{2}$. Then

$$\begin{aligned}\|F(u)\|_{X_T} &\leq \frac{1}{2}\|u_j\|_{X_T} + \frac{1}{2}R \leq R\\ \|F(u_1) - F(u_2)\|_{X_T} &\leq C_0 T^{\frac{1}{4}} R\|u_1 - u_2\|_{X_T} \leq \frac{1}{2}\|u_1 - u_2\|_{X_T}\end{aligned}$$

if $\|(u, u_1, u_2)\|_{X_T} \leq R$. Hence $F\colon \overline{B_R(0)} \cap X_T \to \overline{B_R(0)} \cap X_T$ is a contraction and there is a unique fixed-point $u \in \overline{B_R(0)} \cap X_T$. Uniqueness of the solution among all $u \in X_T$ follows from Proposition 4.8 below.

To prove the second part, let $R := \frac{1}{2}\min(C_0^{-1}, C_2^{-1})$ and let $\varepsilon_0 := \frac{1}{2}C_3^{-1}R$. Then $F\colon \overline{B_R(0)} \cap X_\infty \to \overline{B_R(0)} \cap X_\infty$ is a contraction if $\|f\|_{L^2(0,\infty;H^s)} + \|v_0\|_{H^{s+1}} \leq \varepsilon_0$. Uniqueness of the solution in X_∞ follows again from Proposition 4.8. ∎

As for the usual Navier-Stokes equation and similarly to Definition 1.3 we call $v \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; V_2(\Omega))$ a weak solution of (4.21)-(4.24) if

$$\begin{aligned}&-(v, \partial_t\varphi)_{Q_T} - (v_0, \varphi(0))_\Omega\\ &\quad+(v \cdot \nabla v, \varphi)_{Q_T} + (\nu(c)Dv, D\varphi)_{Q_T} = \int_0^T \langle f(t), \varphi(t)\rangle_{V_2', V_2}\, dt \qquad (4.25)\end{aligned}$$

for all $\varphi \in C^\infty_{(0)}([0, T) \times \Omega)^d$ with $\operatorname{div} v = 0$. Here c is a measurable function, $v_0 \in L^2_\sigma(\Omega)$, $f \in L^2(0, T; V_2(\Omega)')$, and $0 < T \leq \infty$. Moreover, we require that a weak solution is in $BC_w(0, T; L^2_\sigma(\Omega))$ and satisfies the strong energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + (\nu(c)Dv, Dv)_{Q_{(s,t)}} \leq \frac{1}{2}\|v(s)\|_2^2 + (f, v)_{Q_{(s,t)}} \quad \text{for all } t \in [s, T) \qquad (4.26)$$

and for $s = 0$ and almost every $0 < s < T$.

Remark 4.7 We note that, because of the energy equality (3.9) a weak solution of (1.1)-(1.7) is a weak solution of (4.25) with $f = \mu\nabla c$ satisfying (4.26).

Proposition 4.8 (Uniqueness)
Let v, v' be two weak solutions of (4.21)-(4.24) on $(0, T)$ with the same data $(f, v_0) \in L^2(Q_T) \times V_2(\Omega)$ and c. If $\nabla v \in L^2(0, T; L^3(\Omega))$, then v' coincides with v.

Proof: Let v, v' be as in the assumptions and let $w = v - v'$. Since $v \cdot \nabla v \in L^2(0, T; L^{\frac{6}{5}}) \hookrightarrow L^2(0, T; H^{-1})$, $\partial_t v \in L^2(0, T; V_2')$. Thus we can use $w\chi_{[0,t]}$, $0 < t < T$, in (4.25) and (2.8) to conclude

$$\frac{1}{2}\|v(t)\|_2^2 - (\partial_t v, v')_{Q_t} + (\nu(c)Dv, Dw)_{Q_t} = \frac{1}{2}\|v_0\|_2^2 + (f, w)_{Q_t} - (v \cdot \nabla v, w)_{Q_t}.$$

On the other hand choosing $\varphi = v\chi_{[0,t]}$ in the equation for v' we conclude

$$-(v', \partial_t v)_{Q_t} + (v(t), v'(t))_\Omega + (\nu(c)Dv', Dv)_{Q_t} = \|v_0\|_2^2 + (f, v)_{Q_t} + (v', v' \cdot \nabla v)_{Q_t}.$$

Moreover, by the energy inequality for v'

$$\frac{1}{2}\|v'(t)\|_2^2 + (\nu(c)Dv', Dv')_{Q_t} \leq \frac{1}{2}\|v_0\|_2^2 + (f, v')_{Q_t}.$$

Hence

$$\frac{1}{2}\|w(t)\|_2^2 + (\nu(c)Dw, Dw)_{Q_t} \leq -(w \cdot \nabla v, w)_{Q_t} \leq \int_0^t \|w\|_2 \|w\|_{H^1} \|\nabla v\|_{L^3}\, d\tau.$$

Because of Young's inequality, we obtain

$$\frac{1}{2}\|w(t)\|_2^2 + (\nu(c)Dw, Dw)_{Q_t} \leq C \int_0^t \|w\|_2^2 \|\nabla v\|_{L^3}^2\, d\tau.$$

Thus Gronwall's inequality shows $w = v - v' \equiv 0$ since $\nabla v \in L^2(0, T; L^3(\Omega))$. ■

5 Existence of Weak Solutions

We prove existence of weak solutions with the aid of solutions to following approximative system: Let $\Psi_\varepsilon w = P_2 \psi_\varepsilon * w$, where $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$, $\varepsilon > 0$, is a usual smoothing kernel and w is extended by 0 outside of Ω. Then we consider

$$\partial_t v + \Psi_\varepsilon v \cdot \nabla v - \operatorname{div}(\nu(c)Dv) + \nabla p = -\operatorname{div}(\nabla c \otimes \nabla c) \quad \text{in } \Omega \times (0,T), \tag{5.1}$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega \times (0,T), \tag{5.2}$$
$$\partial_t c + v \cdot \nabla c = \Delta \mu \quad \text{in } \Omega \times (0,T), \tag{5.3}$$
$$\mu = \phi(c) - \Delta c \quad \text{in } \Omega \times (0,T), \tag{5.4}$$

where $0 < T < \infty$, together with the boundary and initial conditions

$$v|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{5.5}$$
$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{5.6}$$
$$(v, c)|_{t=0} = (v_0, c_0) \quad \text{in } \Omega. \tag{5.7}$$

The definition of a weak solution is completely analogous to Definition 1.3.

THEOREM 5.1 *Let $0 < T < \infty$, $\varepsilon > 0$, and let $c_0 \in \operatorname{dom} E_0, v_0 \in L^2_\sigma(\Omega)$. Then there is a weak solution (v, c) of (5.1)-(5.7), which satisfies $\nabla^2 c, \phi(c) \in L^2(0, \infty; L^r(\Omega))$, where $r = 6$ if $d = 3$ and $1 < r < \infty$ is arbitrary if $d = 2$. Moreover, (1.16) holds with equality for all $0 \leq s \leq t \leq T$ and c satisfies (3.10)-(3.11).*

Proof: Let $X := L^2(0,T;V_2(\Omega))\cap H^1(0,T;V_2(\Omega)')$. We define a mapping $F\colon X\to X$ as follows: Given $u\in X$, let c be the solution of (3.1)-(3.4) due to Theorem 3.1 with v replaced by u and c_0 as in the assumptions. Then $u\mapsto c$ is continuous from the strong topology of X to the weak topology of

$$Y = L^2(0,T;W^2_r(\Omega))\cap H^1(0,T;H^{-1}_{(0)}(\Omega))$$

as stated in Theorem 3.1. Moreover, $X\ni u\mapsto c\in Y\cap L^\infty(0,T;H^1(\Omega))$ is a bounded mapping and $Y\hookrightarrow\hookrightarrow L^2(0,T;C^1(\overline{\Omega}))$ by (2.1). Interpolation implies that $X\ni c\mapsto \nabla c\in L^4(Q_T)$ and $X\ni c\mapsto \nabla c\otimes\nabla c\in L^2(\Omega_T)$ are completely continuous mappings. Now let $v=F(u)$ be the solution of (4.1)-(4.4) with

$$f = f(u) = -\operatorname{div}(\nabla c\otimes\nabla c) - \Psi_\varepsilon u\cdot\nabla u.$$

Then the mapping $X\ni u\mapsto f(u)\in L^2(0,T;H^{-1}(\Omega))\hookrightarrow L^2(0,T;V_2(\Omega)')$ is completely continuous as well. Because of the continuity statement in Theorem 4.1 $F\colon X\to X$ is completely continuous. In order to apply the Leray-Schauder principle to F, cf. e.g. [25, Chapter II, Lemma 3.1.1], it only remains to show that there is some $R>0$ such that

$$\lambda F(u) = u \text{ for some } u\in X, \lambda\in[0,1] \quad\Rightarrow\quad \|u\|_X\le R.$$

Assume that $\lambda F(u)=u$ for some $u\in X$, $\lambda\in(0,1]$. (The case $\lambda=0$ is trivial). Hence $v=\lambda^{-1}u$ solves (4.1)-(4.4) with right-hand side $f(u)$ as above. Thus taking the L^2-scalar product of (4.1) and v we conclude that

$$\begin{aligned}\frac{1}{2}\|v(T)\|_2^2 &+ (\nu(c)Dv,Dv)_{Q_T}\\ = &\ \frac{1}{2}\|v_0\|_2^2 - \lambda^{-1}(\Psi_\varepsilon u\cdot\nabla u,u)_{Q_T} + (\nabla c\otimes\nabla c,\nabla v)_{Q_T} = \frac{1}{2}\|v_0\|_2^2 + \lambda^{-1}(\mu\nabla c,u)_{Q_T}\end{aligned}$$

where we have used (1.12). Combining this with (3.9) we obtain

$$\frac{1}{2}\|v(T)\|_2^2+\int_{Q_T}\nu(c)|Dv|^2\,d(x,\tau)+\frac{1}{\lambda}E_1(c(T))+\frac{1}{\lambda}\int_{Q_T}|\nabla\mu|^2\,d(x,\tau) = \frac{1}{2}\|v_0\|_2^2+\frac{1}{\lambda}E_1(c_0)$$

and therefore $\|u\|^2_{L^2(0,T;H^1)} = \lambda^2\|v\|^2_{L^2(0,T;H^1)}\le CE(v_0,c_0)$. Because of (3.10), (3.11), there is some $R>0$ such that $\|u\|_X\le\|F(u)\|_X\le C\|u\|_{L^2(0,T;H^1)}\le R$. Hence we can apply the Leray-Schauder principle to conclude the existence of a fixed point $v=F(v)$, $v\in X$. Finally, the energy identity is proved by same calculations as above with $\lambda=1$ and T replaced by $0<t<T$. ■

Proof of Theorem 1.4: It remains to consider the limit $\varepsilon\to 0$ in (5.1)-(5.7). To this end let $(v_\varepsilon,c_\varepsilon,\mu_\varepsilon)$, $\varepsilon>0$, denote the solution of (5.1)-(5.7) with $T=\frac{1}{\varepsilon}$ due to

Theorem 5.1. Because of (1.16), (3.10), (3.11), we can pass to a suitable subsequence $(v_k, c_k, \mu_k) \equiv (v_{\varepsilon_k}, c_{\varepsilon_k}, \mu_{\varepsilon_k})$, $\lim_{k\to\infty} \varepsilon_k = 0$ such that

$$\begin{array}{llll} v_k \rightharpoonup_{k\to\infty} v & \text{in } L^2(0,\infty; H^1(\Omega)), & v_k \rightharpoonup^*_{k\to\infty} v & \text{in } L^\infty(0,\infty; L^2(\Omega)), \\ c_k \rightharpoonup_{k\to\infty} c & \text{in } L^2_{\mathrm{loc}}(0,\infty; W^2_r(\Omega)), & \nabla\mu_k \rightharpoonup_{k\to\infty} \nabla\mu & \text{in } L^2(Q), \end{array}$$

where (v_k, c_k, μ_k) are extended by zero for $t > \frac{1}{\varepsilon_k}$ and r is as in Theorem 5.1. Since

$$\langle \partial_t v_k, \varphi\rangle = -(\nu(c_k) Dv_k, D\varphi)_Q - (\Psi_k v_k \cdot \nabla v_k, \varphi)_Q + (\nabla c_k \otimes \nabla c_k, \nabla\varphi)_Q,$$

for all $\varphi \in C_0^\infty(Q)^d$ with $\operatorname{div}\varphi = 0$, $\partial_t v_k$ is uniformly bounded in $L^2(0,T; H^{-m}_\sigma(\Omega))$, $m > \frac{d}{2}$, for all $0 < T < \infty$, where $H^{-m}_\sigma(\Omega) = (H^m_0(\Omega)^d \cap L^2_\sigma(\Omega))'$. Because of this and $v_k \in L^2(0,T; H^1(\Omega))$, (2.1) implies $v_k \to_{k\to\infty} v$ in $L^2(Q_T)$ for all $T > 0$ and a suitable subsequence. Thus

$$\Psi_k v_k \cdot \nabla v_k \rightharpoonup_{k\to\infty} v\cdot\nabla v \quad \text{in } L^1(Q).$$

Moreover, using (1.14) and (2.1) again $c_k \to_{k\to\infty} c$ strongly in $L^2(0,T; C^1(\overline{\Omega}))$ for every $0 < T < \infty$ and since (c_k) is bounded in $L^\infty(0,T; H^1(\Omega))$, $\nabla c_k \to_{k\to\infty} \nabla c$ in $L^4(Q_T)$ for every $0 < T < \infty$. In particular, $\nabla c_k \otimes \nabla c_k \to_{k\to\infty} \nabla c\otimes\nabla c$ in $L^2_{\mathrm{loc}}(Q)$.

Because of the continuous dependence of the solutions of (3.1)-(3.4), c is the solution of (3.1)-(3.4) with convective term $v\cdot\nabla c$. Furthermore, since (c_k) converges strongly in $L^2(0,T; C^1(\overline{\Omega}))$, we conclude

$$(\nu(c_k) Dv_k, D\varphi)_Q \to_{k\to\infty} (\nu(c) Dv, D\varphi)_Q$$

for all $\varphi \in C^\infty_{(0)}(\overline{Q})^d$. Hence (v, c, μ) solve (1.13)-(1.15), where we use again (1.12).

Furthermore, (1.16) holds for almost all $0 \le s \le t < \infty$ (including $s = 0$) because of the corresponding equality for (v_k, c_k) and by using the fact that $(v_k(s), \nabla c_k(s)) \to_{k\to\infty} (v(s), \nabla c(s))$ strongly in $L^2(\Omega)$ for almost every $s > 0$ as well as

$$\int_{Q_{(s,t)}} \nu(c)|Dv|^2 \, d(x,\tau) \le \liminf_{k\to\infty} \int_{Q_{(s,t)}} \nu(c_k)|Dv_k|^2 \, d(x,\tau)$$

for all $0 \le s \le t < \infty$ since $\nu(c_k)^{\frac12} Dv_k \rightharpoonup_{k\to\infty} \nu(c)^{\frac12} Dv$ in $L^2(Q)$. Using that $v, \nabla c \in BC_w(0,\infty; L^2(\Omega))$ and the weak lower semi-continuity of the L^2-norm one obtains (1.16) for almost all $s > 0$ and $s = 0$ and *all* $s \le t < \infty$.

Moreover, if $d = 2$, then $v \in L^4(0,\infty; H^{\frac12}(\Omega)) \hookrightarrow L^4(Q)$ by (2.2). Therefore $v\cdot\nabla v = \operatorname{div}(v\otimes v) \in L^2(0,\infty; V_2(\Omega)')$ and $\partial_t v \in L^2(0,\infty; V_2(\Omega)')$ because of (1.13) and $P_0\mu\nabla c \in L^2(0,\infty; L^{\frac32}(\Omega)) \hookrightarrow L^2(0,\infty; V_2(\Omega)')$. Hence using $\varphi = v\chi_{[0,t]}$ in (1.13) and using (2.8) we obtain

$$\frac12 \|v(t)\|_2^2 + (\nu(c)Dv, Dv)_{Q_t} = \frac12 \|v_0\|_2^2 + (\mu\nabla c, v)_{Q_t}$$

for all $t > 0$. Together with (3.9) this implies (1.16) with equality for all $t > 0$.

Finally, let (v, c, μ) be any weak solution. We have to show that there is some $q > 3$ such that $\kappa c \in BUC(0,\infty; W^1_q(\Omega))$, where $\kappa(t) \equiv 1$ if $c_0 \in H^2(\Omega)$ and $-\Delta c_0 + \phi(c_0) \in H^1(\Omega)$ and $\kappa(t) = \left(\frac{t}{1+t}\right)^2$ else. To this end we use that $v \in L^{\frac{8}{3}}(0,\infty; L^4(\Omega))$ and therefore $v \cdot \nabla v = \operatorname{div}(v \otimes v) \in L^{\frac{4}{3}}(0,\infty; H^{-1}(\Omega))$. Hence $\partial_t v \in L^{\frac{4}{3}}_{\mathrm{uloc}}([0,\infty); V_2^{-1}(\Omega))$ because of (1.13). This means

$$v \in W^1_{\frac{4}{3},\mathrm{uloc}}([0,\infty); V_2^{-1}(\Omega)) \cap L^2(0,\infty; V_2^1(\Omega)) \hookrightarrow B^s_{q\infty,\mathrm{uloc}}([0,\infty); V_2^{1-2s}(\Omega))$$

for every $0 < s < 1$ and $\frac{1}{q} = \frac{1-s}{2} + \frac{3s}{4}$ because of (2.11). Now let $s \in (\frac{2}{3}, \frac{3}{4})$. In particular this implies $-\frac{1}{2} < 1 - 2s < 0$ and therefore $V_2^{1-2s}(\Omega) = H_\sigma^{1-2s}(\Omega) \subset H^{1-2s}(\Omega)^d$ due to (4.13). Hence we can apply Lemma 3.2 to conclude that $\kappa c \in B^s_{2\infty,\mathrm{uloc}}([0,\infty); H^1(\Omega))$. Next we use that for $\theta \in (0,1)$

$$B^s_{2\infty,\mathrm{uloc}}([0,\infty); H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); W^2_6(\Omega)) \hookrightarrow B^{s\theta}_{2\infty,\mathrm{uloc}}([0,\infty); B^{2-\theta}_{pp}(\Omega)),$$

where $\frac{1}{p} = \frac{1}{6} + \frac{\theta}{3}$, cf. (2.10). Since $s > \frac{2}{3}$, there is some $\theta \in (\frac{1}{2s}, \frac{3}{4})$. Hence

$$B^{2-\theta}_{pp}(\Omega) \hookrightarrow W^1_q(\Omega) \qquad \text{with } \frac{1}{q} = -\frac{1}{6} + \frac{2\theta}{3} < \frac{1}{3}$$

because of Sobolev's embedding theorem, cf. [6, Theorem 6.5.1]. Thus

$$\kappa c \in B^{s\theta}_{2\infty,\mathrm{uloc}}([0,\infty); B^{2-\theta}_{pp}(\Omega)) \hookrightarrow BUC([0,\infty); W^1_q(\Omega))$$

because of $s\theta > \frac{1}{2}$ and the Sobolev embedding theorem for vector-valued Besov spaces, cf. [24, Corollary 26]. This finished the proof. ∎

6 Uniqueness and Regularity of Weak Solutions

Lemma 6.1 *Let $q = 3$ if $d = 3$ and let $q > 2$ if $d = 2$. If (v_j, c_j), $j = 1, 2$, are weak solutions of (1.1)-(1.7) on $(0,T)$, $0 < T \leq \infty$, with $\nabla v_2 \in L^\infty(0,T; L^q(\Omega))$ and $\nabla c_1, \nabla c_2 \in L^\infty(Q_T)$, then $(v_1, c_1) \equiv (v_2, c_2)$.*

Proof: It suffices to consider $T < \infty$. First of all, $\tilde{c} = c_1 - c_2$ solves

$$\partial_t \tilde{c} + \mathcal{A}(c_1) - \mathcal{A}(c_2) = -\alpha \Delta \tilde{c} - w \cdot \nabla c_1 - v_2 \cdot \nabla \tilde{c},$$

where $w = v_1 - v_2$. Hence multiplication with $\tilde{c}$ in $L^2(0,t; H^{-1}_{(0)}(\Omega))$ with $0 < t \leq T$ yields

$$\begin{aligned} &\frac{1}{2}\|\tilde{c}(t)\|^2_{H^{-1}_{(0)}} + \|\nabla \tilde{c}\|^2_{L^2(Q_t)} \\ &\leq C\left(\|\tilde{c}\|^2_{L^2(Q_t)} + \int_0^t \|w \cdot \nabla c_1\|_{H^{-1}_{(0)}} \|\tilde{c}\|_{H^{-1}_{(0)}}\, d\tau + \int_0^t \|v_2 \cdot \nabla \tilde{c}\|_{H^{-1}_{(0)}} \|\tilde{c}\|_{H^{-1}_{(0)}}\, d\tau\right) \\ &\leq C(v_2)\left(\|\tilde{c}\|_{L^2(0,t;H^1)} \|\tilde{c}\|_{L^2(0,t;H^{-1}_{(0)})} + \|w\|^2_{L^2(Q_t)} + \|\tilde{c}\|^2_{L^2(0,t;H^{-1}_{(0)})}\right) \end{aligned}$$

where we have used (2.13), (3.7), (3.8). Hence

$$\frac{1}{2}\|\tilde{c}(t)\|^2_{H^{-1}_{(0)}} + \|\nabla\tilde{c}\|^2_{L^2(Q_t)} \leq C(T, c_1, v_2)\|w\|^2_{L^2(Q_t)} \tag{6.1}$$

by Young's and Gronwall's inequality.

Because of Remark 4.7, one derives, by the same calculations as in Proposition 4.8, that

$$\frac{1}{2}\|w(T)\|_2^2 + (\nu(c_1)Dw, Dw)_{Q_T} \leq (w\cdot\nabla v_2, w)_{Q_T}$$
$$+((\nu(c_1)-\nu(c_2))Dv_2 : Dw)_{Q_T} + (\nabla c_1\otimes\nabla c_1 - \nabla c_2\otimes\nabla c_2, \nabla w)_{Q_T},$$

where we have used again (1.12). Since $\nabla c_j \in L^\infty(Q_T)$ and $\nabla v_2 \in L^\infty(0,T;L^q(\Omega))$,

$$\begin{aligned}
&\frac{1}{2}\|w(T)\|_2^2 + (\nu(c_1)Dw, Dw)_{Q_T}\\
&\leq \; C(c_1,c_2,v_2)\left(\|w\|_{L^2(Q_T)} + \|\nabla\tilde{c}\|_{L^2(Q_T)}\right)\|w\|_{L^2(0,T;H^1)}\\
&\leq \; C'(T,c_1,c_2,v_2)\|w\|_{L^2(Q_T)}\|w\|_{L^2(0,T;H^1)}
\end{aligned}$$

where we have used (6.1) and $\|\nu(c_1(t)) - \nu(c_2(t))\|_{L^r(\Omega)} \leq C\|\nabla\tilde{c}(t)\|_{L^2(\Omega)}$ with $r=6$ if $d=3$ and $\frac{1}{r}=\frac{1}{2}-\frac{1}{q}$ if $d=2$. Hence by Young's inequality

$$\frac{1}{2}\|w(T)\|_2^2 + \|w\|^2_{L^2(0,T;H^1)} \leq C(T,c_1,c_2,v_2)\int_0^T \|w(t)\|_2^2\,dt,$$

which implies $w\equiv 0$ by the lemma of Gronwall. Finally, by (6.1) $c_1\equiv c_2$ follows, which proves uniqueness. ■

Lemma 6.2 *Let $v_0 \in V_2^{1+s}(\Omega)$, $s\in(0,1]$, and let $c_0\in H^2_N(\Omega)$ such that $-\Delta c_0 + \phi_0(c_0) \in H^1(\Omega)$, and let $d=2$. Then every weak solution (v,c,μ) of (1.1)-(1.7) satisfies $c\in L^\infty(0,\infty;W^2_r(\Omega))$ for every $r<\infty$, $\partial_t c\in L^\infty(0,\infty;H^{-1}_{(0)}(\Omega))$, and*

$$v\in L^2(0,\infty;V_2^{2+s'}(\Omega))\cap H^1(0,\infty;V_2^{s'}(\Omega))\cap BUC([0,\infty);H^{1+s-\varepsilon}(\Omega))$$

for all $s'\in[0,\frac{1}{2})\cap[0,s]$ and all $\varepsilon>0$. In particular, the weak solution is unique.

Proof: First we show that the weak solution satisfies $v\in H^{1,2}(Q)$. First of all, because of Theorem 1.4, $c\in BUC([0,\infty);W^1_3(\Omega))$. Hence $P_0\mu\nabla c\in L^2(Q)$. Moreover, we can apply Theorem 4.6 and Proposition 4.8 to conclude that there is some $T>0$ such that the weak solution (v,c) satisfies $v\in H^{1,2}(Q_T)\hookrightarrow BUC([0,T];H^1(\Omega))$. Hence it suffices to prove that there is some $C=C(v_0,c)$ independent of T such that

$$\sup_{0\leq t\leq T}\|\nabla v(t)\|_2^2 + \|v\|^2_{H^{1,2}(Q_T)} \leq C(v_0,c).$$

If this is shown, we can apply again Theorem 4.6 and Proposition 4.8 to conclude that $v \in H^{1,2}(Q)$. To this end we apply Proposition 4.5 with $f = P_0\mu\nabla c - v \cdot \nabla v$ and obtain

$$\begin{aligned}\sup_{0\le t\le T} \|\nabla v(t)\|_2^2 + \|v\|^2_{L^2(0,T;H^2)} &\le C\left(\|v\|^2_{H^{1,2}(Q_T)} + \|\nabla v_0\|_2^2\right)\\ &\le C'(c)\left(\|v\cdot\nabla v\|^2_{L^2(Q_T)} + \|\nabla v_0\|_2^2 + \|P_0\mu\nabla c\|^2_{L^2(Q)}\right)\end{aligned}$$

with a constant C independent of T because of (2.9) and $C'(c)$ depending only on $\|c\|_{BUC([0,\infty);W^1_3(\Omega))}$. Moreover,

$$\begin{aligned}\int_0^T \|v\cdot\nabla v\|_2^2\,dt &\le \|v\|^2_{L^4(0,T;L^\infty)}\|\nabla v\|^2_{L^4(0,T;L^2)}\\ &\le C\|v\|_{L^\infty(0,T;L^2)}\|v\|_{L^2(0,T;H^2)}\|\nabla v\|^2_{L^4(0,T;L^2)}\\ &\le C(v_0,c_0)\|v\|_{L^2(0,T;H^2)}\|\nabla v\|^2_{L^4(0,T;L^2)}\end{aligned}$$

because of (1.16) and (2.16). Hence, applying Young's inequality, we obtain

$$\sup_{0\le t\le T} \|\nabla v(t)\|_2^2 + \|v\|^2_{H^{1,2}(Q_T)} \le C(c,v_0)\left(\int_0^T \|\nabla v\|_2^4\,dt + 1\right).$$

Thus Gronwall's inequality implies

$$\sup_{0\le t\le T} \|\nabla v(t)\|_2^2 + \|v\|^2_{H^{1,2}(Q_T)} \le C(v_0,c_0)\exp\left(C(v_0,c_0)\int_0^T \|\nabla v(t)\|_2^2\,dt\right) \le C'(v_0,c_0)$$

with $C'(v_0,c_0)$ independent of T. Hence $v \in H^{1,2}(Q)$.

Therefore Lemma 3.2 yields

$$\partial_t c \in L^\infty(0,\infty;H^{-1}_{(0)}(\Omega)),\ \mu \in L^\infty(0,\infty;H^1(\Omega)),\ c \in L^\infty(0,\infty;W^2_r(\Omega))$$

for every $1 < r < \infty$, which implies $\mu_0\nabla c \in L^2(0,\infty;H^1(\Omega))$. Hence we can apply Theorem 4.6 and Proposition 4.8 again to conclude that

$$v \in L^2(0,T;V_2^{2+s'}(\Omega)) \cap H^1(0,T;V_2^{s'}(\Omega)) \qquad \text{for all } 0 < T < \infty,$$

where $s' \in [0,\frac{1}{2}) \cap [0,s]$ is arbitrary. Moreover,

$$\begin{aligned}\|v\cdot\nabla v\|^2_{L^2(0,T;H^{s'})} &\le C\int_0^T \|v\|_{H^{s'+\frac{3}{2}}}\|v\|_{H^{s'+1}}\,dt \le C'\int_0^T \|v\|^{\frac{1}{2}}_{H^{s'+2}}\|v\|^{\frac{3}{2}}_{H^{s'+1}}\,dt\\ &\le C\|v\|^{\frac{1}{1}}_{L^2(0,T;H^{s'+2})}\|v\|^{\frac{3}{2}}_{L^2(0,T;H^2)} \le C'(c,v_0)\|v\|^{\frac{1}{2}}_{L^2(0,T;H^{s'+2})}\end{aligned}$$

by (2.18) with a constant independent of T. Therefore Proposition 4.5 yields

$$\|v\|_{L^2(0,T;H^{s'+2})} + \|\partial_t v\|_{L^2(0,T;H^{s'})} \le C(c,v_0)\left(\|v\|^{\frac{1}{4}}_{L^2(0,T;H^{s'+2})} + 1\right).$$

Hence $\|v\|_{L^2(0,T;H^{s'+2})} + \|\partial_t v\|_{L^2(0,T;H^{s'})}$ is uniformly bounded in $0 < T < \infty$ and therefore $v \in L^2(0,\infty; H^{s'+1}(\Omega)) \cap H^1(0,\infty; H^{s'}(\Omega))$, where still $s' \in [0,\frac{1}{2}) \cap [0,s]$ is arbitrary. Furthermore, we can use that $v \in BUC([0,\infty); H^{s'+1}(\Omega)) \hookrightarrow L^\infty(Q)$ for any $0 < s' < \min(s, \frac{1}{2})$. Hence

$$f = -v \cdot \nabla v + \mu_0 \nabla c \in L^\infty(0,\infty; L^2(\Omega))$$

and Proposition 4.5 yields

$$v \in W^1_q(0,\infty; L^2(\Omega)) \cap L^q(0,\infty; H^2(\Omega)) \hookrightarrow BUC([0,\infty); (L^2(\Omega), H^2(\Omega))_{1-\frac{1}{q},q})$$

for all $2 \leq q < \infty$ with $2 - \frac{2}{q} < 1 + s$, where we note that

$$\begin{aligned} V_2^{1+s}(\Omega) &= (L^2_\sigma(\Omega), V_2^2(\Omega))_{\frac{1}{2}(1+s),2} \hookrightarrow (L^2_\sigma(\Omega), V_2^2(\Omega))_{1-\frac{1}{q},q}, \\ (L^2(\Omega), H^2(\Omega))_{1-\frac{1}{q},q} &\hookrightarrow (L^2(\Omega), H^2(\Omega))_{1-\frac{1}{q}+\varepsilon,q} = H^{2-\frac{2}{q}-2\varepsilon}(\Omega) \end{aligned}$$

for all $\varepsilon > 0$ due to (2.3), (2.4), and (4.15). ■

Lemma 6.3 *Let $d = 3$, $v_0 \in V_2^{1+s}(\Omega)$, $\frac{1}{2} \leq s \leq 1$ and let $c_0 \in H^2_N(\Omega)$ such that $-\Delta c_0 + \phi_0(c_0) \in H^1(\Omega)$. Then there is some $T > 0$ such that any weak solution (v, c, μ) with initial values (v_0, c_0) satisfies $\nabla^2 c, \phi(c) \in L^\infty(0,T; L^6(\Omega))$ and*

$$v \in L^2(0,T; V_2^{2+s'}(\Omega)) \cap H^1(0,T; V_2^{s'}(\Omega)) \cap BUC([0,T]; H^{1+s-\varepsilon}(\Omega))$$

for all $s' \in [0, \frac{1}{2})$ and all $\varepsilon > 0$. In particular, the weak solution is unique on $(0,T)$.

Proof: As in the proof of Lemma 6.2, Theorem 4.6 and Proposition 4.8 imply the existence of some T depending only on (c_0, v_0) such that $v \in L^2(0,T; H^{2+s'}(\Omega)) \cap H^1(0,T; H^{s'}(\Omega))$, where $s' \in [0, \frac{1}{2})$. In particular, $\partial_t v \in L^2(Q_T)$ and Lemma 3.2 implies that $c \in L^\infty(0,T; W^2_6(\Omega)$. Again

$$f = -v \cdot \nabla v + \mu_0 \nabla c \in L^\infty(0,T; L^2(\Omega))$$

and Proposition 4.5 yields

$$v \in W^1_q(0,\infty; L^2(\Omega)) \cap L^q(0,\infty; H^2(\Omega)) \hookrightarrow BUC([0,\infty); (L^2(\Omega), H^2(\Omega))_{1-\frac{1}{q},q})$$

for all $2 \leq q < \infty$ with $2 - \frac{2}{q} < 1 + s$, where $V_2^{1+s}(\Omega) \hookrightarrow (L^2_\sigma(\Omega), V_2^2(\Omega))_{1-\frac{1}{q},q}$ and $(L^2(\Omega), H^2(\Omega))_{1-\frac{1}{q},q} \hookrightarrow H^{2-\frac{2}{q}-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ as in the proof of Lemma 6.2.

Finally, the uniqueness follows from Proposition 1.5. ■

Lemma 6.4 *Let $d = 2, 3$. Moreover, let (v, c, μ) be a weak solution of (1.1)-(1.7) on $(0, \infty)$. Then there is some $T > 0$ such that $\nabla^2 c, \phi(c) \in L^\infty(T, \infty; L^r(\Omega))$ with $r = 6$ if $d = 3$ and $1 < r < \infty$ if $d = 2$ and*

$$v \in L^2(T, \infty; V_2^{2+s}(\Omega)) \cap H^1(T, \infty; V_2^s(\Omega)) \cap BUC([T, \infty); H^{2-\varepsilon}(\Omega))$$

for all $s \in [0, \frac{1}{2})$ and all $\varepsilon > 0$. Moreover, (1.16) holds with equality for all $T \leq s \leq t < \infty$.

Proof: First of all,

$$\begin{aligned} \|\mu_0 \nabla c\|_{L^2((T,\infty)\times\Omega)} &\leq \|\mu_0\|_{L^2(T,\infty;L^6(\Omega))} \|\nabla c\|_{L^\infty(T,\infty;L^3(\Omega))} \\ &\leq C(v_0, c_0) \|\nabla \mu\|_{L^2((T,\infty)\times\Omega)} \to_{T\to\infty} 0 \end{aligned}$$

since $\nabla\mu \in L^2(Q)$. Moreover, since $v \in L^2(0, \infty; H^1(\Omega))$, for every $T > 0$ and $\varepsilon > 0$ there is some $t \geq T$ such that $v(t) \in H^1(\Omega)$ and $\|\nabla v(t)\|_2 \leq \varepsilon$. Hence there is some $T > 0$ such that

$$\|\nabla v(T)\|_2 + \|\mu_0 \nabla c\|_{L^2((T,\infty)\times\Omega)} \leq \varepsilon_0,$$

where $\varepsilon_0 > 0$ is the same as in Theorem 4.6 for $s = 0$. Thus $v \in L^2(T, \infty; V_2^2(\Omega)) \cap H^1(T, \infty; L^2_\sigma(\Omega))$ by Theorem 4.6 with $j = 0$ and Proposition 4.8. Hence $c \in L^\infty(T, \infty; W_6^2(\Omega))$ by Lemma 3.2. By the same argument as before there is some $T' \geq T$

$$\|v(T')\|_{H^2} + \|\mu_0 \nabla c\|_{L^2((T',\infty);H^1(\Omega))} \leq \varepsilon_1,$$

where $\varepsilon_1 > 0$ is the same as in Theorem 4.6 for $0 \leq s \leq j = 1$. Hence

$$v \in L^2(T', \infty; V_2^{2+s}(\Omega)) \cap H^1(T', \infty; V_2^s(\Omega))$$

for all $s \in [0, \frac{1}{2})$. By the same argument as at the end of the proof of Lemma 6.2 one shows $v \in BUC([T', \infty); H^{2-\varepsilon}(\Omega))$ for all $\varepsilon > 0$. Since $v \cdot \nabla v \in L^2(T', \infty; L^2(\Omega))$ and $v \in BUC([T', \infty); L^2(\Omega))$ it is easy to prove (1.16) for all $T' \leq s \leq t < \infty$ by the same arguments as in the proof of Theorem 1.4 for $d = 2$. ■

Proof of Theorem 1.6: All statements follow from the Lemmas 6.2-6.4. ■

7 Asymptotic Behavior in Time

Let (v, c) be a weak solution. Then by Lemma 6.4 there is some $T \geq 0$ such that $v \in BUC(T, \infty; H^{2-\varepsilon}(\Omega))$ and $c \in L^\infty(T, \infty; W_6^2(\Omega))$ for all $\varepsilon > 0$. Since we are only discussing the asymptotic behavior of (v, c) we can by a simple shift in time reduce to the case that $v \in L^\infty(0, \infty; H^2(\Omega))$ and $c \in L^\infty(0, \infty; W_6^2(\Omega))$.

Now we define the ω-limit set of (v, c) as

$$\omega(v, c) = \left\{(v', c') \in H^{2r}(\Omega)^{d+1} : \exists (t_n) \nearrow \infty \text{ s.t. } (v(t_n), c(t_n)) \to (v', c') \text{ in } H^{2r}\right\},$$

where $r \in (\frac{3}{4}, 1)$. By the definition and since $(v,c) \in BUC(0,\infty; H^{2r'}(\Omega))^{d+1}$ with $r < r' < 1$, $\omega(v,c)$ is a non-empty, compact, and connected subset of $H^{2r}(\Omega)^{d+1}$.

Since E is a strict Lyaponov functional for (1.1)-(1.7), we are able to prove:

Lemma 7.1 *Let $(v,c) \in L^\infty(0,\infty; H^{2r}(\Omega)^d \times H^2(\Omega))$ be as above. Then*

$$\omega(v,c) \subseteq \left\{(0,c') : c' \in H^2(\Omega) \cap H^1_{(0)}(\Omega) \text{ solves (1.18)-(1.20)}\right\}.$$

Proof: First of all, since (1.16) holds for all $0 \le s \le t < \infty$, cf. Lemma 6.4, $E(v(t),c(t))$ is non-increasing in $t > 0$ and $E_\infty := \lim_{t\to\infty} E(v(t),c(t))$ exists. Let $(t_n) \nearrow \infty$ such that $\lim_{n\to\infty}(v(t_n),c(t_n)) = (v_0', c_0')$ and let $(v_n(t), c_n(t)) := (v(t+t_n), c(t+t_n))$, $t \in [0,\infty)$. Since $v \in L^2(0,\infty; H^1(\Omega))$, $v_n \to_{n\to\infty} 0$ strongly in $L^2(0,\infty; H^1(\Omega))$. Because of $\partial_t v \in L^2(Q)$ and (2.8), $v_n \to_{n\to\infty} 0$ in $BUC(0,\infty; L^2_\sigma(\Omega))$ and $v(t_n) \to_{n\to\infty} 0 = v_0'$ in $L^2_\sigma(\Omega)$. Moreover, due to Theorem 3.1 $(c_n)_{n\in\mathbb{N}}$ converges weakly to a solution c' of (3.1)-(3.4) with $v = 0$ and initial value $c'|_{t=0} = c_0'$ in the sense stated in Theorem 3.1. In particular, $c_n \to_{n\to\infty} c'$ in $L^2_{\text{loc}}(0,\infty; H^1_{(0)}(\Omega))$ and therefore

$$E_1(c_n(t)) \to_{n\to\infty} E_1(c'(t)) \qquad \text{for a.e. } t \in [0,\infty)$$

and a suitable subsequence. On the other hand, since $\lim_{n\to\infty} E(v_n(t), c_n(t)) = \lim_{n\to\infty} E_1(c_n(t)) = E_\infty$,

$$E_1(c'(t)) = E_\infty \qquad \text{for a.e. } t \in [0,\infty).$$

Hence by (3.9) $\nabla\mu(t) = 0$ for almost all $t \in [0,\infty)$. Thus $\partial_t c'(t) = 0$, and $c'(t) \equiv c_0'$ solves the stationary Cahn-Hilliard equation (1.18)-(1.20). ∎

Using Proposition 3.4 we are able to prove:

Lemma 7.2 *Let (v,c) be as above. Then there are some $T > 0, \varepsilon > 0$ such that*

$$a + \varepsilon \le c(t,x) \le b - \varepsilon \qquad \text{for all } t \ge T, x \in \overline{\Omega}.$$

Proof: By Lemma 7.1, Proposition 3.4, $H^{2r}(\Omega) \hookrightarrow C(\overline{\Omega})$, and the compactness of $\omega(v,c)$ there are some $a < M_1 \le M_2 < b$ such that

$$M_1 \le c'(x) \le M_2 \qquad \text{for all } x \in \overline{\Omega}, (0,c') \in \omega(v,c).$$

Since $\lim_{t\to\infty} \text{dist}((v(t),c(t)), \omega(v,c)) = 0$ in the norm of $H^{2r}(\Omega)$, we conclude that there are some $T > 0$ and $a < M_1' \le M_2' < b$ such that

$$M_1' \le c(t,x) \le M_2' \qquad \text{for all } x \in \overline{\Omega}, t \ge T,$$

which proves the lemma. ∎

Now let $\widetilde{\Phi}$ be a smooth and bounded function such that $\widetilde{\Phi}|_{[a+\varepsilon,b-\varepsilon]} = \Phi|_{[a+\varepsilon,b-\varepsilon]}$ and such that Proposition 3.5 holds. In particular this implies that the Lojasiewicz-Simon inequality (3.23) holds as seen in Section 3.2.

Proof of Theorem 1.7: In order to prove convergence as $t \to \infty$, we consider

$$H(t) := \left(E(v(t), c(t)) - E_\infty\right)^\theta,$$

where θ is as in (3.23). Then $H(t)$ is non-increasing and

$$\begin{aligned} -\frac{d}{dt}H(t) &= \theta\left(\|\nabla\mu(t)\|_2^2 + \int_\Omega \nu(c(t))|Dv(t)|^2\,dx\right)\left(E(v(t),c(t)) - E_\infty\right)^{\theta-1} \\ &\geq C\left(\|\nabla\mu(t)\|_2^2 + \int_\Omega \nu(c(t))|Dv(t)|^2\,dx\right)\left(\|D\widetilde{E}_1(c(t))\|_{H^{-1}_{(0)}} + \|v(t)\|_{L^2}\right)^{-1} \end{aligned}$$

by (3.23) and (1.16) with equality. Now we use that

$$D\widetilde{E}_1(c(t)) = -\Delta c(t) + P_0\widetilde{\phi}(c(t)) = -\Delta c(t) + P_0\phi(c(t)) = P_0\mu(t), \quad \widetilde{\phi} = \widetilde{\Phi}',$$

for $t \geq T$ since $M_1 \leq u(t) \leq M_2$. Because of $\|P_0\mu(t)\|_{H^{-1}_{(0)}} \leq C\|\nabla\mu(t)\|_2$ and $\|v(t)\|_2 \leq C\|Dv(t)\|_2$ by Korn's inequality, $-\frac{d}{dt}H(t) \geq C\left(\|\nabla\mu(t)\|_2 + \|Dv(t)\|_2\right)$. This implies

$$\int_T^\infty \|\nabla\mu(t)\|_2\,dt + \int_T^\infty \|Dv(t)\|_2\,dt \leq CH(T) < \infty$$

and therefore

$$\int_T^\infty \|\partial_t c(t)\|_{H^{-1}_{(0)}}\,dt \;\leq\; \int_T^\infty \|\nabla\mu(t)\|_{H^{-1}_{(0)}}\,dt + \int_T^\infty \|v\|_{L^2}\,dt \leq CH(T) < \infty$$

due to (1.1) and (3.8). Hence $\partial_t c \in L^1(T,\infty; H^{-1}_{(0)})$ and therefore

$$c(t) = c(T) + \int_T^t \partial_t c(\tau)\,d\tau \to_{t\to\infty} c_\infty \qquad \text{in } H^{-1}_{(0)}(\Omega).$$

In particular, $\omega(v,c) = \{(0,c_\infty)\}$ and c_∞ solves the stationary Cahn-Hilliard equation (1.18) -(1.20) because of Lemma 7.1. Since $(v(t),c(t)) \in H^2(\Omega)^{d+1}$ is uniformly bounded in $t \geq 0$, we conclude that $(v(t),c(t))$ converges weakly to $(0,c_\infty)$ in $H^2(\Omega)^{d+1}$. ■

References

[1] H. Abels. On generalized solutions of two-phase flows for viscous incompressible fluids. *Interfaces Free Bound.*, 9:31–65, 2007.

[2] H. Abels. On the notion of generalized solutions of two-phase flows for viscous incompressible fluids. *RIMS Kôkyûroku Bessatsu*, B1:1–15, 2007.

[3] H. Abels and M. Wilke. Convergence to equilibrium for the Cahn-Hilliard equation with a logarithmic free energy. *Nonlinear Anal.*, 67(11):3176–3193, 2007.

[4] H. Amann. *Linear and Quasilinear Parabolic Problems, Volume 1: Abstract Linear Theory.* Birkhäuser, Basel - Boston - Berlin, 1995.

[5] D. M. Anderson, G. B. McFadden, and A. A. Wheeler. Diffuse-interface methods in fluid mechanics. In *Annual review of fluid mechanics, Vol. 30*, volume 30 of *Annu. Rev. Fluid Mech.*, pages 139–165. Annual Reviews, Palo Alto, CA, 1998.

[6] J. Bergh and J. Löfström. *Interpolation Spaces.* Springer, Berlin - Heidelberg - New York, 1976.

[7] F. Boyer. Mathematical study of multi-phase flow under shear through order parameter formulation. *Asymptot. Anal.*, 20(2):175–212, 1999.

[8] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[9] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. I. Interfacial energy. *J. Chem. Phys.*, 28, No. 2:258–267, 1958.

[10] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.

[11] J. Diestel and J. J. Uhl, Jr. *Vector measures.* American Mathematical Society, Providence, R.I., 1977.

[12] C. M. Elliott and S. Luckhaus. A generalized equation for phase separation of a multi-component mixture with interfacial free energy. *Preprint SFB 256 Bonn No. 195*, 1991.

[13] Klaus-Jochen Engel and Rainer Nagel. *A short course on operator semigroups.* Universitext. Springer, New York, 2006.

[14] G. P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Volume 1.* Springer, Berlin - Heidelberg - New York, 1994.

[15] M. E. Gurtin, D. Polignone, and J. Viñals. Two-phase binary fluids and immiscible fluids described by an order parameter. *Math. Models Methods Appl. Sci.*, 6(6):815–831, 1996.

[16] P.C. Hohenberg and B.I. Halperin. Theory of dynamic critical phenomena. *Rev. Mod. Phys.*, 49:435–479, 1977.

[17] J. Johnsen. Pointwise multiplication of Besov and Triebel-Lizorkin spaces. *Math. Nachr.*, 175:85–133, 1995.

[18] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

[19] C. Liu and J. Shen. A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method. *Phys. D*, 179(3-4):211–228, 2003.

[20] T. Roubíček. A generalization of the Lions-Temam compact imbedding theorem. *Časopis Pěst. Mat.*, 115(4):338–342, 1990.

[21] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, volume 49 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[22] C. G. Simader and H. Sohr. A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. In *Mathematical problems relating to the Navier-Stokes equation*, volume 11 of *Ser. Adv. Math. Appl. Sci.*, pages 1–35. World Sci. Publ., River Edge, NJ, 1992.

[23] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[24] J. Simon. Sobolev, Besov and Nikol′skiĭ fractional spaces: imbeddings and comparisons for vector valued spaces on an interval. *Ann. Mat. Pura Appl. (4)*, 157:117–148, 1990.

[25] H. Sohr. *The Navier-Stokes equations.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2001. An elementary functional analytic approach.

[26] V. N. Starovoĭtov. On the motion of a two-component fluid in the presence of capillary forces. *Mat. Zametki*, 62(2):293–305, 1997.

[27] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton Hall Press, Princeton, New Jersey, 1970.

[28] P. Sternberg and K. Zumbrun. Connectivity of phase boundaries in strictly convex domains. *Arch. Rational Mech. Anal.*, 141(4):375–400, 1998.

[29] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland Publishing Company, Amsterdam, New York, Oxford, 1978.

[30] E. Zeidler. *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York, 1990. Linear monotone operators, Translated from the German by the author and Leo F. Boron.