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by

Ivano Primi, Angela Stevens, and Juan J.L. Velazquez

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Mass-Selection in Alignment Models with Non-Deterministic Effects

I. Primi[*], A. Stevens [†] and J.J.L. Velázquez [‡]

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Abstract

In this paper we consider a kinetic model for alignment of cells or filaments with probabilistic turning. For this equation existence of solutions is known, see [2]. To understand its qualitative behavior, especially with respect to the selection of orientations and mass distributions for long times, the model is approximated by a diffusion equation in the limit of small deviations of the interactions between the cell bundles. For this new equation existence of steady states is shown. In contrast to the kinetic equation discussed in [2] with deterministic turning, where local stability of two opposite orientations was shown but no selection of mass could be observed, for the new approximating problem with probabilistic turning additionally mass selection takes place. In the limit of small diffusion, steady states can only be constructed, if the aligning masses are either equal or the total mass is concentrated in one direction. By numerical simulations we tested stability of these steady states and for situations with 4 symmetrically placed smooth distributions of alignment. Convergence of the numerical code was proved. The simulations suggest, that only the 2- and the 1-peak steady states can be stable, whereas the 4 peak steady state is always unstable. We conjecture that the noise in the system is responsible for this final selection of masses. There exist other steady states with an arbitrary number of aligned bundles of cells or filaments, but we suspect that, as numerically shown for the 4 peak case, these multi-peak states are all unstable.

Keywords: Mass selection, alignment, integro-differential equation.

1 Introduction

We consider alignment in a two-dimensional geometry. To describe orientational aggregation of bundles of cells or filaments we discuss an integro-differential equation for the evolution of a function f on the unit circle $(\mathbb{R}/\mathbb{Z})$ with arc

[*]Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D - 04103 Leipzig, Germany (ivano.primi@bioquant.uni-heidelberg.de)

[†]Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D - 04103 Leipzig, Germany (angela.stevens@bioquant.uni-heidelberg.de)

[‡]Departamento de Matemática Aplicada, Facultad de Ciencias Matemáticas, 28040 Madrid, Spain (JJ_Velazquez@mat.ucm.es) and Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D - 04103 Leipzig, Germany

length normalized to one. We will choose a representation $\mathcal{I} = [-\frac{1}{2}, \frac{1}{2}]$. In many of the following arguments it is convenient though to think in geometrical terms, namely $u \in \mathcal{I} = [-\frac{1}{2}, \frac{1}{2}] \to (\cos(2\pi u), \sin(2\pi u)) \in \mathbb{S}^1$. We will use this notation freely in some figures, unless confusion is to be expected. Here $f = f(u,t)$ denotes the density distribution of cells or filaments over the orientation $u \in \mathcal{I}$. The temporal evolution of f is given by

$$\partial_t f(u,t) = -\int_{\mathcal{I}} T[f](u,v) f(u,t)\,\mathrm{d}\,v + \int_{\mathcal{I}} T[f](v,u) f(v,t)\,\mathrm{d}\,v. \tag{1}$$

The first term on the right hand side describes the bundles of cells or filaments which reorient away from u, and the second term the bundles orienting themselves into direction u. From equation (1) one can easily derive that the total mass is conserved

$$\int_{\mathcal{I}} f(u,t)\,\mathrm{d}\,u = m := \int_{\mathcal{I}} f(u,0)\,\mathrm{d}\,u \qquad \forall t > 0\,. \tag{2}$$

The stationary version of equation (1) was analyzed in detail in [4] and [5].

For notational convenience we will sometimes omit the explicit t-dependencies in the following. The turning rate T in (1) maps a function f acting on $\mathcal{I}$ to a function $T[f]$ acting on $\mathcal{I} \times \mathcal{I}$ with

$$T[f](u,v) = \int_{\mathcal{I}} G_\sigma(v - M_w(u)) f(w,t)\,\mathrm{d}\,w. \tag{3}$$

Here $G_\sigma : [-1,1] \to \mathbb{R}_+$, $\sigma \geq 0$ is an even, bounded probability density, thus $\int_{\mathcal{I}} G_\sigma = 1$, i.e. the standard periodic Gaussian

$$G_\sigma(u) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{m \in \mathbb{Z}} \exp\left(-\frac{(u+m)^2}{2\sigma^2}\right).$$

So the process of turning is considered to be probabilistic. The smaller σ is, the narrower is G_σ, meaning that reorientation happens with higher accuracy. The limiting case is the Dirac mass $G_0(x) = \delta_0$, which describes deterministic turning.

The function $M_w : \mathcal{I} \to \mathcal{I}$, is called the optimal reorientation, indicating reorientation of bundles of cells or filaments due to their interaction with w. More precisely, if the system is invariant under rotations, we assume

$$M_w(v) = v + V(w - v),$$

where $V : [-1,1] \to \mathbb{R}$ is referred to as the orientational angle, compare fig. 1. A more detailed descriptions of M_w and V will be given in Section 2. In [2], the behavior of solutions for the Dirac mass, $G_0 = \delta_0$, was analyzed. For this case it was shown, that for suitable choices of the orientational angle V, the long time distribution of bundles consists of two Dirac masses pointing in two exactly opposite directions, where the distribution of mass is arbitrary.

Here we assume that the typical deviation σ and the orientational angle V are small, so that the kinetic equation (1) can be approximated by a second order parabolic equation using a Fokker-Planck type of argument, which results in

$$\partial_t f = \frac{\sigma^2 m}{2}\,\partial_{xx} f + \partial_x \left(f(x) \int_{\mathcal{I}} V(x-y) f(y) \mathrm{d}y \right) \qquad \forall x \in \mathbb{R}\,, \tag{4}$$

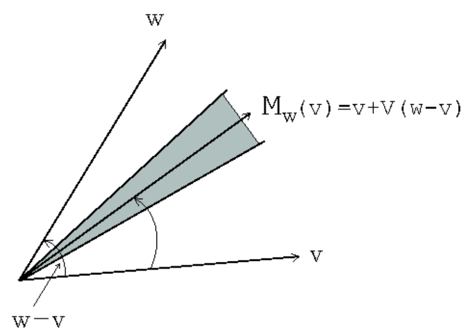

Figure 1: Geometric interpretation of the turning rate v

where m is the total mass, which remains constant for all times. This equation can be derived by eliminating σ via a change of variables, namely $z = \frac{1}{\sigma}(v-u-V(w-u))$ in the first integral term of (1) and $z = \frac{1}{\sigma}(v-u+V(w-v))$ in the second one. Expanding f in a Taylor series up to second order and neglecting terms of order σ^3, $\sigma^2\|V\|$, $\|V\|^2$ and higher we obtain (4). This equation has been justified with different assumptions in [6].

Remark 1.1. Replacing the function $f(x,t)$ by $\frac{1}{m}f(x,\frac{t}{m})$, it is possible to assume $m=1$, without loss of generality.

In section 2 we introduce the steady states of (4) as solutions of a suitable first order integro-differential problem.
In section 3 we present some heuristic arguments suggesting that for double-peaked steady states in the limit of small diffusion the mass is equally distributed between the two opposite peaks. The same kind of heuristics provides a condition for the existence of steady states with the mass concentrated in just one direction.
In section 4 we state the main theorem on the existence of steady states for (4).
In section 5 we prove some preliminary results about the reformulation of the steady state problem for $1/N$-periodic and even functions, where $N=2$ or, more generally, is an arbitrary positive integer.
In section 6 we prove the existence of 1/2-periodic steady states of (4) under the assumptions of the theorem of section 4.
In section 7 we generalize this result giving sufficient conditions for the existence of $1/N$-periodic steady states.
In section 8 we introduce a numerical scheme for the Cauchy problem associated to (4) assuming symmetry for the initial data. We prove convergence of this scheme and we discuss the results of our numerical simulations.

2 An Equivalent Steady State Model

Here we are interested in the steady states of equation (4), i.e. the solutions of

$$\begin{cases} \frac{\sigma^2}{2} f_{xx} + \frac{d}{dx}\left(f(x)\int_{\mathcal{I}} V(x-y)f(y)\mathrm{d}y\right) = 0 \quad \forall x \in \mathbb{R} \\ f\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) \\ \int_{\mathcal{I}} f(x)\,\mathrm{d}\,x = 1\,. \end{cases} \tag{5}$$

To deal with them we first reformulate this problem and then look for an approximating model. In the following we always assume that $V \in C^\infty(\mathbb{R})$ and

(V1) $V(x+1) \equiv V(x)$,

(V2) $V(-x) \equiv -V(x)$,

which are natural conditions.

Lemma 2.1. *f is a solution of problem (5) if and only if it solves*

$$\begin{cases} \frac{\sigma^2}{2} f_x + f(x)\int_{\mathcal{I}} V(x-y)f(y)\mathrm{d}y = 0 \quad \forall x \in \mathbb{R} \\ \int_{\mathcal{I}} f(x)\;\mathrm{d}\,x = 1\,. \end{cases} \tag{6}$$

Moreover, a solution f of (6) fulfills $f \in C^\infty(\mathbb{R})$, $f > 0$, and $f(x+1) \equiv f(x)$.

Proof: Due to Assumption (V2), it is easy to see that

$$\begin{cases} \frac{\sigma^2}{2} f_{xx} + \frac{d}{dx}\left(f(x)\int_{\mathcal{I}} V(x-y)f(y)\mathrm{d}y\right) = 0 \quad \forall x \in \mathbb{R} \\ f\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) \end{cases}$$

is equivalent to

$$\frac{\sigma^2}{2} f_x + f(x)\int_{\mathcal{I}} V(x-y)f(y)\mathrm{d}y = j \qquad \forall x \in \mathbb{R}\,, \tag{7}$$

with

$$j = \int_{\mathcal{I}} f(x)\int_{\mathcal{I}} f(y)V(x-y)dxdy = 0\,. \tag{8}$$

Since every solution of equation (7) can be written in the form

$$f(x) = K\exp\left(-\frac{2}{\sigma^2}\int_0^x \mathrm{d}\,\xi \int_{\mathcal{I}} V(\xi-y)f(y)\mathrm{d}y\right)\,, \tag{9}$$

the equivalence follows trivially by integration/differentiation and taking into account properties (V1) and (V2). From here one immediately deduces the equivalence of problems (5) and (6).

To solve (6) it is sufficient to choose the constant $K \in \mathbb{R}$ in (9) such that the normalization condition $\int_{\mathcal{I}} f(x)\;\mathrm{d}\,x = 1$ is fulfilled. Since $V \in C^\infty(\mathbb{R})$ and

$K > 0$, it follows from (9) that $f \in C^\infty(\mathbb{R})$ and $f > 0$. Also f is 1-periodic since, due to Assumptions (V1) and (V2), the function $\varphi(\xi) = \int_\mathcal{I} V(\xi - y) f(y) \mathrm{d}y$ is 1-periodic and

$$\int_\mathcal{I} \varphi(\xi)\,\mathrm{d}\,\xi = \int_I f(y)\,\mathrm{d}\,y \int_I V(\xi - y)\,\mathrm{d}\,\xi = 0\,,$$

which implies the 1-periodicity of the integral function of φ too. □

3 Heuristics for the Selection Mechanism

Let us first consider the formal solution of (6) for $\sigma = 0$. Then the equation reduces to

$$f(x) \int_\mathcal{I} V(x - y) f(y)\,\mathrm{d}\,y = 0.$$

Let δ_0 denote the Dirac mass concentrated at 0, then any function of the form $f(x) = \alpha\delta_0(x) + \beta\delta_0(x - \frac{1}{2})$ is a solution to this equation, since

$$\left(\alpha\delta_0(x) + \beta\delta_0\left(x - \frac{1}{2}\right)\right)\left[\alpha V(x) + \beta V\left(x - \frac{1}{2}\right)\right] = 0\,,$$

due to $V(0) = V(\frac{1}{2}) = V(-\frac{1}{2}) = 0$. So the choice of α and β is arbitrary, as it is the case for the model discussed in [2].

For $\sigma > 0$ the choice of α and β is not arbitrary, although one could expect that each solution of the limit problem $\sigma = 0$ is an *approximating* solution of (6) if σ is *sufficiently* small. So with respect to these considerations the model for $\sigma > 0$ is more robust. Keeping in mind the normalization condition $\alpha + \beta = 1$ we can show that for σ close to zero we either obtain $\alpha \to 1$ and $\beta \to 0$ or, the other way around, or $\alpha = \beta = \frac{1}{2}$.

The next question is, what the conditions on the orientational angle V are, for either just one peak to occur or two peaks of equal size.

Suppose that for every $\sigma > 0$ *sufficiently* small there exists a peak-like smooth function f, which is mainly concentrated in 0, solves problem (6) and converges to δ_0 for $\sigma \to 0$. Then this function may be approximated by the solution of the following problem

$$\begin{cases} \frac{\sigma^2}{2} f_x(x) + f(x)V(x) = 0 & x \in \mathbb{R} \\ \int_\mathcal{I} f(x)\ \mathrm{d}\,x = 1\,, \end{cases} \tag{10}$$

since

$$\int_\mathcal{I} V(x - y) f(y) \mathrm{d}y \approx \int_\mathcal{I} V(x - y)\delta_0(y) \mathrm{d}y = V(x). \tag{11}$$

Therefore, the solution of problem (10) is supposed to be smooth and positive with the mass strongly concentrated in 0. However, depending on the definition of the potential V, this is not always the case. A straightforward computation gives (10)

$$f(x) = \frac{\mathrm{e}^{-\frac{2}{\sigma^2}\int_0^x V(\xi)\,\mathrm{d}\,\xi}}{\int_\mathcal{I} \mathrm{e}^{-\frac{2}{\sigma^2}\int_0^y V(\xi)\,\mathrm{d}\,\xi}\,\mathrm{d}\,y} \qquad x \in \mathbb{R}. \tag{12}$$

If $\int_0^{1/2} V(x)\,\mathrm{d}\,x > 0$, then for small values of σ this function is smooth and mainly concentrated in 0, something which agrees with the picture of the Dirac mass located at 0 that we mentioned in the beginning. But if $\int_0^{1/2} V(x)\,\mathrm{d}\,x < 0$, then the peak is located in $\pm\frac{1}{2}$, which contradicts our assumption about the location of the peak. Therefore this strongly suggests, that a sufficient condition

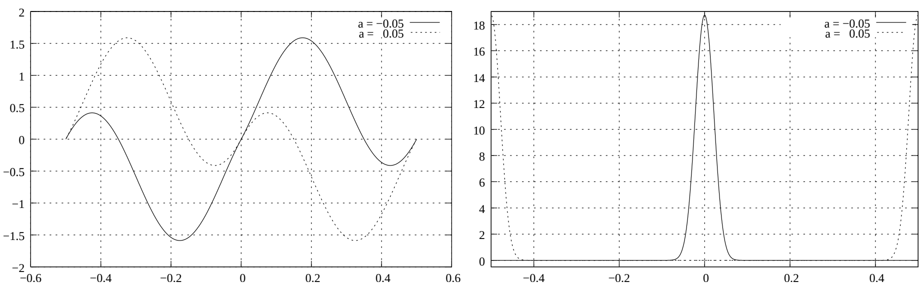

Figure 2: Different potential functions V, on the left; and related solutions f of (10), on the right

for a single peak is close to assuming $\int_0^{\frac{1}{2}} V(x)dx > 0$. The exact condition will be given more precisely later. Figure 2 shows these differences when

$$V(x) = 2\sin(2\pi x)\left(\cos(2\pi x)\cos(4\pi a) - |\sin(2\pi x)|\sin(4\pi a)\right)$$

and the parameter a takes the values -0.05 and 0.05, corresponding to the cases in which $\int_0^{1/2} V(x)\,\mathrm{d}\,x$ is positive, respectively negative.

Therefore, we can say that the presence of a small diffusion coefficient in the integro-differential equation (6) makes a selection between the solutions of the limit problem for $\sigma = 0$. The selection is determined by the nature of the action of the potential V. For instance, if V tends to align the bundles having close orientations then we can have solutions concentrated in a single peak. In contrast, if there is a strong tendency to produce alignments in opposite orientations, then most likely two peaks are to be expected.

Concerning the two peaks-like steady states, it is easy to see that the total mass must be equally distributed between the peaks. For this suppose that the solution f_σ of (6) approaches to $\alpha\delta_0(x)+\beta\delta_{1/2}(x)$ for $\sigma \to 0$ (where $\delta_{1/2}$ denotes the Dirac mass concentrated at 1/2). Then for *small* values of σ the function f_σ can be approximated by the solution of

$$\begin{cases} \frac{\sigma^2}{2} f_x(x) + f(x)V_{\alpha,\beta}(x) = 0 \quad x \in \mathbb{R} \\ \int_{\mathcal{I}} f(x)\,\mathrm{d}\,x = 1\ , \end{cases} \tag{13}$$

where

$$V_{\alpha,\beta}(x) = \int_{\mathcal{I}} V(x-y)(\alpha\delta_0 + \beta\delta_{1/2})(y)dy = \alpha V(x) + \beta V(x-1/2) \quad x \in \mathbb{R}\,.$$

This equation can be solved explicitly, namely

$$f(x) = \frac{\mathrm{e}^{-\frac{2}{\sigma^2}[\alpha\phi(x)+\beta\phi(x-1/2)]}}{\int_{\mathcal{I}} \mathrm{e}^{-\frac{2}{\sigma^2}[\alpha\phi(y)+\beta\phi(y-1/2)]}\,\mathrm{d}\,y} \quad x \in \mathbb{R}\ , \tag{14}$$

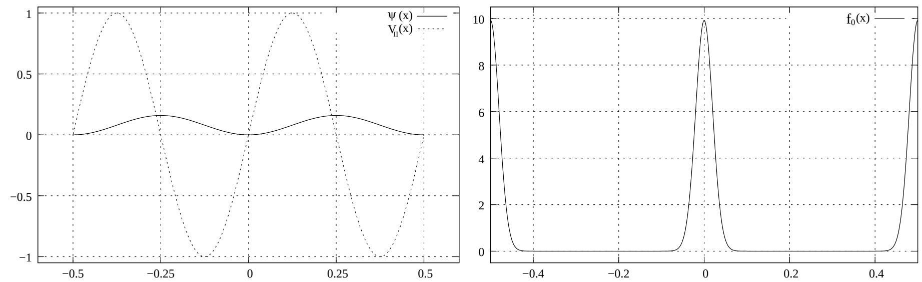

Figure 3: The functions V_{II}, $\psi(x) = \int_0^x V_{II}(\xi)\,\mathrm{d}\,\xi$, and f_0 for (16)

where $\phi(x) = \int_0^x V(\xi)d\xi$. In view of (V2) we already know that $\phi(x) \equiv \phi(-x)$. In addition we assume that $\phi(1/2) \neq 0$, which is true for generic potentials. If we define

$$\psi(x) = \alpha\phi(x) + \beta\phi(x - 1/2)\,,$$

then the condition for having two peaks concentrated at $x = 0$ and $x = 1/2$ is that $\psi(x)$ reaches its minimum value at these points. In particular we must have $\psi(0) = \psi(1/2)$, and this can only happen for $\alpha = \beta = 1/2$.

Therefore, it is natural to restrict further considerations to the case $\alpha = \beta = 1/2$ and $V_{\alpha,\beta} = V_{II}$, where V_{II} is defined by

$$V_{II}(x) = \frac{V(x) + V(x - 1/2)}{2} \qquad x \in \mathbb{R}\,. \tag{15}$$

Because of (V1) and (V2), V_{II} satisfies

(W1) $V_{II}(x + 1/2) \equiv V_{II}(x)$, and

(W2) $V_{II}(-x) \equiv -V_{II}(x)$.

Thus V_{II} must be antisymmetric with respect to 1/4, i.e. $V_{II}(1/2 - x) \equiv -V_{II}(x)$, and $V_{II}(0) = V_{II}(1/4) = V_{II}(1/2) = 0$. If in addition we choose V such that

(W3) $V'_{II}(0) > 0$, and

(W4) $V_{II}(x) \geq 0$ in $(0, 1/4)$, $V_{II}(x) \leq 0$ in $(1/4, 1/2)$,

then the function $\psi(x) \equiv \int_0^x V_{II}(\xi)\,\mathrm{d}\,\xi$ is 1/2-periodic, symmetric with respect to 1/4, and non-negative, increasing in $[0, 1/4]$, decreasing in $[1/4, 1/2]$, and bounded away from zero outside a neighborhood of 0.
Therefore, the solution of problem (13) for $\alpha = \beta = 1/2$ is

$$f_0(x) = \frac{\mathrm{e}^{-\frac{2}{\sigma^2}\int_0^x V_{II}(\xi)\,\mathrm{d}\,\xi}}{\int_{\mathcal{I}} \mathrm{e}^{-\frac{2}{\sigma^2}\int_0^y V_{II}(\xi)\,\mathrm{d}\,\xi}\,\mathrm{d}\,y} \quad x \in \mathbb{R}\,, \tag{16}$$

and has a double-peaked shape with concentrations in 0 and $\frac{1}{2}$. The situation now is similar to the one represented in Figure 3.

These considerations suggest that for *small* values of σ problem (6) can have a two-peaks-like solution concentrated in 0 and $\frac{1}{2}$, provided that the function

V_{II} satisfies (W1)-(W4). We will see, that these assumptions are actually sufficient to construct a 1/2-periodic, two-peaks-like smooth solution of problem (6). However, assumption (W4) is stronger than what we finally need. We will show that it is sufficient to require

(W4') For every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $\int_0^x V_{II}(\xi)\,\mathrm{d}\,\xi > \varepsilon$ for every $x \in [\delta, 1/4]$.

This ensures that f_0 has, at least for *small* values of σ, the double-peaked shape shown in Figure 3.

4 Main Result

In this section we state our theorem about the existence of steady states for equation (4).

Theorem 4.1. *Let $V \in C^\infty(\mathbb{R})$ satisfy (V1),(V2), and*

(V3) $V'(0) + V'(1/2) > 0$,

(V4) there exists $z \in (0, 1/2)$ such that $V(z) = V(0) = V(1/2) = 0$, $V(x) > 0$ in $(0, z)$, $V(x) < 0$ in $(z, 1/2)$, and

(V5) $V(x) \geq V(1/2 - x)$ for every $x \in [\min(z, 1/2 - z), 1/4]$.

Then there exists $\bar{\sigma} = \bar{\sigma}(V) > 0$ such that for every fixed $\sigma \in (0, \bar{\sigma})$ problem (5) has a solution $f \in C^\infty(\mathbb{R})$.

Remark 4.2. Hypothesis (V4) corresponds to the following situation: if the angle between the filaments or cells is close, then they tend to align into the same orientation; if the angle between the filaments is large, then they tend to align themselves in opposite directions. Therefore, in this context, (V4) is a natural assumption.

Hypothesis (V3) is needed to ensure that V_{II} satisfies (W3), otherwise ψ can not have a local minimum in 0 and $\frac{1}{2}$.

Hypothesis (V5) is satisfied in particular when $z = 1/4$ or when the potential V is decreasing between $\min(z, 1/2 - z)$ and $\max(z, 1/2 - z)$. In case (V4) is valid, (V5) is equivalent to (W4).

Since it is harder to prove existence of a solution for problem (5) directly, we will prove Theorem 4.1 by working on an easier problem (see (19)), which is not equivalent to (5), but whose solutions yield a solution of problem (5).

We will deduce this new problem from the first order equivalent formulation of (5), i.e. problem (6).

5 Reformulation of problem (6)

Lemma 5.1. *f is a solution of problem (6) with $f(x+1/2) \equiv f(x)$ if and only if f solves*

$$\begin{cases} \frac{\sigma^2}{4} f_x + f(x) \int_{-\frac{1}{4}}^{\frac{1}{4}} V_{II}(x-y) f(y) \mathrm{d}y = 0 \quad \forall x \in \mathbb{R} \\ \\ \int_{-\frac{1}{4}}^{\frac{1}{4}} f(x)\ \mathrm{d}\,x = \frac{1}{2}\,. \end{cases} \tag{17}$$

Proof: If f is a $\frac{1}{2}$-periodic function, with (V1) we find that

$$\int_{\mathcal{I}} V(x-y)f(y)\mathrm{d}y = \int_{\mathcal{I}} V(x-y)\frac{f(y)+f(y+1/2)}{2}\,\mathrm{d}\,y$$

$$= \int_{\mathcal{I}} \frac{V(x-y)}{2}\,f(y)\,\mathrm{d}\,y + \int_{\mathcal{I}} \frac{V(x-\theta+1/2)}{2}\,f(\theta)\,\mathrm{d}\,\theta$$

$$= \int_{\mathcal{I}} V_{II}(x-y)f(y)\mathrm{d}y = 2\int_{-\frac{1}{4}}^{\frac{1}{4}} V_{II}(x-y)f(y)\mathrm{d}y\,. \tag{18}$$

Therefore, a 1/2-periodic solution of (6) solves (17). Viceversa, if f solves (17), then

$$f(x) = K\exp\left(-\frac{4}{\sigma^2}\int_0^x \mathrm{d}\,\xi \int_{-\frac{1}{4}}^{\frac{1}{4}} V_{II}(\xi-y)f(y)\mathrm{d}y\right),$$

where the constant K must be chosen such that $\int_{-\frac{1}{4}}^{\frac{1}{4}} f(x)\,\mathrm{d}\,x = \frac{1}{2}$. From this formula we immediately deduce that f is 1/2-periodic. Using (18) we find that f solves problem (6). □

Lemma 5.2. *f is a solution of problem (6) with $f(x+1/2)\equiv f(x)\equiv f(-x)$ if and only if $v(x) := f(x)/f_0(x)$ solves*

$$\begin{cases} \frac{\sigma^2}{4}v_x(x) = v(x)\int_{-\frac{1}{4}}^{\frac{1}{4}} f_0(y)\left(V_{II}(x)-V_{II}(x-y)\right)v(y)\ \mathrm{d}\,y \\ v(-x)=v(x) \\ \int_{-\frac{1}{4}}^{\frac{1}{4}} f_0(x)v(x)\ \mathrm{d}\,x = 1/2\,. \end{cases} \tag{19}$$

Proof: Since f_0 is an even function, condition $v(-x)\equiv v(x)$ is equivalent to $f(-x)\equiv f(x)$. Moreover, by Lemma 5.1, f is a 1/2-periodic solution of (6) if and only if f solves problem (17).

If $g = f - f_0$, then a straightforward computation shows that the first equation in (17) is equivalent to

$$\frac{\sigma^2}{4}g_x(x) + \frac{1}{2}g(x)V_{II}(x) - \frac{1}{2}(f_0(x)+g(x))V_{II}(x)$$

$$+(f_0+g)(x)\int_{-\frac{1}{4}}^{+\frac{1}{4}} V_{II}(x-y)(f_0+g)(y)\mathrm{d}y \equiv 0.$$

Dividing this identity by f_0 and taking into account that

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} f_0(x)+g(x)\,\mathrm{d}\,x = \frac{1}{2} \quad \text{and} \quad V_{II}(x) = \frac{-\frac{\sigma^2}{2}(f_0)_x(x)}{f_0(x)},$$

we derive the following equivalent equation for g

$$\frac{\sigma^2}{4}\frac{d}{dx}\left(\frac{g}{f_0}\right)\bigg|_x = \left(1+\frac{g}{f_0}\right)(x)\int_{-\frac{1}{4}}^{\frac{1}{4}}(V_{II}(x)-V_{II}(x-y))f_0(y)\left(1+\frac{g}{f_0}\right)(y)\,\mathrm{d}\,y\,. \tag{20}$$

Since $1+g/f_0 = f/f_0 = v$, our lemma is proved. □

Remark 5.3. The two lemmas just shown can be easily extended to the case of $\frac{1}{N}$-periodic solutions of problem (6) with N being an arbitrary positive integer. If V_N is an $\frac{1}{N}$-periodic and odd function defined by

$$V_N(x) = \frac{1}{N}\sum_{i=0}^{N-1} V\left(x - \frac{i}{N}\right), \tag{21}$$

then, adapting the proofs of Lemma 5.1 and Lemma 5.2, we obtain the following results

Lemma 5.4. *f is a solution of problem (6) with $f(x+1/N) \equiv f(x)$ if and only if f solves*

$$\begin{cases} \frac{\sigma^2}{2N} f_x + f(x)\int_{-\frac{1}{2N}}^{\frac{1}{2N}} V_N(x-y)f(y)\mathrm{d}y = 0 \quad \forall x \in \mathbb{R} \\ \\ \int_{-\frac{1}{2N}}^{\frac{1}{2N}} f(x)\,\mathrm{d}\,x = \frac{1}{N}\,. \end{cases} \tag{22}$$

Lemma 5.5. *Define*

$$f_{0,N}(x) = \frac{\mathrm{e}^{-\frac{2}{\sigma^2}\int_0^x V_N(\xi)\,\mathrm{d}\,\xi}}{\int_{\mathcal{I}} \mathrm{e}^{-\frac{2}{\sigma^2}\int_0^y V_N(\xi)\,\mathrm{d}\,\xi}\,\mathrm{d}\,y} \quad x \in \mathbb{R}\,. \tag{23}$$

f is a solution of problem (6) with $f(x+1/N) \equiv f(x) \equiv f(-x)$ if and only if $v(x) := f(x)/f_{0,N}(x)$ solves

$$\begin{cases} \frac{\sigma^2}{2N} v_x(x) = v(x)\int_{-\frac{1}{2N}}^{\frac{1}{2N}} f_{0,N}(y)\left(V_N(x) - V_N(x-y)\right)v(y)\,\mathrm{d}\,y \\ \\ v(-x) = v(x) \\ \\ \int_{-\frac{1}{2N}}^{\frac{1}{2N}} f_{0,N}(x)v(x)\,\mathrm{d}\,x = \frac{1}{N}\,. \end{cases} \tag{24}$$

6 Construction of a solution for problem (19)

Lemma (5.2) allows to obtain a solution to problem (6), i.e. a steady state of equation (4), by solving problem (19). It is easy to see that (19) can be written in the equivalent form

$$\begin{cases} v'(x) = v(x)\left(-\frac{2}{\sigma^2}\left(\int_{-\frac{1}{4}}^{\frac{1}{4}} f_0(x)v(x)x^2\,\mathrm{d}\,x\right)V_{II}''(x) + \frac{4}{\sigma^2}\int_{-\frac{1}{4}}^{\frac{1}{4}} f_0(y)P(x,y)v(y)\,\mathrm{d}\,y\right) \\ \\ v(-x) \equiv v(x) \\ \\ \int_{-\frac{1}{4}}^{\frac{1}{4}} f_0(x)v(x)\,\mathrm{d}\,x = \frac{1}{2}\,, \end{cases} \tag{25}$$

with $P(x,y) = V_{II}(x) - V_{II}(x-y) + \frac{1}{2}V_{II}''(x)y^2$. Here and in the following $'$ will denote the derivative with respect to x.

We solve (25) iteratively. First we solve

$$\begin{cases} v_0'(x) = -\left(\frac{2}{\sigma^2}\int_{-\frac14}^{\frac14} f_0(x)v_0(x)x^2\,\mathrm{d}\,x\right)V_{II}''(x)v_0(x) \\ \\ \int_{-\frac14}^{\frac14} f_0(x)v_0(x)\,\mathrm{d}\,x = \frac12\,. \end{cases} \tag{26}$$

Because of the properties of V_{II}, v_0 is a 1/2-periodic and even function.
Now we construct a sequence $\{v_n\}_{n\in\mathbb{N}_0}$ of bounded continuous functions by choosing a suitable solution v_{n+1} for

$$\begin{cases} v_{n+1}'(x) = -\left(\frac{2}{\sigma^2}\int_{-\frac14}^{\frac14} f_0(x)v_{n+1}(x)x^2\,\mathrm{d}\,x\right)V_{II}''(x)v_{n+1}(x) + v_n(x)\mathscr{P}(x,v_n) \\ \\ \int_{-\frac14}^{\frac14} f_0(x)v_{n+1}(x)\,\mathrm{d}\,x = \frac12\,, \end{cases} \tag{27}$$

with

$$\mathscr{P}(x,w) = \frac{4}{\sigma^2}\int_{-\frac14}^{\frac14} f_0(y)P(x,y)w(y)\,\mathrm{d}\,y$$

$$= \frac{4}{\sigma^2}\int_{-\frac14}^{\frac14} f_0(y)\left(V_{II}(x) - V_{II}(x-y) + \frac12 V_{II}''(x)y^2\right)w(y)\,\mathrm{d}\,y\,. \tag{28}$$

To prove Theorem (4.1) we will first solve (27) by using variations of constants. With this we show that if v_n is a 1/2-periodic and even function, then v_{n+1} has the same properties. Let $C_p(\mathbb{R})$ be the space of continuous functions with period 1/2. We will show that $v_n \to v$ in $C_p(\mathbb{R})$ for $n \to \infty$. Passing to the limit in the integral representation for v_{n+1}, it follows that v is a mild solution of problem (25) or, equivalently, a classic solution of problem (19).

For the proof of Theorem 4.1 it is therefore sufficient to show the convergence result for the sequence $\{v_n\}$. In order to prove this we first need the following fundamental estimate

Proposition 6.1. *Let $\mathcal{L}$ be a Lipschitz continuous function over $[0,1/4]$,*

$$\mathcal{B} = \sup_{x\in[0,1/4]}|\mathcal{L}(y)|, \quad \mathcal{K} = \sup_{x,y\in[0,1/4],\,x\neq y}\frac{|\mathcal{L}(y)-\mathcal{L}(x)|}{|y-x|}\,,$$

and

$$W(x) = \int_0^x V_{II}(\xi)\,\mathrm{d}\,\xi\,,$$

then there exists $\bar\sigma = \bar\sigma(V) > 0$ such that for every $\sigma \in (0,\bar\sigma(V)]$ and $k \in \mathbb{N}_0$

$$\int_0^{\frac14} \mathcal{L}(y)y^{2k}\,\mathrm{e}^{-\frac{2}{\sigma^2}W(y)}\,\mathrm{d}\,y = \sigma^{2k+1}(\mathcal{L}(0)\mathcal{C}_k(V) + \Omega_k(\sigma)),$$

with

$$\mathcal{C}_k(V) = \sqrt{\frac{\pi}{2}}\frac{(2k-1)!!}{(\sqrt{V'(0)+V'(1/2)})^{2k+1}}$$

and $|\Omega_k(\sigma)| \le \mathcal{M}_k\sigma$, where $\mathcal{M}_k$ is a constant depending only on $\mathcal{B},\mathcal{K},V$ and k.

For the proof of Proposition 6.1 we need

Lemma 6.2. *$W(x) = \int_0^x V_{II}(\xi)\,\mathrm{d}\,\xi$ is 1/2-periodic, even and*

(i) (W4') holds,

(ii) $W(x) = \frac{V'(0)+V'(1/2)}{4}x^2 + R(x) \quad \forall x \in [0,1/4]$,

where R is a bounded function with $|R(x)| \le \frac{\|V'''\|_\infty}{24}x^4$.

Proof: Since (V1)-(V2) imply that V_{II} satisfies (W1)-(W2), we know that W is a 1/2-periodic and even function. (V4) and (V5) imply (W4), and, since $W' = V_{II}$, from (W4) it follows that W is non decreasing in $[0,1/4]$. Moreover, for every $x \in [0,1/4]$

$$W(x) = \frac{1}{2}\int_0^x V(\xi)\,\mathrm{d}\,\xi - \frac{1}{2}\int_{1/2-x}^{1/2} V(\xi)\,\mathrm{d}\,\xi\,.$$

Since for $x \in [0, \min(z, 1/2-z)]$ we have $1/2 - x \ge z \ge x$, (V4) implies that $W(x) \ge \frac{1}{2}\int_0^x V(\xi)\,\mathrm{d}\,\xi > 0$ for every $x \in (0, \min(z, 1/2-z)]$. From this, using the monotonicity of W in $[0,1/4]$, we deduce that $W(x) \ge W(\min(z, 1/2-z)) > 0$ for every $x \in [\min(z, 1/2-z), 1/4]$. Thus W satisfies (W4').

Now using Taylor's expansion and taking (V1) and (V2) into account we obtain for all $x \in [0,1/4]$

$$W(x) = \frac{1}{2}\int_0^x V(\xi) - V(1/2-\xi)\ \mathrm{d}\,\xi\ = \frac{1}{2}\int_0^x \left(V(0) + V'(0)\xi + \frac{V''(0)}{2}\xi^2\right.$$

$$\left. + \frac{V'''(\theta_\xi \xi)}{6}\xi^3 - \left(V(1/2) - V'(1/2)\xi + \frac{V''(1/2)\xi^2}{2} - \frac{V'''(1/2-\psi_\xi\xi)\xi^3}{6}\right)\right)\mathrm{d}\,\xi$$

$$= \int_0^x \frac{V'(0)+V'(1/2)}{2}\xi\ \mathrm{d}\,\xi + R(x) = \frac{x^2}{4}\left(V'(0)+V'(1/2)\right) + R(x)\,,$$

with

$$R(x) = \frac{1}{12}\int_0^x \left(V'''(\theta_\xi\xi) + V'''(1/2-\psi_\xi\xi)\right)\xi^3\ \ \mathrm{d}\,\xi \quad (\theta_\xi, \psi_\xi \in (0,1)).$$

Also (iii) is then proved. □

After this we can now give the

Proof of Proposition 6.1: Let $k \in \mathbb{N}_0$ be arbitrarily fixed. For every $\delta \in (0,1/4)$, we can write

$$\int_0^{1/4} \mathcal{L}(y)y^{2k}\exp\left(-\frac{2}{\sigma^2}W(y)\right)\mathrm{d}\,y\ =\ I_{\delta,1} + I_{\delta,2} + I_{\delta,3} + I_{\delta,4}\,, \qquad (29)$$

with

$$\begin{aligned}
I_{\delta,1} &= \mathcal{L}(0)\int_0^\delta y^{2k}\exp\left(-\tfrac{V'(0)+V'(1/2)}{2\sigma^2}y^2\right)\mathrm{d}\,y\,,\\
I_{\delta,2} &= \mathcal{L}(0)\int_0^\delta y^{2k}\left(\exp\left(-\tfrac{2}{\sigma^2}W(y)\right) - \exp\left(-\tfrac{V'(0)+V'(1/2)}{2\sigma^2}y^2\right)\right)\mathrm{d}\,y\,,\\
I_{\delta,3} &= \int_0^\delta y^{2k}(\mathcal{L}(y) - \mathcal{L}(0))\exp\left(-\tfrac{2}{\sigma^2}W(y)\right)\mathrm{d}\,y\,,\ \text{and}\\
I_{\delta,4} &= \int_\delta^{\frac{1}{4}} \mathcal{L}(y)y^{2k}\exp\left(-\tfrac{2}{\sigma^2}W(y)\right)\mathrm{d}\,y\,.
\end{aligned}$$

We will prove the statement of Proposition 6.1 by providing estimates of $I_{\delta,1}, I_{\delta,2}, I_{\delta,3}$ and $I_{\delta,4}$ for a suitable $\delta = \delta(V)$. First we see

$$|I_{\delta,4}| \leq \mathcal{B} \int_{\delta}^{\frac{1}{4}} y^{2k} \exp\left(-\frac{2}{\sigma^2}\varepsilon(\delta)\right) \mathrm{d}\, y \ \leq \frac{\mathcal{B}}{(2k+1)4^{2k+1}} \exp\left(-\frac{2}{\sigma^2}\varepsilon(\delta)\right) , \quad (30)$$

in view of (W4').
Because of Lemma 6.2,(ii) and (V3), there exist $c = c(V), \bar{\delta} = \bar{\delta}(V) > 0$ such that $W(y) \geq cy^2/2 \ \forall y \in [0, \bar{\delta}]$. Therefore, taking $\delta \leq \bar{\delta}$, we have

$$|I_{\delta,3}| \leq \mathcal{K} \int_0^{\delta} y^{2k+1} \exp\left(-\frac{2}{\sigma^2} W(y)\right) \mathrm{d}\, y \ \leq \mathcal{K} \int_0^{\delta} y^{2k+1} \exp\left(-c\frac{y^2}{\sigma^2}\right) \mathrm{d}\, y$$

$$\leq \mathcal{K}\sigma^{2k+2} \int_0^{\infty} z^{2k+1} \exp(-cz^2)\, \mathrm{d}\, z\,. \quad (31)$$

For $\delta > \sqrt{\sigma}$ we rewrite $I_{\delta,2} = I'_{\delta,2} + I''_{\delta,2}$, with

$$I'_{\delta,2} = \mathcal{L}(0) \int_0^{\sqrt{\sigma}} y^{2k} \left(\exp\left(-\tfrac{2}{\sigma^2} W(y)\right) - \exp\left(-\tfrac{V'(0)+V'(1/2)}{2\sigma^2} y^2\right)\right) \mathrm{d}\, y$$

and

$$I''_{\delta,2} = \mathcal{L}(0) \int_{\sqrt{\sigma}}^{\delta} y^{2k} \left(\exp\left(-\tfrac{2}{\sigma^2} W(y)\right) - \exp\left(-\tfrac{V'(0)+V'(1/2)}{2\sigma^2} y^2\right)\right) \mathrm{d}\, y\,.$$

We estimate both terms separately. If $\delta \leq \sqrt{\sigma}$ the estimate for $|I'_{\delta,2}|$ will be sufficient to control $|I_{\delta,2}|$. Due to Lemma 6.2,(ii) we obtain

$$|I'_{\delta,2}| \leq |\mathcal{L}(0)| \int_0^{\sqrt{\sigma}} y^{2k} \exp\left(-\frac{V'(0)+V'(1/2)}{2\sigma^2} y^2\right) \left|\exp\left(-\frac{2}{\sigma^2} R(y)\right) - 1\right| \mathrm{d}\, y$$

$$\leq \mathcal{B} \int_0^{\sqrt{\sigma}} y^{2k} \exp\left(-\frac{V'(0)+V'(1/2)}{2} \left(\frac{y}{\sigma}\right)^2\right) \left(\exp\left(\frac{\|V'''\| y^4}{12\sigma^2}\right) - 1\right) \mathrm{d}\, y\,,$$

where $\|\cdot\|$ is the L^∞-norm. Using the elementary inequality

$$\mathrm{e}^{ax} - 1 \leq a\, \mathrm{e}^{a}\, x \quad \forall x \in [0,1],$$

we deduce

$$|I'_{\delta,2}| \leq \mathcal{B} \frac{\|V'''\|}{12}\, \mathrm{e}^{\frac{\|V'''\|}{12}} \int_0^{\sqrt{\sigma}} y^{2k} \exp\left(-\frac{V'(0)+V'(1/2)}{2} \left(\frac{y}{\sigma}\right)^2\right) \frac{y^4}{\sigma^2}\ \mathrm{d}\, y\,.$$

Then, by change of variable $z = y/\sigma$ we conclude

$$|I'_{\delta,2}| \leq \mathcal{B} \frac{\|V'''\|}{12}\, \mathrm{e}^{\frac{\|V'''\|}{12}}\ \sigma^{2k+3} \int_0^{\infty} z^{2k+4} \exp\left(-\frac{V'(0)+V'(1/2)}{2} z^2\right) \mathrm{d}\, z\,. \quad (32)$$

If $\delta \leq \bar{\delta}$, we have

$$|I''_{\delta,2}| \leq |\mathcal{L}(0)| \int_{\sqrt{\sigma}}^{\delta} y^{2k} \max\left(\exp\left(-\frac{2}{\sigma^2} W(y)\right), \exp\left(-\frac{V'(0)+V'(1/2)}{2\sigma^2} y^2\right)\right) \mathrm{d}\, y$$

$$\leq \mathcal{B} \int_{\sqrt{\sigma}}^{\delta} y^{2k}\, \mathrm{e}^{-\alpha\frac{y^2}{\sigma^2}}\ \mathrm{d}\, y$$

with $\alpha = \min\left(c, \frac{V'(0)+V'(1/2)}{2}\right)$. Therefore, if $\delta \le \bar\delta$ we obtain

$$|I''_{\delta,2}| \le \mathcal{B}\sigma^{2k+1} \int_{\frac{1}{\sqrt{\sigma}}}^{\infty} z^{2k}\, \mathrm{e}^{-\alpha z^2}\, \mathrm{d}\, z \le \mathcal{B} A_k \sigma^{2k+1} \int_{\frac{1}{\sqrt{\sigma}}}^{\infty} \mathrm{e}^{-\frac{\alpha}{2} z^2}\, \mathrm{d}\, z$$

$$\le \mathcal{B} A_k \sigma^{2k+1} \sqrt{\frac{\pi}{2\alpha}}\, \mathrm{e}^{-\frac{\alpha}{4\sigma}} \quad , \text{with } A_k = \begin{cases} 1 & \text{if } k = 0 \\ \left(\frac{2k}{\alpha\, \mathrm{e}}\right)^k & \text{if } k > 0\,. \end{cases} \tag{33}$$

Here we have used the elementary estimates

$$z^{2k}\, \mathrm{e}^{-\gamma z^2} \le \left(\frac{k}{\gamma e}\right)^k, \quad \int_z^\infty \mathrm{e}^{-\gamma t^2}\, \mathrm{d}\, t \le \frac{1}{2}\sqrt{\frac{\pi}{\gamma}}\, \mathrm{e}^{-\frac{\gamma z^2}{2}} \quad \forall z \ge 0,\ \gamma > 0\,.$$

Finally, by change of variable $z = \frac{y}{\sigma}$ we obtain

$$I_{\delta,1} = \mathcal{L}(0)\sigma^{2k+1}\left(\int_0^\infty z^{2k}\, \mathrm{e}^{-\frac{V'(0)+V'(1/2)}{2} z^2}\, \mathrm{d}\, z - \int_{\delta/\sigma}^\infty z^{2k}\, \mathrm{e}^{-\frac{V'(0)+V'(1/2)}{2} z^2}\, \mathrm{d}\, z\right).$$

On the other hand, setting $\beta = (V'(0) + V'(1/2))/4$, we have

$$\int_{\delta/\sigma}^\infty z^{2k}\, \mathrm{e}^{-\frac{V'(0)+V'(1/2)}{2} z^2}\, \mathrm{d}\, z \le B_k \int_{\delta/\sigma}^\infty \mathrm{e}^{-\beta z^2}\, \mathrm{d}\, z \le \frac{B_k}{2}\sqrt{\frac{\pi}{\beta}}\, \mathrm{e}^{-\frac{\beta\delta^2}{2\sigma^2}}$$

with $B_k = \left(\frac{k}{\beta\, \mathrm{e}}\right)^k$ if $k > 0$, and $B_0 = 1$. Then we can write

$$I_{\delta,1} = \mathcal{L}(0)\sigma^{2k+1}\sqrt{\frac{\pi}{2}}\frac{(2k-1)!!}{(\sqrt{V'(0)+V'(1/2)})^{2k+1}} + J_\delta \tag{34}$$

with

$$|J_\delta| \le \mathcal{B}\frac{B_k}{2}\sqrt{\frac{\pi}{\beta}}\, \mathrm{e}^{-\frac{\beta\delta^2}{2\sigma^2}}\, \sigma^{2k+1}\,.$$

Proposition 6.1 now follows from (29), (30), (31), (32), (33), and (34) by taking $\delta = \bar\delta(V)$. ☐

Remark 6.3. From the proof of Proposition 6.1 we deduce that $\mathcal{M}_k$ is monotone increasing with respect to $\mathcal{B}$ and $\mathcal{K}$.

Remark 6.4. Let $W_N(x) = \int_0^x V_N(\xi)\, \mathrm{d}\, \xi$, where V_N is defined by (21) and N is an arbitrary positive integer, then from (V1) and (V2) it follows that

Lemma 6.5. *W_N is a $\frac{1}{N}$-periodic and even function such that*

$$W_N(x) = \left(\frac{1}{N}\sum_{i=0}^{N-1} V'\left(\frac{i}{N}\right)\right)\frac{x^2}{2} + R(x) \quad \forall x \in \left[0, \frac{1}{2N}\right],$$

where $R(x)$ is bounded with $|R(x)| \le \frac{\|V'''\|_\infty}{24} x^4$.

If we assume that

$$\sum_{i=0}^{N-1} V'\left(\frac{i}{N}\right) > 0 \qquad \text{and} \tag{35}$$

$$\forall\delta > 0 \text{ there exists } \varepsilon = \varepsilon(\delta) > 0 \text{ such that } W_N(x) > \varepsilon \quad \forall x \in \left[\delta, \frac{1}{2N}\right], \tag{36}$$

then a simple adaptation of the proof of Proposition 6.1 allows to prove

Proposition 6.6. *Let $\mathcal{L}$ be a Lipschitz continuous function over $\left[0,\frac{1}{2N}\right]$,*

$$\mathcal{B}=\sup_{x\in\left[0,\frac{1}{2N}\right]}|\mathcal{L}(y)|,\quad \mathcal{K}=\sup_{x,y\in\left[0,\frac{1}{2N}\right],\,x\neq y}\frac{|\mathcal{L}(y)-\mathcal{L}(x)|}{|y-x|}\,.$$

Then there exists $\bar{\sigma}=\bar{\sigma}(V)>0$ such that for every $\sigma\in(0,\bar{\sigma}(V)]$ and $k\in\mathbb{N}_0$

$$\int_0^{\frac{1}{2N}}\mathcal{L}(y)y^{2k}\,\mathrm{e}^{-\frac{2}{\sigma^2}W_N(y)}\,\mathrm{d}\,y=\sigma^{2k+1}(\mathcal{L}(0)\mathcal{C}_k(V)+\Omega_k(\sigma)),$$

with

$$\mathcal{C}_k(V)=\sqrt{\frac{\pi}{2}}\frac{(2k-1)!!}{\left(\sqrt{\frac{2}{N}\sum_{i=0}^{N-1}V'\left(\frac{i}{N}\right)}\right)^{2k+1}}$$

and $|\Omega_k(\sigma)|\leq\mathcal{M}_k\sigma$, where $\mathcal{M}_k$ is a constant depending only on $\mathcal{B},\mathcal{K},V,N$ and k.

In what follows $\mathcal{K}_V$ is the constant $\frac{1}{2}(V_{II}'(0))^{-1}=\frac{1}{V'(0)+V'(1/2)}$ and $\|\cdot\|$ is the norm

$$\|F\|=\sup_{x\in\mathbb{R}}|F(x)|\qquad\forall F\in C^0(\mathbb{R})\,.$$

For continuous and periodic functions this norm is always finite. Proposition 6.1 now allows to show

Proposition 6.7. *There exists a constant $\Sigma_V>0$ such that for every continuous, $1/2$-periodic and even function w, and for every $\sigma\in(0,\Sigma_V\min(1,\|w\|^{-1}))$ problem*

$$\begin{cases}v'(x)=-KV_{II}''(x)v(x)+w(x)\mathscr{P}(x,w)\\ K=\frac{2}{\sigma^2}\int_{-\frac{1}{4}}^{\frac{1}{4}}f_0(x)v(x)x^2\,\mathrm{d}\,x\\ \int_{-\frac{1}{4}}^{\frac{1}{4}}f_0(x)v(x)\,\mathrm{d}\,x=\frac{1}{2}\,,\end{cases}\tag{37}$$

has a unique solution v with the following properties

(i) $v(x+1/2)\equiv v(x)\equiv v(-x)$,

(ii)

$$K_\sigma\equiv\frac{2}{\sigma^2}\int_{-\frac{1}{4}}^{\frac{1}{4}}x^2\,f_0(x)v(x)\,\mathrm{d}\,x\ \in\left(\frac{1}{2}\mathcal{K}_V,\frac{3}{2}\mathcal{K}_V\right).$$

Moreover, v satisfies

(iii)

$$\frac{1}{2}\inf_{x\in[0,1/4]}\mathrm{e}^{K_\sigma(V_{II}'(0)-V_{II}'(x))}\leq v(x)\leq\frac{3}{2}\sup_{x\in[0,1/4]}\mathrm{e}^{K_\sigma(V_{II}'(0)-V_{II}'(x))}$$

for every $x\in\mathbb{R}$, and

(iv) $\|v'\|\leq C_V(\|v\|+\Sigma_V^2)$, *where C_V is a constant depending only on V.*

Proof: By Taylor expansion and since f_0 and w are even functions, we derive from (28)

$$\mathscr{P}(x,w) = -\frac{1}{6\sigma^2}\int_{-\frac14}^{\frac14} f_0(y)w(y)V_{II}^{(4)}(x-\theta_y\, y)y^4 \,\mathrm{d}\, y \quad \text{for a suitable } \theta_y \in (0,1).$$

Therefore

$$|\mathscr{P}(x,w)| \le \frac{\|V^{(4)}\|\|w\|}{3\sigma^2}\,\frac{\int_0^{\frac14} y^4\,\mathrm{e}^{-\frac{2}{\sigma^2}W(y)}\,\mathrm{d}\, y}{4\int_0^{\frac14}\mathrm{e}^{-\frac{2}{\sigma^2}W(y)}\,\mathrm{d}\, y},$$

and, due to Proposition 6.1, from this it follows that for $\sigma \in (0, \bar\sigma(V)]$

$$|\mathscr{P}(x,w)| \le \frac{\|V^{(4)}\|\|w\|}{12\sigma^2}\,\frac{\sigma^5(\mathcal{C}_2(V)+\Omega(\sigma))}{\sigma(\mathcal{C}_0(V)+\Omega(\sigma))},$$

where $\Omega(\sigma)$ denotes a value with modulus less or equal to $\mathcal{M}\sigma$, and the constant $\mathcal{M}$ only depends on V. Thus there exist $G_V, \Sigma_V > 0$ such that $\forall \sigma \in (0, \Sigma_V)$

$$|\mathscr{P}(x,w)| \le G_V\|w\|\sigma^2 \qquad \forall x \in \mathbb{R}\,. \tag{38}$$

To derive this inequality we have only used that w is a continuous and even function. From now on we will assume $\sigma \in (0,\Sigma_V)$, so that (38) is true.

For arbitrary $K \in \mathbb{R}$ define

$$\Psi_w(K,x) = \mathrm{e}^{-KV_{II}'(x)}\int_0^x \mathrm{e}^{KV_{II}'(\xi)}\, w(\xi)\mathscr{P}(\xi,w)\,\mathrm{d}\,\xi\,. \tag{39}$$

Then the general solution of the differential equation in (37) can be written as

$$v(x) = Q\,\mathrm{e}^{-KV_{II}'(x)} + \Psi_w(K,x) \tag{40}$$

with Q being an arbitrary real constant. Also from (38) it follows that

$$|\Psi_w(K,x)| \le G_K(V;w)\sigma^2, \quad \left|\frac{\partial \Psi_w}{\partial K}\right| \le 2G_K(V;w)\|V'\|\sigma^2 \qquad \forall x \in \left[-\frac14, \frac14\right], \tag{41}$$

with $G_K(V;w) = \frac14\,\mathrm{e}^{2|K|\|V'\|}\,G_V\|w\|^2$. The conditions

$$\int_{-\frac14}^{\frac14} f_0(x)v(x)\,\mathrm{d}\,x = \frac12 \quad \text{and} \quad K = \frac{2}{\sigma^2}\int_{-\frac14}^{\frac14} f_0(x)v(x)x^2\,\mathrm{d}\,x$$

of problem (37) can only be satisfied if K is chosen such that

$$\frac{1}{\sigma^2}\left(\frac{1-2\int_{-\frac14}^{\frac14} f_0(x)\Psi_w(K,x)\,\mathrm{d}\,x}{\int_{-\frac14}^{\frac14} f_0(x)\,\mathrm{e}^{-KV_{II}'(x)}\,\mathrm{d}\,x}\right)\int_{-\frac14}^{\frac14} f_0(x)x^2\,\mathrm{e}^{-KV_{II}'(x)}\,\mathrm{d}\,x$$
$$+\frac{2}{\sigma^2}\int_{-\frac14}^{\frac14} f_0(x)x^2\,\Psi_w(K,x)\,\mathrm{d}\,x = K \tag{42}$$

with Q in (40) being

$$Q = \frac{1-2\int_{-\frac14}^{\frac14} f_0(x)\Psi_w(K,x)\,\mathrm{d}\,x}{2\int_{-\frac14}^{\frac14} f_0(x)\,\mathrm{e}^{-KV_{II}'(x)}\,\mathrm{d}\,x}\,. \tag{43}$$

Taking into account that the functions appearing in the integrals of (42) are all symmetric with respect to zero and

$$f_0(x) = \frac{\mathrm{e}^{-\frac{2}{\sigma^2}W(x)}}{\int_{\mathcal{I}} \mathrm{e}^{-\frac{2}{\sigma^2}W(y)}\,\mathrm{d}\,y},$$

we can rewrite (42) as

$$\begin{aligned} \mathcal{H}_\sigma(K) = \sigma^2 K \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ - \left(1 - 4\int_0^{\frac{1}{4}} f_0(x)\Psi_w(K,x)\,\mathrm{d}\,x\right)\int_0^{\frac{1}{4}} x^2\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ -4\int_0^{\frac{1}{4}} f_0(x)x^2\Psi_w(K,x)\,\mathrm{d}\,x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \;=\; 0\,. \end{aligned} \tag{44}$$

To solve problem (37) we have to solve (44) and then use (43) to compute the corresponding value of the constant Q. $\mathcal{H}_\sigma$ is a smooth function of K and

$$\begin{aligned} \frac{d\mathcal{H}_\sigma}{dK} = \sigma^2 \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ -\sigma^2 K \int_0^{\frac{1}{4}} V'_{II}(x)\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x + \int_0^{\frac{1}{4}} x^2 V'_{II}(x)\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ +4\int_0^{\frac{1}{4}} f_0(x)\frac{\partial \Psi_w}{\partial K}(K,x)\,\mathrm{d}\,x \int_0^{\frac{1}{4}} x^2\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ -4\int_0^{\frac{1}{4}} f_0(x)\Psi_w(K,x)\,\mathrm{d}\,x \int_0^{\frac{1}{4}} x^2 V'_{II}(x)\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ -4\int_0^{\frac{1}{4}} f_0(x)x^2\frac{\partial \Psi_w}{\partial K}(K,x)\,\mathrm{d}\,x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \\ +4\int_0^{\frac{1}{4}} f_0(x)x^2\Psi_w(K,x)\,\mathrm{d}\,x \int_0^{\frac{1}{4}} V'_{II}(x)\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x\,. \end{aligned}$$

Up to redefining Σ_V, Proposition 6.1 and (41) imply that for every $K \in \left[\frac{1}{2}\mathcal{K}_V, \frac{3}{2}\mathcal{K}_V\right]$ we can write

(a0) $\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x)}\,\mathrm{d}\,x = \sigma(\mathcal{C}_0(V) + \Omega_{0,V}(\sigma))$,

(a1) $\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x = \sigma\,\mathrm{e}^{-KV'_{II}(0)}(\mathcal{C}_0(V) + \Omega_{0,K,V}(\sigma))$,

(a2) $\int_0^{\frac{1}{4}} x^2\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x = \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}(\mathcal{C}_1(V) + \Omega_{1,K,V}(\sigma))$,

(a3) $\int_0^{\frac{1}{4}} V'_{II}(x)\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x = \sigma\,\mathrm{e}^{-KV'_{II}(0)}(V'_{II}(0)\mathcal{C}_0(V) + \Omega_{0,K,V}(\sigma))$,

(a4) $\int_0^{\frac{1}{4}} x^2 V'_{II}(x)\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x = \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}(V'_{II}(0)\mathcal{C}_1(V) + \Omega_{1,K,V}(\sigma))$,

(a5) $\left|\int_0^{\frac{1}{4}} f_0(x)\Psi_w(K,x)\,\mathrm{d}\,x\right| \le \frac{G_K(V;w)\sigma^2}{4}$, (a6) $\left|\int_0^{\frac{1}{4}} f_0(x)x^2\Psi_w(K,x)\,\mathrm{d}\,x\right| \le G_K(V;w)\frac{\sigma^4(\mathcal{C}_1(V)+\Omega_{1,V}(\sigma))}{4(\mathcal{C}_0(V)+\Omega_{0,V}(\sigma))}$,

(a7) $\left|\int_0^{\frac{1}{4}} f_0(x)\frac{\partial\Psi_w}{\partial K}(K,x)\,\mathrm{d}\,x\right| \le \frac{G_K(V;w)\|V'\|\sigma^2}{2}$, and

(a8) $\left|\int_0^{\frac{1}{4}} f_0(x)x^2\frac{\partial\Psi_w}{\partial K}(K,x)\,\mathrm{d}\,x\right| \le G_K(V;w)\|V'\|\frac{\sigma^4(\mathcal{C}_1(V)+\Omega_{1,V}(\sigma))}{2(\mathcal{C}_0(V)+\Omega_{0,V}(\sigma))}$,

where $|\Omega_{0,K,V}(\sigma)|, |\Omega_{1,K,V}(\sigma)|, |\Omega_{0,V}(\sigma)|$ and $|\Omega_{1,V}(\sigma)|$ can all be controlled by $\mathcal{M}_V\cdot\sigma$, where $\mathcal{M}_V$ is a constant depending only on V.

From (a1)-(a8) and the definition of $G_K(V;w)$ it follows, up to redefining G_V and $\mathcal{M}_V$, that for all $K\in\left[\frac{1}{2}\mathcal{K}_V,\frac{3}{2}\mathcal{K}_V\right]$

$$\mathcal{H}_\sigma(K) \le \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}\Bigg(K\mathcal{C}_0(V)-\mathcal{C}_1(V)+\mathcal{M}_V\sigma$$

$$+G_V\sigma^2\|w\|^2\left(\mathcal{C}_1(V)+\mathcal{M}_V\sigma+\frac{\mathcal{C}_1(V)+\mathcal{M}_V\sigma}{\mathcal{C}_0(V)-\mathcal{M}_V\sigma}(\mathcal{C}_0(V)+\mathcal{M}_V\sigma)\right)\Bigg),$$

$$\mathcal{H}_\sigma(K) \ge \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}\Bigg(K\mathcal{C}_0(V)-\mathcal{C}_1(V)-\mathcal{M}_V\sigma$$

$$-G_V\sigma^2\|w\|^2\left(\mathcal{C}_1(V)+\mathcal{M}_V\sigma+\frac{\mathcal{C}_1(V)+\mathcal{M}_V\sigma}{\mathcal{C}_0(V)-\mathcal{M}_V\sigma}(\mathcal{C}_0(V)+\mathcal{M}_V\sigma)\right)\Bigg),$$

and

$$\frac{d\mathcal{H}_\sigma}{dK} \ge \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}\Bigg(\frac{\mathcal{C}_0(V)}{2}\Big(3-K(V'(0)+V'(1/2))\Big)-\mathcal{M}_V\sigma-\sigma^2\|w\|^2$$

$$\cdot(\mathcal{G}_V+\mathcal{M}_V\sigma)\Bigg)$$

$$\ge \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}\left(\frac{3}{4}\mathcal{C}_0(V)-\mathcal{M}_V\sigma-\sigma^2\|w\|^2(\mathcal{G}_V+\mathcal{M}_V\sigma)\right)$$

where $\mathcal{G}_V$ is a constant depending on V. Since $\frac{\mathcal{C}_1(V)}{\mathcal{C}_0(V)}=\mathcal{K}_V$, it finally follows that, up to taking Σ_V smaller, $\forall\sigma\in\big(0,\Sigma_V\min(1,\|w\|^{-1})\big)$ we have $\mathcal{H}_\sigma(\frac{1}{2}\mathcal{K}_V)<0$, $\mathcal{H}_\sigma(\frac{3}{2}\mathcal{K}_V)>0$, and

$$\frac{d\mathcal{H}_\sigma}{dK} \ge \sigma^3\,\mathrm{e}^{-KV'_{II}(0)}\,\frac{\mathcal{C}_0(V)}{2}>0 \qquad \forall K\in\left[\frac{\mathcal{K}_V}{2},\frac{3\mathcal{K}_V}{2}\right]. \tag{45}$$

Therefore, for every $\sigma\in\big(0,\Sigma_V\min(1,\|w\|^{-1})\big)$ there is only one value K_σ of K in the interval $\left(\frac{1}{2}\mathcal{K}_V,\frac{3}{2}\mathcal{K}_V\right)$ for which problem (37) has a solution.

When we replace K with K_σ in (43) we obtain a value Q_σ for which

$$v(x)=Q_\sigma\,\mathrm{e}^{-K_\sigma V'_{II}(x)}+\Psi_w(K_\sigma,x) \tag{46}$$

is a solution to (37). Then $v(x)$ in (46) trivially satisfies properties (i) and (ii) and is uniquely determined by property (ii).
From (a0), (a1), and (a5) we derive that

$$\frac{1-G_V\sigma^2\|w\|^2}{\mathcal{C}_0(V)+\mathcal{M}_V\sigma}(\mathcal{C}_0(V)-\mathcal{M}_V\sigma)\le Q_\sigma\, \mathrm{e}^{-K_\sigma V_{II}'(0)} \le \frac{1+G_V\sigma^2\|w\|^2}{\mathcal{C}_0(V)-\mathcal{M}_V\sigma}(\mathcal{C}_0(V)+\mathcal{M}_V\sigma)$$

and thus, up to redefining Σ_V, for every $\sigma \in \big(0, \Sigma_V \min(1, \|w\|^{-1})\big)$ we have $Q_\sigma = q_\sigma(V)\, \mathrm{e}^{K_\sigma V_{II}'(0)}$ for some $q_\sigma(V) \in [3/4, 5/4]$. Because of (38), $v(x)$ as defined by (46) satisfies for every $x \in [0, \frac{1}{4}]$

$$\begin{aligned} v(x) &\le q_\sigma(V)\, \mathrm{e}^{K_\sigma(V_{II}'(0)-V_{II}'(x))} + G_V\sigma^2\|w\|^2 \int_0^x \mathrm{e}^{K_\sigma(V_{II}'(\xi)-V_{II}'(x))}\, \mathrm{d}\,\xi \\ &\le \mathrm{e}^{K_\sigma(V_{II}'(0)-V_{II}'(x))} \left(q_\sigma(V) + \frac{G_V}{4}\sigma^2\|w\|^2\, \mathrm{e}^{K_\sigma(\|V_{II}'\|-V_{II}'(0))} \right) \end{aligned} \tag{47}$$

and

$$\begin{aligned} v(x) &\ge q_\sigma(V)\, \mathrm{e}^{K_\sigma(V_{II}'(0)-V_{II}'(x))} - G_V\sigma^2\|w\|^2 \int_0^x \mathrm{e}^{K_\sigma(V_{II}'(\xi)-V_{II}'(x))}\, \mathrm{d}\,\xi \\ &\ge \mathrm{e}^{K_\sigma(V_{II}'(0)-V_{II}'(x))} \left(q_\sigma(V) - \frac{G_V}{4}\sigma^2\|w\|^2\, \mathrm{e}^{K_\sigma(\|V_{II}'\|-V_{II}'(0))} \right). \end{aligned} \tag{48}$$

Since $q_\sigma(V) \in [3/4, 5/4]$, property (iii) follows from (47), (48), and property (i) up to a redefinition of Σ_V. Finally, (iv) follows from the differential equation in problem (37) because of (ii) and (38). □

Taking $w \equiv 0$ in (37) we obtain (26). Then the first consequence of our last proposition for $\sigma < \Sigma_V$, is that (26) has a solution v_0 whose C^0-norm is controlled by a positive constant C_V, which only depends on V. Another consequence of Proposition 6.7, for arbitrary $n \in \mathbb{N}_0$ is, that if $\|v_n\| \le C_V$ and $\sigma \in (0, \Sigma_V \min(1, 1/C_V))$, then problem (27) has a solution v_{n+1} whose C^0-norm is less than or equal to C_V. Therefore, for $\sigma \in (0, \Sigma_V \min(1, 1/C_V))$, an easy inductive argument shows that we can construct a sequence $\{v_n\} \subset C^1(\mathbb{R})$ of 1/2-periodic and even functions whose C^0-norms are all less than or equal to C_V. To prove Theorem 4.1 it is sufficient to show that, up to taking Σ_V smaller, there exists a constant $L < 1$ such that

$$\|v_{n+1} - v_n\| \le L\|v_n - v_{n-1}\| \qquad \forall n \in \mathbb{N}. \tag{49}$$

To show this we need some preliminary results. In what follows, Σ_V will be used to denote the constant appearing in the statement of Proposition 6.7.

Proposition 6.8. *If $\sigma \in (0, \Sigma_V)$, then for every $K_1, K_2 > 0$ and for every couple w_1, w_2 of continuous and even functions the following estimate holds*

$$\begin{aligned} |\Psi_{w_1}(K_1,x) - \Psi_{w_2}(K_2,x)| \le & \frac{G_V\sigma^2}{4}\Big((\|w_1\|+\|w_2\|)\, \mathrm{e}^{2K_1\|V'\|}\, \|w_1-w_2\| \\ & +2\|V'\|\|w_2\|^2\, \mathrm{e}^{2\max(K_1,K_2)\|V'\|}\, |K_1-K_2| \Big) \quad \forall x \in [0, 1/4], \end{aligned}$$

where G_V is given in (38).

Proof: For every $x \in [0, 1/4]$ we have

$$\Psi_{w_1}(K_1, x) - \Psi_{w_2}(K_2, x) = \int_0^x \mathrm{e}^{K_1(V'_{II}(\xi) - V'_{II}(x))}\, w_1(\xi)\mathscr{P}(\xi, w_1 - w_2)\,\mathrm{d}\,\xi$$

$$+ \int_0^x \mathrm{e}^{K_1(V'_{II}(\xi) - V'_{II}(x))}(w_1(\xi) - w_2(\xi))\mathscr{P}(\xi, w_2)\,\mathrm{d}\,\xi$$

$$+ \int_0^x \left(\mathrm{e}^{K_1(V'_{II}(\xi) - V'_{II}(x))} - \mathrm{e}^{K_2(V'_{II}(\xi) - V'_{II}(x))}\right) w_2(\xi)\mathscr{P}(\xi, w_2)\,\mathrm{d}\,\xi\,.$$

Since $\sigma \in (0, \Sigma_V)$, we can apply (38) and obtain

$$|\Psi_{w_1}(K_1, x) - \Psi_{w_2}(K_2, x)| \leq \frac{G_V \sigma^2}{4}\|w_1 - w_2\|(\|w_1\| + \|w_2\|)\,\mathrm{e}^{K_1(\|V'_{II}\| - V'_{II}(x))}$$

$$+ \frac{G_V \sigma^2}{4}\|w_2\|^2 |K_1 - K_2| \int_0^x \mathrm{e}^{(K_1 + \vartheta(K_2 - K_1))(V'_{II}(\xi) - V'_{II}(x))}\,|V'_{II}(\xi) - V'_{II}(x)|\,\mathrm{d}\,\xi\,, \tag{50}$$

with $\vartheta = \vartheta(x, \xi) \in (0, 1)$. This completes the proof. □

Proposition 6.9. *Let $\tilde{w}, \bar{w}$ be continuous, $1/2$-periodic and even functions and let $\sigma \in (0, \Sigma_V \min(1, \|\tilde{w}\|^{-1}, \|\bar{w}\|^{-1}))$. Let $\tilde{v}, \bar{v}$ denote the solutions of problem (37) for $w = \tilde{w}$ and $w = \bar{w}$ which satisfy Proposition (6.7),(i),(ii). Let $K_\sigma(\tilde{v}), K_\sigma(\bar{v})$ be*

$$K_\sigma(\tilde{v}) := \frac{2}{\sigma^2}\int_{-\frac14}^{\frac14} x^2 f_0(x)\tilde{v}(x)\,\mathrm{d}\,x, \quad K_\sigma(\bar{v}) := \frac{2}{\sigma^2}\int_{-\frac14}^{\frac14} x^2 f_0(x)\bar{v}(x)\,\mathrm{d}\,x,$$

and $Q_\sigma(\tilde{v}), Q_\sigma(\bar{v})$ constants such that
$\tilde{v}(x) = Q_\sigma(\tilde{v})\,\mathrm{e}^{-K_\sigma(\tilde{v})V'_{II}(x)} + \Psi_{\tilde{w}}(K_\sigma(\tilde{v}), x)$,
$\bar{v}(x) = Q_\sigma(\bar{v})\,\mathrm{e}^{-K_\sigma(\bar{v})V'_{II}(x)} + \Psi_{\bar{w}}(K_\sigma(\bar{v}), x)$.
Then, up to replacing Σ_V by a smaller positive constant which depends on V, we obtain

(i) $|K_\sigma(\tilde{v}) - K_\sigma(\bar{v})| \leq \alpha_V \sigma^2(\|\tilde{w}\| + \|\bar{w}\|)\|\tilde{w} - \bar{w}\|$,

(ii) $|Q_\sigma(\tilde{v}) - Q_\sigma(\bar{v})| \leq \sigma^2\|\tilde{w} - \bar{w}\|(\beta_V + \gamma_V(\|\tilde{w}\| + \|\bar{w}\|))$,

where α_V, β_V, and γ_V are constants also only depending on V.

Proof: (i) If $\mathcal{H}_\sigma(K; \tilde{w})$ and $\mathcal{H}_\sigma(K; \bar{w})$ are defined by (44) for $K \in \mathbb{R}$ with $w = \tilde{w}$ and $w = \bar{w}$ respectively, then $K_\sigma(\tilde{v})$ and $K_\sigma(\bar{v})$ are uniquely determined by

$$K_\sigma(\tilde{v}), K_\sigma(\bar{v}) \in \left(\frac12\mathcal{K}_V, \frac32\mathcal{K}_V\right), \quad \mathcal{H}_\sigma(K_\sigma(\tilde{v}); \tilde{w}) = \mathcal{H}_\sigma(K_\sigma(\bar{v}); \bar{w}) = 0. \tag{51}$$

A straightforward computation shows that

$$\mathcal{H}_\sigma(K; \tilde{w}) - \mathcal{H}_\sigma(K; \bar{w})$$

$$= 4\int_0^{\frac14} x^2\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x) - KV'_{II}(x)}\,\mathrm{d}\,x \int_0^{\frac14} f_0(x)\left(\Psi_{\tilde{w}}(K, x) - \Psi_{\bar{w}}(K, x)\right)\mathrm{d}\,x$$

$$-4\int_0^{\frac14} \mathrm{e}^{-\frac{2}{\sigma^2}W(x)-KV'_{II}(x)}\,\mathrm{d}\,x\int_0^{\frac14} f_0(x)x^2\left(\Psi_{\tilde w}(K,x)-\Psi_{\bar w}(K,x)\right)\mathrm{d}\,x \qquad \forall K\in\mathbb{R}^+\,.$$

Taking $K_1 = K_2 = K$ and $w_1 = \tilde w, w_2 = \bar w$, it follows from (50) that

$$|\Psi_{\tilde w}(K,x)-\Psi_{\bar w}(K,x)|\,\mathrm{e}^{KV'_{II}(x)} \le \frac{G_V\sigma^2}{4}\|\tilde w-\bar w\|(\|\tilde w\|+\|\bar w\|)\,\mathrm{e}^{K\|V'\|}\ .$$

Thus we can deduce that for every $K > 0$

$$4\left|\int_0^{\frac14} f_0(x)\left(\Psi_{\tilde w}(K,x)-\Psi_{\bar w}(K,x)\right)\mathrm{d}\,x\right| \le \lambda(K)\frac{\int_0^{\frac14}\mathrm{e}^{-\frac{2}{\sigma^2}W(x)-KV'_{II}(x)}\,\mathrm{d}\,x}{\int_0^{\frac14}\mathrm{e}^{-\frac{2}{\sigma^2}W(x)}\,\mathrm{d}\,x}\|\tilde w-\bar w\|\,,$$

and

$$4\left|\int_0^{\frac14} f_0(x)x^2\left(\Psi_{\tilde w}(K,x)-\Psi_{\bar w}(K,x)\right)\mathrm{d}\,x\right| \le \lambda(K)\frac{\int_0^{\frac14}x^2\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x)-KV'_{II}(x)}\,\mathrm{d}\,x}{\int_0^{\frac14}\mathrm{e}^{-\frac{2}{\sigma^2}W(x)}\,\mathrm{d}\,x}\|\tilde w-\bar w\|\,,$$

for $\lambda(K) = \frac{G_V\sigma^2}{4}(\|\tilde w\|+\|\bar w\|)\,\mathrm{e}^{K\|V'\|}$. Therefore

$$|\mathcal{H}_\sigma(K;\tilde w)-\mathcal{H}_\sigma(K;\bar w)|$$
$$\le 2\lambda(K)\,\frac{\int_0^{\frac14}\mathrm{e}^{-\frac{2}{\sigma^2}W(x)-KV'_{II}(x)}\,\mathrm{d}\,x\int_0^{\frac14}x^2\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x)-KV'_{II}(x)}\,\mathrm{d}\,x}{\int_0^{\frac14}\mathrm{e}^{-\frac{2}{\sigma^2}W(x)}\,\mathrm{d}\,x}\,\|\tilde w-\bar w\|$$

for every $K > 0$. In view of (a0), (a1) and (a2), this inequality implies, up to a redefinition of Σ_V, that

$$|\mathcal{H}_\sigma(K;\tilde w)-\mathcal{H}_\sigma(K;\bar w)| \le D_V\sigma^5(\|\tilde w\|+\|\bar w\|)\|\tilde w-\bar w\| \quad \forall K\in\left[\frac12\mathcal{K}_V,\frac32\mathcal{K}_V\right]\,,$$

where D_V is a constant only depending on V. From (51) and (45) we finally deduce

$$\sigma^3\,\mathrm{e}^{-\frac32\mathcal{K}_VV'_{II}(0)}\,\frac{\mathcal{C}_0(V)}{2}|K_\sigma(\tilde v)-K_\sigma(\bar v)| \le |\mathcal{H}_\sigma(K_\sigma(\tilde v);\tilde w)-\mathcal{H}_\sigma(K_\sigma(\bar v);\tilde w)|$$
$$= |\mathcal{H}_\sigma(K_\sigma(\bar v);\bar w)-\mathcal{H}_\sigma(K_\sigma(\bar v);\tilde w)| \le D_V\sigma^5(\|\tilde w\|+\|\bar w\|)\|\tilde w-\bar w\|\,.$$

With this (i) follows immediately.
(ii) From the definition of $\Psi_{\tilde w}(K_\sigma(\tilde v),x)$ and $\Psi_{\bar w}(K_\sigma(\bar v),x)$ (see (39)) we obtain

$$\frac{\partial}{\partial x}\left(\Psi_{\bar w}(K_\sigma(\bar v),x)-\Psi_{\tilde w}(K_\sigma(\tilde v),x)\right) = -K_\sigma(\bar v)V''_{II}(x)\Psi_{\bar w}(K_\sigma(\bar v),x)$$
$$+K_\sigma(\tilde v)V''_{II}(x)\Psi_{\tilde w}(K_\sigma(\tilde v),x)+\bar w(x)\mathscr{P}(x,\bar w)-\tilde w(x)\mathscr{P}(x,\tilde w)\,.$$

Then for every $x\in[0,1/4]$

$$\left|\frac{\partial}{\partial x}\left(\Psi_{\bar w}(K_\sigma(\bar v),x)-\Psi_{\tilde w}(K_\sigma(\tilde v),x)\right)\right| \le K_\sigma(\tilde v)|V''_{II}(x)||\Psi_{\tilde w}(K_\sigma(\tilde v),x)-\Psi_{\bar w}(K_\sigma(\bar v),x)|$$
$$+|K_\sigma(\tilde v)-K_\sigma(\bar v)||V''_{II}(x)||\Psi_{\bar w}(K_\sigma(\bar v),x)|+|\bar w(x)-\tilde w(x)||\mathscr{P}(x,\bar w)|+|\tilde w(x)||\mathscr{P}(x,\bar w-\tilde w)|$$
$$\le A_V\sigma^2(\|\tilde w\|+\|\bar w\|)\|\tilde w-\bar w\|+B_V\|\bar w\|^2\sigma^2|K_\sigma(\tilde v)-K_\sigma(\bar v)| \le L_V\sigma^2(\|\tilde w\|+\|\bar w\|)\|\tilde w-\bar w\|$$

$$\leq L_V \sigma \|\tilde{w} - \bar{w}\| \tag{52}$$

because of (i), Proposition 6.7-(ii), Proposition 6.8, inequality (41), and the choice of σ. Here A_V, B_V, and L_V are constants all depending on V.

From (43) and (16) it follows that

$$\begin{aligned}
&Q_\sigma(\tilde{v}) - Q_\sigma(\bar{v}) \\
&= \Bigg(\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \left(\mathrm{e}^{-K_\sigma(\bar{v}) V_{II}'(x)} - \mathrm{e}^{-K_\sigma(\tilde{v}) V_{II}'(x)} \right) \mathrm{d}\, x \\
&\quad - \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,\Psi_{\tilde{w}}(K_\sigma(\tilde{v}), x) \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\bar{v}) V_{II}'(x)} \,\mathrm{d}\, x \\
&\quad + \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,\Psi_{\bar{w}}(K_\sigma(\bar{v}), x) \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\tilde{v}) V_{II}'(x)} \,\mathrm{d}\, x \Bigg) \\
&\cdot \left(\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\tilde{v}) V_{II}'(x)} \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\bar{v}) V_{II}'(x)} \,\mathrm{d}\, x \right)^{-1},
\end{aligned} \tag{53}$$

since all functions in the integrals are even. Using (a0), (a1) and Proposition 6.7-(ii) we find that

$$\begin{aligned}
&\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\tilde{v}) V_{II}'(x)} \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\bar{v}) V_{II}'(x)} \,\mathrm{d}\, x \\
&\geq \sigma^2 (\mathcal{C}_0(V)^2 - \mathcal{M}_V \sigma) \,\mathrm{e}^{-(K_\sigma(\tilde{v}) + K_\sigma(\bar{v})) V_{II}'(0)} \geq \sigma^2 (\mathcal{C}_0(V)^2 - \mathcal{M}_V \sigma) \,\mathrm{e}^{-3 V_{II}'(0) \mathcal{K}_V}
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
&\left| \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \left(\mathrm{e}^{-K_\sigma(\bar{v}) V_{II}'(x)} - \mathrm{e}^{-K_\sigma(\tilde{v}) V_{II}'(x)} \right) \mathrm{d}\, x \right| \\
&\qquad \leq \sigma^2 (\mathcal{C}_0(V)^2 + \mathcal{M}_V \sigma) \,\mathrm{e}^{-\frac{1}{2} V_{II}'(0) \mathcal{K}_V} \,|K_\sigma(\tilde{v}) - K_\sigma(\bar{v})| V_{II}'(0),
\end{aligned} \tag{55}$$

where $\mathcal{M}_V$ is a constant only depending on V. Also

$$\begin{aligned}
&\Bigg| \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,\Psi_{\bar{w}}(K_\sigma(\bar{v}), x) \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\tilde{v}) V_{II}'(x)} \,\mathrm{d}\, x \\
&\quad - \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,\Psi_{\tilde{w}}(K_\sigma(\tilde{v}), x) \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\bar{v}) V_{II}'(x)} \,\mathrm{d}\, x \Bigg| \\
&\leq \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x) - K_\sigma(\tilde{v}) V_{II}'(x)} \,\mathrm{d}\, x \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \,|\Psi_{\bar{w}}(K_\sigma(\bar{v}), x) - \Psi_{\tilde{w}}(K_\sigma(\tilde{v}), x)| \,\mathrm{d}\, x \\
&+ \mathfrak{C} \left| \int_0^{\frac{1}{4}} f_0(x) \Psi_{\tilde{w}}(K_\sigma(\tilde{v}), x) \,\mathrm{d}\, x \right| \left| \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2} W(x)} \left(\mathrm{e}^{-K_\sigma(\tilde{v}) V_{II}'(x)} - \mathrm{e}^{-K_\sigma(\bar{v}) V_{II}'(x)} \right) \mathrm{d}\, x \right|,
\end{aligned} \tag{56}$$

where $\mathfrak{C} = 4\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x)}\,\mathrm{d}\,x$.
Because of (52) and since $\Psi_{\tilde{w}}(K_\sigma(\tilde{v}),0) = \Psi_{\bar{w}}(K_\sigma(\bar{v}),0) = 0$, it turns out that

$$\int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{2}{\sigma^2}W(x)} \left|\Psi_{\bar{w}}(K_\sigma(\bar{v}),x) - \Psi_{\tilde{w}}(K_\sigma(\tilde{v}),x)\right| \mathrm{d}\,x$$

$$\le L_V\sigma\|\tilde{w}-\bar{w}\| \int_0^{\frac{1}{4}} x\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x)}\,\mathrm{d}\,x \le \sigma^3\|\tilde{w}-\bar{w}\|(D_V + E_V\sigma)\,, \qquad (57)$$

with $D_V, E_V > 0$ depending on V. The last inequality follows trivially since, taking Σ_V smaller,

$$\int_0^{\frac{1}{4}} x\,\mathrm{e}^{-\frac{2}{\sigma^2}W(x)} \le \sigma^2(\mathcal{C}(V) + \mathcal{M}_V\sigma)\,. \qquad (58)$$

Now (58) can be proved with the same arguments as we have used in the proof of Proposition 6.1 for (30) and (31). Putting together (53), (54), (55), (56) and using (a0), (a1), (a5), and (57), we find

$$\begin{aligned}|Q_\sigma(\tilde{v}) - Q_\sigma(\bar{v})| \le \Bigg(&\sigma^2(\mathcal{C}_0(V)^2 + \mathcal{M}_V\sigma)\,\mathrm{e}^{-\frac{1}{2}V_{II}'(0)\mathcal{K}_V}\,|K_\sigma(\tilde{v}) - K_\sigma(\bar{v})|V_{II}'(0)\\ &+\sigma\,\mathrm{e}^{-K_\sigma(\bar{v})V_{II}'(0)}(\mathcal{C}_0(V) + \mathcal{M}_V\sigma)\sigma^3\|\tilde{w}-\bar{w}\|(D_V + E_V\sigma)\\ &+\sigma^2(\mathcal{C}_0(V)^2 + \mathcal{M}_V\sigma)G_V\sigma^2\|\tilde{w}\|^2\left|\mathrm{e}^{-K_\sigma(\tilde{v})V_{II}'(0)} - \mathrm{e}^{-K_\sigma(\bar{v})V_{II}'(0)}\right|\Bigg)\\ &\cdot\left(\sigma^2(\mathcal{C}_0(V)^2 - \mathcal{M}_V\sigma)\,\mathrm{e}^{-3V_{II}'(0)\mathcal{K}_V}\right)^{-1}\end{aligned}$$

Up to redefinition of Σ_V, we can deduce that

$$|Q_\sigma(\tilde{v}) - Q_\sigma(\bar{v})| \le \mathcal{N}_V\left(|K_\sigma(\tilde{v}) - K_\sigma(\bar{v})| + \sigma^2\|\tilde{w}-\bar{w}\|\right) \qquad (59)$$

for a suitable constant $\mathcal{N}_V$. Now (ii) is an immediate consequence of (59) and (i). □

Finally we show the last result of this section, which concludes the proof of Theorem 4.1.

Proposition 6.10. *There exists $T_V \in (0,\Sigma_V)$ such that for $\sigma \in (0,T_V)$*

$$\|v_{n+1} - v_n\| \le \frac{1}{2}\|v_n - v_{n-1}\| \qquad \forall n \in \mathbb{N}\,.$$

Proof: Here we assume that $\sigma \in (0, \Sigma_V\min(1, 1/C_V))$, where Σ_V and C_V are such that we can construct the sequence $\{v_n\}$ with $\sup_{n\in\mathbb{N}_0}\|v_n\| \le C_V$. For every $n \in \mathbb{N}_0$, let

$$K_n = \frac{2}{\sigma^2}\int_{-\frac{1}{4}}^{\frac{1}{4}} x^2 f_0(x)v_n(x)\,\mathrm{d}\,x \in \left[\frac{1}{2}\mathcal{K}_V, \frac{3}{2}\mathcal{K}_V\right]$$

and let Q_n be the constant for which

$$v_n(x) \equiv Q_n\,\mathrm{e}^{-K_nV_{II}'(x)} + \Psi_{v_{n-1}}(K_n,x)\,.$$

Here by definition $v_{-1} \equiv 0$. Applying Proposition 6.9 with $\tilde{w} = v_n$ and $\bar{w} = v_{n-1}$ for $n \in \mathbb{N}$, we obtain

$$|K_{n+1} - K_n| \leq D_V \sigma^2 \|v_n - v_{n-1}\|, \tag{60}$$

$$|Q_{n+1} - Q_n| \leq D_V \sigma^2 \|v_n - v_{n-1}\| \quad \forall n \in \mathbb{N}\,, \tag{61}$$

for a suitable constant D_V depending only on V. Since $Q_n = v_n(0)\, \mathrm{e}^{K_n V'_{II}(0)}$, we have

$$|Q_n| \leq C_V\, \mathrm{e}^{\frac{3}{4}}\ . \tag{62}$$

Using (60), (62) and Proposition 6.8, we find that for every $n \in \mathbb{N}$ and $x \in [0, 1/4]$

$$|v_{n+1}(x) - v_n(x)| \leq |Q_{n+1} - Q_n|\, \mathrm{e}^{-K_{n+1} V'_{II}(x)}$$

$$+|Q_n| \left| \mathrm{e}^{-K_{n+1} V'_{II}(x)} - \mathrm{e}^{-K_n V'_{II}(x)} \right| + |\Psi_{v_n}(K_{n+1}, x) - \Psi_{v_{n-1}}(K_n, x)|$$

$$\leq \sigma^2 \|v_n - v_{n-1}\| (A_V + B_V \sigma^2)\,,$$

where A_V and B_V only depend on V.
Choosing $T_V \in (0, \Sigma_V \min(1, 1/C_V))$ suitably, we have $\sigma^2(A_V + B_V\sigma^2) \leq \frac{1}{2}$ for all $\sigma \in (0, T_V)$. Using that the functions v_n are even and 1/2-periodic our theorem is proved. □

Remark 6.11. The key result to prove Propositions 6.7, 6.8, 6.9, 6.10 and then Theorem 4.1 is Proposition 6.1, which follows from (V3) and Lemma 6.2. The properties (V4) and (V5) of the orientational angle V are only used in the proof of Lemma 6.2,(i) and then they can be replaced in the statement of Theorem 4.1 by assumption (W4'). Thus the minimal hypotheses under which Theorem 4.1 holds are (V1), (V2), (V3) and (W4'). However, in our context (V4) is a natural assumption.

7 $1/N$-periodic, N-peaks-like steady states

In the preceding section we have shown that, if the orientational angle V satisfies (V1)-(V5), or better (V1)-(V3) and (W4'), then there exists a 1/2-periodic steady state of (4). To prove this statement we constructed a solution of problem (19) using a contractive argument. Now, if we want to obtain a $1/N$-periodic steady state of (4) with $N \in \mathbb{N}$ arbitrary, it is natural to try the same approach to construct a solution of (24). The key result which makes the contractive argument actually work is Proposition 6.6. This one can be proved using properties (V1) and (V2), from which Lemma 6.5 follows, and the assumptions (35), (36). Leaving to the reader to check all the details, we state the following theorem about the existence of $1/N$-periodic steady states:

Theorem 7.1. *If the orientational angle $V \in C^\infty(\mathbb{R})$ satisfies (V1), (V2), (35), and (36), then there exists $\bar{\sigma} = \bar{\sigma}(V) > 0$ such that for every fixed $\sigma \in (0, \bar{\sigma})$ problem (5) has a $1/N$-periodic solution $f \in C^\infty(\mathbb{R})$.*

8 Numerical simulations

In order to get some insight into stability and instability of the N-peaks-like steady states, we do some numerical simulations on the evolution of the solution of (4), starting from a small 1-periodic perturbation of the steady state. The numerical simulations are done for the Cauchy problem

$$\begin{cases} \partial_t f = \frac{\sigma^2 m}{2}\,\partial_{xx} f + \partial_x \left(f(x) \int_{\mathcal{I}} V(x-y) f(y) \mathrm{d}y \right) & \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \\ f(x,0) = f_0(x) & \forall x \in \mathbb{R}\,, \end{cases} \tag{63}$$

where $f_0 \in C^\infty(\mathbb{R})$ is a positive and 1-periodic function with mass $\int_{\mathcal{I}} f_0(x)\,\mathrm{d}\,x = m$. The orientational angle V is assumed to be smooth ($C^\infty(\mathbb{R})$) and to satisfy (V1)-(V4).

Remark 8.1. If $V \in C^\infty(\mathbb{R})$ fulfills (V1) and f is a classical solution of (63), bounded on $\mathbb{R} \times [0,T]$ for every $T > 0$, then f is 1-periodic with respect to x. Defining $h = f(x+1,t)$, then f and h both solve the parabolic problem

$$\begin{cases} \partial_t u = \varepsilon\, \partial_{xx} u + \alpha\, \partial_x u + \beta\, u & \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = f_0(x) & \forall x \in \mathbb{R}\,, \end{cases} \tag{64}$$

where $\varepsilon = \sigma^2 m/2 > 0$, and $\alpha(x,t) = \int_{\mathcal{I}} V(x-y) f(y,t)\,\mathrm{d}\,y$ and $\beta(x,t) = \partial_x \alpha(x,t) = \int_{\mathcal{I}} V'(x-y) f(y,t)\,\mathrm{d}\,y$ are smooth and bounded functions on every strip $\mathbb{R} \times [0,T]$ ($T > 0$). For (64) there exists a unique solution, smooth and bounded on every strip $\mathbb{R} \times [0,T]$. Thus $h \equiv f$ on each of such stripes. Since $f_0 \geq 0$, it follows from the parabolic comparison principle that $f \geq 0$. Because $V, f_0 \in C^\infty(\mathbb{R})$, also $f \in C^\infty(\mathbb{R} \times [0,\infty))$.

From the previous arguments it follows easily that problem (63) has at most one classical solution being bounded on every strip $\mathbb{R} \times [0,T]$. If f_1 and f_2 are two such solutions, then their difference $g = f_2 - f_1$ solves

$$\begin{cases} \partial_t g = \varepsilon\, \partial_{xx} g + \mathcal{G}_{0,f_2}\, \partial_x g + \mathcal{G}_{0,g} \partial_x f_1 & \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+\,, \\ g(x+1,t) \equiv g(x,t) & \forall t > 0\,, \\ g(x,0) = 0 & \forall x \in \mathbb{R}\,, \end{cases} \tag{65}$$

where $\mathcal{G}_{i,f}$ is defined by

$$\mathcal{G}_{i,f}(x,t) = \int_{\mathcal{I}} \frac{d^i V}{dx^i}(x-y) f(y,t)\,\mathrm{d}\,y$$

for every $i \in \mathbb{N}_0$ and every function f such that $f(\cdot,t) \in L^1(\mathcal{I})$ for all $t \geq 0$. If we multiply the differential equation in (65) by g and integrate all terms over $\mathcal{I}$, then, after integration by parts, we obtain that for every $t > 0$

$$\frac{d}{dt}\left(\int_{\mathcal{I}} \frac{g^2}{2}\,\mathrm{d}\,x \right) = -\varepsilon \left(\int_{\mathcal{I}} g_x^2\,\mathrm{d}\,x \right) - \int_{\mathcal{I}} \frac{g^2}{2} \mathcal{G}_{1,f_2}\,\mathrm{d}\,x + \int_{\mathcal{I}} \partial_x f_1\, g\, \mathcal{G}_{0,g}\,\mathrm{d}\,x\,.$$

Fixing an arbitrary time $T > 0$, and defining

$$M_{1,T} = \max_{\mathcal{I} \times [0,T]} |\partial_x f_1|\,, \qquad M_{2,T} = \max_{\mathcal{I} \times [0,T]} |f_2|\,,$$

then from the previous equality for every $t \in (0,T)$ follows

$$\frac{d}{dt}\left(\int_{\mathcal{I}} \frac{g^2}{2}\,\mathrm{d}\,x\right) \le (M_{2,T}\|V'\|_\infty + 2M_{1,T}\|V\|_\infty)\left(\int_{\mathcal{I}} \frac{g^2}{2}\,\mathrm{d}\,x\right).$$

Since $g(x,0) \equiv 0$ and T is arbitrary, we finally obtain $f_1 \equiv f_2$ on $\mathbb{R}\times\mathbb{R}^+$.

If f is the solution of (63) being bounded on every strip $\mathbb{R}\times[0,T]$ with $T \in (0,\infty)$, then, using the 1-periodicity of f with respect to x and integrating the differential equation over $\mathcal{I}$, we immediately obtain $\int_{\mathcal{I}} f(x,t)dx \equiv \int_{\mathcal{I}} f_0(x)dx = m$. Thus for every $i \in \mathbb{N}_0$ the inequality $|\mathcal{G}_{i,f}| \le m \left\|\frac{d^iV}{dx^i}\right\|_\infty$ holds. Multiplying the differential equation in (63) by f and integrating all terms over $\mathcal{I}$, it follows that

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathcal{I}} \frac{f^2}{2}\,\mathrm{d}\,x\right) &= -\varepsilon\left(\int_{\mathcal{I}} f_x^2\,\mathrm{d}\,x\right) + \int_{\mathcal{I}} \partial_x\left(\frac{f^2}{2}\right)\mathcal{G}_{0,f}\,\mathrm{d}\,x + \int_{\mathcal{I}} f^2\mathcal{G}_{1,f}\,\mathrm{d}\,x\\ &\le \int_{\mathcal{I}} \frac{f^2}{2}\mathcal{G}_{1,f}\,\mathrm{d}\,x \le m\|V'\|_\infty \int_{\mathcal{I}} \frac{f^2}{2}\,\mathrm{d}\,x \quad \forall t>0\,.\end{aligned}$$

So

$$\int_{\mathcal{I}} f^2(x,t)\,\mathrm{d}\,x \le \left(\int_{\mathcal{I}} f_0^2(x)\,\mathrm{d}\,x\right)\mathrm{e}^{m\|V'\|_\infty t} \quad \forall t \ge 0\,. \tag{66}$$

Differentiating the differential equation in (63) with respect to x, multiplying all terms by f_x and integrating over $\mathcal{I}$, then, after some integrations by parts and other standard arguments, we arrive at

$$\frac{d}{dt}\left(\int_{\mathcal{I}} f_x^2\,\mathrm{d}\,x\right) \le 3\int_{\mathcal{I}} f_x^2\mathcal{G}_{1,f}\,\mathrm{d}\,x - \int_{\mathcal{I}} f^2\mathcal{G}_{3,f}\,\mathrm{d}\,x \qquad \forall t>0\,.$$

Therefore

$$\frac{d}{dt}\left(\int_{\mathcal{I}} f_x^2\,\mathrm{d}\,x\right) \le 3m\|V'\|_\infty\int_{\mathcal{I}} f_x^2\,\mathrm{d}\,x + m\|V'''\|_\infty\int_{\mathcal{I}} f^2\,\mathrm{d}\,x \qquad \forall t>0\,.$$

From here it follows immediately that for every $t \ge 0$

$$\int_{\mathcal{I}} f_x^2\,\mathrm{d}\,x \le \mathrm{e}^{3m\|V'\|_\infty t}\left(\int_{\mathcal{I}} (f_0)_x^2\,\mathrm{d}\,x + m\|V'''\|_\infty t\int_{\mathcal{I}} f_0^2\,\mathrm{d}\,x\right). \tag{67}$$

Taking higher derivatives of the differential equation in (63) one obtains similar $L^2(\mathcal{I})$-estimates for these. Therefore, the $L^2(\mathcal{I})$-norms of $f, \partial_x f, \partial_{xx} f, \ldots$ can all be controlled by the norms of f_0 and its derivatives.

It is straightforward to obtain similar estimate for $\partial_t f$ and, by sequentially taking derivatives with respect to x, also the analogous $L^2(\mathcal{I})$-estimates for $\partial_{xt} f$, $\partial_{xxt} f$, $\partial_{xxxt} f$, and so on.

Finally, taking the derivative with respect to t, one obtains an estimate for the $L^2(\mathcal{I})$-norm of $\partial_{tt} f$. Going further with these arguments, it is easy to see that for every derivative of f the $L^2(\mathcal{I})$-norm at time $t>0$ can be controlled by a value depending only on t, on the $L^2(\mathcal{I})$-norms of f_0 and on a finite number of its derivatives. These a-priori estimates allow to prove existence of a smooth solution of (63) slightly modifying the arguments used for usual parabolic initial-boundary problems (*end Remark*).

Problem (63) has a unique smooth solution f which is 1-periodic in x (see last Remark): $f(x+1,t) \equiv f(x,t)$. Moreover, $f \in C^\infty(\mathbb{R} \times [0,\infty))$, $f > 0$, and $\int_{\mathcal{I}} f(x,t)\,\mathrm{d}\,x = m$ for every $t > 0$, as follows by integration of the integro-differential equation over $\mathcal{I}$. The uniqueness of the solution of (63) implies that, if $f_0(-x) \equiv f_0(x)$, then $f(-x,t) \equiv f(x,t)$ for every $t > 0$.
Our numerical simulations are performed under this symmetry assumption. They show that the two-peaks-like steady state is unstable if $\int_0^{\frac{1}{2}} V(x)\,\mathrm{d}\,x > 0$, i.e. when we take V such that we can also construct a one-peak-like steady state, whereas it seems stable, at least with respect to symmetric perturbations, if $\int_0^{\frac{1}{2}} V(x)\,\mathrm{d}\,x < 0$. This is exactly what we expect according to the different ways the orientational angle works in these two cases. Our simulations also show that a four-peaks-like steady state, if existent, is always unstable.
From now on we suppose that the initial data f_0 of problem (63) are even, $f_0(-x) \equiv f_0(x)$. The numerical algorithm we have used to calculate an approximate solution of (63) is a finite difference scheme for convection-diffusion equations, suitably modified to compute the convolution term appearing in our equation.

The algorithm has been set up in the following way. First observe that f is a solution to (63) if and only if the function

$$F(x,t) := \int_{-\frac{1}{2}}^{x} \Big(f(\xi,t) - m\Big)\,\mathrm{d}\,\xi \equiv \int_{-\frac{1}{2}}^{x} f(\xi,t)\,\mathrm{d}\,\xi - m(x+1/2)$$

solves

$$\begin{cases} \partial_t F = \frac{\sigma^2 m}{2}\,\partial_{xx}F + (\partial_x F + m)\int_{\mathcal{I}} V'(x-y)F(y,t)\,\mathrm{d}\,y & \forall (x,t) \in \mathbb{R}\times\mathbb{R}^+\,, \\ F(x,0) = F_0(x) & \forall x \in \mathbb{R}\,, \end{cases} \tag{68}$$

where

$$F_0(x) := \int_{-\frac{1}{2}}^{x} \Big(f_0(\xi) - m\Big)\,\mathrm{d}\,\xi \equiv \int_{-\frac{1}{2}}^{x} f_0(\xi)\,\mathrm{d}\,\xi - m(x+1/2) \equiv \int_0^x f_0(\xi)\,\mathrm{d}\,\xi - mx\,. \tag{69}$$

The equivalence of (63) and (68) is proved by rewriting (63) for $\bar{f}(x,t) := f(x,t) - m$ and then integrating both, the equation and the initial condition, over the interval $[-1/2, x]$, for $x \in \mathbb{R}$. In doing that, we use the assumption $f(-x,t) \equiv f(x,t)$, i.e. $F(-x,t) \equiv -F(x,t)$, to show

$$\begin{aligned} &\frac{\sigma^2 m}{2}\,\partial_{xx}F(-1/2,t) + (\partial_x F(-1/2,t) + m)\int_{\mathcal{I}} V'(-1/2-y)F(y,t)\,\mathrm{d}\,y \\ &\quad = \frac{\sigma^2 m}{2}\,\partial_x f(-1/2,t) + f(-1/2,t)\int_{\mathcal{I}} V'(1/2+y)F(y,t)\,\mathrm{d}\,y = 0\,. \end{aligned}$$

Because of our assumptions, we have that $F_0 \in C^\infty(\mathbb{R})$ and $F_0(x+1) \equiv F_0(x) \equiv -F_0(-x)$. The unique solution of (68) is then 1-periodic with respect to x, odd and solves

$$\begin{cases} \partial_t F = \frac{\sigma^2 m}{2}\,\partial_{xx}F + (\partial_x F + m)\int_0^{\frac{1}{2}} V_1(x,y)F(y,t)\,\mathrm{d}\,y & \forall (x,t) \in \big(0,\tfrac{1}{2}\big)\times\mathbb{R}^+\,, \\ F(x,0) = F_0(x) & \forall x \in \big[0,\tfrac{1}{2}\big]\,, \\ F(0,t) = F(1/2,t) = 0 & \forall t > 0\,, \end{cases} \tag{70}$$

with $V_1(x,y) = V'(x-y) - V'(x+y)$. Viceversa, if we know the solution F of (70) in the strip $[0,1/2]\times[0,\infty)$, we obtain the solution of (68) by extending F to $\mathbb{R}\times[0,\infty)$ by oddness and 1-periodicity with respect to x. If F_0 is given by (69), then

$$f(x,t) := \partial_x F(x,t) + m \tag{71}$$

is the solution of (63).

The numerical scheme we used computes an approximate solution of (70) and then supplies an approximate solution of (63) through a discrete version of (71).

In the following we denote the diffusion coefficient $\frac{\sigma^2 m}{2}$ by ε and define

$$c(x,t) = \int_0^{\frac{1}{2}} V_1(x,y)F(y,t)\,\mathrm{d}\,y\,. \tag{72}$$

Let T be a positive time. On the rectangle $[0,1/2]\times[0,T]$ we introduce a rectangular grid with space step $\Delta x = \frac{1}{2M}$ and time step $\Delta t = T/N$, where $M,N \gg 1$ are two positive integer values. The nodes (x_i,t_n) of the grid are given by

$$x_i = i\Delta x, \quad t_n = n\Delta t \qquad \forall i = 0,\dots,M\,, n = 0,\dots,N\,.$$

If $F_{i,n}$ and $c_{i,n}$ are supposed to approximate the values $F(x_i,t_n)$ and $c(x_i,t_n)$, then, using a finite difference scheme implicit in time, we set

$$\begin{cases} \frac{F_{i,n+1}-F_{i,n}}{\Delta t} = \varepsilon\frac{F_{i+1,n+1}-2F_{i,n+1}+F_{i-1,n+1}}{\Delta x^2} + c_{i,n}\left(m + \frac{F_{i+1,n+1}-F_{i-1,n+1}}{2\Delta x}\right) \\ c_{i,n} = \mathfrak{I}(\{F_{j,n}\},i,\Delta x) \qquad i = 1,\dots,M-1,\ n = 0,\dots,N-1 \\ F_{i,0} = F_0(x_i) \\ F_{0,n} = F_{M,n} = 0 \\ c_{0,n} = c_{M,n} = 0 \end{cases} \tag{73}$$

Here $\mathfrak{I}(\{F_{j,n}\},i,\Delta x)$ denotes an approximation of $c(x_i,t_n)$ obtained by using a suitable integration formula with the values $F(x_j,t_n)$ replaced by $F_{j,n}$. For instance, if we use the so called trapezoidal rule, then

$$\mathfrak{I}(\{F_{j,n}\},i,\Delta x) = \Delta x \sum_{j=1}^{M-1} \left(V'(x_i-x_j) - V'(x_i+x_j)\right) F_{j,n}\,. \tag{74}$$

Because of (V1)-(V2), the integral on the right hand of (72) vanishes and for $x = x_0 = 0$ and $x = x_M = \frac{1}{2}$ we directly set $c_{0,n} = c_{M,n} = 0$.
The approximations $f_{i,n}$ of the values $f(x_i,t_n)$ can be computed from $\{F_{i,n}\}$ by

$$\begin{cases} f_{0,n} = \frac{F_{1,n}}{\Delta x} + m \\ \\ f_{i,n} = \frac{F_{i+1,n}-F_{i-1,n}}{2\Delta x} + m \quad \text{for } i = 1,\dots,M-1 \\ \\ f_{M,n} = \frac{-F_{M-1,n}}{\Delta x} + m\,. \end{cases} \tag{75}$$

It follows immediately that

$$\frac{\Delta x}{2}\left(f_{0,n} + 2\sum_{i=1}^{M-1} f_{i,n} + f_{M,n}\right) = \frac{m}{2}.$$

Since the left-hand side of this equality is the approximation of $\int_0^{\frac{1}{2}} f(x,t_n)\,\mathrm{d}\,x$ given by the trapezoidal rule when replacing $f(x_i,t_n)$ by $f_{i,n}$, we can say that (73), (75) conserve the mass of the initial data, just as the continuous problem (63) does.

The first relation of (73) can be rewritten in the following way:

$$\left(\frac{c_{i,n}\lambda}{2} - \varepsilon\nu\right) F_{i-1,n+1} + \Big(1 + 2\varepsilon\nu\Big) F_{i,n+1} + \left(-\frac{c_{i,n}\lambda}{2} - \varepsilon\nu\right) F_{i+1,n+1}$$
$$= F_{i,n} + mc_{i,n}\Delta t \qquad \text{for } i = 1,\ldots,M-1,\ n = 0,\ldots,N-1\,, \tag{76}$$

if we define $\lambda = \Delta t/\Delta x$ and $\nu = \Delta t/\Delta x^2$. Using matrix notation and taking into account that $F_{0,n} = F_{M,n} = 0$, we can formulate (76) as a linear system in the unknowns $F_{1,n+1},\ldots,F_{M-1,n+1}$

$$A_n F_{n+1} = b_n\,. \tag{77}$$

Here F_{n+1} and b_n denote the vectors $(F_{1,n+1},\ldots,F_{M-1,n+1})^t$ and $(F_{1,n} + mc_{1,n}\,\Delta t,\ldots,F_{M-1,n} + mc_{M-1,n}\,\Delta t)^t$, while $A_n = (a^n_{i,j})$ is a square tridiagonal matrix of order $M-1$ defined by

$$\begin{array}{rcll} a^n_{i,i} & = & 1 + 2\varepsilon\nu & i = 1,\ldots,M-1\,, \\ a^n_{i,i+1} & = & -\frac{c_{i,n}\lambda}{2} - \varepsilon\nu & i = 1,\ldots,M-2\,, \\ a^n_{i,i-1} & = & \frac{c_{i,n}\lambda}{2} - \varepsilon\nu & i = 2,\ldots,M-1\,. \end{array}$$

For convenience, we sometimes will work with the original formulation of the numerical algorithm, otherwise we will use (76) or (77).

Replacing $F_{i,n}$ and $c_{i,n}$ in the first relation of (73) by $F(x_i,t_n)$ and $c(x_i,t_n)$, and using smoothness of F and V, we find that (73) is a consistent scheme.

Proposition 8.2. *There exist $\alpha,\beta > 0$, only depending on V, T, and the solution F of (70), such that the absolute value of*

$$\ell_{i,n} := \frac{F(x_i,t_{n+1}) - F(x_i,t_n)}{\Delta t} - \varepsilon\frac{F(x_{i+1},t_{n+1}) - 2F(x_i,t_{n+1}) + F(x_{i-1},t_{n+1})}{\Delta x^2}$$
$$-c(x_i,t_n)\left(m + \frac{F(x_{i+1},t_{n+1}) - F(x_{i-1},t_{n+1})}{2\Delta x}\right) \tag{78}$$

is less or equal to $\alpha\Delta x^2 + \beta\Delta t$ *for every* $i = 1,\ldots,m-1$ *and* $n = 0,\ldots,N-1$.

Proof: A straightforward computation shows that for every $i = 1,\ldots,m-1$ and $n = 0,\ldots,N-1$

$$\ell_{i,n} = \left(c_t(x_i,\tau)(m + F_x(x_i,t_{n+1})) + \frac{F_{tt}(x_i,t_{n+1})}{2}\right)\Delta t$$
$$+\left(c(x_i,t_n)F_{xxx}(\xi,t_{n+1}) + \varepsilon\frac{F_{xxxx}(\psi,t_{n+1})}{2}\right)\frac{\Delta x^2}{6}\,,$$

where $\xi, \psi \in (x_{i-1}, x_{i+1})$, $\tau \in (t_n, t_{n+1})$. Since $\|c\| \leq \|V'\|\|F\|$ and $\|c_t\| \leq \|V'\|\|F_t\|$, the proposition follows. □

We now choose Δx such that

$$\Delta x \sup_{i=1,\ldots,M-1} |c_{i,n}| \leq 2\varepsilon \qquad \forall n = 0, \ldots, N-1 \tag{79}$$

holds. With this it is possible to prove that (73) is stable and convergent.

Remark 8.3. Since $|c_{i,n}| \approx |c(x_i, t_n)| \leq \|V\|m$, it is sufficient to choose Δx such that $\Delta x \|V\| m < 2\varepsilon$, i.e. $\Delta x \|V\| < \sigma^2$, to fulfill (79). The validity of (79) can be checked for every time step while running the numerical algorithm and, if (79) is violated, the algorithm can be restarted with a smaller value of Δx, i.e. choosing a larger value for M. In our simulations Δx is always chosen such that (79) is fulfilled.

Proposition 8.4. *If (79) is satisfied, then for every $n = 0, \ldots, N-1$ we have*

1. *A_n is an invertible matrix and*
2. $$\max_{i=1,\ldots,M-1} |F_{i,n}| \leq \mathrm{e}^{m\|V'\|T} \max_{i=1,\ldots,M-1} |F_{i,0}| \leq \frac{m}{2} \mathrm{e}^{m\|V'\|T} \,.$$

Proof: If (79) is true, then for every $n = 0, \ldots, N-1$ the matrix A_n of (77) is a diagonal dominant M-matrix and thus invertible (see [3]). Let $\mathbf{1}$ denote the vector having only ones as components, then (79) also implies that $A_n \mathbf{1} \geq \mathbf{1}$, which has to be understood componentwise. Then (see [3])

$$\|A_n^{-1}\|_\infty := \sup_{v \neq 0} \frac{\|A_n^{-1} v\|_\infty}{\|v\|_\infty} \leq 1$$

and from (74) and (77) it follows that

$$\max_{i=1,\ldots,M-1} |F_{i,n+1}| \leq \max_{i=1,\ldots,M-1} (|F_{i,n}| + m|c_{i,n}|\Delta t) \leq \max_{i=1,\ldots,M-1} |F_{i,n}|(1 + m\|V'\|\Delta t) \,.$$

Therefore, for every $n = 0, \ldots, N$

$$\max_{i=1,\ldots,M-1} |F_{i,n}| \leq (1 + m\|V'\|\Delta t)^n \max_{i=1,\ldots,M-1} |F_{i,0}|$$

$$\leq \left(1 + m\|V'\|\frac{T}{N}\right)^N \max_{i=1,\ldots,M-1} |F_{i,0}| \leq \mathrm{e}^{m\|V'\|T} \max_{i=1,\ldots,M-1} |F_{i,0}|,$$

and, since

$$\max_{i=1,\ldots,M-1} |F_{i,0}| \leq \|F_0\| \leq \frac{m}{2} \,,$$

compare (69), the proposition follows. □

For every $i = 0, \ldots, M$ and $n = 0, \ldots, N$ let $E_{i,n}$ and $e_{i,n}$ be defined by

$$F_{i,n} = F(x_i, t_n) + E_{i,n}, \quad c_{i,n} = c(x_i, t_n) + e_{i,n} \,.$$

In particular

$$\begin{aligned} &E_{i,0} = 0 && \forall i = 1, \ldots, M-1, \\ &E_{0,n} = E_{M,n} = e_{0,n} = e_{M,n} = 0 && \forall n = 0, \ldots, N \,. \end{aligned} \tag{80}$$

Theorem 8.5. *If (79) is satisfied, then there exists a constant $K > 0$ depending on F, T, and V such that*

$$\max_{i=1,\dots,M-1} |E_{i,n}| \le K(\Delta x^2 + \Delta t) \qquad \forall n = 0,\dots,N\,.$$

Proof: Due to (74), for every $i = 1,\dots,M-1$ and $n = 0,\dots,N$ we have

$$e_{i,n} = \Delta x \sum_{j=1}^{M-1} V_1(x_i,x_j)E_{j,n} + \Delta x \sum_{j=1}^{M-1} V_1(x_i,x_j)F(x_j,t_n)$$

$$- \int_0^{\frac{1}{2}} V_1(x_i,y)F(y,t_n)\,\mathrm{d}\,y = \Delta x \sum_{j=1}^{M-1} V_1(x_i,x_j)E_{j,n} + \omega_{i,n}\,, \tag{81}$$

where

$$|\omega_{i,n}| \le \frac{\Delta x^2}{24}\left\| \frac{d^2}{dy^2}\Big(V_1(x_i,y)F(y,t_n)\Big)\right\| \le C\Delta x^2, \tag{82}$$

with C being a constant only depending on the final time T and on the data (σ, m, V_1 and F_0) of (70). Writing $F_{i,n}$ as $F(x_i,t_n) + E_{i,n}$ in (76), we obtain that for every $i = 1,\dots,M-1$ and $n = 0,\dots,N-1$

$$\left(\frac{c_{i,n}\lambda}{2} - \varepsilon\nu\right)E_{i-1,n+1} + \Big(1+2\varepsilon\nu\Big)E_{i,n+1} + \left(-\frac{c_{i,n}\lambda}{2} - \varepsilon\nu\right)E_{i+1,n+1}$$

$$= E_{i,n} + me_{i,n}\Delta t + \frac{e_{i,n}\lambda}{2}(F(x_{i+1},t_{n+1}) - F(x_{i-1},t_{n+1})) - \ell_{i,n}\Delta t\,,$$

where $\ell_{i,n}$ is given by (78). Because of (80), we can write more compactly $A_nE_{n+1} = d_n$, where E_{n+1} is the vector $(E_{1,n+1},\dots,E_{M-1,n+1})^t$ and d_n is the column vector whose i-th element is

$$d_{n,i} := E_{i,n} + me_{i,n}\Delta t + \frac{e_{i,n}\lambda}{2}(F(x_{i+1},t_{n+1}) - F(x_{i-1},t_{n+1})) - \ell_{i,n}\Delta t\,.$$

Since (79) implies that A_n is a diagonal dominant M-matrix with $\|A_n^{-1}\|_\infty \le 1$ (see the proof of Proposition (8.4) for a reference), we have

$$\max_{i=1,\dots,M-1} |E_{i,n+1}| \le \max_{i=1,\dots,M-1} |d_{i,n}|\,. \tag{83}$$

Now, in view of Proposition (8.2) and because

$$e_{i,n}\left(m\Delta t + \frac{\lambda}{2}(F(x_{i+1},t_{n+1}) - F(x_{i-1},t_{n+1}))\right)$$

$$= \Delta t e_{i,n}\left(m + F_x(x_i,t_{n+1}) + \frac{\Delta x^2}{6}F_{xxx}(\xi,t_{n+1})\right)$$

for a suitable $\xi \in (x_{i-1},x_{i+1})$, we may write
$d_{i,n} = E_{i,n} + \Delta t\Big(\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t) + \mathcal{O}(1)e_{i,n}\Big)$.

Using (81) and (82) to estimate $e_{i,n}$, from (83) we finally derive that

$$\max_{i=1,\dots,M-1} |E_{i,n+1}| \le (1+\gamma\Delta t\|V'\|) \max_{i=1,\dots,M-1} |E_{i,n}| + \Delta t(\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t))$$

for every $n \in \{0, \ldots, N-1\}$. Since $E_{i,0} = 0$ for all $i \in \{1, \ldots, M-1\}$, it easily follows that for every $n \in \{0, \ldots, N\}$

$$\max_{i=1,\ldots,M-1} |E_{i,n}| \leq \frac{(1+\gamma\Delta t\|V'\|)^n - 1}{\gamma\Delta t\|V'\|}\Delta t(\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t))$$

$$\leq \frac{\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t)}{\gamma\|V'\|}\left(\left(1+\gamma\|V'\|\frac{T}{N}\right)^N - 1\right) \leq \frac{\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t)}{\gamma\|V'\|}(\mathrm{e}^{\gamma\|V'\|T} - 1).$$

□

Remark 8.6. To apply the numerical scheme (73) the linear system (77) has to be solved at every time step. Luckily the matrix of the coefficients of this linear system is tridiagonal, which allows to solve (77) by the so called Thomas algorithm (see [1]). This algorithm is extremely convenient since its execution time is of the same order as the size of the matrix.

Remark 8.7. The non-zero coefficients of the matrix A_n are

$$\begin{array}{rcll} a_{1,1}^n = a_{2,2}^n = \cdots = a_{M-1,M-1}^n = & 1+2\delta & (\delta := \varepsilon\nu > 0), \\ a_{1,2}^n = a_{2,3}^n = \cdots = a_{M-2,M-1}^n = & -\alpha_i^n - \delta & \left(\alpha_i^n := \frac{c_{i,n}\lambda}{2}\right), \\ a_{2,1}^n = a_{3,2}^n = \cdots = a_{M-1,M-2}^n = & \alpha_i^n - \delta & . \end{array}$$

With assumption (79), we have $|\alpha_i^n| \leq \delta$ for every $i \in \{1, \ldots, M-1\}$, which implies that the Thomas algorithm is well conditioned.

In the following $V_{A,\alpha}$ is defined by $(A > 0)$

$$V_{A,\alpha}(x) = \begin{cases} A\sin(2\pi x + \alpha\sin(2\pi x)) & \text{if } \alpha > 0 \\ -A\sin(2\pi x + \alpha\sin(2\pi x)) & \text{if } \alpha < 0\,. \end{cases}$$

$V_{A,\alpha}$ satisfies (V1) and (V2) and, if $1 < |\alpha| \leq \pi$, also (35) and (36) both for $N = 2$ and $N = 4$. Thus, if we take $V = V_{A,\alpha}$ with $1 < |\alpha| \leq \pi$, the existence of 2- and 4-peaks-like steady states for equation (4) is ensured by Theorem 7.1. For $\alpha \in (1, \pi]$ we also know that there exists a one-peak-like steady state, since

$$\alpha \int_0^{\frac{1}{2}} V_{A,\alpha}(x)\,\mathrm{d}\,x > 0 \qquad \forall \alpha \neq 0\,. \tag{84}$$

Figure 4 shows the graph of $V_{A,\alpha}$ for $A = 4.6$ and for $\alpha = 1.2$ (continuous line), $\alpha = -1.2$ (dashed line) respectively. In our simulations we have set $\sigma = 0.32$ and chosen $V = V_{A,1.2}$ or $V = V_{A,-1.2}$ with suitable values for A. According to the statement of Theorem 4.1, we expect to have to adjust the values of σ and A to allow for the existence of 1-, 2- and 4-peaks-like steady states. Figure 5 shows a two-peaks-like steady state in the case $A = 4.6$. This figure has been obtained by taking initial data f_0 as given in (16) and running the simulation for a sufficiently large time T, to see the density distribution stabilizing (it seems sufficient to take $T = 1.2$). Since $\frac{1}{2}(V_{A,1.2}(x) + V_{A,1.2}(x - 1/2))$ and $\frac{1}{2}(V_{A,-1.2}(x) + V_{A,-1.2}(x - 1/2))$ coincide, the two-peaks-like steady state is the same for $\alpha = 1.2$ and $\alpha = -1.2$.

In figures 6-8 the symbol $+$ is used to outline the shape of the initial data, while the density distribution at the specified final time T is represented by a continuous line for $\alpha = +1.2$, and by a dashed line for $\alpha = -1.2$.

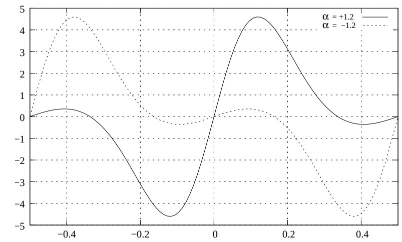

Figure 4: The function $V_{A,\alpha}$ for $A = 4.6$ and $\alpha = \pm 1.2$.

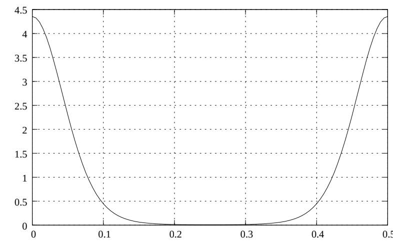

Figure 5: A two-peaks-like steady state for $V = V_{4.6,\pm 1.2}$

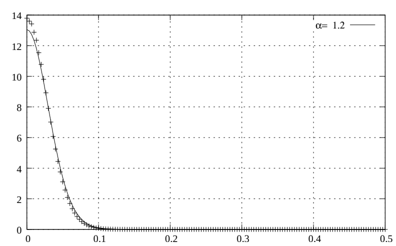

Figure 6: Evolution of a one-peak-like density distribution

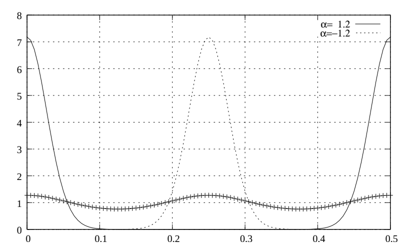

Figure 7: Evolution of a four-peaks-like density distribution

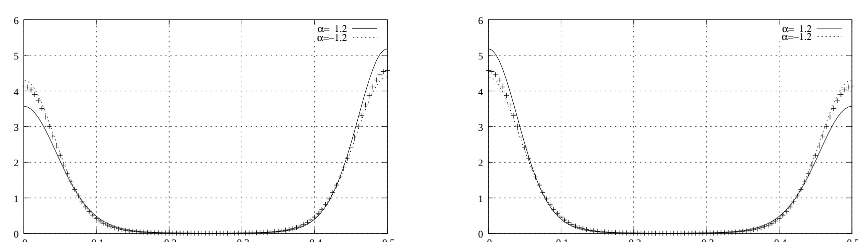

Figure 8: Long time evolution of small perturbations of a two-peaks-like steady state

Figure 6 refers to the case of a one-peak-like steady state. We take $A = 4.6$, $\alpha = 1.2$ and as initial data the even function given by (12), so that $m = 1$. Even if (12) is not exactly a steady state of (4), due to (84) and since the diffusion coefficient is small, we expect that it is a small perturbation of a one-peak-like steady state. Confirming our intuition about the stability, the density distribution at time $T = 1$ is close to the initial data.

Figure 8 shows the long time evolution ($T = 5$) of small perturbations of the two-peaks-like steady state represented in Figure 5. These perturbations, which are positive, 1-periodic and even functions with mass equal to 1, just as the steady state itself, are obtained through a redistribution of the total mass between the two peaks, such that one of them becomes bigger than the other one. The relative difference between the perturbation and the steady state is always less or equal to 5%. The simulations show that the two-peaks-like steady state is stable with respect to such perturbations if $\alpha < 0$, and is unstable for $\alpha > 0$. In view of (84), this behavior agrees with the different action of the function V in the two cases.

Figure 7 shows that the four-peaks-like steady state is unstable, independently of V. The initial data are now given by

$$f_0(x) = \frac{\mathrm{e}^{-\frac{2}{\sigma^2}\int_0^x \psi(\xi)\,\mathrm{d}\,\xi}}{\int_{\mathcal{I}} \mathrm{e}^{-\frac{2}{\sigma^2}\int_0^y \psi(\xi)\,\mathrm{d}\,\xi}\,\mathrm{d}\,y} \quad x \in \mathbb{R}\,, \tag{85}$$

where $\psi(x) = \frac{1}{4}\big(V(x) + V(x - 1/2) + V(x - 1/4) + V(x + 1/4)\big)$. The function ψ is the same for $\alpha = 1.2$ and $\alpha = -1.2$, its total mass is equal to 1. Due to our choice of V, we know that, if the ratio between the diffusion coefficient $\sigma^2/2$ and the magnitude of V is sufficiently small, then equation (4) has a four-peaks-like steady state of which (85) is a small perturbation. Figure 7 has been obtained by computing the density distribution at time $T = 2$ for $A = 10$. It is worth to point out that when taking $A = 4.6$, which is the value used in the one- and two-peaks cases, then the simulations always show a density distribution which rapidly converges to the constant steady state 1. In this case the value of σ has been chosen large enough, in comparison to the function ψ, so that the diffusion term completely overtakes the action of the orientational angle. Since the function V_N of problem (22) (see (21) for its definition) converges uniformly over $\mathcal{I}$ to $\int_{\mathcal{I}} V(x)\,\mathrm{d}\,x = 0$ as $N \to \infty$, we tend to believe that the maximum value of σ for which an N-peaks-like steady state of equation (4) can be constructed, decreases as N becomes larger.

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