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On Global Attraction to Quantum Stationary
States II. Several Nonlinear Oscillators Coupled
to Massive Scalar Field

by

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On Global Attraction to Quantum Stationary States II. Several Nonlinear Oscillators Coupled to Massive Scalar Field

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Abstract

The global attraction is established for all finite energy solutions to a model $\mathbf{U}(1)$-invariant nonlinear Klein-Gordon equation in one dimension coupled to a finite number of nonlinear oscillators: We prove that *each finite energy solution* converges as $t \to \pm \yen$ to the set of all "nonlinear eigenfunctions" of the form $f(x)e^{-i\omega t}$ if all oscillators are strictly nonlinear, and the distances between all neighboring oscillators are sufficiently small. The *global attraction* is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersive radiation. This result for one oscillator was obtained in [KK07].

We construct counterexamples showing that the convergence to the solitary waves may break down if the distance between some of the neighboring oscillators is sufficiently large or if some of the oscillators are harmonic. In these cases, the global attractor can contain "multifrequency solitary waves" or linear combinations of distinct solitary waves.

1 Introduction

This is the second paper where we establish the global attraction to solitary waves in $\mathbf{U}(1)$-invariant dispersive systems. In [KK07], we proved such an attraction for the Klein-Gordon field coupled to one anharmonic oscillator. Here we generalize this result to several anharmonic oscillators.

The long time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers by Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and the local attraction to zero solution were proved. Local attraction to solitary waves, or *asymptotic stability*, in $\mathbf{U}(1)$-invariant dispersive systems was addressed in [SW90, BP93, SW92, BP95] and then developed in [PW97, SW99, Cuc01a, Cuc01b, BS03, Cuc03]. Global attraction to *static*, stationary solutions in the dispersive systems *without* $\mathbf{U}(1)$ *symmetry* was established in [Kom91, Kom95, KV96, KSK97, Kom99, KS00]. The first result about the global attraction to solitary waves in $\mathbf{U}(1)$-invariant dispersive systems was obtained in [KK06, KK07].

In this paper, we establish the global attraction for the complex Klein-Gordon field $y(x,t)$, interacting with N nonlinear oscillators located at the points $X_1 < X_2 < \ldots < X_N$:

$$\ddot{y} = y'' - m^2 y + \sum_J \delta(x - X_J) F_J(y(X_J, t)), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $m > 0$ and F_J are nonlinear functions describing anharmonic oscillators at the points X_J. The dots stand for the derivatives in t, and the primes for the derivatives in x. All derivatives and the equation are understood in the sense of

distributions. We assume that equation (1.1) is $\mathbf{U}(1)$-invariant; that is,

$$F_J(e^{iq}y) = e^{iq}F_J(y), \qquad q \in \mathbb{R}, \quad y \in \mathbb{C}, \quad 1 \le J \le N. \tag{1.2}$$

This symmetry leads to the charge conservation and to the existence of the solitary wave solutions, which are finite energy solutions of the following form:

$$y_w(x,t) = f_w(x)e^{-iwt}, \quad w \in \mathbb{C}, \quad f_w \in H^1(\mathbb{R}). \tag{1.3}$$

Above, $H^1(\mathbb{R})$ denotes the Sobolev space.

Definition 1.1. S is the set of all functions $f_w(x) \in H^1(\mathbb{R})$ with $w \in \mathbb{C}$, so that $f_w(x)e^{-iwt}$ is a solution to (1.1).

Note that S also contains the zero solution.

Generically, the quotient $S/\mathbf{U}(1)$ is isomorphic to a finite union of one-dimensional intervals. The set of all solitary waves for equation (1.1) is described in Proposition 2.8. Typically, such solutions exist for w from an interval or a collection of intervals of the real line.

Our main result is the following long-time asymptotics: In the case when all oscillators are polynomial and strictly anharmonic (see Assumptions 2.1 and 2.2 below) and all distances $|X_{J+1} - X_J|$ are sufficiently small, we prove that any finite energy solution converges to the set S of all solitary waves:

$$y(\cdot, t) \longrightarrow S, \qquad t \to \pm\maltese, \tag{1.4}$$

where the convergence holds in local energy seminorms. For a similar result in the case $N = 1$ and for the motivation and relation to problems of Quantum Mechanics, see [KK06, KK07]. In the case $N > 1$, the general plan of the proof is similar to $N = 1$: Separation of dispersive components, absolute continuity of spectrum outside a bounded interval, compactness of time shifts of the bound component, and a nonlinear spectral analysis of omega-limit trajectories by the Titchmarsh Convolution Theorem. However, the justifications of all steps are based on new arguments.

The requirement that the nonlinearities F_J are polynomial allows us to apply the Titchmarsh theorem which is vital in the proof. We construct counterexamples showing the sharpness of our assumptions for the global attraction to the solitary waves. Namely, in the case $N = 2$, we construct multifrequency solitary waves if the distance $|X_2 - X_1|$ is sufficiently large or one of the oscillators is linear. For $N = 1$, a counterexample given by a superposition of two different solitary waves is constructed in [KK07].

Our paper is organized as follows. In Section 2, we formulate our main results. In Section 3, we separate first dispersive component. In Sections 4 and 5, we construct spectral representation for the remaining component, and prove absolute continuity of its spectrum for high frequencies. In Sections 6, we separate first dispersive component corresponding to the high frequencies and establish compactness for the remaining *bound* component with the bounded spectrum. In Section 7, we study omega-limit trajectories of the solution: (*i*) first, we prove that any omega-limit trajectory also is a solution to the nonlinear Klein-Gordon equation, (*ii*) second, we reduce the spectrum of the trajectory to a bounded set, (*iii*) finally, we reduce the spectrum to a single point using the Titchmarsh Convolution Theorem. This means that any omega-limit trajectory is a solitary wave, and proves the global attraction to the set of all solitary waves. In Section 8 we collect counterexamples, and in Appendix A we establish global well-posedness.

2 Main results

Model

We consider the Cauchy problem for the Klein-Gordon equation with the nonlinearity concentrated at the points $X_1 < X_2 < \ldots < X_N$:

$$\begin{cases} \ddot{y}(x,t) = y''(x,t) - m^2 y(x,t) + {}_J d(x - X_J)F_J(y(X_J,t)), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ y|_{t=0} = y_0(x), \qquad \dot{y}|_{t=0} = p_0(x). \end{cases} \tag{2.1}$$

If we identify a complex number $y = u + iv \in \mathbb{C}$ with the two-dimensional vector $(u,v) \in \mathbb{R}^2$, then, physically, equation (2.1) describes small crosswise oscillations of the infinite string in three-dimensional space (x,u,v) stretched along the

x-axis. The string is subject to the action of an "elastic force" $-m^2 y(x,t)$ and coupled to anharmonic oscillators of forces $F_J(y)$ attached at the points X_J. We denote by $\mathscr{X}$ the set of all the locations of oscillators:

$$\mathscr{X} = \{X_1, X_2, \ldots, X_N\}. \tag{2.2}$$

We will assume that the oscillator forces F_J admit real-valued potentials:

$$F_J(y) = -\nabla U_J(y), \quad y \in \mathbb{C}, \qquad U \in C^2(\mathbb{C}), \tag{2.3}$$

where the gradient is taken with respect to $\operatorname{Re} y$ and $\operatorname{Im} y$. We define $\Psi(t) = \begin{bmatrix} y(x,t) \\ p(x,t) \end{bmatrix}$ and write the Cauchy problem (2.1) in the vector form:

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi(t) + \sum_J d(x - X_J) \begin{bmatrix} 0 \\ F_J(y) \end{bmatrix}, \qquad \Psi\big|_{t=0} = \Psi_0 \equiv \begin{bmatrix} y_0 \\ p_0 \end{bmatrix}. \tag{2.4}$$

Equation (2.4) formally can be written as a Hamiltonian system,

$$\dot{\Psi}(t) = \mathscr{J} D\mathscr{H}(\Psi), \qquad \mathscr{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.5}$$

where $D\mathscr{H}$ is the variational derivative of the Hamilton functional

$$\mathscr{H}(\Psi) = \frac{1}{2} \int_{\mathbb{R}} \left(|p|^2 + |y'|^2 + m^2|y|^2 \right) dx + \sum_J U_J(y(X_J)), \qquad \Psi = \begin{bmatrix} y(x) \\ p(x) \end{bmatrix}. \tag{2.6}$$

We assume that the potentials $U_J(y)$ are $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group e^{iq}, $q \in \mathbb{R} \bmod 2p$. Namely, we assume that there exist $u_J \in C^2(\mathbb{R})$ such that

$$U_J(y) = u_J(|y|^2), \qquad y \in \mathbb{C}, \quad 1 \le J \le N. \tag{2.7}$$

Remark 2.1. In the context of the model of the infinite string in $\mathbb{R}^3$ that we described after (2.1), the assumption (2.7) means that the potentials $U_J(y)$ are rotation-invariant with respect to the x-axis.

Conditions (2.3) and (2.7) imply that

$$F_J(y) = a_J(|y|^2)y, \qquad y \in \mathbb{C}, \tag{2.8}$$

where $a_J(\cdot) = -2u_J'(\cdot) \in C^1(\mathbb{R})$ are real-valued. Therefore, (1.2) holds. Since (2.4) is $\mathbf{U}(1)$-invariant, the Nöther theorem formally implies that the *charge functional*

$$\mathscr{Q}(\Psi) = \frac{i}{2} \int_{\mathbb{R}} (\overline{y}p - \overline{p}y) \, dx, \qquad \Psi = \begin{bmatrix} y(x) \\ p(x) \end{bmatrix}, \tag{2.9}$$

is conserved for solutions $\Psi(t)$ to (2.4).

Let us introduce the phase space $\mathscr{E}$ of finite energy states for equation (2.1). Denote by L^2 the complex Hilbert space $L^2(\mathbb{R})$ with the norm $\|\cdot\|_{L^2}$, and denote by $\|\cdot\|_{L_R^2}$ the norm in $L^2(-R,R)$ for $R > 0$.

Definition 2.2. *(i)* $\mathscr{E}$ is the Hilbert space of the states $\Psi = (y,p)$, with the norm

$$\|\Psi\|_{\mathscr{E}}^2 := \|p\|_{L^2}^2 + \|y'\|_{L^2}^2 + m^2\|y\|_{L^2}^2. \tag{2.10}$$

(ii) $\mathscr{E}_F$ is the space $\mathscr{E}$ endowed with the Fréchet topology defined by local energy seminorms

$$\|\Psi\|_{\mathscr{E},R}^2 := \|p\|_{L^2(-R,R)}^2 + \|y'\|_{L^2(-R,R)}^2 + m^2\|y\|_{L^2(-R,R)}^2, \qquad R > 0. \tag{2.11}$$

Remark 2.3. The space $\mathscr{E}_F$ is metrizable. The metric could be introduced by

$$\operatorname{dist}(\Psi, \Phi) = \sum_{R=1}^{\infty} 2^{-R} \|\Psi - \Phi\|_{\mathscr{E},R}.$$

Equation (2.4) is formally a Hamiltonian system with the phase space $\mathscr{E}$ and the Hamilton functional $\mathscr{H}$. Both $\mathscr{H}$ and $\mathscr{Q}$ are continuous functionals on $\mathscr{E}$. Let us note that $\mathscr{E} = H^1 \oplus L^2$, where H^1 denotes the Sobolev space

$$H^1 = H^1(\mathbb{R}) = \{y(x) \in L^2(\mathbb{R}) : y'(x) \in L^2(\mathbb{R})\}.$$

We introduced into (2.10) the factor $m^2 > 0$, to have a convenient relation $\mathscr{H}(y,\dot{y}) = \frac{1}{2}\|(y,\dot{y})\|_{\mathscr{E}}^2 + \sum_J U_J(y(X_J))$.

3

Global well-posedness

To have a priori estimates available for the proof of the global well-posedness, we assume that

$$U_J(y) \geq A_J - B_J|y|^2 \quad \text{for } y \in \mathbb{C}, \quad \text{where} \quad A_J \in \mathbb{R}, \quad B_J \geq 0, \quad 1 \leq J \leq N; \quad \sum_J B_J < m. \tag{2.12}$$

Theorem 2.4. *Let $F_J(y)$ satisfy conditions (2.3) and (2.7):*

$$F_J(y) = -\nabla U_J(y), \qquad U_J(y) = u_J(|y|^2), \qquad u_J(\cdot) \in C^2(\mathbb{R}).$$

Additionally, assume that (2.12) holds. Then:

    (i) For every $Y_0 \in \mathscr{E}$ the Cauchy problem (2.4) has a unique solution $Y(t)$ such that $Y \in C(\mathbb{R}, \mathscr{E})$.

    (ii) The map $W(t): Y_0 \mapsto Y(t)$ is continuous in $\mathscr{E}$ for each $t \in \mathbb{R}$.

    (iii) The energy and charge are conserved: $\mathscr{H}(Y(t)) = \text{const}, \mathscr{Q}(Y(t)) = \text{const}, t \in \mathbb{R}$.

    (iv) The following a priori *bound holds: $\|Y(t)\|_{\mathscr{E}} \leq C(Y_0), t \in \mathbb{R}$.*

We prove this Theorem in Appendix A.

Solitary waves and the main theorem

Definition 2.5.   *(i)* The solitary waves of equation (2.1) are solutions of the form

$$y(x,t) = \phi_w(x)e^{-iwt}, \qquad \text{where} \quad w \in \mathbb{C}, \quad \phi_w \in H^1(\mathbb{R}). \tag{2.13}$$

    (ii) The solitary manifold is the set $\mathbf{S} = \left\{ (\phi_w, -iw\,\phi_w) \colon w \in \mathbb{C}, \phi_w \in H^1(\mathbb{R}) \right\} \subset \mathscr{E}$.

Remark 2.6.   *(i)* Identity (1.2) implies that the set $\mathbf{S}$ is invariant under multiplication by $e^{iq}, q \in \mathbb{R}$.

    (ii) Let us note that for any $w \in \mathbb{C}$ there is a zero solitary wave with $\phi_w(x) \equiv 0$ since $F_J(0) = 0$ by (2.8).

    (iii) According to (2.8), $a_J(|C|^2) = F_J(C)/C \in \mathbb{R}$ for any $C \in \mathbb{C}\backslash 0$.

Definition 2.7. The function $F_J(y)$ is *strictly nonlinear* if the equation $a_J(C^2) = a$ has a discrete (or empty) set of positive roots C for each particular $a \in \mathbb{R}$.

The following proposition provides a concise description of all solitary waves. Formally this proposition is not necessary for our exposition.

Proposition 2.8. *Assume that $F_J(y)$ satisfy (1.2) and that $F_J(y)$, $1 \leq J \leq N$, are strictly nonlinear in the sense of Definition 2.7. Then all solitary wave solutions to (2.1) are given by (2.13) with*

$$\phi_w(x) = \sum_J C_J e^{-k(w)|x-X_J|}, \qquad k(w) = \sqrt{m^2 - w^2}, \tag{2.14}$$

where $w \in [-m, m]$ and $C_J \in \mathbb{C}$, $1 \leq J \leq N$, satisfy the following relations:

$$2k(w)C_J = F_J\left(\sum_K C_K e^{-k(w)|X_J - X_K|} \right). \tag{2.15}$$

Remark 2.9. By (2.14), $w = \pm m$ can only correspond to zero solution.

The proof of this Proposition repeats the proof of a similar result for the case $N = 1$ in [KK07].

As we mentioned before, we need to assume that the nonlinearities are nonlinear polynomials. This condition is crucial in our argument: It will allow to apply the Titchmarsh convolution theorem. Now all our conditions on F_J can be summarized as following two assumptions

Assumption 2.1. For all $1 \le J \le N$,

$$F_J(y) = - \ U_J(y), \quad \text{where} \quad U_J(y) = \sum_{n=0}^{p_J} u_{J,n} |y|^{2n}, \qquad u_{J,n} \in \mathbb{R}. \tag{2.16}$$

Assumption 2.2. For all $1 \le J \le N$, we have

$$u_{J,p_J} > 0 \quad \text{and} \quad p_J \ge 2. \tag{2.17}$$

These Assumptions guarantee that all nonlinearities F_J are strictly nonlinear and satisfy (2.3), (2.7), and also that the bound (2.12) takes place. We introduce the following quantities:

$$m_1 = m, \quad m_{J+1} = (2p_J - 1)m_J; \qquad m'_N = m, \quad m'_{J-1} = (2p_J - 1)m'_J, \tag{2.18}$$

where p_J are exponentials from (2.16) We also denote

$$M_J = \min(m_J, m'_J), \qquad \mathtt{L} = \max_{1 \le J \le N} (2p_J - 1)M_J. \tag{2.19}$$

We will show later that the spectrum of any omega-limit trajectory belongs to the intervals $[-\mathtt{L}, \mathtt{L}]$. We also denote

$$w_{J,n} := \sqrt{\frac{p^2 n^2}{|X_{J+1} - X_J|^2} + m^2}, \qquad 1 \le J \le N - 1, \quad n \in \mathbb{N}, \tag{2.20}$$

and introduce the set of frequencies

$$\mathscr{W} = \{\pm w_{J,n} \colon 1 \le J \le N - 1, \quad n \in \mathbb{N}\}. \tag{2.21}$$

The frequencies $\pm w_{J,n}$ correspond to the trapped modes, vanishing at the endpoints of the interval $[X_J, X_{J+1}]$. We will prove that if all the frequencies $\pm w_{J,n}$ are outside of $[-\mathtt{L}, \mathtt{L}]$, then the global attractor consists of the solitary waves only. This will be the case if the intervals $[X_J, X_{J+1}]$ are sufficiently small.

Assumption 2.3. We assume that the intervals $[X_J, X_{J+1}]$, $1 \le J \le N - 1$, are small enough so that

$$w_{J,1} = \sqrt{\frac{p^2}{|X_{J+1} - X_J|^2} + m^2} > \mathtt{L}, \qquad 1 \le J \le N - 1. \tag{2.22}$$

Under this assumption, we have

$$[-\mathtt{L}, \mathtt{L}] \cap \mathscr{W} = \emptyset. \tag{2.23}$$

Remark 2.10. The condition (2.22) guarantees that there are no trapped modes of frequencies smaller than $\mathtt{L}$ that vanish at the adjacent points X_J, X_{J+1}.

Our main result is the following theorem.

Theorem 2.11 (Main Theorem)**.** *Let Assumptions 2.1, 2.2 and 2.3 hold. Then for any* $\mathtt{Y}_0 \in \mathscr{E}$ *the solution* $\mathtt{Y}(t)$ *to the Cauchy problem (2.4) converges to* **S**:

$$\lim_{t \to \pm \maltese} \text{dist}(\mathtt{Y}(t), \mathbf{S}) = 0, \tag{2.24}$$

where $\text{dist}(\mathtt{Y}, \mathbf{S}) := \inf_{\mathtt{F} \in \mathbf{S}} \text{dist}(\mathtt{Y}, \mathtt{F})$.

Remark 2.12. (*i*) It suffices to prove Theorem 2.11 for $t \to +\maltese$.

(*ii*) In Sections 8.1 and 8.2, we construct counterexamples to the convergence (2.24) in the case when (2.22) or (2.17) are not satisfied.

3 Separation of dispersive component

Let us split the solution $y(x,t)$ into two components, $y(x,t) = c(x,t) + j(x,t)$, which are defined for all $t \in \mathbb{R}$ as solutions to the following Cauchy problems:

$$\ddot{c}(x,t) = c''(x,t) - m^2 c(x,t), \qquad (c,\dot{c})|_{t=0} = (y_0(x), p_0(x)), \tag{3.1}$$

$$\ddot{j}(x,t) = j''(x,t) - m^2 j(x,t) + \sum_J d(x - X_J) f_J(t), \qquad (j,\dot{j})|_{t=0} = (0,0), \tag{3.2}$$

where $(y_0(x), p_0(x))$ is the initial data from (2.1), and

$$f_J(t) := F_J(y(X_J,t)), \qquad t \in \mathbb{R}. \tag{3.3}$$

The following lemma is proved in [KK07, Lemma 3.1].

Lemma 3.1. *There is a local energy decay for c:*

$$\lim_{t \to \mp\infty} \|(c(\cdot,t), \dot{c}(\cdot,t))\|_{\mathscr{E},R} = 0, \qquad \forall R > 0. \tag{3.4}$$

Let $k(w)$ be the analytic function with the domain $D := \mathbb{C} \setminus ((-\infty, -m] \cup [m, +\infty))$ such that

$$k(w) = \sqrt{w^2 - m^2}, \qquad \operatorname{Im} k(w) > 0, \qquad w \in D. \tag{3.5}$$

Let us also denote its limit values for $w \in \mathbb{R}$ by

$$k_\pm(w) := k(w \pm i0), \qquad w \in \mathbb{R}. \tag{3.6}$$
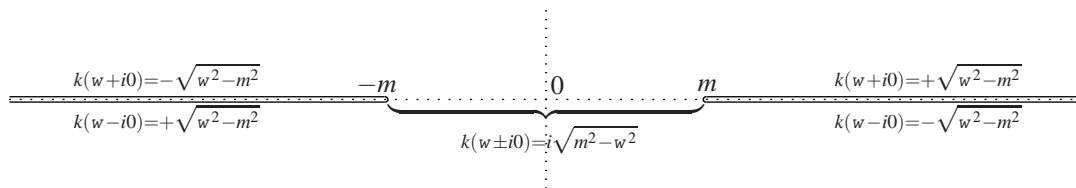

Figure 1: Domain D and the values of $k_\pm(w) := k(w \pm i0)$, $w \in \mathbb{R}$.

As illustrated on Figure 1 (where all square roots take positive values), we have

$$k_-(w) = k_+(w) \quad \text{for} \quad -m \le w \le m, \qquad k_-(w) = -k_+(w) \quad \text{for} \quad w \in \mathbb{R} \setminus [-m,m], \tag{3.7}$$

and also

$$w\, k_+(w) \ge 0 \qquad \text{for} \quad w \in \mathbb{R} \setminus [-m,m]. \tag{3.8}$$

Let us study the Fourier transform $\hat{c}(x,w) := \mathscr{F}_{t \to w}[c(x,t)]$ in the sense of tempered distributions. For test functions $j(t)$, from the Schwarz space, we set $\mathscr{F}_{t \to w}[j(t)] = \int_{\mathbb{R}} e^{iwt} j(t)\, dt$.

Lemma 3.2. • $\hat{c}(x,w)$ *is a continuous function of $x \in \mathbb{R}$ with values in $L^1_{\mathrm{loc}}(\mathbb{R})$, and*

$$\hat{c}(x,w) = 0, \qquad |w| < m. \tag{3.9}$$

• *The following bound holds:*

$$\sup_{x \in \mathbb{R}} \int_{|w| > m} |\hat{c}(x,w)|^2 w\, k_+(w)\, dw < \infty. \tag{3.10}$$

Proof. Set $w(k) = \operatorname{sgn} k\sqrt{m^2 + k^2}$ for $k \in \mathbb{R}$. Note that the function $k_+(w)$ for $|w| > m$ is inverse to the function $w(k)$, $k \neq 0$. We have:

$$c(x,t) = \frac{1}{2p}\int_{\mathbb{R}} e^{-ikx}\Big[\hat{y}_0(k)\cos(w(k)t) + \hat{p}_0(k)\frac{\sin(w(k)t)}{w(k)}\Big]\,dk. \tag{3.11}$$

Hence, for the Fourier transform of $c(x,t)$, we *formally* obtain:

$$\hat{c}(x,w) = \int_{\mathbb{R}} e^{-ikx}\Big[\hat{y}_0(k)\frac{d(w - w(k)) + d(w + w(k))}{2} + \hat{p}_0(k)\frac{d(w - w(k)) - d(w + w(k))}{2iw(k)}\Big]\,dk$$

$$= \int_{|w'|>m} e^{-ik_+(w')x}\Big[\hat{y}_0(k_+(w'))\frac{d(w - w') + d(w + w')}{2} + \hat{p}_0(k_+(w'))\frac{d(w - w') - d(w + w')}{2iw'}\Big]\frac{w'\,dw'}{k_+(w')}.$$

Above, we used the substitution $k = k_+(w')$. Now (3.9) is obvious. Evaluating the last integral, we get:

$$\hat{c}(x,w) = \frac{w}{2k_+(w)}\left\{e^{-ik_+(w)x}\hat{y}_0(k_+(w)) + e^{ik_+(w)x}\hat{y}_0(-k_+(w)) + e^{-ik_+(w)x}\frac{\hat{p}_0(k_+(w))}{iw} - e^{ik_+(w)x}\frac{\hat{p}_0(-k_+(w))}{iw}\right\}, \qquad |w| > m.$$

We took into account that $k_+(-w) = -k_+(w)$ for $w \in \mathbb{R}\setminus[-m,m]$ (see (3.7)). Thus, we have:

$$\int_{|w|>m} |\hat{c}(x,w)|^2 w\,k_+(w)\,dw \leq \int_{|w|>m}\Big[\frac{w^2|\hat{y}_0(k_+(w))|^2}{k^2(w)} + \frac{|\hat{p}_0(k_+(w))|^2}{k^2(w)}\Big]w\,k_+(w)\,dw = \int_{\mathbb{R}}\Big[|\hat{y}_0(k)|^2 + \frac{|\hat{p}_0(k)|^2}{w^2(k)}\Big]w^2(k)\,dk.$$

The finiteness of the right-hand side follows from the finiteness of the energy of the initial data (y_0, p_0):

$$\|(y_0, p_0)\|_{\mathscr{E}}^2 = \frac{1}{2p}\int_{\mathbb{R}}\Big[w^2(k)|\hat{y}_0(k)|^2 + |\hat{p}_0(k)|^2\Big]\,dk < ¥.$$

$\square$

4 Spectral representation

The function $j(x,t) = y(x,t) - c(x,t)$ satisfies the following Cauchy problem:

$$\ddot{j}(x,t) = j''(x,t) - m^2 j(x,t) + \sum_J d(x - X_J)f_J(t), \qquad (j, \dot{j})|_{t=0} = (0,0), \tag{4.1}$$

with $f_J(t)$ defined in (3.3). Note that $y(X_J, \cdot) \in C_b(\mathbb{R})$ for $1 \leq J \leq N$ by the Sobolev embedding, since $(y(x,t), \dot{y}(x,t)) \in C_b(\mathbb{R}, \mathscr{E})$ by Theorem 2.4 (*iv*). Hence, $f_J(t) \in C_b(\mathbb{R})$. On the other hand, since $c(x,t)$ is a finite energy solution to the free Klein-Gordon equation, we also have

$$(c(x,t), \dot{c}(x,t)) \in C_b(\mathbb{R}, \mathscr{E}). \tag{4.2}$$

Therefore, the function $j(x,t) = y(x,t) - c(x,t)$ satisfies

$$(j(x,t), \dot{j}(x,t)) \in C_b(\mathbb{R}, \mathscr{E}). \tag{4.3}$$

The Fourier transform

$$\hat{j}(x,w) = \mathscr{F}_{t \to w}[j(x,t)], \qquad (x,w) \in \mathbb{R}^2, \tag{4.4}$$

is continuous function of $x \in \mathbb{R}$ with values in tempered distribution of $w \in \mathbb{R}$. It satisfies the following equation (Cf. (4.1)):

$$-w^2 \hat{j}(x,w) = \hat{j}''(x,w) + \sum_J d(x - X_J)\hat{f}_J(w), \qquad (x,w) \in \mathbb{R}^2. \tag{4.5}$$

We are going to construct a representation for the solution $\hat{j}(x,w)$ in a form suitable for our purposes.

Lemma 4.1. $\hat{\jmath}$ *is a smooth function of* $x \in \mathbb{R} \setminus \mathcal{X}$ *(where* $\mathcal{X} = \{X_1, X_2, \ldots, X_N\}$*), with values in tempered distributions of* $w \in \mathbb{R}$*, and there exist quasimeasures* $\hat{F}_J^{\pm}$*,* $1 \leq J \leq N$*, and* $\hat{Q}_J$*,* $1 \leq J \leq N-1$*, so that*

$$
\hat{\jmath}(x,w) = \begin{cases} \hat{F}_1^+(w) e^{-ik_+(w)(x-X_1)} + \hat{F}_1^-(w) e^{-ik_-(w)(x-X_1)}, & x \leq X_1, \\ \hat{F}_J(w) \cos(k_+(w)(x-X_J)) + \hat{Q}_J(w) \frac{\sin(k_+(w)(x-X_J))}{k_+(w)}, & x \in [X_J, X_{J+1}], \quad 1 \leq J \leq N-1, \\ \hat{F}_N^+(w) e^{ik_+(w)(x-X_N)} + \hat{F}_N^-(w) e^{ik_-(w)(x-X_N)}, & x \geq X_N, \end{cases} \tag{4.6}
$$

where $\hat{F}_J(w) := \hat{F}_J^+(w) + \hat{F}_J^-(w)$.

Remark 4.2. A tempered distribution $m(w) \in \mathscr{S}'(\mathbb{R})$ is called a *quasimeasure* if $\check{m}(t) = \mathscr{F}_{w \to t}^{-1}[m(w)] \in C_b(\mathbb{R})$. For more details, see [KK07, Appendix B].

Remark 4.3. The representation (4.6) implies that

$$
\hat{F}_J(w) = \hat{\jmath}(X_J, w), \quad 1 \leq J \leq N, \tag{4.7}
$$

$$
\hat{F}_1^+(w) + \hat{F}_1^-(w) = \hat{F}_1(w) = \hat{\jmath}(X_1, w), \qquad \hat{F}_N^+(w) + \hat{F}_N^-(w) = \hat{\jmath}(X_N, w), \tag{4.8}
$$

and also that

$$
\hat{\jmath}'(X_J + 0, w) = \hat{Q}_J(w), \quad 1 \leq J \leq N-1. \tag{4.9}
$$

Proof. **Step 1: Complex Fourier-Laplace transform.** We denote

$$
f_J^{\pm}(t) := q(\pm t) f_J(t) = q(t) F_J(y(X_J, t)) \tag{4.10}
$$

and split $\jmath(x,t)$ into

$$
\jmath(x,t) = \jmath^+(x,t) + \jmath^-(x,t), \qquad \text{where} \quad \jmath^{\pm}(x,t) := q(\pm t) \jmath(x,t). \tag{4.11}
$$

Then $\jmath^{\pm}(x,t)$ satisfy

$$
\ddot{\jmath}^{\pm}(x,t) = \partial_x^2 \jmath^{\pm}(x,t) - m^2 \jmath^{\pm}(x,t) + \sum_J d(x - X_J) f_J^{\pm}(t), \qquad t \in \mathbb{R}, \tag{4.12}
$$

since $(\jmath^{\pm}, \dot{\jmath}^{\pm})|_{t=0} = (0,0)$. Let us analyze the complex Fourier-Laplace transforms of $\jmath^{\pm}(x,t)$:

$$
\tilde{\jmath}^{\pm}(x, w) = \mathscr{F}_{t \to w}[q(\pm t) \jmath(x,t)] := \int_0^{\pm \infty} q(\pm t) e^{iwt} \jmath(x,t)\, dt, \quad w \in \mathbb{C}^{\pm}, \tag{4.13}
$$

where $\mathbb{C}^{\pm} := \{z \in \mathbb{C} : \pm \mathrm{Im}\, z > 0\}$. Due to (4.3), $\tilde{\jmath}^{\pm}(\cdot, w)$ are H^1-valued analytic functions of $w \in \mathbb{C}^{\pm}$. In what follows, we will consider $\jmath^+$; the function $\jmath^-$ considered in the same way.

Equation (4.12) implies that $\jmath^+$ satisfies

$$
-w^2 \tilde{\jmath}^+(x, w) = \partial_x^2 \tilde{\jmath}^+(x, w) - m^2 \tilde{\jmath}^+(x, w) + \sum_J d(x - X_J) \tilde{f}_J^+(w), \quad w \in \mathbb{C}^+. \tag{4.14}
$$

The fundamental solutions $G_{\pm}(x, w) = \dfrac{e^{\pm ik(w)|x|}}{\pm 2ik(w)}$ satisfy

$$
G_{\pm}''(x, w) + (w^2 - m^2) G_{\pm}(x, w) = d(x), \qquad w \in \mathbb{C}^+.
$$

The solution $\tilde{\jmath}(x, w)$ could be written as a linear combination of these fundamental solutions. We use the standard "limiting absorption principle" for the selection of the fundamental solution: Since $\tilde{\jmath}^+(\cdot, w) \in H^1$ for $w \in \mathbb{C}^+$, only G_+ is appropriate, because for $w \in \mathbb{C}^+$ the function $G_+(\cdot, w)$ is in H^1 by definition (3.5), while G_- is not. This suggests the following representation:

$$
\tilde{\jmath}^+(x, w) = - \sum_J \tilde{f}_J^+(w) G_+(x - X_J, w) = - \sum_J \tilde{f}_J^+(w) \frac{e^{ik(w)|x-X_J|}}{2ik(w)}, \qquad w \in \mathbb{C}^+. \tag{4.15}
$$

The proof is straightforward since (4.15) belongs to $H^1(\mathbb{R})$ for $w \in \mathbb{C}^+$ while the solution to (4.14) which is an H^1-valued analytic function in w is unique. For $x \leq X_1$, the relation (4.15) yields

$$
\tilde{\jmath}^+(x, w) = - \sum_J \tilde{f}_J^+(w) \frac{e^{-ik(w)(x-X_J)}}{2ik(w)} = e^{-ik(w)(x-X_1)} \tilde{\jmath}^+(X_1, w), \qquad x \leq X_1, \quad w \in \mathbb{C}^+. \tag{4.16}
$$

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For $x \in [X_J, X_{J+1}]$, the relation (4.15) implies that

$$\tilde{\jmath}^+(w) = \tilde{F}_J^+(w)\cos(k(w)(x-X_J)) + \tilde{Q}_J^+(w)\frac{\sin(k(w)(x-X_J))}{k(w)}, \qquad x \in [X_J, X_{J+1}], \quad w \in \mathbb{C}^+, \qquad (4.17)$$

where $\tilde{F}_J^+$ and $\tilde{Q}_J^+$, $1 \le J \le N-1$, are analytic functions of $w \in \mathbb{C}^+$. We note that, by (4.15),

$$\tilde{F}_J^+(w) = \tilde{\jmath}^+(X_J, w), \qquad \tilde{Q}_J^+(w) = \P_x \tilde{\jmath}^+(X_J+0, w) = -\sum_{J'} \mathrm{sgn}(X_J - X_{J'})\tilde{f}_{J'}^+(w)\frac{e^{ik(w)|X_J - X_{J'}|}}{2}. \qquad (4.18)$$

Step 2: Traces on real line. The Fourier transform $\hat{\jmath}^+(x, w) := \mathscr{F}_{t \to w}[q(t)\jmath(x,t)]$ is a tempered H^1-valued distribution of $w \in \mathbb{R}$ by (4.3). It is the boundary value of the analytic function $\tilde{\jmath}^+(x, w)$, in the following sense:

$$\hat{\jmath}^+(x, w) = \lim_{e \to 0+} \tilde{\jmath}^+(x, w + ie), \qquad w \in \mathbb{R}, \qquad (4.19)$$

where the convergence is in the space of tempered distributions $\mathscr{S}'(\mathbb{R}, H^1(\mathbb{R}))$. Indeed,

$$\tilde{\jmath}^+(x, w+ie) = \mathscr{F}_{t \to w}[q(t)\jmath(x,t)e^{-et}], \qquad q(t)\jmath(x,t)e^{-et} \xrightarrow[e \to 0+]{} q(t)\jmath(x,t)$$

where the convergence holds in $\mathscr{S}'(\mathbb{R}, H^1(\mathbb{R}))$. Therefore, (4.19) holds by the continuity of the Fourier transform $\mathscr{F}_{t \to w}$ in $\mathscr{S}'(\mathbb{R})$.

Similarly to (4.19), the distributions $\hat{F}_J^+(w)$, $\hat{Q}_J^+(w) \in \mathscr{S}'(\mathbb{R})$, $w \in \mathbb{R}$, are the boundary values of the functions $\tilde{F}_J^+(w)$ and $\tilde{Q}_J^+(w)$ analytic in $w \in \mathbb{C}^+$:

$$\hat{F}_J^+(w) = \lim_{e \to 0+} \tilde{F}_J^+(w + ie), \qquad w \in \mathbb{R}, \quad 0 \le J \le N, \qquad (4.20)$$

$$\hat{Q}_J^+(w) = \lim_{e \to 0+} \tilde{Q}_J^+(w + ie), \qquad w \in \mathbb{R}, \quad 1 \le J \le N-1. \qquad (4.21)$$

The above convergence holds in the space of quasimeasures by (4.18), since $\tilde{\jmath}^+(X_J, w)$ and $\tilde{f}_J^+(w)$ are quasimeasures (see Remark 4.2) while the exponential factors are multiplicators in the space of quasimeasures [KK07, Appendix B]. Therefore, the formulas (4.17) with $1 \le J \le N-1$ imply, in the limit $\mathrm{Im}\, w \to 0+$, that

$$\hat{\jmath}^+(x, w) = \hat{F}_J^+(w)\cos(k(w+i0)(x-X_J)) + \hat{Q}_J^+(w)\frac{\sin(k(w+i0)(x-X_J))}{k(w+i0)}, \qquad x \in [X_J, X_{J+1}], \quad w \in \mathbb{R}, \qquad (4.22)$$

since $\cos(k(w+i0)(x-X_J))$ and $\frac{\sin(k(w+i0)(x-X_J))}{k(w+i0)}$ are smooth functions of $w \in \mathbb{R}$. Similar representation holds for $\hat{\jmath}^-(x, w)$. Therefore, the representation (4.6) follows for $X_1 \le x \le X_N$.

The formula (4.6) for $x \le X_1$ follows from taking the limit $\mathrm{Im}\, w \to 0+$ in the expression (4.16) for $\tilde{\jmath}^+(x, w)$ and the limit $\mathrm{Im}\, w \to 0-$ in a similar expression for $\tilde{\jmath}^-(x, w)$:

$$\tilde{\jmath}^-(x, w) = -\sum_J \tilde{f}_J^-(w)\frac{e^{-ik(w)(x-X_J)}}{2ik(w)} = e^{-ik(w)(x-X_1)}\tilde{\jmath}^-(X_1, w), \qquad x \le X_1, \quad w \in \mathbb{C}^-, \qquad (4.23)$$

and then taking the sum of the resulting expressions. This justifies (4.6) for $x \le X_1$. Similarly we justify (4.6) for $x \ge X_N$. $\qquad \square$

5 Absolute continuity of the spectrum

Lemma 5.1. *The distributions $\hat{F}_1^\pm(w)$, $\hat{F}_N^\pm(w)$ are absolutely continuous for $|w| > m$, and moreover*

$$\int_{|w|>m} \left[|\hat{F}_1^\pm(w)|^2 + |\hat{F}_N^\pm(w)|^2\right] w\, k_+(w)\, dw < \yen, \qquad (5.1)$$

where $w\, k_+(w) \ge 0$ by (3.8).

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The bound for each of $\hat{F}_1^{\pm}(w)$, $\hat{F}_N^{\pm}(w)$ is obtained verbatim by applying the proof of [KK07, Proposition 3.3].

Proposition 5.2. *The distributions $\hat{F}_J(w)$, $1 \le J \le N$, and $\hat{Q}_J(w)$, $1 \le J \le N-1$, are absolutely continuous for $|w| > m_J$ and $|w| > (2p_J - 1)m_J$, respectively, with m_J defined in (2.18). Moreover, for any $e > 0$,*

$$\int_{|w|>m_J+e} |\hat{F}_J(w)|^2 w^2\, dw < \infty, \quad 1 \le J \le N; \qquad \int_{|w|>(2p_J-1)m_J+e} |\hat{Q}_J(w)|^2\, dw < \infty, \quad 1 \le J \le N-1. \tag{5.2}$$

Proof. We will use induction, proving the absolute continuity of $\hat{\jmath}(X_J, w)$ and $\partial_x \hat{\jmath}(X_J \pm 0, w)$ starting with $J = 1$ and going to $J = N$. By Lemma 4.1, $\hat{\jmath}(X_1, w) = \hat{F}_1(w) = \hat{F}_1^+(w) + \hat{F}_1^-(w)$ and $\partial_x \hat{\jmath}(X_1 - 0, w) = -ik_+(w)\hat{F}_1^+(w) - ik_-(w)\hat{F}_1^-(w)$. Hence, Lemma 5.1 implies that, for any $e > 0$,

$$\int_{|w|>m+e} |\hat{\jmath}(X_1, w)|^2 w^2\, dw < \infty, \qquad \int_{|w|>m+e} |\hat{\jmath}'(X_1 - 0, w)|^2\, dw < \infty. \tag{5.3}$$

Now assume that for some $1 \le J < N$ and for any $e > 0$ we have:

$$\int_{|w|>m_J+e} |\hat{\jmath}(X_J, w)|^2 w^2\, dw < \infty, \qquad \int_{|w|>m_J+e} |\hat{\jmath}'(X_J - 0, w)|^2\, dw < \infty. \tag{5.4}$$

Lemma 4.1 and equation (4.5) yield the jump condition

$$\hat{Q}_J(w) = \hat{\jmath}'(X_J + 0, w) = \hat{\jmath}'(X_J - 0, w) - \hat{f}_J(w), \qquad w \in \mathbb{R}, \tag{5.5}$$

where $f_J(t) = F_J(y(X_J, t))$ by (3.3).

Lemma 5.3. *For any $e > 0$ the following inequality holds:*

$$\int_{|w|>(2p_J-1)(m_J+2e)} |\hat{f}_J(w)|^2\, dw < \infty. \tag{5.6}$$

Proof. Let $z_J(w) \in C_0^{\infty}(\mathbb{R})$ be such that $z_J(w) \equiv 1$ for $|w| \le m_J + e$ and $z_J(w) \equiv 0$ for $|w| \ge m_J + 2e$. We denote $y(X_J, t)$ by $\psi_J(t)$, and split it into

$$\psi_J(t) = \psi_{J,b}(t) + \psi_{J,d}(t), \tag{5.7}$$

where the functions in the right-hand side are defined by their Fourier transforms:

$$\hat{\psi}_{J,b}(w) = z_J(w)\hat{\psi}_J(w) = z_J(w)\hat{y}(X_J, w), \qquad \hat{\psi}_{J,d}(w) = (1 - z_J(w))\hat{\psi}_J(w) = (1 - z_J(w))\hat{y}(X_J, w). \tag{5.8}$$

By Lemma 3.2 and by (5.4), we have

$$\int_{\mathbb{R}} |(1 - z_J(w))\hat{c}(X_J, w)|^2 w^2\, dw < \infty, \qquad \int_{\mathbb{R}} |(1 - z_J(w))\hat{\jmath}(X_J, w)|^2 w^2\, dw < \infty. \tag{5.9}$$

Since $\hat{\psi}_{J,d}(w) = (1 - z_J(w))(\hat{c}(X_J, w) + \hat{\jmath}(X_J, w))$, we also have

$$\int_{\mathbb{R}} \left| (1 - z_J(w))\hat{\psi}_J(w) \right|^2 w^2\, dw < \infty,$$

proving that

$$\psi_{J,d}(t) \in H^1(\mathbb{R}). \tag{5.10}$$

For $\hat{f}_J(w) = \mathscr{F}_{t \to w}[F_J(\psi_J(t))] = \mathscr{F}_{t \to w}[F_J(y(X_J, t))]$, taking into account (2.16) and (5.7), we have:

$$\hat{f}_J(w) = -\sum_{n=1}^{p_J} 2n u_{J,n} \underbrace{(\hat{\psi}_J * \bar{\hat{\psi}}_J) * \ldots * (\hat{\psi}_J * \bar{\hat{\psi}}_J)}_{n-1} * \hat{\psi}_J$$

$$= \ldots\ldots - \sum_{n=1}^{p_J} 2n u_{J,n} \underbrace{(\hat{\psi}_{J,b} * \bar{\hat{\psi}}_{J,b}) * \ldots * (\hat{\psi}_{J,b} * \bar{\hat{\psi}}_{J,b})}_{n-1} * \hat{\psi}_{J,b}, \tag{5.11}$$

where the dots in the right-hand side denote the convolutions of $\hat{\psi}_{J,b}$, $\bar{\hat{\psi}}_{J,b}$, $\hat{\psi}_{J,d}$, and $\bar{\hat{\psi}}_{J,d}$ that contain at least one of $\hat{\psi}_{J,d}$, $\bar{\hat{\psi}}_{J,d}$. Since $\psi_{J,b}(t)$, $\psi_{J,d}(t)$ are bounded while $\psi_{J,d}(t) \in H^1(\mathbb{R})$ by (5.10), all these terms belong to $L^2(\mathbb{R})$. Finally, since $\mathrm{supp}\,\hat{\psi}_{J,b} \subset [-m_J - 2e, m_J + 2e]$, the convolutions under the summation sign in the right-hand side of (5.11) are supported inside $[-(2p_J - 1)(m_J + 2e), (2p_J - 1)(m_J + 2e)]$ and do not contribute into the integral (5.6). $\square$

Using (5.4) and Lemma 5.3 to estimate the norms of $\partial_x \hat{\jmath}(X_J - 0, w)$ and $\hat{f}_J(w)$ in the right-hand side in the relation (5.5), we conclude that

$$\int\limits_{|w|>(2p_J-1)(m_J+2e)} |\hat{\jmath}'(X_J + 0, w)|^2\, dw < \infty. \tag{5.12}$$

Now the inequalities

$$\int\limits_{|w|>(2p_J-1)(m_J+2e)} |\hat{\jmath}(X_{J+1}, w)|^2 w^2\, dw < \infty, \qquad \int\limits_{|w|>(2p_J-1)(m_J+2e)} |\hat{\jmath}'(X_{J+1} - 0, w)|^2\, dw < \infty \tag{5.13}$$

follow from the representation (4.6) for $x \in [X_J, X_{J+1}]$, where we apply the first inequality from (5.4) and the inequality (5.12). Therefore, starting with (5.3), one shows by induction that (5.4) holds for all $1 \le J \le N$. The estimates on $\hat{F}_J(w) = \hat{\jmath}(X_J, w)$ and $\hat{Q}_J(w) = \hat{\jmath}'(X_J + 0, w)$ stated in the Proposition follow from (5.4) and (5.12), respectively. This finishes the proof of Proposition 5.2. $\qquad\square$

Corollary 5.4. *The distributions* $\hat{F}_J(w) = \hat{\jmath}(X_J, w)$, $1 \le J \le N$, *are absolutely continuous for* $|w| > M_J$, *while* $\hat{Q}_J(w) = \partial_x \hat{\jmath}(X_J + 0, w)$, $1 \le J \le N - 1$, *are absolutely continuous for* $|w| > (2p_J - 1)M_J$, *where* $M_J := \min(m_J, m_J')$ *is defined in (2.19).*

Proof. In the proof of Proposition 5.2, we could as well proceed from $J = N$ to $J = 1$, proving the result stated in the Corollary. $\qquad\square$

6 Compactness

Second dispersive component

Let $z(w) \in C_0^\infty(\mathbb{R})$ be such that $z(w) \equiv 1$ for $|w| < L$, where L is from (2.19). Define $\jmath_d(x,t)$ by its Fourier transform:

$$\hat{\jmath}_d(x, w) := (1 - z(w))\hat{\jmath}(x, w) \quad x \in \mathbb{R}, \ w \in \mathbb{R}. \tag{6.1}$$

Lemma 6.1. $\jmath_d(x,t)$ *is a bounded continuous function of* $t \in \mathbb{R}$ *with values in* $H^1(\mathbb{R})$:

$$\jmath_d(x,t) \in C_b(\mathbb{R}, H^1(\mathbb{R})). \tag{6.2}$$

The local energy decay holds for $\jmath_d(x,t)$:

$$\lim_{t\to\infty} \|(\jmath_d, \dot{\jmath}_d)\|_{\mathscr{E},R} = 0, \qquad \forall R > 0. \tag{6.3}$$

Proof. We generalize the proof of [KK07, Proposition 3.6]. By Lemma 4.1,

$$\hat{\jmath}_d(x,w) = \begin{cases} (1 - z(w))\left[\hat{F}_1^+(w)e^{-ik_+(w)(x-X_1)} + \hat{F}_1^-(w)e^{-ik_-(w)(x-X_1)}\right], & x \le X_1, \\ (1 - z(w))\hat{F}_J(w)\cos(k_+(w)(x-X_J)) + (1 - z(w))\hat{Q}_J(w)\frac{\sin(k_+(w)(x-X_J))}{k_+(w)(x-X_J)}, & x \in [X_J, X_{J+1}], \\ (1 - z(w))\left[\hat{F}_N^+(w)e^{ik_+(w)(x-X_N)} + \hat{F}_N^-(w)e^{ik_-(w)(x-X_N)}\right], & x \ge X_N. \end{cases} \tag{6.4}$$

Each of the functions entering the above expression, considered on the whole real line, corresponds to a finite energy solution to a linear Klein-Gordon equation, satisfying the properties stated in the lemma. For example, define $u(x,t)$ by its Fourier transform:

$$\hat{u}(x,w) := (1 - z(w))\hat{F}_1(w)\cos(k_+(w)(x-X_1)), \qquad x \in \mathbb{R}.$$

Then $u(x,t)$ is a solution to a linear Klein-Gordon equation, and by Proposition 5.2 the corresponding initial data are of finite energy:

$$(u(x,0), \dot{u}(x,0)) \in \mathscr{E}.$$

Hence $u(x,t) \in C_b(\mathbb{R}, H^1(\mathbb{R}))$ and satisfies the local energy decay of the form (6.3) (see [KK07, Lemma 3.1]. This finishes the proof. $\qquad\square$

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Compactness for the bound component

We introduce the bound component of $j(x,t)$ by

$$j_b(x,t) = j(x,t) - j_d(x,t) = y(x,t) - c(x,t) - j_d(x,t), \qquad x \in \mathbb{R}, \ t \in \mathbb{R}. \tag{6.5}$$

By Lemma 6.1,

$$j_b(x,t) \in C_b(\mathbb{R}, H^1(\mathbb{R})). \tag{6.6}$$

Lemma 4.1 and (6.1), (6.5) imply the multiplicative relation

$$\hat{j}_b(x,w) = \begin{cases} z(w)\big[\hat{F}_1^+(w)e^{-ik_+(w)(x-X_1)} + \hat{F}_1^-(w)e^{-ik_-(w)(x-X_1)}\big], & x \le X_1, \\ z(w)\big[\hat{F}_J(w)\cos(k_+(w)(x-X_J)) + \hat{Q}_J(w)\frac{\sin(k_+(w)(x-X_J))}{k_+(w)}\big], & x \in [X_J, X_{J+1}], \\ z(w)\big[\hat{F}_N^+(w)e^{ik_+(w)(x-X_N)} + \hat{F}_N^-(w)e^{ik_-(w)(x-X_N)}\big], & x \ge X_N. \end{cases} \tag{6.7}$$

By (6.6), the functions

$$j_{b,J}(t) := j_b(X_J,t) = j(X_J,t) - j_d(X_J,t)$$

are bounded and continuous. Therefore, $\hat{j}_b(X_J,\cdot) \in \mathscr{S}'(\mathbb{R})$ are quasimeasures (see Remark 4.2).

Proposition 6.2. *(i) The function $j_b(x,t)$ is smooth for $x \in \mathbb{R} \setminus \mathscr{X}$ (where $\mathscr{X} = \{X_1, X_2, \ldots, X_N\}$) and $t \in \mathbb{R}$.*

(ii) For any $R > 0$,

$$\sup_{|x| \le R, x \notin \mathscr{X}} \sup_{t \in \mathbb{R}} |\partial_x^m \partial_t^n j_b(x,t)| < \infty. \tag{6.8}$$

The argument repeats the proof of Proposition [KK07, Proposition 4.1].

Remark 6.3. Let us note that the bounds (6.8) are independent of x and remain valid for $x \notin \mathscr{X}$, although the derivatives $\partial_x^m \partial_t^n j_b(x,t)$ with $m \ne 0$ may have jumps at $x = X_J$. (Note that this is the case for the solitary waves in (2.14).)

We now may deduce the compactness of the set of translations of the bound component, $\{j_b(x,s+t) : s \ge 0\}$.

Corollary 6.4. *(i) By the Ascoli-Arzelà Theorem, for any sequence $s_j \to \infty$ there exists a subsequence $s_{j'} \to \infty$ such that for any nonnegative integers m and n,*

$$\partial_x^m \partial_t^n j_b(x, s_{j'}+t) \to \partial_x^m \partial_t^n b(x,t), \qquad x \notin \mathscr{X}, \quad t \in \mathbb{R}, \tag{6.9}$$

for some $b(x,t) \in C_b(\mathbb{R}, H^1(\mathbb{R}))$. The convergence in (6.9) is uniform in x and t as long as $|x| + |t| \le R$, for any $R > 0$.

(ii) By the Fatou Lemma,

$$\sup_{t \in \mathbb{R}} \|b(\cdot,t)\|_{H^1} < \infty. \tag{6.10}$$

We call *omega-limit trajectory* any function $b(x,t)$ that can appear as a limit in (6.9).

Remark 6.5. Previous analysis demonstrates that the long-time asymptotics of the solution $y(x,t)$ in $\mathscr{E}_F$ depends only on the singular component $j(x,t)$. Due to Corollary 6.4, to conclude the proof of Theorem 2.11, it suffices to check that every omega-limit trajectory belongs to the set of solitary waves; that is,

$$b(x,t) = f_{w_+}(x)e^{-iw_+t} \qquad \text{for some } w_+ \in [-m,m]. \tag{6.11}$$

7 Nonlinear spectral analysis

Bounds for the spectrum

By Lemmas 3.1 and 6.1, the dispersive components $c(\cdot,t)$ and $j_d(\cdot,t)$ converge to zero in $\mathscr{E}_F$ as $t \to \infty$. On the other hand, by Corollary 6.4, the bound component $j_b(x, t+s_{j'})$ converges to $b(x,t)$ as $j' \to \infty$, uniformly in every compact set of the plane $\mathbb{R}^2$. Hence, $y(x, t+s_{j'}) = j_b(x, t+s_{j'}) + c(x, t+s_{j'}) + j_d(x, t+s_{j'})$ also converges to $b(x,t)$ uniformly in every compact set of the plane $\mathbb{R}^2$. Therefore, taking the limit in equation (2.1), we conclude that the omega-limit trajectory $b(x,t)$ also satisfies the same equation:

$$\ddot{b}(x,t) = b''(x,t) - m^2 b(x,t) + \sum_J d(x-X_J)F_J(b). \tag{7.1}$$

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Remark 7.1. Note that the bound component $j_b(x,t)$ itself generally does not satisfy equation (7.1).

Taking the Fourier transform of b in time, we see by (6.9) that $\hat{b}(x,w)$ is a continuous function of $x \in \mathbb{R}$, smooth for $x \in \mathbb{R}\setminus\mathcal{X}$, with values in tempered distributions of $w \in \mathbb{R}$, and that it satisfies the corresponding stationary equation

$$-w^2\hat{b}(x,w) = \hat{b}''(x,w) - m^2\hat{b}(x,w) + \sum_J d(x - X_J)\hat{g}_J(w), \qquad (x,w) \in \mathbb{R}^2, \tag{7.2}$$

valid in the sense of tempered distributions of $(x,w) \in \mathbb{R}^2$, where $\hat{g}_J(w)$ are the Fourier transforms of the functions

$$g_J(t) := F_J(b(X_J,t)), \qquad 1 \le J \le N. \tag{7.3}$$

We also denote

$$\beta_J(t) := b(X_J,t), \qquad \mathsf{S}_J := \operatorname{supp}\hat{\beta}_J, \qquad 1 \le J \le N. \tag{7.4}$$

From (6.7), we know that the spectrum of $j_b(x,t)$ is bounded for all $x \in \mathbb{R}$. Hence, the convergence (6.9) implies that the spectrum of $b(x,t)$ is also bounded. We will need more precise bounds on the size of the spectrum of b:

Lemma 7.2. *(i)* $\mathsf{S}_J \subset [-M_J, M_J], \quad 1 \le J \le N;$

(ii) $\operatorname{supp}\hat{b}'(X_J + 0, w) \subset [-(2p_J - 1)M_J, (2p_J - 1)M_J], \quad 1 \le J \le N - 1$, with $M_J > 0$ defined in (2.19).

Proof. For every $x \in \mathbb{R}$, we have formally the relations

$$j(x,s_j+t) = \frac{1}{2p}\int_{\mathbb{R}} e^{-iwt}e^{-iws_j}\hat{j}(x,w)\,dw, \qquad \P_x j(x,s_j+t) = \frac{1}{2p}\int_{\mathbb{R}} e^{-iwt}e^{-iws_j}\P_x\hat{j}(x,w)\,dw, \qquad t \in \mathbb{R}.$$

Then the convergence (6.9) implies that

$$e^{-iws_{j'}}\hat{j}(x,w) \to \hat{b}(x,w), \qquad e^{-iws_{j'}}\P_x\hat{j}(x,w) \to \P_x\hat{b}(x,w), \qquad s_{j'} \to ¥, \tag{7.5}$$

in the sense of quasimeasures. Since $\hat{j}(X_J,w)$ are locally L^2 for $|w| > M_J$ while $\hat{j}'(X_J + 0, w)$ are locally L^2 for $|w| > (2p_J - 1)M_J$ by Corollary 5.4, the convergence (7.5) at $x = X_J$ shows that $\hat{\beta}_J(w)$ and $\hat{b}'(X_J + 0, w)$ vanish for $|w| > M_J$ and $|w| > (2p_J - 1)M_J$, respectively. $\qquad\square$

We denote

$$k(w) := -ik_+(w), \qquad w \in \mathbb{R}, \tag{7.6}$$

where $k_+(w)$ was introduced in (3.6). We then have $\operatorname{Re} k(w) \ge 0$, and also

$$k(w) = \sqrt{w^2 - m^2} > 0 \quad \text{for} \quad -m < w < m,$$

in accordance with (2.14).

Proposition 7.3. *The distribution* $\hat{b}(x,w)$ *admits the following representation:*

$$\hat{b}(x,w) = \begin{cases} \hat{\beta}_1(w)e^{k(w)(x-X_1)}, & x \le X_1, \\ \hat{\beta}_J(w)\cosh(k(w)(x-X_J)) + \hat{b}'(X_J + 0, w)\frac{\sinh(k(w)(x-X_J))}{k(w)}, & x \in [X_J, X_{J+1}], \quad 1 \le J \le N-1, \\ \hat{\beta}_N(w)e^{-k(w)(x-X_N)}, & x \ge X_N. \end{cases} \tag{7.7}$$

Proof. By (7.5), the middle line in (7.7) follows from the representation (4.6) since the multiplicators are smooth bounded functions of $w \in \mathbb{R}$. Taking the limit in the first line of (4.6), we obtain the first line in (7.7) since $\mathsf{S}_1 \subset [-m,m]$ by Lemma 7.2, while $k_+(w) = k_-(w) = ik_+(w)$ for $-m \le w \le m$ (Cf. (3.7), (7.6)). Similarly we explain the last line in (7.7). $\qquad\square$

Reduction to point spectrum

Proposition 7.4. *Any omega-limit trajectory $b(x,t)$ is a solitary wave, i.e. $b(x,t) = f(x)e^{-iw_+ t}$ with $w_+ \in [-m,m]$ and $f(x) \in H^1(\mathbb{R})$.*

Proof. The proof is based on the following lemmas.

Lemma 7.5. *If $S_1 = \emptyset$, then $b(x,t) \equiv 0$.*

Proof. According to equation (7.2), the function $\hat{b} \in C(\mathbb{R}, \mathscr{S}'(\mathbb{R}))$ satisfies the following continuity and jump conditions at the point X_1:

$$\hat{b}(X_1 + 0, w) = \hat{b}(X_1 - 0, w) = \hat{\beta}_1(w), \qquad \hat{b}'(X_1 + 0, w) = \hat{b}'(X_1 - 0, w) + \hat{g}_1(w), \qquad w \in \mathbb{R}. \tag{7.8}$$

$S_1 = \emptyset$ means that $\hat{\beta}_1(w) \equiv 0$, that is, $\beta_1(t) \equiv 0$. Hence, $g_1(t) \equiv F_1(\beta_1(t)) \equiv 0$, and $\hat{g}_1(w) \equiv 0$. On the other hand, first line of (7.7) implies that $\hat{b}(x, w) \equiv 0$ for $x \leq X_1$, and in particular $\hat{b}'(X_1 - 0, w) \equiv 0$. Therefore, the jump condition (7.8) implies that $\hat{b}'(X_1 + 0, w) \equiv 0$. Hence, $\hat{b}(x, w) \equiv 0$ for $x \in [X_1, X_2]$ by the middle line of (7.7). By induction, $\hat{\beta}_J(x, w) \equiv 0$. $\qquad\square$

Now we consider the case $S_1 \neq \emptyset$.

Lemma 7.6. *If $S_1 \neq \emptyset$, then $S_1 = \{w_+\}$ for some $w_+ \in [-m, m]$.*

Proof. By Lemma 7.2, we know that $S_1 \subset [-m, m]$. To show that S_1 consists of a single point, we assume that, on the contrary, $\inf S_1 < \sup S_1$. By (2.16), the Fourier transform $\hat{g}_1(w)$ of $g_1(t) := F_1(b(X_1, t))$ is given by

$$\hat{g}_1 = - \sum_{n=1}^{p_1} 2n u_{1,n} \underbrace{(\hat{\beta}_1 * \hat{\bar{\beta}}_1) * \ldots * (\hat{\beta}_1 * \hat{\bar{\beta}}_1)}_{n-1} * \hat{\beta}_1. \tag{7.9}$$

Applying the Titchmarsh Convolution Theorem [Tit26] (see also [Lev96, p.119] and [Hör90, Theorem 4.3.3]) to the convolutions in (7.9), we obtain the following equalities:

$$\inf \operatorname{supp} \hat{g}_1 = \inf \operatorname{supp} \hat{\beta}_1 + (p_1 - 1) \inf \operatorname{supp}(\hat{\beta}_1 * \hat{\bar{\beta}}_1) = \inf S_1 + (p_1 - 1)(\inf S_1 - \sup S_1), \tag{7.10}$$

$$\sup \operatorname{supp} \hat{g}_1 = \sup \operatorname{supp} \hat{\beta}_1 + (p_1 - 1) \sup \operatorname{supp}(\hat{\beta}_1 * \hat{\bar{\beta}}_1) = \sup S_1 + (p_1 - 1)(\sup S_1 - \inf S_1), \tag{7.11}$$

where we used the relations $\inf \operatorname{supp} \hat{\bar{\beta}}_1 = -\sup \operatorname{supp} \hat{\beta}_1$, $\sup \operatorname{supp} \hat{\bar{\beta}}_1 = -\inf \operatorname{supp} \hat{\beta}_1$. Note that the Titchmarsh theorem is applicable since $\operatorname{supp} \hat{\beta}_1$ is compact by Lemma 7.2. Since we assumed that $\inf S_1 < \sup S_1$, (7.10) and (7.11) imply that $\inf \operatorname{supp} \hat{g}_1 < \inf S_1$, $\sup \operatorname{supp} \hat{g}_1 > \sup S_1$. Therefore, the jump condition (7.8) with $J = 1$ implies that

$$\inf \operatorname{supp} \hat{b}'(X_1 + 0, \cdot) = \inf \operatorname{supp} \hat{g}_1 < \inf S_1, \qquad \sup \operatorname{supp} \hat{b}'(X_1 + 0, \cdot) = \sup \operatorname{supp} \hat{g}_1 > \sup S_1. \tag{7.12}$$

The ratio $\sinh(k(w)(X_2 - X_1))/k(w)$ could only vanish at certain points from $\mathscr{W}$ (see (2.21)), while $\operatorname{supp} \hat{b}'(X_1 + 0, w) \cap \mathscr{W} = \emptyset$ due to Lemma 7.2 and the condition (2.23). Hence, the middle line of (7.7) at $x = X_2 - 0$ and the inequalities (7.12) imply that

$$\inf S_2 = \inf \operatorname{supp} \hat{g}_1 < \inf S_1, \qquad \sup S_2 = \sup \operatorname{supp} \hat{g}_1 > \sup S_1. \tag{7.13}$$

We proceed by induction, proving that

$$\inf S_1 > \inf S_2 > \ldots > \inf S_N, \qquad \sup S_1 < \sup S_2 < \ldots < \sup S_N. \tag{7.14}$$

It then follows that $\inf S_N < \sup S_N$. Starting from $J = N$ and going to the left, we also prove the opposite inequalities:

$$\inf S_1 < \inf S_2 < \ldots < \inf S_N, \qquad \sup S_1 > \sup S_2 > \ldots > \sup S_N. \tag{7.15}$$

The contradiction of (7.14) and (7.15) shows that our assumption that $\inf S_1 < \sup S_1$ was false, hence $S_1 = \{w_+\}$ for some $w_+ \in [-m, m]$. $\qquad\square$

Thus, $\operatorname{supp} \hat{b}_1(w) = S_1 \subset \{w_+\}$, with $w_+ \in [-m, m]$. Therefore,

$$\hat{b}_1(w) = a_1 d(w - w_+), \qquad \text{with some } a_1 \in \mathbb{C}. \tag{7.16}$$

Note that the derivatives $d^{(k)}(w - w_+)$, $k \geq 1$ do not enter the expression for $\hat{b}_1(w) = \mathscr{F}_{t \to w}[b(X_1, t)]$ since $b(x, t)$ is a bounded continuous function of $(x, t) \in \mathbb{R}^2$ due to the bound (6.10).

Lemma 7.7. $\hat{b}(x, w) = a(x) d(w - w_+)$, *where $a(x)$ is a bounded continuous function.*

Proof. For $x \leq X_1$, the representation stated in the lemma follows from the first line in (7.7) and from (7.16). Let us prove this representation for $X_1 \leq x \leq X_2$. By (7.16), we have $b_1(t) := b(X_1, t) = a_1 e^{-iw_+ t}/2p$, hence $g_1(t) := F_1(b_1(t)) = b_1 e^{-iw_+ t}$ for some $b_1 \in \mathbb{C}$ due to the $U(1)$-invariance (1.2). Therefore, $\hat{g}_1(w) = 2pb_1 d(w - w_+)$. Moreover, by (7.7), we have $\hat{b}'(X_1 - 0, w) = k(w_+)a_1 d(w - w_+)$. Hence, the jump condition (7.8) implies that $\hat{b}'(X_1 + 0, w) = c_1 d(w - w_+)$, for some $c_1 \in \mathbb{C}$. Finally, (7.7) implies that $\hat{b}(x, w) = a(x)d(w - w_+)$ for $x \in [X_1, X_2]$, with $a(x)$ a continuous complex-valued function of x. Proceeding by induction, we obtain similar representation for $\hat{b}(x, w)$ for all $x \in \mathbb{R}$. $\qquad\square$

Now we can finish the proof of Proposition 7.4. Lemma 7.7 implies that $b(x, t) = f(x)e^{-iwt}$, where $f(x) = a(x)/2p$. We conclude from (6.10) that $f \in H^1(\mathbb{R})$, finishing the proof of Proposition 7.4. Note that $w = \pm m$ could only correspond to the zero solution (see Remark 2.9). $\qquad\square$

According to Remark 6.5, Proposition 7.4 completes the proof of Theorem 2.11.

8 Multifrequency solitary waves

We will show that when the assumptions of Theorem 2.11 are not satisfied, then the attractor could be more complicated because the equation admits multifrequency solitary wave solutions.

8.1 Wide gaps

Let us consider equation (2.1) with $N = 2$, under Assumptions 2.1 and 2.2.

Proposition 8.1. *If the Assumption 2.3 is violated, then the conclusion of Theorem 2.11 may no longer be correct.*

Proof. We will show that if $L := X_2 - X_1$ is sufficiently large, then one can take $F_1(y)$ and $F_2(y)$ satisfying Assumptions 2.1 and 2.2 such that the global attractor of the equation contains the multifrequency solutions which do not converge to solitary waves of the form (2.13). For our convenience, we assume that $X_1 = 0$, $X_2 = L$. We consider the model (2.1) with the nonlinearity

$$F_1(y) = F_2(y) = F(y), \qquad \text{where} \quad F(y) = a y + b|y|^2 y, \qquad a, b \in \mathbb{R}. \tag{8.1}$$

In terms of the condition (2.16), $p_1 = p_2 = 2$. We take L to be large enough:

$$L > \frac{p}{2^{3/2}m}. \tag{8.2}$$

Consider the function

$$y(x, t) = A(e^{-k(w)|x|} + e^{-k(w)|x-L|}) \sin(wt) + B c_{[0,L]}(x) \sin(k(3w)x) \sin(3wt), \qquad A, B \in \mathbb{C}. \tag{8.3}$$

Then $y(x, t)$ solves (2.1) for x away from the points X_J. We require that

$$k(3w) = \frac{p}{L}, \tag{8.4}$$

so that $y(x, t)$ is continuous in $x \in \mathbb{R}$ and symmetric with respect to $x = L/2$:

$$y(x, t) = y\left(\frac{L}{2} - x, t\right), \qquad x \in \mathbb{R}.$$

We need $|w| < m$ to have $k(w) > 0$, and $3|w| > m$ to have $k(3w) \in \mathbb{R}$. We take $w > 0$, and thus $m < 3w < 3m$. By (8.4), this means that we need

$$m < \sqrt{\frac{p^2}{L^2} + m^2} < 3m.$$

The second inequality is satisfied by (8.2).

Due to the symmetry of $y(x,t)$ with respect to $x = L/2$, the jump condition (7.8) both at $x = 0$ and at $x = L$ takes the following identical form:

$$2Ak(w)\sin wt - Bk(3w)\sin 3wt = F\left(A(1 + e^{-k(w)L})\sin(wt)\right). \tag{8.5}$$

Using the identity

$$\sin^3 q = \frac{3}{4}\sin q - \frac{1}{4}\sin 3q, \tag{8.6}$$

we see that

$$F(A(1 + e^{-k(w)L})\sin wt) = \left(aA(1 + e^{-k(w)L}) + \frac{3}{4}b|A|^2 A(1 + e^{-k(w)L})^3\right)\sin(wt) - \frac{1}{4}b|A|^2 A(1 + e^{-k(w)L})^3\sin(3wt). \tag{8.7}$$

Collecting in (8.5) the terms at $\sin wt$ and at $\sin 3wt$, we obtain the following system:

$$\begin{cases} 2Ak(w) = aA(1 + e^{-k(w)L}) + \frac{3}{4}b|A|^2 A(1 + e^{-k(w)L})^3, \\ Bk(3w) = \frac{1}{4}b|A|^2 A(1 + e^{-k(w)L})^3. \end{cases} \tag{8.8}$$

Assuming that $A \neq 0$, we divide the first equation by A:

$$2k(w) = a(1 + e^{-k(w)L}) + \frac{3}{4}b|A|^2(1 + e^{-k(w)L})^3. \tag{8.9}$$

The condition for the existence of a solution $A \neq 0$ is

$$\left(\frac{2k(w)}{1 + e^{-k(w)L}} - a\right)b > 0. \tag{8.10}$$

Once we found A, the second equation in (8.8) can be used to express B in terms of A.

Remark 8.2. Condition (8.10) shows that we can choose $b < 0$ taking large $a > 0$. The corresponding potential $U(y) = -a|y|^2/2 - b|y|^4/4$ satisfies (2.12) and Assumptions 2.1 and 2.2.

$\square$

8.2 Linear degeneration

Let us consider equation (2.1) with $N = 2$, under Assumptions 2.1 and 2.3.

Proposition 8.3. *If the Assumption 2.2 is violated, then the conclusion of Theorem 2.11 may no longer be correct.*

Proof. Again, we construct multifrequency solutions. Consider the equation

$$\ddot{y} = y'' - m^2 y + d(x)F_1(y) + d(x - L)F_2(y), \tag{8.11}$$

where

$$F_1(y) = ay + b|y|^2 y, \qquad F_2(y) = gy, \qquad a, b, g \in \mathbb{R}. \tag{8.12}$$

Note that the function F_2 is linear, failing to satisfy Assumption 2.2. The function

$$y(x,t) = \begin{cases} (A + B)e^{k(w)x}\sin(wt), & x \leq 0, \\ \left(Ae^{-k(w)x} + Be^{k(w)x}\right)\sin(wt) + C\sinh(k(3w)x)\sin(3wt), & x \in [0, L], \\ \left(Ae^{-k(w)} + Be^{k(w)(2L-x)}\right)\sin(wt) + \frac{C}{\sinh(k(3w)L)}e^{-k(3w)(x-L)}\sin(3wt), & x \geq L, \end{cases}$$

where $w \in (0, m/3)$, will be a solution if the jump conditions are satisfied at $x = 0$ and at $x = L$:

$$-y'(0+,t) + y'(0-,t) = a\, y(0,t) + b\, y^3(0,t),\tag{8.13}$$

$$-y'(L+,t) + y'(L-,t) = a\, y(L,t) + b\, y^3(L,t).\tag{8.14}$$

We use the identity

$$a\,(A+B)\sin(wt) + b((A+B)\sin(wt))^3 = \left(a\,(A+B) + b\frac{3(A+B)^3}{4}\right)\sin(wt) - b\frac{(A+B)^3}{4}\sin(3wt)$$

which follows from (8.6). Collecting the terms at $\sin(wt)$ and at $\sin(3wt)$, we see that the condition (8.13) takes the form

$$2k(w)A = \left(a\,(A+B) + b\frac{3(A+B)^3}{4}\right),\tag{8.15}$$

$$-k(3w)C = -b\frac{(A+B)^3}{4}.\tag{8.16}$$

The condition (8.14) takes the form

$$2Bk(w)e^{k(w)L} = g(Ae^{-k(w)L} + Be^{k(w)L}),\tag{8.17}$$

$$\frac{k(3w)C}{\sinh(k(3w)L)} + k(3w)C\cosh(k(3w)L) = gC\sinh(k(3w)L).\tag{8.18}$$

Equations (8.15), (8.16), (8.17), and (8.18) could be satisfied for arbitrary $L > 0$. Namely, for any $w \in (0, m/3)$, one uses (8.18) to determine g. For any $b \neq 0$, there is always a solution A, and B to the nonlinear system (8.15), (8.17). Finally, C is obtained from (8.16).

$\square$

A Global well-posedness

Here we prove Theorem 2.4. We first need to adjust the nonlinearity F so that it becomes bounded, together with its derivatives. Define

$$1_0 = \sqrt{\frac{\mathcal{H}(y_0, p_0) - {}_J A_J}{m - {}_J B_J}},\tag{A.1}$$

where $(y_0, p_0) \in \mathscr{E}$ is the initial data from Theorem 2.4 and A_J, B_J are constants from (2.12). Then we may pick a modified potential function $\widetilde{U}_J \in C^2(\mathbb{C}, \mathbb{R})$, $\widetilde{U}_J(y) = \widetilde{U}_J(|y|)$, $j = 1, 2$, so that

$$\widetilde{U}_J(y) = U_J(y) \qquad \text{for } |y| \leq 1_0, \quad y \in \mathbb{C},\tag{A.2}$$

$\widetilde{U}_J(y)$ satisfy (2.12) with the same constants A_J, B_J as $U_J(y)$ do:

$$\widetilde{U}_J(y) \geq A_J - B_J|y|^2, \quad \text{for } y \in \mathbb{C}, \quad \text{where} \quad A_J \in \mathbb{R}, \quad B_J \geq 0, \quad 1 \leq J \leq N, \quad {}_J B_J < m,\tag{A.3}$$

and so that $|\widetilde{U}_J(y)|$, $|\widetilde{U}'_J(y)|$, and $|\widetilde{U}''_J(y)|$ are bounded for $y \geq 0$. We define

$$\widetilde{F}_J(y) = -\ \widetilde{U}_J(y), \qquad y \in \mathbb{C},\tag{A.4}$$

where $\ $ denotes the gradient with respect to $\mathrm{Re}\, y$, $\mathrm{Im}\, y$; Then $\widetilde{F}_J(e^{is}y) = e^{is}\widetilde{F}_J(y)$ for any $y \in \mathbb{C}$, $s \in \mathbb{R}$.

We consider the Cauchy problem of type (2.1) with the modified nonlinearity,

$$\begin{cases} \ddot{y}(x,t) = y''(x,t) - m^2 y(x,t) + {}_J d(x - X_J)\widetilde{F}_J(y(X_J,t)), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ y|_{t=0} = y_0(x), \qquad \dot{y}|_{t=0} = p_0(x). \end{cases}\tag{A.5}$$

Equation (A.5) formally can be written as the following Hamiltonian system (Cf. (2.5)):

$$\dot{Y}(t) = \mathscr{J} D\widetilde{\mathscr{H}}(Y), \qquad \mathscr{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{A.6}$$

where $D\widetilde{\mathscr{H}}$ is the variational derivative of the Hamilton functional

$$\widetilde{\mathscr{H}}(Y) = \int_{\mathbb{R}} \left(|p|^2 + |\ y\ |^2 + m^2|y|^2 \right) dx + \sum_J \widetilde{U}_J(y(X_J,t)), \quad Y = \begin{bmatrix} y(x) \\ p(x) \end{bmatrix} \in \mathscr{E}, \tag{A.7}$$

which is Fréchet differentiable in the space $\mathscr{E} = H^1 \times L^2$. By the Sobolev embedding theorem, $L^{\maltese}(\mathbb{R}) \subset H^1(\mathbb{R})$, and there is the following inequality:

$$\|y\|_{L^{\maltese}}^2 \le \frac{1}{2m}(\|y'\|_{L^2}^2 + m^2\|y\|_{L^2}^2) \le \frac{1}{2m}\|Y\|_{\mathscr{E}}^2. \tag{A.8}$$

Thus, (A.3) leads to

$$\widetilde{U}_J(y(0)) \ge A_J - B_J\|y\|_{L^{\maltese}}^2 \ge A_J - \frac{B_J}{2m}\|Y\|_{\mathscr{E}}^2. \tag{A.9}$$

Taking into account (A.7), we obtain the inequality

$$\|Y\|_{\mathscr{E}}^2 = 2\widetilde{\mathscr{H}}(Y) - 2\sum_J \widetilde{U}_J(y(X_J)) \le 2\widetilde{\mathscr{H}}(Y) - 2\sum_J A_J + \frac{\sum_J B_J}{m}\|Y\|_{\mathscr{E}}^2, \qquad Y \in \mathscr{E}. \tag{A.10}$$

It follows that

$$\|Y\|_{\mathscr{E}}^2 \le \frac{2m}{m - \sum_J B_J}\left(\widetilde{\mathscr{H}}(Y) - \sum_J A_J\right), \qquad Y \in \mathscr{E}. \tag{A.11}$$

Lemma A.1. *(i) There is the identity $\widetilde{\mathscr{H}}(Y_0) = \mathscr{H}(Y_0)$.*

(ii) If $Y = \begin{bmatrix} y(x) \\ p(x) \end{bmatrix} \in \mathscr{E}$ satisfies $\widetilde{\mathscr{H}}(Y) \le \widetilde{\mathscr{H}}(Y_0)$, then $\widetilde{U}_J(y(x)) = U_J(y(x))$ for any $x \in \mathbb{R}$.

Proof. According to (A.11), the Sobolev embedding (A.8), and the choice of l_0 in (A.1),

$$\|y_0\|_{L^{\maltese}}^2 \le \frac{1}{2m}\|Y_0\|_{\mathscr{E}}^2 \le \frac{\mathscr{H}(Y_0) - \sum_J A_J}{m - \sum_J B_J} = l_0^2. \tag{A.12}$$

Thus, by (A.2), $\widetilde{U}(y_0(x)) = U(y_0(x))$ for all $x \in \mathbb{R}$. This proves (i).

By (A.8), the relation (A.11), the condition $\widetilde{\mathscr{H}}(Y) \le \widetilde{\mathscr{H}}(Y_0)$, and part (i) of the Lemma, we have:

$$\|y\|_{L^{\maltese}}^2 \le \frac{1}{2m}\|Y\|_{\mathscr{E}}^2 \le \frac{\widetilde{\mathscr{H}}(Y) - \sum_J A_J}{m - \sum_J B_J} \le \frac{\widetilde{\mathscr{H}}(Y_0) - \sum_J A_J}{m - \sum B_J} = \frac{\mathscr{H}(Y_0) - \sum_J A_J}{m - \sum_J B_J} = l_0^2.$$

Now the statement (ii) follows by (A.2). $\qquad\square$

If $Y(t)$ solves (A.6), then $\widetilde{\mathscr{H}}(Y(t)) = \widetilde{\mathscr{H}}(Y_0)$, By Lemma A.1 (ii), $\widetilde{U}_J(y(x,t)) = U_J(y(x,t))$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}$. Hence, $\widetilde{F}_J(y(x,t)) = F_J(y(x,t))$ for all $x \in \mathbb{R}$, $t \ge 0$, allowing us to conclude that $y(t)$ solves (2.1) as well as (A.5). The rest of the proof of Theorem 2.4 repeats the proof of a similar result for the case $N = 1$ [KK07, Theorem 2.3].

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