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by

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Abstract

We study approximation of continuous linear functionals I_d defined over reproducing kernel weighted Hilbert spaces of d-variate functions. Let $n(\varepsilon, I_d)$ denote the minimal number of function values needed to solve the problem to within ε. There are many papers studying polynomial tractability for which $n(\varepsilon, I_d)$ is to be bounded by a polynomial in ε^{-1} and d. We study *generalized tractability* for which we want to guarantee that either $n(\varepsilon, I_d)$ is not exponentially dependent on ε^{-1} and d, which is called *weak tractability*, or is bounded by a power of $T(\varepsilon^{-1}, d)$ for $(\varepsilon^{-1}, d) \in \Omega \subseteq [1, \infty) \times \mathbb{N}$, which is called (T, Ω)-tractability. Here, the *tractability* function T is non-increasing in both arguments and does not depend exponentially on ε^{-1} and d.

We present necessary conditions on generalized tractability for arbitrary continuous linear functionals I_d defined on weighted Hilbert spaces whose kernel has a decomposable component, and sufficient conditions on generalized tractability for multivariate integration for general reproducing kernel Hilbert spaces. For some weighted Sobolev spaces these necessary and sufficient conditions coincide. They are expressed in terms of necessary and sufficient conditions on the weights of the underlying spaces.

1 Introduction

The study of approximation of continuous linear functionals I_d over spaces of d-variate functions has recently been a popular research subject especially for large d. The primary example of I_d is multivariate integration which occurs in many applications for huge d.

Let $n(\varepsilon, I_d)$ be the minimal number of function values needed to reduce the initial error by a factor $\varepsilon \in (0,1)$ for functions from the unit ball of a given space. The initial error is the minimal error which can be achieved without sampling the functions and is equal to the norm of I_d. There are many papers, see [3] for a survey, studying *polynomial tractability* for which $n(\varepsilon, I_d)$ is to be bounded by a polynomial in ε^{-1} and d for all $(\varepsilon^{-1}, d) \in [1,\infty) \times \mathbb{N}$. By now we know that polynomial tractability of multivariate integration holds for reproducing kernel weighted Hilbert spaces for sufficiently fast decaying weights. Let $\gamma_{d,j}$ be a weight which controls the influence of the jth variable for the d-dimensional case. A typical result is that multivariate integration is polynomially tractable iff

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln\, d} \;<\; \infty.$$

There is also the notion of *strong polynomial tractability* for which $n(\varepsilon, I_d)$ is bounded by a polynomial only in ε^{-1} for all d. A typical result is that multivariate integration is strongly polynomially tractable iff

$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} \;<\; \infty.$$

In this paper we study *generalized tractability*, see [1]. First of all, we possibly limit the set of all pairs (ε^{-1}, d) of our interest, by assuming that $(\varepsilon^{-1}, d) \in \Omega$ where $\Omega \subseteq [1,\infty) \times \mathbb{N}$. To have a meaningful problem, the set Ω is chosen such that at least one of the arguments ε^{-1} or d may go to infinity. Since our main emphasis is on large d, in many cases we assume that $[1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for some $\varepsilon_0 \in (0,1)$ which allows to take arbitrary large d.

We study *weak tractability* in Ω for which we want to check when $n(\varepsilon, I_d)$ is *not* exponentially dependent on ε^{-1} and d for $(\varepsilon^{-1}, d) \in \Omega$. We also study (T, Ω)-tractability. Here T is called a *tractability* function which means that T is non-increasing in both arguments and does not depend exponentially on ε^{-1} and d. In this case, we want to check when $n(\varepsilon, I_d)$ is bounded by

a power of $T(\varepsilon^{-1}, d)$ for all $(\varepsilon^{-1}, d) \in \Omega$. *Strong* (T, Ω)-tractability means that $n(\varepsilon, I_d)$ is bounded by a power of $T(\varepsilon^{-1}, 1)$ for all $(\varepsilon^{-1}, d) \in \Omega$.

We present necessary and sufficient conditions on generalized tractability for $I = \{I_d\}$. Necessary conditions are obtained for arbitrary continuous linear functionals I_d defined over reproducing kernel weighted Hilbert spaces whose kernel has a decomposable component. We make heavy use of [4] where this concept was introduced and polynomial tractability was studied for weights independent of d. We generalize the approach of [4] by studying generalized tractability and weights which may depend on d. Sufficient conditions are obtained only for multivariate integration defined over general reproducing kernel weighted or unweighted Hilbert spaces. Sufficient conditions easily follow from upper bounds on $n(\varepsilon, I_d)$ which are obtained by using a known proof technique which can be found in, e.g., [6].

We prove that for some reproducing kernel weighted Hilbert spaces, such as some weighted Sobolev spaces, necessary and sufficient conditions for generalized tractability of multivariate integration coincide. These conditions are expressed in terms of the weights of the underlying space. A typical result is that weak tractability holds iff

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} \;=\; 0.$$

If we compare this with polynomial tractability, we see that $\ln d$ is now replaced by d but the corresponding limit must be zero. Hence, the unweighted case, $\gamma_{d,j} = \text{constant} > 0$, leads to the lack of weak tractability which is called *strong intractability*. To guarantee weak tractability we must take decaying weights. For example, for $\gamma_{d,j} = j^{-\beta}$ we have polynomial tractability iff $\beta \ge 1$ whereas we have weak tractability iff $\beta > 0$.

We obtain (T, Ω)-tractability iff

$$\limsup_{(\varepsilon^{-1},d)\in\Omega,\ \varepsilon^{-1}+d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j} \;+\; \ln \varepsilon^{-1}}{\ln\left(1 + T(\varepsilon^{-1}, d)\right)} \;<\; \infty.$$

and strong (T, Ω)-tractability iff

$$\limsup_{(\varepsilon^{-1},d)\in\Omega,\ \varepsilon^{-1}+d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j} \;+\; \ln \varepsilon^{-1}}{\ln\left(1 + T(\varepsilon^{-1}, 1)\right)} \;<\; \infty.$$

We illustrate these conditions for $T(x, y) = x^{\beta_1} \exp(y^{\beta_2})$ with non-negative β_i and $\beta_2 < 1$ which is needed to guarantee that T is non-exponential. We

consider two tractability domains $\Omega = \Omega_1 = [1,\infty) \times \mathbb{N}$ and $\Omega = \Omega_2 = [1,2] \times \mathbb{N}$. Then (T,Ω_1)-tractability holds iff

$$\beta_1 > 0 \quad \text{and} \quad \limsup_d \frac{\sum_{j=1}^d \gamma_{d,j}}{d^{\beta_2}} < \infty,$$

and strong (T,Ω_1)-tractability holds iff

$$\beta_1 > 0 \quad \text{and} \quad \limsup_d \sum_{j=1}^d \gamma_{d,j} < \infty.$$

For Ω_2, the dependence on ε is not important since $\varepsilon \geq \frac{1}{2}$, and (T,Ω_2)-tractability holds as before without assuming that $\beta_1 > 0$. That is, it holds even for $\beta_1 = 0$.

2 Approximation of Linear Functionals

For $d \in \mathbb{N}$, let F_d be a normed linear space of functions $f : D_d \subseteq \mathbb{R}^d \to \mathbb{R}$ for a Lebesgue measurable set D_d. Let

$$I_d : F_d \to \mathbb{R}$$

be a continuous linear functional. The primary example of I_d is *multivariate integration*. In this case, we assume that F_d is a space of Lebesgue measurable functions and

$$I_d f = \int_{D_d} \rho_d(x)\, f(x)\, \mathrm{d}x,$$

where ρ_d is a weight, i.e., $\rho_d \geq 0$ and $\int_{D_d} \rho_d(x)\, \mathrm{d}x = 1$. Obviously I_d is a linear functional. The norm of F_d is chosen such that I_d is also continuous.

Without loss of generality, see e.g., [7], we approximate I_d by linear algorithms using function values, i.e., by algorithms of the form

$$Q_{n,d} f := \sum_{i=1}^n a_i f(z_i)$$

for some real coefficients a_i and deterministic sample points $z_i \in D_d$. Let

$$e(n, I_d) = \inf_{a_i, z_i\,;\, i=1,2,\ldots,n} \; \sup_{f \in F_d,\ \|f\|_{F_d} \leq 1} \left| I_d f - \sum_{i=1}^n a_i f(z_i) \right|$$

be the nth minimal worst case error when we use n function values. In particular, for $n = 0$ we do not use function values and approximate I_d by zero. We then have the *initial error*,

$$e(0, I_d) \;=\; \|I_d\|\,,$$

where $\|\cdot\|$ is the operator norm induced by the norm of F_d. Let

$$n(\varepsilon, I_d) = \min\{\, n \mid \; e(n, F_d) \le \varepsilon\, e(0, F_d)\,\}$$

denote the smallest number of sample points for which there exists an algorithm $Q_{n,d}$ such that $e(Q_{n,d})$ reduces the initial error by a factor at least ε.

3 Generalized Tractability

For polynomial tractability, we assume that $(\varepsilon^{-1}, d) \in [1,\infty)\times\mathbb{N}$. Sometimes it is natural, see [1], to assume that $(\varepsilon^{-1}d) \in \Omega$, where Ω is a proper subset of $[1,\infty)\times\mathbb{N}$. As motivated in [1], it is natural to assume that Ω satisfies the following condition.

Let us define $[k] := \{1, 2, \ldots, k\}$ for arbitrary $k \in \mathbb{N}$ and $[0] := \emptyset$. A *tractability domain* Ω is a subset of $[1,\infty)\times\mathbb{N}$ satisfying

$$[1,\infty)\times[d^*]\cup[1,\varepsilon_0^{-1})\times\mathbb{N} \;\subseteq\; \Omega \tag{1}$$

for some $d^* \in \mathbb{N}\cup\{0\}$ and some $\varepsilon_0 \in (0,1]$ such that $d^* + (1-\varepsilon_0) > 0$. This implies that at least one of the arguments ε^{-1} or d may go to infinity within Ω.

For polynomial tractability, $n(\varepsilon, I_d)$ is to be bounded by a polynomial in ε^{-1} and d. For generalized tractability, we replace this polynomial dependence by a tractability function T which does *not* depend exponentially on ε^{-1} and d. More precisely, as in [1], a function $T:\,[1,\infty)\times[1,\infty) \;\to\; [1,\infty)$ is called a *tractability function* if T is non-decreasing in x and y and

$$\lim_{(x,y)\,\in\,\Omega,\;\; x+y\to\infty} \frac{\ln T(x,y)}{x+y} = 0\,. \tag{2}$$

Let now Ω be a tractability domain and T a tractability function. The multivariate problem $I = \{I_d\}$ is (T,Ω)-*tractable* if there exist non-negative numbers C and t such that

$$n(\varepsilon, I_d) \;\le\; C\,T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \tag{3}$$

The *exponent* t^{tra} *of* (T,Ω)*-tractability* is defined as the infimum of all non-negative t for which there exists a $C = C(t)$ such that (3) holds.

The multivariate problem I is *strongly* (T,Ω)*-tractable* if there exist non-negative numbers C and t such that

$$n(\varepsilon, I_d) \;\le\; C\,T(\varepsilon^{-1},1)^t \quad \text{for all } (\varepsilon^{-1},d) \in \Omega. \tag{4}$$

The *exponent* t^{str} *of strong* (T,Ω)*-tractability* is the infimum of all non-negative t for which there exists a $C = C(t)$ such that (4) holds.

An extensive motivation of the notion of generalized tractability and many examples of tractability domains and functions can be found in [1].

Similarly as in [5], we say that $I = \{I_d\}$ is *weakly tractable* in Ω iff

$$\lim_{(\varepsilon^{-1},d)\in\Omega,\,\varepsilon^{-1}+d\to\infty} \frac{\ln\, n(\varepsilon, I_d)}{\varepsilon^{-1}+d} \;=\; 0.$$

If I is not weakly tractable in Ω then I is called *strongly intractable* in Ω.

The essence of weak tractability in Ω is to guarantee that $n(\varepsilon, I_d)$ is *not* exponential in ε^{-1} and d without specifying a bound on $n(\varepsilon, I_d)$. Note that if I is (T,Ω)-tractable then I is weakly tractable in Ω. Or equivalently, if I is strongly intractable in Ω then I is also not (T,Ω)-tractable for any tractability function T.

4 Hilbert Spaces with Reproducing Kernels

In this section we make specific assumptions on the spaces F_d. We slightly modify the approach proposed in [4].

Let D_1 be a Lebesgue measurable subset of $\mathbb{R}$. Let $H(R_i)$, $i = 1,2$, denote Hilbert spaces with reproducing kernels $R_i : D_1^2 \to \mathbb{R}$. We assume that

$$H(R_1) \cap H(R_2) \;=\; \{0\},$$

and define the reproducing kernel K_1 by $K_1 = R_1 + R_2$.

Let F_1 be the Hilbert space with reproducing kernel K_1. That is, for all $f \in F_1$ there exist uniquely determined $f_1 \in H(R_1)$ and $f_2 \in H(R_2)$ such that $f = f_1 + f_2$, and the inner product of F_1 is given by

$$\langle f, g\rangle_{F_1} = \langle f_1, g_1\rangle_{H(R_1)} + \langle f_2, g_2\rangle_{H(R_2)}\,.$$

Let I_1 be a continuous linear functional on F_1. Then there exists a $h_1 \in F_1$ such that

$$I_1 f \;=\; \langle f, h_1\rangle_{F_1} \quad \text{for all} \;\; f \;\in\; F_1.$$

The function h_1 has the unique decomposition

$$h_1 = h_{1,1} + h_{1,2} \quad \text{with } h_{1,i} \in H(R_i).$$

We assume that R_2 is *decomposable*, i.e., there exists an $a^* \in \mathbb{R}$ such that

$$R_2(x,t) \;=\; 0 \quad \text{for all} \;\; (x,t) \;\in\; D_{(0)} \times D_{(1)} \cup D_{(1)} \times D_{(0)},$$

where

$$D_{(0)} := \{x \in D_1 \,|\, x \le a^*\} \quad \text{and} \quad D_{(1)} := \{x \in D_1 \,|\, x \ge a^*\}\,.$$

Let $h_{1,2,(0)}$ and $h_{1,2,(1)}$ be functions defined on D_1 such that they are the restrictions of $h_{1,2}$ to the sets $D_{(0)}$ and $D_{(1)}$, respectively, and take zero values otherwise.

We also consider weighted tensor product problems. We define $F_{1,\gamma}$ as the Hilbert space determined by the weighted reproducing kernel $K_{1,\gamma}$,

$$K_{1,\gamma} = R_1 + \gamma R_2\,, \tag{5}$$

where γ is positive. The case $\gamma = 0$ can be obtained by taking the limit of positive γ. Since the norms of $F_{1,\gamma} = F_1$ are equivalent, $I_{1,\gamma} = I_1$ is also a continuous linear functional on $F_{1,\gamma}$. It is easy to show that for

$$h_{1,\gamma} \;:=\; h_{1,1} + \gamma\, h_{1,2}$$

we have

$$I_{1,\gamma} f \;=\; I_1 f \;=\; \langle f, h_{1,\gamma}\rangle_{F_{1,\gamma}} \quad \text{for all } f \in F_{1,\gamma}$$

and

$$\|h_{1,\gamma}\|^2_{F_{1,\gamma}} = \|h_{1,1}\|^2_{H(R_1)} + \gamma \|h_{1,2}\|^2_{H(R_2)}\,. \tag{6}$$

For $d \ge 2$, let $F_{d,\gamma} = F_{1,\gamma_{d,1}} \otimes \cdots \otimes F_{1,\gamma_{d,d}}$ be the tensor product Hilbert space of the $F_{1,\gamma_{d,j}}$ for some positive weights $\gamma := \{\gamma_{d,j}\}$, $d \in \mathbb{N}$, $j \in [d]$. The case of a zero weight can be obtained, as before, by taking the limit of positive weights. Without loss of generality, we assume that

$$\gamma_{d,1} \ge \gamma_{d,2} \ge \cdots \ge \gamma_{d,d} \quad \text{for all} \;\; d \in \mathbb{N}.$$

To relate weights for different d, we assume that for fixed j the sequence $\{\gamma_{d,j}\}_{d=1}^{\infty}$ is non-increasing. Hence, for all $d \ge j$ we have

$$\gamma_{d,j} \;\le\; \gamma_{j,j} \;\le\; \gamma_{j,1} \;\le\; \gamma_{1,1}.$$

Examples of weights include $\gamma_{d,j} = \gamma_j$ for $\gamma_j \geq \gamma_{j+1}$, as considered in [4], or $\gamma_{d,j} = d^{-\beta}$ with $\beta \geq 0$.

Let us define the sequence of spaces $F_\gamma := \{F_{d,\gamma}\}$, where

$$K_{d,\gamma}(x,t) = \prod_{j=1}^{d} [R_1(x_j,t_j) + \gamma_{d,j} R_2(x_j,t_j)]$$

is the reproducing kernel of $F_{d,\gamma}$. We define the linear functional

$$I_{d,\gamma} \,:\, F_{d,\gamma} \;\to\; \mathbb{R}$$

as the d-fold tensor product of I_1. Then $I_{d,\gamma} f = \langle f, h_{d,\gamma} \rangle_{F_{d,\gamma}}$ with

$$h_{d,\gamma}(x) = \prod_{j=1}^{d} h_{1,\gamma_{d,j}}(x_j) = \prod_{j=1}^{d} [h_{1,1}(x_j) + \gamma_{d,j} h_{1,2}(x_j)] \;.$$

From (6) we have

$$e^2(0, I_{d,\gamma}) = \|h_{d,\gamma}\|^2_{F_{d,\gamma}} = \prod_{j=1}^{d} \left[\|h_{1,1}\|^2_{H(R_1)} + \gamma_{d,j} \|h_{1,2}\|^2_{H(R_2)} \right] \;.$$

Let $\alpha_1 := \|h_{1,1}\|^2_{H(R_1)}$, $\alpha_2 := \|h_{1,2}\|^2_{H(R_2)}$, $\alpha_3 := \|h_{1,2}\|^2_{H(R_2)} \|h_{1,1}\|^{-2}_{H(R_1)}$, and

$$\alpha := \frac{\max\{\|h_{1,2,(0)}\|^2_{H(R_2)}, \|h_{1,2,(1)}\|^2_{H(R_2)}\}}{\|h_{1,2,(0)}\|^2_{H(R_2)} + \|h_{1,2,(1)}\|^2_{H(R_2)}} \;.$$

Furthermore, let us define for $k = 1, 2, \ldots, d$,

$$C_{d,k} := \sum_{\mathfrak{u} \subseteq [d]; |\mathfrak{u}| = k} \prod_{j \in \mathfrak{u}} \gamma_{d,j} \;,$$

and, by convention, $C_{d,0} := 1$. Then, Theorem 2 of [4] states that

$$e^2(n, I_{d,\gamma}) \geq \sum_{k=0}^{d} C_{d,k} (1 - n\alpha^k)_+ \, \alpha_1^{d-k} \alpha_2^k \;, \tag{7}$$

where, by convention, $0^0 = 1$.

5 Necessary Conditions

We are ready to study generalized tractability of $I_\gamma = \{I_{d,\gamma}\}$. In this section, we consider lower bounds on the minimal errors $e(n, I_{d,\gamma})$ from which we obtain necessary conditions on generalized tractability.

The following theorem extends Theorem 3 from [4].

Theorem 5.1. *Assume that $H(R_1) \cap H(R_2) = \{0\}$ and R_2 is decomposable. Assume that both $h_{1,2,(0)}$ and $h_{1,2,(1)}$ are non-zero. Let T be an arbitrary tractability function, and let Ω be a tractability domain with $[1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for some $\varepsilon_0 \in (0,1)$.*

1. *Let $h_{1,1} = 0$. Then $I_\gamma = \{I_{d,\gamma}\}$ is strongly intractable in Ω and is not (T, Ω)-tractable.*

2. *Let $h_{1,1} \neq 0$ and*
$$\lim_{d\to\infty}{}^{*} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(1 + f(d))} = \infty$$
for some non-decreasing function $f : \mathbb{N} \to [1, \infty)$, where $\lim^* \in \{\lim, \limsup\}$. Then
$$\lim_{d\to\infty}{}^{*} \frac{e\left(\lfloor f(d)^q \rfloor, I_{d,\gamma}\right)}{e(0, I_{d,\gamma})} = 1 \quad \text{for all } q \in \mathbb{N}. \tag{8}$$
In particular,
$$\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} = \infty$$
implies that I_γ is not strongly (T, Ω)-tractable, and
$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(1 + T(\varepsilon^{-1}, d))} = \infty$$
for some $\varepsilon \in (\varepsilon_0, 1)$ implies that I_γ is not (T, Ω)-tractable.

3. *Let $h_{1,1} \neq 0$ and $\gamma_{d,d} \geq \gamma^* > 0$ for all $d \in \mathbb{N}$. Then*
$$\lim_{d\to\infty} \frac{e(\lfloor b^d \rfloor, I_{d,\gamma})}{e(0, I_{d,\gamma})} = 1 \quad \text{for all } b \in (1, \alpha^{-c}), \tag{9}$$
where $c \in (0,1)$ satisfies the following two inequalities
$$c \leq \alpha_3 \gamma^* \quad \text{and} \quad (1 + \ln(\alpha_3 \gamma^*) - \ln c)c < \ln(1 + \alpha_3 \gamma^*)\,.$$
Hence, I_γ is strongly intractable in Ω.

4. Let $h_{1,1} \neq 0$ and

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} \;>\; 0.$$

Then I_γ is strongly intractable in Ω.

Proof. We follow here the lines of the proof of [4, Thm. 3]. Note that just now $\alpha \in [1/2, 1)$ since $h_{1,2,(0)} \neq 0 \neq h_{1,2,(1)}$.

Let us first prove statement 1. For $h_{1,1} = 0$, we have $\alpha_1 = 0$ and the only non-zero term in (7) is for $k = d$. Then

$$e(n, I_{d,\gamma}) \geq (1 - n\alpha^d)_+^{1/2}\, e(0, I_{d,\gamma})\,. \tag{10}$$

From (10) we conclude that

$$n(\varepsilon, I_{d,\gamma}) \;\geq\; (1 - \varepsilon^2)\alpha^{-d}.$$

Note that $[1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for $\varepsilon_0 \in (0,1)$ implies that, say, $((1+\varepsilon_0^{-1})/2, d) \in \Omega$ for all d. Therefore

$$\limsup_{(\varepsilon^{-1},d)\in\Omega,\, \varepsilon^{-1}+d\to\infty} \frac{\ln\, n(\varepsilon, I_{d,\gamma})}{\varepsilon^{-1} + d} \;\geq\; \ln\, \alpha^{-1} \;> 0\,,$$

which means that I_γ is strongly intractable in Ω and not (T, Ω)-tractable.

Let us now prove statement 2. For $h_{1,1} \neq 0$ we get from (7)

$$1 \geq \frac{e^2(n, I_{d,\gamma})}{e^2(0, I_{d,\gamma})} \geq \frac{\sum_{k=0}^{d} C_{d,k}\alpha_3^k (1 - n\alpha^k)_+}{\sum_{k=0}^{d} C_{d,k}\alpha_3^k}\,.$$

Define $\gamma'_{d,j} := \alpha_3\gamma_{d,j}$ and $C'_{d,k} := \alpha_3^k C_{d,k}$. Then we have

$$1 \geq \frac{e^2(n, I_{d,\gamma})}{e^2(0, I_{d,\gamma})} \geq \frac{\sum_{k=0}^{d} C'_{d,k}(1 - n\alpha^k)_+}{\sum_{k=0}^{d} C'_{d,k}}\,. \tag{11}$$

Now take $n = \lfloor f(d)^q \rfloor$ for an arbitrary $q \in \mathbb{N}$. For any $a \in (0,1)$ there exist non-negative β_1 and β_2 such that

$$n\alpha^k \;\leq\; a \;\text{ for }\; k \in [k(d,\beta), d], \;\text{ where }\; k(d,\beta) \;=\; \lceil \beta_2 + \beta_1 \ln f(d) \rceil.$$

Let us denote

$$s_d \;:=\; \sum_{j=1}^{d} \gamma'_{d,j} \;=\; C'_{d,1}.$$

Since $k(d,\beta) = O(\ln(1+f(d)))$, the conditions

$$\lim_{d\to\infty}{}^{*} \frac{s_d}{\ln(1+f(d))} = \infty \quad \text{and} \quad s_d = O(d)$$

imply that

$$s_d \geq k(d,\beta) \quad \text{and} \quad d \geq k(d,\beta)$$

for infinitely many d. We confine our analysis to those values of d. From (11) we conclude

$$\frac{e^2(n, I_{d,\gamma})}{e^2(0, I_{d,\gamma})} \geq (1-a)\, \frac{\sum_{k=k(d,\beta)+1}^{d} C'_{d,k}}{\sum_{k=0}^{d} C'_{d,k}} = (1-a)(1-\alpha_{d,\beta})\,,$$

where

$$\alpha_{d,\beta} = \frac{\sum_{k=0}^{k(d,\beta)} C'_{d,k}}{\sum_{k=0}^{d} C'_{d,k}}\,.$$

To prove (8) it is enough to show that $\alpha_{d,\beta}$ goes to zero as d tends to infinity since a can be arbitrarily small.

It is easy to see that $C'_{d,k} \leq s_d^k/k!$. Thus we have

$$\sum_{k=0}^{k(d,\beta)} C'_{d,k} \leq \sum_{k=0}^{k(d,\beta)} \frac{s_d^k}{k!}\,. \tag{12}$$

Observe that

$$\sum_{k=0}^{d} C'_{d,k} = \prod_{j=1}^{d} (1+\gamma'_{d,j}) = \exp\left(\sum_{j=1}^{d} \ln(1+\gamma'_{d,j})\right)\,.$$

Assume for a moment that there exists a positive γ^* such that $\gamma_{d,d} \geq \gamma^* > 0$ for all $d \in \mathbb{N}$. Since $\ln(1+f(d)) = o(d)$ we have $\lfloor f(d)^q \rfloor \leq \lfloor b^d \rfloor$ for $b > 1$ and sufficiently large d, and therefore (8) follows from (9) which will be addressed in a moment.

Thus we may consider here only the case where $\lim_{d\to\infty} \gamma_{d,d} = 0$. Then for an arbitrary $\vartheta \in (0,1)$ there exists a positive constant c_ϑ with

$$\exp\left(\sum_{j=1}^{d} \ln(1+\gamma'_{d,j})\right) \geq c_\vartheta \exp\big(s_d(1-\vartheta)\big) \quad \text{for sufficiently large } d. \tag{13}$$

Indeed, let τ be such that $\vartheta = \tau/(1+\tau)$. It is easily seen that

$$x(1-\vartheta) \leq \ln(1+x) \quad \text{for all } x \in [0,\tau].$$

Since $\gamma'_{d,j} \le \gamma'_{j,j}$ for $d > j$, there is an index j_τ such that $\gamma'_{d,j} \in [0,\tau]$ for all $d \ge j \ge j_\tau$. For $d \ge j_\tau$, we have

$$\begin{aligned}\sum_{j=1}^{d} \ln(1+\gamma'_{d,j}) &\ge \sum_{j=1}^{j_\tau-1} \ln(1+\gamma'_{d,j}) + \left(\sum_{j=j_\tau}^{d} \gamma'_{d,j}\right)(1-\vartheta) \\ &= \sum_{j=1}^{j_\tau-1} \ln(1+\gamma'_{d,j}) - (1-\vartheta)\sum_{j=1}^{j_\tau-1} \gamma'_{d,j} + s_d(1-\vartheta)\,.\end{aligned}$$

Hence (13) holds for $d \ge j_\tau$ with

$$c_\vartheta = \exp\left(-(1-\vartheta)\sum_{j=1}^{j_\tau-1} \gamma'_{j_\tau,j}\right).$$

Observe that $s_d^k/k!$ is an increasing function of k over the interval $[0,k^*]$ as long as $s_d \ge k^*$. Since $s_d \ge k(d,\beta)$ we get

$$\sum_{k=0}^{k(d,\beta)} \frac{s_d^k}{k!} \le k(d,\beta)\frac{s_d^{k(d,\beta)}}{k(d,\beta)!} = \exp\left[k(d,\beta)\ln s_d - \ln((k(d,\beta)-1)!)\right]\,.$$

Using the formula $k! \ge k^k e^{-k}$ and $k(d,\beta) = O(\ln(1+f(d)))$, we get

$$\begin{aligned}\sum_{k=0}^{k(d,\beta)} C'_{d,k} &\le \exp\left[k(d,\beta)\ln s_d - (k(d,\beta)-1)(\ln(k(d,\beta)-1)-1))\right] \\ &\le \exp\left[k(d,\beta)\ln\left(\frac{s_d}{k(d,\beta)-1}\right) + O(\ln(1+f(d)))\right] \\ &\le \exp\left[O\left(\ln(1+f(d))\ln\left(\frac{s_d}{\ln(1+f(d))}\right)\right)\right]\,.\end{aligned}$$

Let $\vartheta \in (0,1)$. Then for $d \ge j_\tau$,

$$\begin{aligned}\alpha_{d,\beta} &\ge c_\vartheta^{-1}\exp\left[-s_d(1-\vartheta) + O\left(\ln(1+f(d))\ln\left(\frac{s_d}{\ln(1+f(d))}\right)\right)\right] \\ &= c_\vartheta^{-1}\exp\left[-\ln(1+f(d))\left(\frac{s_d(1-\vartheta)}{\ln(1+f(d))} + O\left(\ln\left(\frac{s_d}{\ln(1+f(d))}\right)\right)\right)\right]\,.\end{aligned}$$

Thus $\lim^*_{d\to\infty} \alpha_{d,\beta} = 0$, and the proof of (8) is completed.

Let $\lim^*_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} = \infty$. Then $\lim^*_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j}/\ln(1+f(d)) = \infty$ for an arbitrary constant function f. According to (8), this results in

$$\lim_{d\to\infty}{}^* \frac{e(n, I_{d,\gamma})}{e(0, I_{d,\gamma})} = 1 \quad \text{for all } n \in \mathbb{N}.$$

Since $\{\varepsilon\} \times \mathbb{N} \subset \Omega$ for arbitrary $\varepsilon \in (\varepsilon_0, 1)$, we conclude that $n(\varepsilon, I_{d,\gamma})$ must go to infinity with d which means that I_d is not strongly (T, Ω)-tractable.

Finally assume that $\lim^*_{d\to\infty} \sum_{j=1}^d \gamma_{d,j}/\ln(1 + T(\varepsilon^{-1}, d)) = \infty$ for some $\varepsilon \in (\varepsilon_0, 1)$. This corresponds to $f(d) = T(\varepsilon^{-1}, d)$. If $T(\varepsilon^{-1}, d) = 1$ for all d, we are in the preceeding case and I_γ is not (T, Ω)-tractable. If $T(\varepsilon^{-1}, d) > 1$ for some d, then (8) implies that for arbitrary positive constants C and t there exists a positive constant q and infinitely many d with

$$n(\varepsilon, I_{d,\gamma}) > T(\varepsilon^{-1}, d)^q \geq CT(\varepsilon^{-1}, d)^t \,.$$

This implies again that I_γ is not (T, Ω)-tractable.

We now address statement 3, which, apart from the slightly more general weights, is actually the fourth statement of [4, Thm. 3]. If one checks the proof there and makes the obvious small modifications, one easily sees that (9) also holds. From it, we have $n(\varepsilon, I_{d,\gamma}) \geq \lfloor b^d \rfloor$ for a fixed $\varepsilon \in (\varepsilon_0, 1)$ and sufficiently large d. Hence, $n(\varepsilon, I_{d,\gamma})$ is exponential in d which implies strong intractability in Ω.

We turn to the last statement 4 and reduce it to statement 3. We now know that there exists a sequence $\{d_k\}$, with $\lim_k d_k = \infty$, and a positive c_1 such that

$$\sum_{j=1}^{d_k} \gamma_{d_k,j} \;\geq\; c_1 d_k \quad \text{for all} \quad k \in \mathbb{N}.$$

For $s \in [d_k]$ we have

$$\sum_{j=1}^{d_k} \gamma_{d_k,j} \;=\; \sum_{j=1}^{s-1} \gamma_{d_k,j} \;+\; \sum_{j=s}^{d_k} \gamma_{d_k,j} \;\leq\; (s-1)\gamma_{1,1} \;+\; (d_k - s + 1)\gamma_{s,s}.$$

For all $s \leq 1 + c_1 d_k/(2\gamma_{1,1})$ we obtain

$$\gamma_{s,s} \;\geq \gamma^* \;:=\; c_1/2.$$

Since d_k goes to infinity this proves that $\gamma_{d,d} \geq \gamma^*$ for all d and the assumptions of statement 3 are satisfied. This completes the proof.

□

6 Sufficient Conditions for Integration

In this section we analyze sufficient conditions for generalized tractability of multivariate integration for a Hilbert space F_d with a general reproducing

kernel $K_d : D_d \times D_d \to \mathbb{R}$. We assume that K_d is Lebesgue measurable and

$$\int_{D_d}\int_{D_d} \rho_d(x)\rho_d(y)\, K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\le\; \int_{D_d} \rho_d(x)\, K_d(x,x)\,\mathrm{d}x \;<\; \infty,$$

where $\rho_d \ge 0$ and $\int_{D_d} \rho_d(x)\,\mathrm{d}x \;=\; 1$. Then multivariate integration

$$I_d f \;=\; \int_{D_d} \rho_d(x)\, f(x)\,\mathrm{d}x \quad \text{for all} \;\; f \;\in\; F_d,$$

is a continuous linear functional, and $I_d f = \langle f, h_d\rangle_{F_d}$ with

$$h_d(x) \;=\; \int_{D_d} \rho_d(y)\, K_d(x,y)\,\mathrm{d}y.$$

Without loss of generality we assume that $h_d \neq 0$ since otherwise multivariate integration is trivial. The initial error is now of the form

$$\begin{aligned} e(0, I_d) \;&=\; \|I_d\| \;=\; \|h_d\|_{F_d} \;=\; \left(\int_{D_d} \rho_d(x)\, h_d(x)\,\mathrm{d}x\right)^{1/2} \\ &=\; \left(\int_{D_d}\int_{D_d} \rho_d(x)\rho_d(y)\, K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \;>\; 0. \end{aligned}$$

For the algorithm $Q_{n,d} f \;=\; \sum_{i=1}^n a_i f(z_i)$ we have

$$I_d f - Q_{n,d} f \;=\; \left\langle f, h_d \;-\; \sum_{i=1}^n a_i K_d(\cdot, z_i)\right\rangle_{F_d}.$$

This yields a well know formula for the worst case error of $Q_{n,d}$,

$$\begin{aligned} e(Q_{n,d}) \;&=\; \sup_{f\in F_d,\, \|f\|_{F_d}\le 1} \left| I_d f \;-\; \sum_{i=1}^n a_i f(z_i)\right| \;=\; \left\| h_d \;-\; \sum_{i=1}^n a_i K_d(\cdot, z_i)\right\|_{F_d} \\ &=\; \left(\|h_d\|_{F_d}^2 - 2\sum_{i=1}^n a_i h_d(z_i) + \sum_{i,j=1}^n a_i a_j K_d(z_i, z_j)\right)^{1/2}. \end{aligned}$$

We now assume that $Q_{n,d}$ is a QMC algorithm, i.e., $a_i \;=\; n^{-1}$, and treat the sample points z_i as independent and identically distributed points over D_d with the density function ρ_d. We use the notation $e(Q_{n,d}) \;=\; e(Q_{n,d}, \{z_i\})$ to stress the dependence on the sample points z_i. Let

$$E(n,d) \;=\; \int_{D_d^n} e^2(Q_{n,d}, \{z_i\})\rho_d(z_1)\,\rho_d(z_2)\,\cdots\,\rho_d(z_n)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\cdots\,\mathrm{d}z_n$$

denote the average of the square of the worst case error of $Q_{n,d}$. It is easy to obtain an explicit formula for $E(n,d)$ which is also well known, see e.g., [6],

$$\begin{aligned} E(n,d) &= \|h_d\|^2 - 2\|h_d\|^2 + \frac{n^2-n}{n^2}\|h_d\|^2 + \frac{1}{n}\int_{D_d} \rho_d(x)\, K_d(x,x)\,\mathrm{d}x \\ &= \frac{\int_{D_d} \rho_d(x)\, K_d(x,x)\,\mathrm{d}x - \int_{D_d}\int_{D_d} \rho_d(x)\rho_d(y)\, K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y}{n}. \end{aligned}$$

Here, $\|h_d\| = \|h_d\|_{F_d}$. By the mean value theorem we know that there are sample points z_i for which the square of the worst case error is at most $E(n,d)$. This proves that the square of the nth minimal error $e(n,I_d)$ is at most $E(n,d)$ and we have

$$\frac{e(n,I_d)}{e(0,I_d)} \le \frac{1}{\sqrt{n}} \left(\frac{\int_{D_d} \rho_d(x)\, K_d(x,x)\,\mathrm{d}x}{\int_{D_d}\int_{D_d} \rho_d(x)\rho_d(y)\, K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y} - 1 \right)^{1/2}. \tag{14}$$

From this estimate it is easy to conclude sufficient conditions on generalized tractability of multivariate integration.

Theorem 6.1. *Consider multivariate integration $I = \{I_d\}$ defined as in this section. Let T be an arbitrary tractability function, and let Ω be a tractability domain with $[1,\varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for some $\varepsilon_0 \in (0,1)$. Let*

$$\eta_d = \frac{\int_{D_d} \rho_d(x)\, K_d(x,x)\,\mathrm{d}x}{\int_{D_d}\int_{D_d} \rho_d(x)\rho_d(y)\, K_d(x,y)\,\mathrm{d}x\,\mathrm{d}y} - 1.$$

1. *We have*
$$n(\varepsilon, I_d) \le \left\lceil \frac{\eta_d}{\varepsilon^2} \right\rceil.$$

2. *If*
$$\lim_{d\to\infty} \frac{\ln \max(1,\eta_d)}{d} = 0$$
then I is weakly tractable in Ω.

3. *If*
$$t^* := \limsup_{(\varepsilon^{-1},d)\in\Omega,\ \varepsilon^{-1}+d\to\infty} \frac{\ln \max(1,\eta_d) + 2\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},d))} < \infty$$
then I is (T,Ω)-tractable with the exponent of (T,Ω)-tractability equal to at most t^*.

4. If

$$t^* := \limsup_{(\varepsilon^{-1},d)\in\Omega,\ \varepsilon^{-1}+d\to\infty} \frac{\ln\,\max(1,\eta_d)\ +\ 2\,\ln\,\varepsilon^{-1}}{\ln\,(1+T(\varepsilon^{-1},1))}\ <\ \infty$$

then I is strongly (T,Ω)-tractable with the exponent of strong (T,Ω)-tractability equal to at most t^*.

Proof. The proof is obvious. The bound on $n(\varepsilon,d)$ directly follows from (14). We have

$$n(\varepsilon,I_d)\ \le\ \lceil \max(1,\eta_d)\,\varepsilon^{-2}\rceil\ \le\ 2\,\max(1,\eta_d)\,\varepsilon^{-2}$$

since $\lceil x\rceil \le 2x$ for $x\ge 1$. Since pairs (ε,d) for $\varepsilon\in(\varepsilon_0,1)$ and $d\in\mathbb{N}$ belong to Ω, we have

$$\frac{\ln\,n(\varepsilon,I_d)}{\varepsilon^{-1}+d}\ \le\ \frac{\ln\,\max(1,\eta_d)}{d}\ +\ \frac{2\ln\,\varepsilon^{-1}}{\varepsilon^{-1}+d}\ +\ \frac{\ln 2}{\varepsilon^{-1}+d}$$

which goes to 0 if $\lim_{d\to\infty}\ln(\max(1,\eta_d))/d=0$. This yields weak tractability in Ω. The rest follows from the fact that $n(\varepsilon,I_d)\ \le\ C\,T(\varepsilon^{-1},k_d)^t$, with $k_d=d$ when we consider (T,Ω)-tractability and $k_d=1$ when we consider strong (T,Ω)-tractability, if

$$\frac{\ln\,\max(1,\eta_d)\ +\ 2\ln\,\varepsilon^{-1}}{\ln\,(1+T(\varepsilon^{-1},k_d))}\ \le\ \frac{\ln\,(C/2)}{\ln\,(1+T(\varepsilon^{-1},k_d))}\ +\ t.$$

For any $\delta\in(0,1)$ there exists C_δ such that for all $(\varepsilon^{-1},d)\in\Omega$ with $\varepsilon^{-1}+d\ge C_\delta$, the left hand side is at most $t^*+\delta$. Hence, we can take $t=t^*+\delta$ and C sufficiently large so that the last inequality holds for all $(\varepsilon^{-1},d)\in\Omega$. This proves (T,Ω)-tractability or strong (T,Ω)-tractability as well as the needed bounds on the exponents. □

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7 Examples

We illustrate Theorems 5.1 and 6.1 by a number of examples of spaces for multivariate integration.

Example 1: Sobolev Space for Bounded Domain

In this example we consider multivariate integration for the bounded domain, $D_d=[0,1]^d$ and for a specific Sobolev space. More precisely for

$d = 1$, as in [4], let $F_{1,\gamma}$ be the Sobolev space of absolutely continuous functions defined on $D_1 = [0,1]$ whose first derivatives are in $L_2([0,1])$ with the inner product

$$\langle f, g\rangle_{F_{1,\gamma}} \;=\; f(\tfrac{1}{2})g(\tfrac{1}{2}) \;+\; \gamma^{-1} \int_0^1 f'(x)g'(x)\,\mathrm{d}x.$$

The reproducing kernel is $K_{1,\gamma} = R_1 + \gamma\, R_2$ with

$$R_1 \;=\; 1 \;\text{ and }\; R_2(x,y) \;=\; 1_M(x,y)\,\min\big(|x-\tfrac{1}{2}|, |y-\tfrac{1}{2}|\big)$$

with the characteristic function 1_M of $M = [0,\frac{1}{2}] \times [0,\frac{1}{2}] \;\cup\; [\frac{1}{2},1] \times [\frac{1}{2},1]$. Clearly, R_2 is decomposable with $a^* = \frac{1}{2}$. The kernel R_2 can also be written as

$$R_2(x,y) \;=\; \tfrac{1}{2}\left(|x-\tfrac{1}{2}| \;+\; |y-\tfrac{1}{2}| \;-\; |x-y|\right).$$

We have

$$\int_0^1\int_0^1 K_{1,\gamma}(x,y)\,\mathrm{d}x\,\mathrm{d}y \;=\; 1 + \tfrac{1}{12}\gamma \;\text{ and }\; \int_0^1 K_{1,\gamma}(x,x)\,\mathrm{d}x \;=\; 1 + \tfrac{1}{4}\gamma.$$

For $d \geq 2$, we obtain the Sobolev space $F_{d,\gamma}$ with the inner product

$$\langle f, g\rangle_{F_{d,\gamma}} \;=\; \sum_{\mathfrak{u}\subseteq[d]} \gamma_{\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial\, x_{\mathfrak{u}}} f(x_{\mathfrak{u}}, \tfrac{1}{2}) \frac{\partial^{|\mathfrak{u}|}}{\partial\, x_{\mathfrak{u}}} g(x_{\mathfrak{u}}, \tfrac{1}{2})\,\mathrm{d}x_{\mathfrak{u}}.$$

Here $\gamma_\emptyset = 1$ and $\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j}$ for non-empty $\mathfrak{u}$, and $x_{\mathfrak{u}}$ is the vector from $[0,1]^{|\mathfrak{u}|}$ whose components corresponding to indices in $\mathfrak{u}$ are the same as for the vector $x \in [0,1]^d$, and $(x_{\mathfrak{u}}, \frac{1}{2})$ is the vector $x \in [0,1]^d$ with all components whose indices are not in $\mathfrak{u}$ replaced by $\frac{1}{2}$. Furthermore, we use the convention $\int_{[0,1]^{|\emptyset|}} dx_\emptyset = 1$.

Consider multivariate integration, $I_{d,\gamma} f = \int_{[0,1]^d} f(x)\,\mathrm{d}x = \langle f, h_{d,\gamma}\rangle_{F_{d,\gamma}}$ with

$$h_{d,\gamma}(x) \;=\; \prod_{j=1}^{d} \left[1 + \frac{1}{2}\,\gamma_{d,j}\left(|x_j - \tfrac{1}{2}| - \tfrac{1}{4} + x_j - x_j^2\right)\right].$$

We now have

$$\begin{aligned} h_{1,1} \;&=\; 1, \\ h_{1,2,(0)}(x) \;&=\; \tfrac{1}{2}(\tfrac{1}{2}-x)(\tfrac{1}{2}+x)\,1_{[0,\frac{1}{2}]}(x), \\ h_{1,2,(1)}(x) \;&=\; \tfrac{1}{2}(x-\tfrac{1}{2})(2-\tfrac{1}{2}-x)\,1_{[\frac{1}{2},1]}(x). \end{aligned}$$

Hence, $\|h_{1,2,(0)}\|_{H(R_2)} = \|h_{1,2,(1)}\|_{H(R_2)} = \frac{1}{24}$ and $\alpha = \frac{1}{2}$. Hence, the assumptions of Theorem 5.1 are satisfied. Furthermore, the initial error is

$$e(0, I_{d,\gamma}) \;=\; \prod_{j=1}^{d} \left(1 + \tfrac{1}{12}\,\gamma_{d,j}\right)^{1/2},$$

whereas

$$\int_{[0,1]^d} K_{d,\gamma}(x,x)\,\mathrm{d}x \;=\; \prod_{j=1}^{d} \left(1 + \tfrac{1}{4}\,\gamma_{d,j}\right).$$

Hence, the assumptions of Theorem 6.1 are also satisfied and

$$\eta_d \;=\; \prod_{j=1}^{d} \frac{1 + \frac{1}{4}\,\gamma_{d,j}}{1 + \frac{1}{12}\,\gamma_{d,j}} \;-1 \;\le\; \prod_{j=1}^{d} \left(1 + \tfrac{1}{6}\,\gamma_{d,j}\right)$$

since $(1 + b\,x)/(1 + a\,x) \le 1 + (b - a)x$ for $b \ge a$ and $x \ge 0$.

Since $\ln(1 + x) \le x$ for $x \ge 0$, we now have

$$\ln \max(1, \eta_d) \;\le\; \tfrac{1}{6} \sum_{j=1}^{d} \gamma_{d,j}. \tag{15}$$

Combining Theorems 5.1 and 6.1 we see that $\lim_d \sum_{j=1}^{d} \gamma_{d,j}/d = 0$ is necessary and sufficient for weak tractability.

For $d = 1$, it is known that the minimal error $e(n, I_{1,\gamma}) = \Theta(n^{-1})$, see e.g., [7]. Hence, if $[1,\infty) \times [d^*] \subseteq \Omega$ with $d^* \ge 1$ then the limit superior of $(\ln \varepsilon^{-1})/\ln(1 + T(\varepsilon^{-1}, 1))$ must be finite if we want to have (T, Ω)-tractability.

We are ready to summarize results for generalized tractability of this multivariate integration problem.

Theorem 7.1. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined for the Sobolev space $F_{d,\gamma}$ as in this example. Let T be an arbitrary tractability function, and let Ω be a tractability domain with $[1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for some $\varepsilon_0 \in (0,1)$. Then*

$$I_\gamma \text{ is weakly tractable in } \Omega \quad \text{iff} \quad \lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} \;=\; 0.$$

Assume additionally that $[1,\infty) \times [d^*] \subseteq \Omega$ for $d^* \ge 1$ and that

$$S(\varepsilon) := \sup_{d\in\mathbb{N}} \frac{\ln(1 + T(\varepsilon^{-1}, d))}{\ln(1 + T(1, d))} < \infty \quad \text{for some } \varepsilon \in (\varepsilon_0, 1).$$

Then

1. *I_γ is (T,Ω)-tractable iff*

$$\begin{aligned} t_1^* &:= \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(1,d))} < \infty, \\ t_2^* &:= \limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty. \end{aligned}$$

If this holds then for arbitrary $t > \frac{1}{6}t_1^* + 2t_2^*$ there exists a positive $C = C_t$ such that

$$n(\varepsilon, I_{d,\gamma}) \le C\,T(\varepsilon^{-1},d)^t \quad \text{for all} \quad (\varepsilon^{-1},d) \in \Omega.$$

The exponent t^{tra} of (T,Ω)-tractability is in $[t_2^*, \frac{1}{6}t_1^* + 2t_2^*]$.

2. *I_γ is strongly (T,Ω)-tractable iff*

$$\begin{aligned} t_1^* &:= \limsup_{d\to\infty} \sum_{j=1}^d \gamma_{d,j} < \infty, \\ t_2^* &:= \limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty. \end{aligned}$$

If this holds then for arbitrary $t > 2t_2^*$ there exists a positive $C = C_t$ such that

$$n(\varepsilon, I_{d,\gamma}) \le C\,T(\varepsilon^{-1},1)^t \quad \text{for all} \quad (\varepsilon^{-1},d) \in \Omega\,.$$

The exponent t^{str} of strong (T,Ω)-tractability is in $[t_2^*, 2t_2^*]$.

Proof. The statement on weak tractability follows from Theorems 5.1 and 6.1, and (15). Let us now assume that $[1,\infty)\times[d^*] \subseteq \Omega$ for some $d^* \ge 1$ and $S(\varepsilon) < \infty$ for some $\varepsilon \in (\varepsilon_0, 1)$.

We now address (T,Ω)-tractability. Let I_γ be (T,Ω)-tractable. Due to Theorem 5.1 we have

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(\varepsilon^{-1},d))} < \infty\,.$$

From $S(\varepsilon) < \infty$ it follows that $t_1^* < \infty$. From $e(n, I_{1,\gamma}) = \Theta(n^{-1})$, we get $n(\varepsilon, I_{1,\gamma}) = \Theta(\varepsilon^{-1})$. Thus we have $t_2^* < \infty$.

Let us now assume that t_1^* and t_2^* are finite. Let us first consider the case, where $\limsup_{d\to\infty}\sum_{j=1}^d \gamma_{d,j}$ is finite. Then, due to (15), η_d is uniformly

bounded and Theorem 6.1 implies that for (T,Ω)-tractability it is sufficient to bound ε^{-2} by $CT(\varepsilon^{-1},1)^t$ which is possible if $t > 2t_2^*$. Let us now consider the case where $\limsup_{d\to\infty}\sum_{j=1}^d \gamma_{d,j}$ is infinite. Observe that this and the finiteness of t_1^* imply that $\lim_{d\to\infty} T(1,d)=\infty$. From Theorem 6.1 and (15) we know that for (T,Ω)-tractability it is sufficient to show

$$t' := \limsup_{(\varepsilon^{-1},d)\in\Omega,\,\varepsilon^{-1}+d\to\infty} \left\{ \frac{1}{6}\frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(\varepsilon^{-1},d))} + 2\frac{\ln\varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},d))} \right\} < \infty\,.$$

Let us consider a sequence $\{(\varepsilon_k^{-1},d_k)\}$ with $\varepsilon_k^{-1}\to\infty$ and $\{d_k\}$ bounded. For convenience we omit the indices k. Then

$$\limsup_{\{(\varepsilon^{-1},d)\}} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(\varepsilon^{-1},d))} = \limsup_{\{(\varepsilon^{-1},d)\}} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(1,d))}\frac{\ln(1+T(1,d))}{\ln(1+T(\varepsilon^{-1},d))} = 0\,,$$

since $\ln(1+T(\varepsilon^{-1},d)) \ge \ln(1+T(\varepsilon^{-1},1)) \to \infty$ as $\varepsilon^{-1}\to\infty$. Furthermore,

$$\limsup_{\{(\varepsilon^{-1},d)\}} \frac{\ln\varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},d))} \le t_2^*\,.$$

Let us now consider a sequence $\{(\varepsilon^{-1},d)\}$ with $d\to\infty$. Then

$$\limsup_{\{(\varepsilon^{-1},d)\}} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(\varepsilon^{-1},d))} \le \limsup_{\{(\varepsilon^{-1},d)\}} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(1,d))} \le t_1^*\,,$$

and

$$\limsup_{\{(\varepsilon^{-1},d)\}} \frac{\ln\varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},d))} = \limsup_{\{(\varepsilon^{-1},d)\}} \frac{\ln\varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))}\frac{\ln(1+T(\varepsilon^{-1},1))}{\ln(1+T(\varepsilon^{-1},d))}\,.$$

Let us denote the last quantity by C_1. If $\varepsilon^{-1}\to\infty$ then obviously $C_1\le t_2^*$. If $\{\varepsilon^{-1}\}$ is bounded from above by, say, C_2 then

$$\frac{\ln(1+T(\varepsilon^{-1},1))}{\ln(1+T(\varepsilon^{-1},d))} \le \frac{\ln(1+T(C_2,1))}{\ln(1+T(1,d))}\,,$$

which converges to zero as d tends to infinity; hence $C_1=0$. This shows that $t' \le \frac{1}{6}t_1^* + 2t_2^*$. The statement concerning the exponents t and t^{tra} follows then from Theorem 6.1 and from the univariate case showing that $t^{\mathrm{tra}} \ge t_2^*$.

We now turn to strong (T,Ω)-tractability. If I_γ is strongly (T,Ω)-tractable then Theorem 5.1 implies that t_1^* is finite, whereas the univariate case $d=1$ implies that t_2^* is finite. Assume then that both t_1^* and t_2^* are

finite. Then η_d is uniformly bounded and Theorem 6.1 implies that it is enough to bound ε^{-2} by $C\,T(\varepsilon^{-1},1)^t$ which is possible if $t > 2t_2^*$. The univariate case yields that the exponent t^{str} of strong (T,Ω)-tractability must be at least t_2^*. This completes the proof. $\square$

Remark 7.2. In Theorem 7.1 we made the additional assumptions $[1,\infty)\times[d^*]\subseteq\Omega$ for $d^*\geq 1$ and $S(\varepsilon)\leq\infty$ to ensure that the conditions $t_1^*\leq\infty$ and $t_2^*\leq\infty$ in statement 1 and 2 are not only sufficient but also necessary for (T,Ω)-tractability and strong (T,Ω)-tractability, respectively. Observe that for a tractability function T of product form, i.e., $T(x,y)=f_1(x)f_2(y)$ with non-decreasing functions $f_i:[1,\infty)\to[1,\infty)$, and $\lim_{x\to\infty}\ln(f_i(x))/x=0$ for $i=1,2$, we have $S(\varepsilon)<\infty$ for every $\varepsilon\in(\varepsilon_0,1)$. On the other hand $S(\varepsilon)=\infty$ for all $\varepsilon\in(\varepsilon_0,1)$ if, e.g., $T(x,y)=1+g_1(x)g_2(y)$, where the non-negative and non-decreasing functions g_i satisfy $g_1(1)=0$, $g_1(x)>0$ for $x>0$, and $\lim_{y\to\infty}g_2(y)=\infty$. Observe also that one can always modify a tractability function T by putting $T(1,d)=1$ for all d—it still remains a tractability function. In this case $S(\varepsilon)<\infty$ is equivalent to $\lim_{d\to\infty}T(\varepsilon^{-1},d)<\infty$.

For this Sobolev space $F_{d,\gamma}$ and QMC algorithms $Q_{n,d}$ with $a_i=n^{-1}$, it is known that the worst case errors of $Q_{n,d}$ are equal to the centered discrepancy, see [2, 4]. Since our upper bounds are based on QMC algorithms, the same estimates and conditions on generalized tractability presented in Theorem 7.1 are also valid for the centered discrepancy.

Example 2: Sobolev Space for Unbounded Domain

In this example we consider multivariate integration for the unbounded domain, $D_d=\mathbb{R}$, and for Sobolev spaces of smooth functions. More precisely let r be a positive integer. For $d=1$, similarly as in [4], let $F_{1,\gamma}$ be the Sobolev space of functions defined on $\mathbb{R}$ whose $(r-1)$st derivatives are absolutely continuous and whose rth derivatives belong to $L_2(\mathbb{R})$, and satisfy the conditions

$$f'(0)\;=\;f''(0)\;=\;\ldots\;=\;f^{(r-1)}(0)\;=\;0.$$

The inner product of $F_{1,\gamma}$ is given by

$$\langle f,g\rangle_{F_{1,\gamma}}\;=\;f(0)g(0)\;+\;\gamma^{-1}\int_{\mathbb{R}}f^{(r)}(x)g^{(r)}(x)\,\mathrm{d}x.$$

The reproducing kernel of $F_{1,\gamma}$ is $K_{1,\gamma}=R_1+\gamma\,R_2$ with $R_1=1$ and

$$R_2(x,y)\;=\;1_M(x,y)\int_{\mathbb{R}_+}\frac{(|x|-u)_+^{r-1}}{(r-1)!}\frac{(|y|-u)_+^{r-1}}{(r-1)!}\,\mathrm{d}u,$$

where 1_M is the characteristic function of $M = \{(x,y) \,:\, xy \geq 0\}$. Clearly, R_2 is decomposable with $a^* = 0$. Consider univariate integration

$$I_{1,\gamma} f \;=\; \int_{\mathbb{R}} \rho(x)\, f(x)\, \mathrm{d}x,$$

where $\rho(x) = \rho(-x) \geq 0$, $\int_{\mathbb{R}} \rho(x)\, \mathrm{d}x = 1$, and $\int_{\mathbb{R}} \rho(x)\, |x|^{2r-1}\, \mathrm{d}x \;<\; \infty$. We now have $h_{1,1} \;=\; 1$ and $h_{1,2} = h_{1,2,(0)} + h_{1,2,(1)}$ with

$$h_{1,2,(0)}(x) \;=\; \int_{-\infty}^{0} \rho(y)\, R_2(x,y)\, \mathrm{d}y \quad \text{and} \quad h_{1,2,(1)}(x) \;=\; \int_{0}^{\infty} \rho(y)\, R_2(x,y)\, \mathrm{d}y.$$

Note that both $h_{1,2,(i)}$ are well defined since it can be checked that $R_2(x,x) = O(|x|^{2r-1})$ and therefore even the integral $\int_{\mathbb{R}} \rho(x)\, R_2(x,x)\, \mathrm{d}x < \infty$. Furthermore, the functions $h_{1,2,(i)}$ are not zero since ρ is symmetric and non zero. Hence, the assumptions of Theorem 5.1 are satisfied, and symmetry of ρ and R_2 yield that $\alpha = \frac{1}{2}$.

For $d \geq 2$, we obtain the Sobolev space $F_{d,\gamma}$ with the inner product

$$\langle f, g \rangle_{F_{d,\gamma}} \;=\; \sum_{\mathfrak{u} \subseteq [d]} \gamma_{\mathfrak{u}}^{-1} \int_{\mathbb{R}^{|\mathfrak{u}|}} \frac{\partial^{\,r|\mathfrak{u}|}}{\partial\, x_{\mathfrak{u}}^{r}} f(x_{\mathfrak{u}}, 0) \frac{\partial^{\,r|\mathfrak{u}|}}{\partial\, x_{\mathfrak{u}}^{r}} g(x_{\mathfrak{u}}, 0)\, \mathrm{d}x_{\mathfrak{u}}$$

with the same notation as in the previous example with the obvious exchange of $\frac{1}{2}$ to 0.

Consider multivariate integration,

$$I_{d,\gamma}(f) \;=\; \int_{\mathbb{R}^d} \rho_d(x)\, f(x)\, \mathrm{d}x = \langle f, h_d \rangle_{F_{d,\gamma}}$$

with $\rho_d(x) = \prod_{j=1}^{d} \rho(x_j)$ and

$$h_d(x) \;=\; \prod_{j=1}^{d} \left(1 \,+\, \gamma_{d,j}\, h_{1,2}(x)\right) .$$

Furthermore, the initial error is

$$e(0, I_{d,\gamma}) \;=\; \prod_{j=1}^{d} \left(1 \,+\, \gamma_{d,j}\, A\right)^{1/2} ,$$

where

$$A \;:=\; \int_{\mathbb{R}^2} \rho(x)\, \rho(y)\, R_2(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

We also have

$$\int_{\mathbb{R}^d} \rho_d(x)\, K_{d,\gamma}(x,x)\,\mathrm{d}x \;=\; \prod_{j=1}^{d}(1 \,+\, \gamma_{d,j}\, B) \;<\; \infty,$$

with

$$B \;:=\; \int_{\mathbb{R}} \rho(x)\, R_2(x,x)\,\mathrm{d}x \;<\; \infty.$$

Hence, the assumptions of Theorem 6.1 are also satisfied and

$$\eta_d \;=\; \prod_{j=1}^{d} \frac{1 \,+\, \gamma_{d,j}\, B}{1 \,+\, \gamma_{d,j}\, A} \,-\; 1 \;\le\; \prod_{j=1}^{d} \left(1 \,+\, \gamma_{d,j}\,(B-A)\right).$$

For $d = 1$ it is known that $e(n, I_{1,\gamma}) = \Theta(n^{-r})$, see again e.g., [7]. Combining Theorems 5.1 and 6.1 and proceeding as for the previous example we obtain necessary and sufficient conditions on generalized tractability of this multivariate integration problem.

Theorem 7.3. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined for the Sobolev space $F_{d,\gamma}$ as in this example. Let T be an arbitrary tractability function, and let Ω be a tractability domain with $[1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for some $\varepsilon_0 \in (0,1)$. Then*

$$I_\gamma \text{ is weakly tractable in } \Omega \quad \text{iff} \quad \lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{d} \;=\; 0.$$

Assume additionally that $[1,\infty) \times [d^*] \subseteq \Omega$ for $d^* \ge 1$ and that

$$S(\varepsilon) := \sup_{d\in\mathbb{N}} \frac{\ln(1 + T(\varepsilon^{-1}, d))}{\ln(1 + T(1,d))} < \infty \quad \text{for some } \varepsilon \in (\varepsilon_0, 1).$$

Then

1. *I_γ is (T,Ω)-tractable iff*

$$\begin{aligned} t_1^* \;&:=\; \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(1 + T(1,d))} \;<\; \infty, \\ t_2^* \;&:=\; \limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1 + T(\varepsilon^{-1}, 1))} \;<\; \infty. \end{aligned}$$

   If this holds then for arbitrary $t \;>\; (B-A)\, t_1^* \,+\, 2t_2^*$ there exists a positive $C = C_t$ such that

$$n(\varepsilon, I_{d,\gamma}) \;\le\; C\, T(\varepsilon^{-1}, d)^t \quad \text{for all} \quad (\varepsilon^{-1}, d) \;\in\; \Omega.$$

   The exponent t^{tra} of (T,Ω)-tractability is in $[r^{-1} t_2^*, (B-A)\, t_1^* \,+\, 2t_2^*]$.

2. I_γ is strongly (T,Ω)-tractable iff

$$\begin{aligned} t_1^* &:= \limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_{d,j} < \infty, \\ t_2^* &:= \limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty. \end{aligned}$$

If this holds then for arbitrary $t > 2t_2^*$ there exists a positive $C = C_t$ such that

$$n(\varepsilon, I_{d,\gamma}) \le C\, T(\varepsilon^{-1},1)^t \quad \textit{for all} \quad (\varepsilon^{-1},d) \in \Omega.$$

The exponent t^{str} of strong (T,Ω)-tractability is in $[r^{-1}t_2^*, 2t_2^*]$.

Example 3: General Case with $R_1 = 1$

Based on the two previous examples, it is easy to see that we can obtain necessary and sufficient conditions for generalized tractability of multivariate integration for general spaces if we assume that the kernel $R_1 = 1$. Then $H(R_1) = \mathrm{span}(1)$ and $H(R_1) \cap H(R_2) = \{0\}$ holds iff $1 \notin H(R_2)$. We now assume that R_2 is Lebesgue measurable and that

$$A := \int_{D_1^2} \rho(x)\,\rho(y)\,R_2(x,y)\,\mathrm{d}x\,\mathrm{d}y \le B := \int_{D_1} \rho(x)\,R_2(x,x)\,\mathrm{d}x < \infty.$$

As in Section 4 we assume that R_2 is decomposable and consider integration for the space $F_{1,\gamma}$,

$$I_{1,\gamma} f = \int_{D_1} \rho(x)\,f(x)\,\mathrm{d}x = \langle f, h_{1,\gamma}\rangle_{F_{1,\gamma}}$$

with

$$h_{1,\gamma}(x) = 1 + \gamma\,\left(h_{1,2,(0)}(x) + h_{1,2,(1)}(x)\right),$$

where

$$h_{1,2,(i)}(x) = 1_{D_{(i)}}(x) \int_{D_{(i)}} \rho(y)\,R_2(x,y)\,\mathrm{d}y.$$

We assume that both $h_{1,2,(i)}$ are non-zero. For $d = 1$ we assume that $e(n, I_{1,\gamma}) = \Theta(n^{-r})$ for some $r > 0$.

Then Theorems 5.1 and 6.1 and the analysis of the previous examples yield the following theorem.

Theorem 7.4. *Consider multivariate integration $I_\gamma = \{I_{d,\gamma}\}$ defined for the space $F_{d,\gamma}$ with $R_1 = 1$ as in this example. Let T be an arbitrary tractability function, and let Ω be a tractability domain with $[1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega$ for some $\varepsilon_0 \in (0,1)$. Then*

$$I_\gamma \text{ is weakly tractable in } \Omega \quad \text{iff} \quad \lim_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{d} = 0.$$

Assume additionally that $[1,\infty) \times [d^*] \subseteq \Omega$ for $d^* \geq 1$ and that

$$S(\varepsilon) := \sup_{d\in\mathbb{N}} \frac{\ln(1+T(\varepsilon^{-1},d))}{\ln(1+T(1,d))} < \infty \quad \text{for some } \varepsilon \in (\varepsilon_0, 1).$$

Then

1. *I_γ is (T,Ω)-tractable iff*

$$\begin{aligned} t_1^* &:= \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_{d,j}}{\ln(1+T(1,d))} < \infty, \\ t_2^* &:= \limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty. \end{aligned}$$

   If this holds then for arbitrary $t > (B-A)\, t_1^* + 2t_2^*$ there exists a positive $C = C_t$ such that

$$n(\varepsilon, I_{d,\gamma}) \leq C\, T(\varepsilon^{-1}, d)^t \quad \text{for all} \quad (\varepsilon^{-1}, d) \in \Omega.$$

   The exponent t^{tra} of (T,Ω)-tractability is in $[r^{-1}t_2^*, (B-A)\, t_1^* + 2t_2^*]$.

2. *I_γ is strongly (T,Ω)-tractable iff*

$$\begin{aligned} t_1^* &:= \limsup_{d\to\infty} \sum_{j=1}^d \gamma_{d,j} < \infty, \\ t_2^* &:= \limsup_{\varepsilon^{-1}\to\infty} \frac{\ln \varepsilon^{-1}}{\ln(1+T(\varepsilon^{-1},1))} < \infty. \end{aligned}$$

   If this holds then for arbitrary $t > 2t_2^*$ there exists a positive $C = C_t$ such that

$$n(\varepsilon, I_{d,\gamma}) \leq C\, T(\varepsilon^{-1}, 1)^t \quad \text{for all} \quad (\varepsilon^{-1}, d) \in \Omega.$$

   The exponent t^{str} of strong (T,Ω)-tractability is in $[r^{-1}t_2^*, 2t_2^*]$.

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