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SPECTRAL STABILITY OF POST-FERTILIZATION TRAVELING WAVES

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Abstract. This paper studies the stability of a family of post-fertilization traveling waves on eggs proposed by Lane *et. al* [19], consisting of an elastic deformation pulse on the egg's surface, and a calcium concentration profile. The family is indexed by a coupling parameter $\epsilon > 0$ that measures the contraction stress effects on the egg's surface. It is shown that the corresponding family of Evans functions for $\epsilon > 0$ converges uniformly to the non-vanishing Evans function of the decoupled waves when $\epsilon = 0$, proving spectral stability.

1. Introduction

In this paper we address the spectral stability of post-fertilization waves on eggs, a family of traveling wave solutions to the following one-dimensional version of the model proposed by Lane, Murray and Manoranjan [19],

$$\begin{aligned} \mu u_{xxt} + u_{xx} - \tau(c)_x - su &= 0, \\ c_t - Dc_{xx} - R(c) - \epsilon u_x &= 0, \end{aligned} \tag{1}$$

where u denotes the elastic deformation on the egg's surface, and c is the concentration of free calcium. The nonlinear terms $\tau(c)$ and $R(c)$ represent contractile forces acting on the egg's surface and autocatalytic effects, respectively. In addition, $\mu = \mu_1 + \mu_2 > 0$ measures the combined shear and bulk viscosities; the parameter $1 \gg s > 0$ is the restoring force of the egg's surface; $D > 0$ represents the Fick's diffusion constant of calcium; and $\epsilon > 0$ accounts for the contraction stress effects on the increase of c (see [12, 19] for details). Functions R and τ are assumed to be smooth, and R has a bistable shape; more precisely we assume that

$$R(0) = R(1) = R(c_0) = 0, \quad \text{for some} \;\; c_0 \in (0,1), \tag{2}$$

$$\int_0^1 R(c)\, dc > 0, \tag{3}$$

and,

$$R'(0) < 0, \quad R'(1) < 0. \tag{4}$$

Consider a traveling wave solution to (1), of form $(\bar{u}, \bar{c})(x+\theta t)$. Make the galilean transformation $x \to x + \theta t$, and denote "$\,'\,$" as differentiation with respect to the moving variable. Hence $(\bar{u}, \bar{c})(x)$ satisfies the following system of equations

$$\begin{aligned} \mu\theta\bar{u}''' + \bar{u}'' - (\tau(\bar{c}))' - s\bar{u} &= 0, \\ \theta\bar{c}' - D\bar{c}'' - R(\bar{c}) - \epsilon\bar{u}' &= 0. \end{aligned} \tag{5}$$

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We are interested in bounded solutions $(\bar{u}, \bar{c})$ to (5) with $\bar{u}(\pm\infty) = 0$ and $\bar{c}(+\infty) = 1, \bar{c}(-\infty) = 0$; in other words, $\bar{u}$ is an elastic pulse, $\bar{c}$ is a calcium concentration profile, and the pair $(\bar{u}, \bar{c})$ represents an heteroclinic orbit of system (5). As described in [12] the restoring force of the egg's surface $s > 0$ is crucial for the existence of the elastic pulse (otherwise absent).

In [12], the first author and collaborators proved the existence of a unique heteroclinic connection for the uncoupled system with $\epsilon = 0$,

$$\begin{aligned} \mu\theta\bar{u}''' + \bar{u}'' - (\tau(\bar{c}))' - s\bar{u} = 0, \\ \theta\bar{c}' - D\bar{c}'' - R(\bar{c}) = 0. \end{aligned} \tag{6}$$

consisting of a previously well-studied bistable calcium profile [10], and an elastic pulse which depends, in turn, on the calcium wave. Furthermore, they showed that this heteroclinic connection persists for small values of the coupling parameter $\epsilon > 0$, using Melnikov's integral method [14, 15], and thus proving the existence of wave solutions $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ to (5).

This paper studies the stability of these post-fertilization traveling waves under small perturbations, for which a natural approach is to linearize the equations around the wave. In this context, the first stability analysis must yield *spectral stability*, that is, the conditions under which the linearized operator is "well-behaved" and precludes the existence of bounded solutions with explosive behavior in time of form $U(x)e^{\lambda t}$, Re $\lambda > 0$. This approach provides clues as to the stability of the full nonlinear system. The Evans function (first introduced by J. W. Evans in the study of nerve axon waves [6, 7, 8, 9], and formalized by Alexander, Gardner and Jones in [1]) is a powerful tool to locate isolated eigenvalues in the point spectrum and near the essential spectrum. Recently, Evans function tools have been the subject of great improvements in the context of viscous and relaxation shock profiles [13, 26, 20, 25] and integrable PDEs [17].

In this work we present a direct application of Evans function theory to prove that traveling wave solutions to (1) for $\epsilon > 0$ sufficiently small are spectrally stable, that is, the spectrum of the linearized system around the wave is located in the stable right half plane $\{\text{Re } \lambda < 0\}$. See Section 2.2 for the precise definition of spectral stability considered here. For that purpose, we apply a recent result from Evans function theory, developed in the context of viscous shocks [23], which assures that, under suitable structural but rather general conditions, the Evans functions for $\epsilon > 0$ converge uniformly to that of the uncoupled traveling wave with $\epsilon = 0$. By analiticity and uniform convergence, the non-vanishing property of the Evans function for $\epsilon = 0$ persists for $\epsilon > 0$ sufficiently small, proving spectral stability. A similar approach was implemented to prove the existence of an unstable eigenvalue for the slow pulse of the Fithugh-Nagumo system in [11].

Plan of the paper. In Section 2 we recall the main existence results of [12] and the basic properties of the traveling waves. In Section 4 we show that our spectral problem satisfies the main assumptions for the application of the Evans function machinery, in particular the consistent splitting condition of [1] and hyperbolicity with uniform spectral separation. In Section 5 we prove that the coupled spectral problem with $\epsilon > 0$ satisfies the conditions for unifom convergence of the associated family of Evans functions.

2. Preliminaries

2.1. **Structure of traveling waves.** Let us recall the existence results of [12, 10] which will be needed later.

Proposition 2.1. [10, 2, 3, 12] *Under assumptions* (2)-(4), *equations* (6) *have a unique solution* $(\bar{u}, \bar{c})(x)$ *satisfying* $\bar{u}(\pm\infty) = 0$, $\bar{c}(+\infty) = 1$, $\bar{c}(-\infty) = 0$, *where the wave speed* $\theta = \theta_*$ *is determined uniquely by*

$$\theta_* := \frac{\int_0^1 R(c)\,dc}{\int_{\mathbb{R}} (\bar{c}'(x))^2\,dx} > 0. \tag{7}$$

Moreover, $\bar{c}$ *is strictly monotone,* $\bar{c}' > 0$*. Here* $s > 0$ *must satisfy* $0 < \mu\sqrt{s} < 2\theta_*/\sqrt{27}$*.*

Remark 2.2. The global monotonicity of $\bar{c}$ and the uniqueness of θ_* are discussed in [10]. Details of the existence of the calcium profile can be found in [2, 3]. Being θ_* independent of s, we may choose s in the region $0 < \mu\sqrt{s} < 2\theta_*/\sqrt{27}$ to preclude waves oscillating around $c \equiv 1$ (see [12] for details). We keep this assumption for the rest of the paper.

Proposition 2.3. [12] *Assume* $\epsilon > 0$ *is sufficiently small. Then system* (1) *admits a traveling wave solution* $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ *satisfying* (5) *and* $\bar{u}^\epsilon(\pm\infty) = 0, \bar{c}^\epsilon(+\infty) = 1, \bar{c}^\epsilon(-\infty) = 0$*. The wave speed is given by* $\theta(\epsilon) = \theta_* + o(1)$ *for* $\epsilon \sim 0^+$*.*

As a by-product of the existence theorems we have uniform exponential decay of the traveling waves.

Lemma 2.4. *For all* $\epsilon \geq 0$ *sufficiently small, traveling wave solutions* $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ *satisfy*

$$\begin{aligned} |\partial_x^j \bar{u}^\epsilon(x)| &\lesssim e^{-|x|/C_1}, \quad \text{as } |x| \to +\infty, \quad j = 0, 1, 2, \\ |\partial_x^i (\bar{c}^\epsilon(x) - 1)| &\lesssim e^{-x/C_1}, \quad \text{as } x \to +\infty, \quad i = 0, 1, \\ |\partial_x^i \bar{c}^\epsilon(x)| &\lesssim e^{+x/C_1}, \quad \text{as } x \to -\infty, \quad i = 0, 1, \end{aligned} \tag{8}$$

with some uniform $C_1 > 0$*.*

Proof. This is a direct consequence of hyperbolicity of the non-degenerate equilibrium points. The traveling wave pair $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ is a heteroclinic connection between the hyperbolic points $P_0 = 0$ and $P_1 = (0, 0, 0, 1, 0)^\top$, when equations (5) are written as a first order system for $(u, u', u'', c, c')^\top$ (see [12]). The linearization of the system around P_n, $n = 0, 1$, is given by $\mathbb{A}^\epsilon_\pm(0)$ (see (12) - (14) below), and has three roots with Re $\kappa^\epsilon_j < 0$, and two roots with Re $\kappa^\epsilon_j > 0$ for all $\epsilon \geq 0$ (see Theorem 1 in [12]). Whence, the dimensions of the stable and unstable spaces of $\mathbb{A}^\epsilon_\pm(0)$ are $\dim S^\epsilon_\pm(0) = 3$ and $\dim U^\epsilon_\pm(0) = 2$, respectively, and there is no center eigenspace. Since $|\text{Re}\,\sigma(\mathbb{A}^0_\pm(0))| \geq \delta_0$ for some fixed $\delta_0 > 0$, by continuity on ϵ of the coefficients $\mathbb{A}^\epsilon_\pm(0)$ there exists $\epsilon_0 > 0$ such that there holds the uniform bound

$$|\text{Re}\,\sigma(\mathbb{A}^\epsilon_\pm(0))| \geq 1/C_1 > 0,$$

on $\epsilon \in [0, \epsilon_0]$, with $1/C_1 = \delta_0/2$. Exponential decay (8) follows by standard ODE estimates. $\square$

2.2. **Perturbation equations and the stability problem.** Let us consider solutions to (1) of form $u + \bar{u}, c + \bar{c}$, being u and c perturbations. Making again the change of variables $x \to x + \theta t$ we obtain, in view of the profile equations (5), the following nonlinear system for u and c,

$$\begin{aligned} \mu\theta u_{xxx} + \mu u_{xxt} + u_{xx} - su - (\tau(c + \bar{c}) - \tau(\bar{c}))_x &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - (R(c + \bar{c}) - R(\bar{c})) &= 0. \end{aligned} \tag{9}$$

Linearizing (9) around the wave, we obtain the system

$$\begin{aligned} \mu\theta u_{xxx} + \mu u_{xxt} + u_{xx} - su - (\tau'(\bar{c})c)_x &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c &= 0. \end{aligned} \tag{10}$$

In order to define a suitable spectral problem, specialize (10) to perturbations of form $(e^{\lambda t}u(x), e^{\lambda t}c(x))$ with $\lambda \in \mathbb{C}$ to obtain the following linear system of equations parametrized by λ

$$\begin{aligned} \mu\theta u''' + (\mu\lambda + 1)u'' - su - \tau'(\bar{c})c' - (\tau'(\bar{c}))'c &= 0, \\ \lambda c + \theta c' - Dc'' - \epsilon u' - R'(\bar{c})c &= 0. \end{aligned} \tag{11}$$

It is clear that a necessary condition for stability of the traveling wave is the absence of non-trivial L^2 solutions $(u, c)(x)$ to system (11) with $\text{Re}\ \lambda > 0$. Note that system (10) has not the usual form $U_t = \mathcal{L}U$, with $\mathcal{L}$ a linear operator, for which the spectral condition is simply that the unstable half plane $\{\text{Re}\ \lambda \geq 0\}$ is contained in the resolvent set of $\mathcal{L}$. Therefore, we must make precise the notion of spectral stability suitable for our needs.

Denoting $v := u'$, $w := u''$ and $e := c'$, we can write (11) as a first order ODE system in the frequency regime (according to custom in the Evans function literature [1, 24]) of form

$$W' = \mathbb{A}^\epsilon(x, \lambda)W, \tag{12}$$

where

$$W := (u, v, w, c, e)^\top,$$

$$\mathbb{A}^\epsilon(x, \lambda) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s/\mu\theta & 0 & -(1 + \mu\lambda)/\mu\theta & \tau''(\bar{c})\bar{c}'/\mu\theta & \tau'(\bar{c})/\mu\theta \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\epsilon/D & 0 & (\lambda - R'(\bar{c}))/D & \theta/D \end{pmatrix}, \tag{13}$$

where $\theta = \theta(\epsilon)$ and $\bar{c} = \bar{c}^\epsilon$ are the speed and wave from Proposition 2.3. By standard considerations [16] the asymptotic coefficient matrices at $x = \pm\infty$,

$$\mathbb{A}^\epsilon_\pm(\lambda) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s/\mu\theta & 0 & -(1 + \mu\lambda)/\mu\theta & 0 & \tau'(n)/\mu\theta \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\epsilon/D & 0 & (\lambda - R'(n))/D & \theta/D \end{pmatrix}, \tag{14}$$

determine the localization of the essential spectrum. Here $n = 0, 1$, for $x = +\infty, -\infty$, respectively.

Consider the following family of linear, closed, densely defined operators in $\mathcal{H} = L^2(\mathbb{R};\mathbb{C}^5)$ with domain $\mathcal{D} = H^1(\mathbb{R};\mathbb{C}^5)$,

$$\begin{aligned} \mathcal{T}^\epsilon(\lambda) &: \mathcal{D} \longrightarrow \mathcal{H}, \\ W &\mapsto W' - \mathbb{A}^\epsilon(x,\lambda)W, \end{aligned} \tag{15}$$

indexed by $\epsilon \geq 0$ and $\lambda \in \mathbb{C}$.

Definition 2.5. For fixed $\epsilon \geq 0$ we define the *resolvent* ρ, the *point spectrum* $\sigma_{\rm pt}$ and the *essential spectrum* $\sigma_{\rm ess}$ of problem (11) as

$$\begin{aligned} \rho &:= \{\lambda \in \mathbb{C} \,:\, \mathcal{T}^\epsilon(\lambda) \text{ is one-to-one and onto, and } (\mathcal{T}^\epsilon(\lambda))^{-1} \text{ is bounded}\}, \\ \sigma_{\rm pt} &:= \{\lambda \in \mathbb{C} \,:\, \mathcal{T}^\epsilon(\lambda) \text{ is Fredholm with index 0 and has a non-trivial kernel}\}, \\ \sigma_{\rm ess} &:= \{\lambda \in \mathbb{C} \,:\, \mathcal{T}^\epsilon(\lambda) \text{ is either not Fredholm or has index different from 0}\}. \end{aligned}$$

We call the *spectrum* σ of (11) as the union of the point and essential spectrum, $\sigma = \sigma_{\rm ess} \cup \sigma_{\rm pt}$. Note that since each $\mathcal{T}^\epsilon(\lambda)$ is closed, then $\rho = \mathbb{C}\backslash\sigma$ (see Kato [18], pg. 167). If $\lambda \in \sigma_{\rm pt}$ we say λ is an *eigenvalue* of (11).

Remark 2.6. There are different definitions of the essential spectrum [5, 18]. Our choice allows us to apply Palmer's results directly [21, 22]. Note that this definition of spectra for (11) coincides with the definition for a linearized operator $\mathcal{L}$ when the equations can be written as $U_t = \mathcal{L}U$ (this holds because the Fredholm properties of $\mathcal{L} - \lambda$ and $\mathcal{T}(\lambda)$ are the same [24]).

Remark 2.7. Since operators (15) and $\bar{\mathcal{T}}(\lambda) := d/dx - \bar{\mathbb{A}}(\cdot,\lambda)$, with

$$\bar{\mathbb{A}}(\cdot,\lambda) = \begin{cases} \mathbb{A}^\epsilon_-(\lambda) & \text{if } x < 0, \\ \mathbb{A}^\epsilon_+(\lambda) & \text{if } x \geq 0, \end{cases}$$

differ only by a relatively compact operator [16], then the boundary of the essential spectrum depends only on the asymptotic rest states $\mathbb{A}^\epsilon_\pm$. Hence, the stability of the essential spectrum is a consequence of the hyperbolicity of the asymptotic matrices in a connected region containing $\{\mathrm{Re}\,\lambda \geq 0\}$ (see Corollary 4.2 below).

Remark 2.8. Often the point spectrum is defined as the set of *isolated eigenvalues with finite multiplicity*, that is,

$$\begin{aligned} \tilde{\sigma}_{\rm pt} := \{\lambda \in \mathbb{C} \,:\, &\mathcal{T}^\epsilon(\lambda) \text{ is Fredholm with } \operatorname{ind}\mathcal{T}^\epsilon(\lambda) = 0, \ker\mathcal{T}^\epsilon(\lambda) \neq \{0\}, \\ &\text{and } \mathcal{T}^\epsilon(\tilde{\lambda}) \text{ is invertible for all } \tilde{\lambda} \text{ in a deleted neighborhood of } \lambda\}. \end{aligned}$$

Clearly $\tilde{\sigma}_{\rm pt} \subset \sigma_{\rm pt}$, and if $\tilde{\Omega}$ is a connected component of the open set $\mathbb{C}\backslash\sigma_{\rm ess}$, then either $\mathcal{T}^\epsilon(\lambda)$ is invertible for all but a discrete set of points in $\tilde{\Omega}$, or $\mathcal{T}^\epsilon(\lambda)$ has a non-trivial kernel for all $\lambda \in \tilde{\Omega}$ and, in this case, $\tilde{\Omega} \subset \sigma_{\rm pt}$. This holds because of analiticity of the Evans function, which either vanishes identically on $\tilde{\Omega}$, or has a discrete set of finite order zeroes on $\tilde{\Omega}$ [24]. If, for instance, $\tilde{\Omega} = \{\mathrm{Re}\ \lambda > 0\}$ (assuming stability of $\sigma_{\rm ess}$), then *it suffices to rule out isolated eigenvalues with finite multiplicity in the unstable half plane to prove spectral stability.*

Therefore, we define the spectral stability of the waves in terms of the point spectrum, meaning no loss of generality.

Definition 2.9. We define *strong spectral stability* of the traveling wave solutions of (5) as

$$\sigma_{\rm pt} \subset \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\ \lambda < 0\} \cup \{0\}, \tag{16}$$

or equivalently [16], there are no L^2 solutions to the eigenvalue equations (11) for $\mathrm{Re}\,\lambda \geq 0$ and $\lambda \neq 0$.

2.3. **The Evans function.** If Ω is an open connected region in the complement of the essential spectrum, the Evans function [1, 9, 24] is an analytic function of $\lambda \in \Omega$ with the property that its zeroes coincide with isolated eigenvalues of finite multiplicity; furthermore, the order of the zero is the algebraic multiplicity (a.m.) of the eigenvalue. Eigenfunctions $W \in L^2$ of (11) are characterized by non-trivial intersection between the stable/unstable manifolds at $+\infty/-\infty$ of (12). The Evans function measures then the angle of this intersection. It is usually constructed by means of the wronskian of ordered bases $W_1^-(x,\lambda), \ldots, W_j^-(x,\lambda)$ spanning the solutions to (12) that decay at $-\infty$, and $W_{j+1}^+(x,\lambda), \ldots, W_n^+(x,\lambda)$ spanning the solutions that decay at $+\infty$. The dimensions of these manifolds remain constant for $\lambda \in \Omega$. The Evans function is therefore given by

$$D^\epsilon = \det\left(W_1^-(x,\lambda), \ldots, W_j^-(x,\lambda), W_{j+1}^+(x,\lambda), \ldots, W_n^+(x,\lambda)\right)_{|x=0},$$

(see [26, 24]), measuring the transversality of the initial conditions that provide solutions decaying at both ends $x = \pm\infty$. The Evans function is analytic in the region of constant splitting Ω and $D^\epsilon(\lambda) = 0$ if and only if there exists a non-trivial $W \in L^2$ solving (12). See [1, 24] and the references therein. Notice that the Evans function is highly non-unique, but they all differ modulo a non-vanishing analytic factor.

In addition, by translation invariance of the wave, $\lambda = 0$ is an eigenvalue with the derivative of the wave as eigenfunction, and thus $D^\epsilon(0) = 0$ if the essential spectrum does not touch zero. Proving stability amounts to showing that the Evans function does not vanish in the closed right half plane except possibly for a simple zero at $\lambda = 0$, which is equivalent to condition (16).

In this paper we show:

Theorem 1. *The traveling wave solutions $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ of Proposition 2.3 are strong spectrally stable for $\epsilon > 0$ sufficiently small.*

3. Spectral stability for $\epsilon = 0$

Consider the spectral problem (11) with $\epsilon = 0$,

$$\begin{aligned} \mu\theta_* u''' + (\mu\lambda + 1)u'' - su - \tau'(\bar{c})c' - (\tau'(\bar{c}))'c &= 0, \\ \lambda c + \theta_* c' - Dc'' - R'(\bar{c})c &= 0, \end{aligned} \tag{17}$$

where $(\bar{u}, \bar{c})$ denote the wave solutions to (6) of Proposition 2.1. The spectral stability these waves is a direct consequence of the results of Fife and McLeod [10]. For completeness, we include a proof in the spirit of the present (spectral) analysis, following Henry [16] closely. Let us start with an observation.

Lemma 3.1. *For $Re\,\lambda \geq 0$, the only bounded $L^2(\mathbb{R})$ solution of*

$$(1 + \mu\lambda)u'' + \mu\theta_* u''' - su = 0, \tag{18}$$

with $u(\pm\infty) = 0$ is $u = 0$ a. e.

Proof. Equation (18) can be written as a linear first order system $U' = \mathbb{A}_1(\lambda)U$, with $U = (u, u', u'')^\top$, and whose coefficient matrix

$$\mathbb{A}_1(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s/\mu\theta_* & 0 & -(1+\mu\lambda)/\mu\theta_* \end{pmatrix}$$

is hyperbolic for all $\mathrm{Re}\,\lambda \geq 0$. Indeed, if $\kappa = ia$ is an eigenvalue with $a \in \mathbb{R}$, then a solves $s + a^2(ia\mu\theta_* + (1+\mu\lambda)) = 0$. Taking real part we arrive at $s + a^2(1+\mu\mathrm{Re}\,\lambda) = 0$, which is a contradiction with $\mathrm{Re}\,\lambda \geq 0$, $s > 0$, $\mu > 0$. Thus, $\mathbb{A}_1$ has no center eigenspace for $\mathrm{Re}\,\lambda \geq 0$, and every solution in $L^2(\mathbb{R})$ decaying at both $x = \pm\infty$ is the trivial one, $u = 0$ a. e. $\square$

Lemma 3.2. *Traveling wave solutions* $(\bar{u}, \bar{c})$ *of system* (6) *are spectrally stable. Moreover,* $\lambda = 0$ *is an eigenvalue with geometric multiplicity* $g.m. = 1$.

Proof. It suffices to look at the point spectral problem for the decoupled bistable calcium equation

$$Dc'' - \theta_* c' + R'(\bar{c})c = \lambda c. \tag{19}$$

Suppose (u, c) is a non-trivial L^2 solution to (17) with $\mathrm{Re}\,\lambda \geq 0$. Then c solves (19), and we can assume $c \not\equiv 0$; indeed, if $c = 0$ a. e., then u is a solution to (18) and by Lemma 3.1, $u = 0$ a. e. Whence, we look exclusively at (19) with non-trivial solution c. Define $w = ce^{-\theta_* x/2}$ and $\phi = \bar{c}' e^{-\theta_* x/2}$, and use (5) and (19) to obtain

$$\begin{aligned} Dw'' + \Big(R'(\bar{c}) - \frac{\theta_*^2}{4D}\Big)w &= \lambda w, \\ D\phi'' + \Big(R'(\bar{c}) - \frac{\theta_*^2}{4D}\Big)\phi &= 0. \end{aligned}$$

Notice that $\phi > 0$ by monotonicity of the calcium profile, $\bar{c}' > 0$. Combining the equations for w and ϕ we arrive at

$$w'' - \frac{\phi''}{\phi} w = \frac{\lambda}{D} w. \tag{20}$$

Take the complex L^2 product of last equation with w and integrate by parts to get

$$\begin{aligned} \frac{\lambda}{D}\|w\|_{L^2}^2 &= \langle w, w'' \rangle_{L^2} - \langle w, (\phi''/\phi)w \rangle_{L^2} \\ &= -\int_{\mathbb{R}} |w'|^2\,dx + \int_{\mathbb{R}} \phi' \Big(\frac{|w|^2}{\phi}\Big)'\,dx \\ &= -\int_{\mathbb{R}} \phi^2 \Big|\Big(\frac{w}{\phi}\Big)'\Big|^2\,dx \;\leq\; 0, \end{aligned}$$

showing not only that λ is real but stable $\lambda \leq 0$. Moreover, if $\lambda = 0$ then $w/\phi = k$ a. e., for some constant k, that is, $c = k\bar{c}'$, and $\lambda = 0$ is an eigenvalue of (19) with eigenspace spanned by the sole eigenfunction $\bar{c}'$. For $k \neq 0$ (the case $k = 0$ is trivial) we multiply the equation for $\bar{u}$ with $\lambda = 0$ by k, and take the difference with the equation for u; the result is

$$(u - k\bar{u}')' + \mu\theta_*(u - k\bar{u}')''' - s(u - k\bar{u}') = 0.$$

An elementary L^2 estimate and integration by parts yield $\|u - k\bar{u}'\|_{L^2}^2 = 0$, which implies $u = k\bar{u}'$ a. e. We conclude that $\lambda = 0$ is also an eigenvalue of (17) with eigenfunction $(\bar{u}', \bar{c}')$ and geometric multiplicity $g.m. = 1$, as claimed. This proves the lemma. $\square$

Corollary 3.3. *The Evans function associated to the spectral problem* (17) *satisfies*

$$D^0(\lambda) \neq 0, \quad \textit{for } \; Re\,\lambda \geq 0, \, \lambda \neq 0. \tag{21}$$

4. Hyperbolicity and consistent splitting

We now look at the asymptotic coefficients (14). From the existence result (Proposition 2.3) we know that

$$\theta = \theta_* + \delta(\epsilon), \tag{22}$$

with $\delta(\epsilon) \to 0$ as $\epsilon \to 0^+$. The function $\delta(\cdot)$ will determine the rate of convergence of the family of Evans functions. Plugging (22) into the coefficients (14) and expanding θ around θ_* we obtain

$$\mathbb{A}^\epsilon_\pm(\lambda) = \mathbb{A}^0_\pm(\lambda) + \mathbb{S}^\epsilon_\pm(\lambda), \tag{23}$$

with

$$\mathbb{S}^\epsilon_\pm := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s\mathcal{O}(\delta(\epsilon))/\mu & 0 & -(1+\mu\lambda)\mathcal{O}(\delta(\epsilon))/\mu & 0 & \tau'(n)\mathcal{O}(\delta(\epsilon))/\mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon/D & 0 & 0 & \mathcal{O}(\delta(\epsilon))/D \end{pmatrix}. \tag{24}$$

$\mathbb{A}^\epsilon_\pm(\lambda)$ is linear (analytic) in λ and continuous in ϵ. The perturbation matrix $\mathbb{S}^\epsilon_\pm$ converges to zero at a rate $\mathcal{O}(|\delta(\epsilon)| + \epsilon)$, and has the form

$$\mathbb{S}^\epsilon_\pm(\lambda) = -\mathcal{O}(\delta(\epsilon)) \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \lambda & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + \mathcal{O}(\delta(\epsilon) + \epsilon),$$

so that

$$|\mathbb{S}^\epsilon_\pm(\lambda)| \leq \mathcal{O}(|\delta(\epsilon)| + \epsilon)(1 + |\lambda|). \tag{25}$$

Let us denote the characteristic polynomial of $\mathbb{A}^\epsilon_\pm$ as

$$\pi^\epsilon_\pm(\kappa) := \det\,(\mathbb{A}^\epsilon_\pm - \kappa I). \tag{26}$$

Notice that the coefficients (14) can be expressed as

$$\mathbb{A}^\epsilon_\pm := \begin{pmatrix} \mathbb{A}_1 & \mathbb{A}_2 \\ \mathbb{A}_3 & \mathbb{A}_4 \end{pmatrix},$$

with,

$$\begin{aligned} \mathbb{A}_1 &:= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s/\mu\theta & 0 & -(1+\mu\lambda)/\mu\theta \end{pmatrix}, \\ \mathbb{A}_2 &:= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \tau'(n)/\mu\theta \end{pmatrix}, \\ \mathbb{A}_3 &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon/D & 0 \end{pmatrix}, \\ \mathbb{A}_4 &:= \begin{pmatrix} 0 & 1 \\ (\lambda - R'(n))/D & \theta/D \end{pmatrix}. \end{aligned} \tag{27}$$

Define $\Lambda_0 := \min\{|R'(1)|, |R'(0)|, 1/\mu\} > 0$ and consider the following open connected region in $\mathbb{C}$

$$\Omega := \{\lambda \in \mathbb{C} \, : \mathrm{Re}\, \lambda > -\tfrac{1}{2}\Lambda_0\}. \tag{28}$$

Note that Ω is contained in the complement of the essential spectrum and contains the unstable half plane. Ω is the region of consistent splitting, and it was chosen in such a way that coefficient asymptotic matrices are hyperbolic and the dimensions of the respective unstable spaces agree for all ϵ in a neighborhood of zero, as we shall see. Also notice that *there is a positive spectral gap between the essential spectrum and the imaginary axis.*

Denote $S^\epsilon_\pm(\lambda)$ (resp. $U^\epsilon_\pm(\lambda)$) as the stable (resp. unstable) eigenspaces of $\mathbb{A}^\epsilon_\pm(\lambda)$. The following lemma is the main observation of this section.

Lemma 4.1. *For all $\lambda \in \Omega$ and all $\epsilon \geq 0$ sufficiently small, the coefficient matrices $\mathbb{A}^\epsilon_\pm(\lambda)$ have no center eigenspace and, moreover, $\dim U^\epsilon_\pm(\lambda) = 2$ and $\dim S^\epsilon_\pm(\lambda) = 3$.*

Proof. First observe that $\mathbb{A}_4$ has no center eigenspace for all $\epsilon \geq 0$ and all $\lambda \in \Omega$, as

$$\begin{aligned} \mathrm{Re}\,\det(\mathbb{A}_4 - ia) &= \mathrm{Re}\,(ia(ia - \theta/D) - (\lambda - R'(n))/D) \\ &= -(a^2 + (\mathrm{Re}\,\lambda - R'(n))/D) < 0, \end{aligned}$$

for all $a \in \mathbb{R}$ and all $\epsilon \geq 0$. We denote then

$$p^\epsilon(ia) := \det(\mathbb{A}_4 - ia) \neq 0, \quad a \in \mathbb{R}.$$

First, suppose that $\tau'(n) = 0$. This implies $\mathbb{A}_2 = 0$. Suppose that (26) has a purely imaginary root ia with $a \in \mathbb{R}$. Hence

$$\pi^\epsilon_\pm(ia) = p^\epsilon(ia)\det(\mathbb{A}_1 - ia) = 0$$

if and only if

$$\det(\mathbb{A}_1 - ia) = ia^3 + a^2(1 + \mu\lambda)/\mu\theta + s\mu\theta = 0. \tag{29}$$

Taking the real part of (29) leads to a contradiction with $\lambda \in \Omega$, namely

$$0 = a^2(1 + \mu \mathrm{Re}\,\lambda) + s > 0,$$

for all a. Therefore, in the case $\tau'(n) = 0$, $\mathbb{A}^\epsilon_\pm$ has no center eigenspace for all $\epsilon \geq 0$.

By continuity and connectedness of the region of consistent splitting Ω, the dimensions of the stable (resp. unstable) eigenspaces $S^\epsilon_\pm$ (resp. $U^\epsilon_\pm$) must remain constant on Ω. To compute the dimensions it suffices to set $\lambda = \eta \in \mathbb{R}$ and let $\eta \to +\infty$. The characteristic polynomial becomes

$$\begin{aligned} \pi^\epsilon_\pm(\kappa) &= \det(\mathbb{A}_1(\eta) - \kappa)\det(\mathbb{A}_4(\eta) - \kappa) \\ &= \left(-\kappa^2(\kappa + (1 + \mu\eta)/\mu\theta) + s/\mu\theta\right)\left(\kappa(\kappa - \theta/D) - (\eta - R'(n))/D\right). \end{aligned} \tag{30}$$

The quadratic factor in (30) produces two roots

$$\kappa^\epsilon_{1,2} = \frac{1}{2D}\left(\theta \pm \sqrt{\theta^2 + 4D(\eta - R'(n))}\right)$$

Since $\eta - R'(n) > 0$, $D > 0$, then clearly one root is positive, say $\kappa^\epsilon_1 > 0$, and the other is negative, $\kappa^\epsilon_2 < 0$. The cubic factor in (30)

$$H(\kappa) := s - \kappa^2(\kappa\mu\theta + (1 + \mu\lambda)),$$

produces three roots; a local maximum of H occurs at $\kappa = 0$, with value $H(0) = s > 0$, and a local minimum occurs at $\kappa_m = -2(1 + \mu\eta)/(3\mu\theta)$ with value $H(\kappa_m) =$

$s - 4(1+\mu\eta)^3/(27\mu^2\theta^2)$. Letting η be sufficiently large, we can make the value of H at the local minimum negative, so that we have 2 real negative roots $\kappa^\epsilon_{3,4} < 0$ and one positive real root $\kappa^\epsilon_5 > 0$. This shows that, on Ω, there exist 3 roots $\kappa^\epsilon_{2,3,4}(\lambda)$ with $\mathrm{Re}\,\kappa < 0$, and 2 roots $\kappa^\epsilon_{1,5}(\lambda)$ with $\mathrm{Re}\,\kappa > 0$. Thus $\dim S^\epsilon_\pm = 3$ and $\dim U^\epsilon_\pm = 2$, and the result holds when $\tau'(n) = 0$.

Consider now what happens when $\tau'(n) \neq 0$. For fixed $\lambda \in \Omega$, the dimensions will change only if the eigenvalues cross the imaginary axis, that is, if for some value of $\tau'(n)$ there exists $a \in \mathbb{R}$ such that $\pi^\epsilon_\pm(ia) = 0$. Note that $\mathbb{A}_4 - ia$ is invertible as $p^\epsilon(ia) \neq 0$ for all $a \in \mathbb{R}$. Thus one can write

$$\begin{aligned}\pi^\epsilon_\pm(ia) &= \det\begin{pmatrix}\mathbb{A}_1 - ia & \mathbb{A}_2\\ \mathbb{A}_3 & \mathbb{A}_4 - ia\end{pmatrix}\\ &= \det\left(\begin{pmatrix}I & \mathbb{A}_2\\ 0 & \mathbb{A}_4 - ia\end{pmatrix}\begin{pmatrix}\mathbb{A}_1 - ia - \mathbb{A}_2(\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3 & 0\\ (\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3 & I\end{pmatrix}\right)\\ &= \det(\mathbb{A}_4 - ia)\det(\mathbb{A}_1 - ia - \mathbb{A}_2(\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3).\end{aligned}$$

Perform the matrix computations to find that

$$(\mathbb{A}_4 - ia)^{-1} = p^\epsilon(ia)^{-1}\begin{pmatrix}\theta/D - ia & -1\\ (R'(n) - \lambda)/D & -ia\end{pmatrix},$$

$$\mathbb{A}_2(\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3 = p^\epsilon(ia)^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & ia\epsilon\tau'(n)/(\mu\theta D) & 0\end{pmatrix},$$

and,

$$\pi^\epsilon_\pm(ia) = p^\epsilon(ia)\det\begin{pmatrix}-ia & 1 & 0\\ 0 & -ia & 1\\ s/\mu\theta & -ia\epsilon\tau'(n)/(\mu\theta D p^\epsilon(ia)) & -ia - (1+\mu\lambda)/\mu\theta\end{pmatrix}.$$

Since $p^\epsilon(ia) \neq 0$, then $\pi^\epsilon_\pm(ia) = 0$ if and only if

$$i\mu\theta a^3 + s + a^2\left(\frac{\epsilon\tau'(n)}{Dp^\epsilon(ia)} + 1 + \mu\lambda\right) = 0.$$

In view of $s > 0$ and $\mathrm{Re}\,\lambda > -1/\mu$, taking real part of last equation and for $\epsilon \geq 0$ sufficiently small we arrive at

$$\frac{\epsilon\tau'(n)\mathrm{Re}\,p^\epsilon(ia)}{D|p^\epsilon(ia)|^2} + 1 + \mu\mathrm{Re}\,\lambda > 0,$$

which is a contradiction. This shows that the signs of the real parts of the eigenvalues of $\mathbb{A}^\epsilon_\pm$ are independent of $\tau'(n)$ for each $\lambda \in \Omega$ and $\epsilon \geq 0$ sufficiently small, yielding the result. □

An immediate consequence of hyperbolicity and consistent splitting (the dimensions of the stable/unstable spaces at both ends $x = \pm\infty$ agree) is the stability of the essential spectrum.

Corollary 4.2. *For $\epsilon \geq 0$ sufficiently small, the essential spectrum of* (11) *is contained in the stable half plane.*

Proof. Fix $\lambda \in \Omega$ and $\epsilon \geq 0$ in a neighborhood of zero. Being $\mathbb{A}^\epsilon_+(\lambda)$ and $\mathbb{A}^\epsilon_-(\lambda)$ hyperbolic, by standard exponential dichotomies theory [4] (see also Theorem 3.3 in [24]), system (12) has exponential dichotomies on both $x \in \mathbb{R}^+$ and $x \in \mathbb{R}^-$, with Morse indices $i_+(\lambda) = \dim U^\epsilon_+(\lambda) = 2$ and $i_-(\lambda) = \dim U^\epsilon_-(\lambda) = 2$, respectively.

Henceforth, Palmer's results (Lemma 3.4 in [21]; see also Theorem 3.2 in [24]) imply that the operators $\mathcal{T}^\epsilon(\lambda)$ are Fredholm with index

$$\operatorname{ind} \mathcal{T}^\epsilon(\lambda) = i_+(\lambda) - i_-(\lambda) = 0,$$

showing that $\Omega \subset \mathbb{C}\backslash\sigma_{\rm ess}$, or in other words, that $\sigma_{\rm ess} \subset \mathbb{C}\backslash\Omega \subset \{\mathrm{Re}\,\lambda < 0\}$. □

5. Convergence of Evans functions

Consider the family of first order systems (12), with $\lambda \in \Omega$ and ϵ varying within a set $\mathcal{V} = [0, \epsilon_0]$, where $\epsilon_0 > 0$ is chosen sufficiently small such that the conclusions of Proposition 2.3 and Lemmas 2.4 and 4.1 hold. If we regard coefficients (13) as functions from (λ, ϵ) into $L^\infty(x)$, then they are analytic in λ (linear), and continuous in ϵ (this follows by continuity on ϵ of θ and of the waves $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ by construction [12]). Moreover, $\mathbb{A}^\epsilon(\cdot, \lambda)$ approach exponentially to limits $\mathbb{A}_\pm$ as $x \to \pm\infty$, with uniform exponential decay estimates

$$|\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm| \leq C e^{-|x|/C_1}, \quad \text{for } x \gtrless 0 \tag{31}$$

on compact subsets of $\Omega \times \mathcal{V}$. In addition, on $\Omega \times \mathcal{V}$ the limiting coefficient matrices $\mathbb{A}^\epsilon_+$ and $\mathbb{A}^\epsilon_-$ are both hyperbolic and the dimensions of their unstable subspaces agree. Finally, the geometric separation hypothesis of Gardner and Zumbrun [13] holds trivially (continuous limits of $S^\epsilon_\pm$ and $U^\epsilon_\pm$ along λ-rays, $\lambda = \rho\lambda_0$ as $\rho \to 0^+$, $\lambda_0 \in \Omega$; see condition (A2) in [23]) because of hyperbolicity of the coefficients (with same dimensions) at $\lambda = 0$ as discussed in [12].

Hence, systems (12) belongs to the generic class considered in Section 2 of [23], for which the convergence of approximate flows (Proposition 2.4 in [23]) applies. The proof of Theorem 1 is then a direct consequence of the following

Lemma 5.1. *Suppose $(\lambda, \epsilon) \in \Omega \times \mathcal{V}$. Hence, as $\epsilon \to 0^+$, the asymptotic spaces U^ϵ_- and S^ϵ_+ converge uniformly in angle to U^0_-, S^0_+ with rate $\eta(\epsilon) = \mathcal{O}(\epsilon + |\delta(\epsilon)|)$, that is, for all $\epsilon \in \mathcal{V}$ their spanning bases satisfy*

$$|v^\epsilon_{j\pm} - v^0_{j\pm}| \leq \eta(\epsilon). \tag{32}$$

Moreover, the coefficient matrices (13) *converge uniformly exponentially to limiting values $\mathbb{A}^0$, in the sense that, for all $\epsilon \in \mathcal{V}$,*

$$|(\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm) - (\mathbb{A}^0 - \mathbb{A}^0_\pm)| \leq C_2 \eta(\epsilon) e^{-|x|/C_1}. \tag{33}$$

Proof. To prove (32), it suffices to show that the projections

$$\mathcal{P}^\epsilon_\pm(\lambda) := \frac{1}{2\pi i} \oint_{\Gamma^\epsilon_\pm} (z - \mathbb{A}^\epsilon_\pm)^{-1}\, dz,$$

are uniformly bounded as $\epsilon \to 0^+$ in a closed Ω-neighborhood of λ, with

$$\mathcal{P}^\epsilon_\pm = \mathcal{P}^0_\pm + \mathcal{O}(\epsilon + |\delta(\epsilon)|).$$

Here Γ^ϵ_- (resp. Γ^ϵ_+) is a rectifiable contour enclosing the unstable (resp. stable) eigenvalues of $\mathbb{A}^\epsilon_-$ (resp. $\mathbb{A}^\epsilon_+$). Compactness then gives a uniform resolvent bound

$$|R^0_\pm(z)| := |(\mathbb{A}^0_\pm - zI)^{-1}| \leq C \quad \text{on } \Gamma^\epsilon_\pm.$$

Here C depends only on $|\mathbb{A}^0_\pm|$. Recall that $\mathbb{A}^\epsilon_\pm = \mathbb{A}^0_\pm + \mathbb{S}^\epsilon_\pm$, with perturbation matrix $\mathbb{S}^\epsilon_\pm$ given by (24). Whence one may expand

$$\begin{aligned}(\mathbb{A}^\epsilon_\pm - zI)^{-1} &= (\mathbb{A}^0_\pm - zI + \mathbb{S}^\epsilon_\pm)^{-1}\\ &= ((\mathbb{A}^0_\pm - zI)(I + (\mathbb{A}^0_\pm - zI)^{-1}\mathbb{S}^\epsilon_\pm))^{-1}\\ &= (I + R^0_\pm(z)\mathbb{S}^\epsilon_\pm)^{-1} R^0_\pm(z)\\ &= (I + \mathcal{O}(|\delta(\epsilon)| + \epsilon))R^0_\pm(z),\end{aligned}$$

where $\eta(\epsilon) := \mathcal{O}(|\delta(\epsilon)| + \epsilon)$ depends on $|\mathbb{A}^0|$. We then get

$$\mathcal{P}^\epsilon_\pm = \mathcal{P}^0_\pm + \mathcal{O}\left(\int_{\Gamma^0_\pm} |R^0_\pm(z)||\eta(\epsilon)|\, dz\right) = \mathcal{P}^0_\pm + \mathcal{O}(\eta(\epsilon)),$$

as claimed.

The second assertion (33) is an immediate consequence of exponential decay (Lemma 2.4). Indeed, a direct computation shows that

$$\begin{aligned}|(\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm) - (\mathbb{A}^0 - \mathbb{A}^0_\pm)| = \mathcal{O}\left((|(\bar{c}^\epsilon)'(x)| + |\tau'(\bar{c}^\epsilon) - \tau'(n)|)/\theta\right) +\\ + \mathcal{O}\left((|(\bar{c})'(x)| + |\tau'(\bar{c}) - \tau'(n)|)/\theta_*\right),\end{aligned}$$

and using (8),

$$\begin{aligned}|\tau'(\bar{c}^\epsilon) - \tau'(1)| &= \mathcal{O}(|\bar{c}^\epsilon - 1|) = \mathcal{O}(e^{-|x|/C_1}),\\ |\tau'(\bar{c}^\epsilon) - \tau'(0)| &= \mathcal{O}(|\bar{c}^\epsilon|) = \mathcal{O}(e^{-|x|/C_1}),\\ |(\bar{c}^\epsilon)'(x)| &= \mathcal{O}(e^{-|x|/C_1}),\end{aligned}$$

uniformly in $\epsilon \in \mathcal{V}$ (including $\epsilon = 0$), and $1/\theta = 1/\theta_* + \mathcal{O}(\delta(\epsilon))$; therefore (33) follows directly. □

Proof of Theorem 1. In view of Lemma (5.1), systems (12) satisfy the hypotheses of Proposition 2.4 in [23]. Therefore for any $\epsilon \in \mathcal{V}$ and in an Ω-neighborhood of some λ, the local Evans functions $D^\epsilon(\lambda)$ converge uniformly to $D^0(\lambda)$ in a possibly smaller neighborhood of λ as $\epsilon \to 0^+$, with rate

$$|D^\epsilon - D^0| \leq C\eta(\epsilon) = \mathcal{O}(\epsilon + |\delta(\epsilon)|).$$

Since on $\{\text{Re } \lambda \geq 0\}$, $D^0(\lambda)$ does not vanish except at $\lambda = 0$ (Corollary 3.3), by analyticity and uniform convergence, the same holds for each D^ϵ for all ϵ near zero. This shows that there are no isolated eigenvalues with finite multiplicity in $\{\text{Re } \lambda \geq 0\}$ except $\lambda = 0$, proving the result. □

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