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Hopf Algebra Structure of the Character Rings of Orthogonal and Symplectic Groups

by

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Abstract

We study the character rings Char-O and Char-Sp of the orthogonal and symplectic subgroups of the general linear group, within the framework of symmetric functions. We show that Char-O and Char-Sp admit natural Hopf algebra structures, and Hopf algebra isomorphisms from the general linear group character ring Char-GL (that is, the Hopf algebra of symmetric functions with respect to outer product) are determined. A major structural change is the introduction of new orthogonal and symplectic Schur-Hall scalar products. Standard bases for Char-O and Char-Sp (symmetric functions of orthogonal and symplectic type) are defined, together with additional bases which generalise different attributes of the standard bases of the Char-GL case. Significantly, the adjoint with respect to outer multiplication no longer coincides with the Foulkes derivative (symmetric function 'skew'), which now acquires a separate definition. The properties of the orthogonal and symplectic Foulkes derivatives are explored. Finally, the Hopf algebras Char-O and Char-Sp are not self-dual, and the dual Hopf algebras Char-O* and Char-Sp* are identified.

Keywords: Orthogonal group, symplectic group, irreducible characters, symmetric functions, representation rings, Hopf algebra, group characters

AMS Subject Classification: 16W30; 11E57.

Contents

1 Introduction

1.1 Motivation

It is hardly possible to overestimate the importance of group representation theory and the associated calculus of group characters. It plays a role in many areas of physics, chemistry, biology and not least in pure mathematics. For that reason, new techniques which deal with group characters in a unified and structural way are not only of interest in their own right, but also may be of great help in more applied work.

A common problem involving the application of group representation theory is that of *symmetry breaking*, whereby the symmetries of some idealised system are more realistically limited to some subset of the original symmetries. This process manifests itself group theoretically by way of *restriction from group to subgroup* and the corresponding *branching*, or *reduction* of representations. A related problem is the study of the potentially enlarged symmetries of objects formed from other objects possessing their own intrinsic symmetries. This inverse process of *combining symmetries* is sometimes referred to as *subduction*, and involves the *inverse of the branching* or reduction procedures As an example, the whole philosophy of *particle physics* is based on the principle that smaller building blocks form larger entities, as for example quarks forming hadrons, or nucleons forming clusters. Indeed, such aspects have been a traditional homeland for group theory-powered insights, like the multiplet organization of particles in $SU(3)$, Wigner's $SU(4)$ nuclear multiplet theory, or the nuclear interacting boson model.

In the present paper we study group representations via the Hopf algebraic structure of their characters, within the framework of symmetric functions. It was already observed in earlier work [11, 14] that Hopf algebra techniques allowed symmetric function methods to be organized and generalized in an elegant way (the approach was developed in part by applying methods borrowed from quantum field theory [7, 4], in a simplified group theoretical setting). In group theory terms, this earlier symmetric function work concerns the characters of the general linear group. In the present paper, we pursue these investigations by turning to the *classical subgroups* of the general linear group. We show how the character rings of the orthogonal and symplectic groups admit natural Hopf algebraic structures. We obtain these Hopf algebras as isomorphic images of the Hopf algebra of the character ring of the general linear group, which is in turn isomorphic to the Hopf algebra of symmetric functions. The isomorphy is defined by the underlying branching, which establishes an isomorphism between the module of characters of the general linear group, and those of its classical subgroups. This module map induces a map of Hopf algebras, as we are going to show.

Despite their isomorphism as Hopf algebras, the different character Hopf algebras do encode different information. This stems partly from the fact that we are interested in *canonical bases*, which *differ* for the different character modules. The prime example concerns the Schur functions, which establish irreducible characters of the general linear group GL. If we branch from GL say to the orthogonal group O, or the symplectic group Sp, the Schur functions are no longer the irreducible characters, and they lose, in part, their important and singular meaning. The orthogonality of irreducible GL characters (corresponding to irreducible representations, labelled by integer partitions λ), is expressed formally by the *Schur-Hall scalar product* with respect to which the Schur functions s_λ are orthonormal,

$$\langle s_\lambda \mid s_\mu \rangle = \delta_{\lambda,\mu}. \tag{1}$$

Branching a GL-character to orthogonal (symplectic) characters means finding a decomposition of that character in terms of irreducible orthogonal (symplectic) characters. These characters will be called Schur functions of orthogonal (symplectic) type. Since orthogonal and symplectic groups are completely reducible, we can find a basis of irreducible characters. It is hence a group-theoretical necessity to introduce, on these character Hopf algebras, *new Schur-Hall scalar products* which express the fact that Schur functions of orthogonal (symplectic) type o_λ (sp_λ)

are mutually orthonormal (Schur's lemma):

$$\langle o_\lambda \mid o_\mu \rangle_2 = \delta_{\lambda,\mu} \qquad \text{and} \qquad \langle sp_\lambda \mid sp_\mu \rangle_{11} = \delta_{\lambda,\mu} \, . \tag{2}$$

The indexing stems from the plethystic origin of these particular branchings (see below). These scalar products are the new structural elements which distinguish the otherwise isomorphic Hopf algebras.

The general case of symmetric function branchings was discussed in [14]. There we considered module isomorphisms between the module of characters of a group G and the module of characters of a subgroup H. Specifically, an algebraic subgroup H_π of GL was taken, consisting of matrix transformations fixing an arbitrary tensor of symmetry type π – the orthogonal and symplectic cases correspond to the weight two symmetric and antisymmetric cases, $\{2\}$ and $\{1^2\}$ respectively, of nonsingular bilinear forms. However, generically, the symmetric function bases obtained by branchings with respect to higher order invariants are no longer irreducible, but only indecomposable at best. For this reason we study, in a first attempt, the orthogonal and symplectic cases.

Even these classical cases reveal some novel features when treated in this formal setting. We need to introduce new classes of Schur functions, as described above, as is well-understood classically and was used at least implicitly already by Weyl. Complete and elementary symmetric functions now have different expansions in terms of orthogonal and symplectic irreducibles; also it turns out that power sums pick up an extra additive term. More significantly, we need to separate the notion of multiplicative adjoint (which we denote by $s^\dagger_\lambda$), which leads to skew Schur functions in the GL case, from that of the Foulkes derivative, which we denote by $s^\perp_\lambda$. This stems from the fact that the adjoint of multiplication depends on the Schur-Hall scalar product adopted, and that the branched Hopf algebras are no longer self dual.

This work is partly preparatory in so far as it establishes the necessary tools to deal with vertex operator algebras of orthogonal and symplectic type, which will be studied elswhere [13], see also for example [1]. Of course, a key motivation for studying the orthogonal and symplectic cases is to pave the way for the general case of the subgroups H_π studied in [14]. These subgroups include not only finite groups but also affine groups, with the latter possessing reducible but indecomposable representations, that is to say non-semisimple modules. These pose particular difficulties.

The organisation of the paper is as follows. After recalling some facts about the symmetric function Hopf algebra [11] and branchings [14], we prove the main statement that the orthogonal and symplectic character rings are actually Hopf algebras isomorphic to the universal Hopf algebra of symmetric functions. Then we discuss the *different realizations* of the *power sums*, *complete* and *elementary symmetric functions* in the orthogonal and symplectic cases, and note where differences due to the underlying groups occur. In particular, new bases arise for the O and Sp cases which generalise different aspects of the canonical symmetric function bases. As an overview of the main developments of the paper, we also provide in an early section a detailed directory of the most important notations and definitions. In the last section we discuss the adjoint and the Foulkes derivative, and provide the correct Hopf-theoretical definition of the latter, which allows applications to generic branchings. An informal discussion of the abstract setting in category-theoretical terms is given in the concluding section, which also surveys further work and open questions. Finally, an appendix is included, in which the structure of the dual Hopf algebras $\mathsf{Char}\text{-}\mathsf{O}^*$ and $\mathsf{Char}\text{-}\mathsf{Sp}^*$ is identified.

2 Symmetric functions and their Hopf algebra

2.1 The ring of symmetric functions Λ

We use standard notation as in Macdonald's book [24]. Symmetric functions are conveniently indexed by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The $\lambda_i \in \mathbb{N}$ are called parts of the partition, $|\lambda| = \sum_{i=1}^{\ell} \lambda_i$ is the weight of the partition, that is the sum of its parts, while $\ell(\lambda) = \ell$ is called its length. Each partition λ specifies a corresponding Ferrers or Young diagram, F^λ, with row lengths given by the parts λ_i of λ. The partition λ', conjugate to λ, has parts λ'_j equal to the column lengths of F^λ. Partitions may alternatively be displayed by using multiplicities, that is writing $\lambda = [1^{r_1}, 2^{r_2}, \ldots, n^{r_n}]$ where $m_i(\lambda) = r_i$ is the multiplicity of i in λ. In this notation the conjugacy class within the symmetric group $S_{|\lambda|}$ associated with λ has cardinality $z_\lambda = \prod_i i!\, i^{r_i}$. A third way to describe partitions uses Frobenius notation, that is the list of 'arm' and 'leg' lengths, $\lambda_i - i$ and $\lambda'_j - j$ respectively, of F^λ for $i, j = 1, 2, \ldots, r$, where r, the Frobenius rank of λ, is the length of the main diagonal of F^λ.

By way of an example we have

$$(5,5,3,2,2,1,1,1) = [1^3, 2^2, 3^1, 4^0, 5^2, 6^0, 7^0, \ldots] = \begin{pmatrix} 4 & 3 & 0 \\ 7 & 3 & 0 \end{pmatrix}$$

and its conjugate

$$(8,5,3,2,2) = [1^0, 2^2, 3^1, 4^0, 5^1, 6^0, 7^0, 8^0, \ldots] = \begin{pmatrix} 7 & 3 & 0 \\ 4 & 3 & 0 \end{pmatrix}.$$

Partitions are used to specify a number of objects of interest in the present work. Amongst these are the Schur functions s_λ. These form an orthonormal $\mathbb{Z}$-basis for the ring Λ of symmetric functions. To be more precise, let $\mathbb{Z}[x_1, \ldots, x_N]$ be the polynomial ring, or the ring of formal power series, in N commuting variables. The symmetric group $\mathcal{S}_N$ acting on N letters acts on this ring by permuting the variables. For $\pi \in \mathcal{S}_N$ and $f \in \mathbb{Z}[x_1, \ldots, x_N]$ we have

$$\pi f(x_1, \ldots, x_N) = f(x_{\pi(1)}, \ldots, x_{\pi(N)}) . \tag{3}$$

We are interested in the *subring of functions* invariant under this action, $\pi f = f$, that is to say the ring of symmetric polynomials in N variables:

$$\Lambda_N[x_1, \ldots, x_N] = \mathbb{Z}[x_1, \ldots, x_N]^{S_N}. \tag{4}$$

This ring may be graded by the degree of the polynomials, so that

$$\Lambda_N[x_1, \ldots, x_N] = \oplus_n \Lambda_N^{(n)}[x_1, \ldots, x_N] , \tag{5}$$

where $\Lambda_N^{(n)}[x_1, \ldots, x_N]$ consists of homogenous symmetric polynomials in $x_1, \ldots, x_N$ of total degree n.

In order to work with an arbitrary number of variables, following Macdonald [24], we define the ring of symmetric functions $\Lambda = \lim_{N \to \infty} \Lambda_N$ in its stable limit ($N \to \infty$) where $\Lambda_N = \Lambda_M[x_1, \ldots, x_N, 0, \ldots, 0]$ for all $M \geq N$. This ring of symmetric functions inherits the grading $\Lambda = \oplus_n \Lambda^{(n)}$, with $\Lambda^{(n)}$ consisting of homogeneous symmetric polynomials of degree n.

A $\mathbb{Z}$ basis of $\Lambda^{(n)}$ is provided by the monomial symmetric functions m_λ where λ is any partition of n. There exist further (integral and rational) bases for $\Lambda^{(n)}$ that are indexed by the partitions λ of n. These are the complete, elementary and power sum symmetric function bases defined *multiplicatively* in terms of corresponding one part functions by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}\,, \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}\,, \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}. \quad (6)$$

The one part functions are defined via their generating functions:

$$\begin{aligned} H_t &\equiv \prod_{i\geq 1} \frac{1}{1-x_i t} = \sum_{n\geq 0} h_n t^n, \\ E_t &\equiv \prod_{i\geq 1} (1+x_i t) = \sum_{n\geq 0} e_n t^n, \\ t\frac{\mathrm{d}}{\mathrm{d}t} \log H_t &= \sum_{n\geq 0} p_n t^n, \qquad (\, p_n = \sum_{i\geq 1} x_i^n\,). \end{aligned} \quad (7)$$

The most important *non-multiplicative* basis of $\Lambda^{(n)}$ is provided by the Schur functions s_λ with λ running over all partitions of n. For a finite number of variables the Schur function $s_\lambda(x_1,\dots,x_N)$ may be defined as a ratio of alternants. It is a homogeneous polynomial of total degree n, and is stable in the sense that $s_\lambda(x_1,\dots,x_N,0,\dots,0) = s_\lambda(x_1,\dots,x_N)$ regardless of how many 0's are appended to the list of variables. Taking the limit as $N\to\infty$ of $s_\lambda(x_1,\dots,x_N)$ serves to define the required $s_\lambda \in \Lambda^{(n)}$ [24].

Varying λ over all partitions, the Schur functions s_λ provide a $\mathbb{Z}$-basis of Λ. We can go further. There exists a bilinear form on Λ, the Schur-Hall scalar product $\langle \cdot \mid \cdot \rangle$. With respect to this scalar product, the Schur functions form an orthonormal basis of Λ. In fact we have:

$$\langle s_\lambda \mid s_\mu \rangle = \delta_{\lambda,\mu}\,, \qquad \langle p_\lambda \mid p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}\,, \qquad \langle m_\lambda \mid h_\mu \rangle = \delta_{\lambda,\mu}\,, \qquad \langle f_\lambda \mid e_\mu \rangle = \delta_{\lambda,\mu}. \quad (8)$$

These relations serve to define the ‘monomial’ symmetric functions m_λ, and the so-called ‘forgotten’ symmetric functions f_λ (for details see [24]).

In what follows we make use of various notation for Schur functions, including for example $s_\lambda(x_1,\dots,x_N)$, $s_\lambda(x)$, $s_\lambda(x,y)$ or s_λ, depending on whether or not it is necessary to be explicit about the number of variables or the sets of variables under consideration. Here, a single symbol x may often stand for an alphabet, $x_1, x_2, \dots$, finite or otherwise, while a pair x, y signifies a pair of such alphabets $x_1, x_2, \dots, y_1, y_2, \dots$.

2.2 The Hopf algebra Symm-Λ

The graded ring of symmetric functions Λ spanned by the Schur functions s_λ affords a graded self-dual, bicommutative Hopf algebra, which we denote by Symm-Λ, as can be seen once we have identified the appropriate product, coproduct, unit, counit, antipode and self-duality condition. This can be done as follows.

The **outer product** of Schur functions is given by (linear map form, infix dot product form, first without and then with variables, and finally the explicit linear form):

$$\begin{aligned} m(s_\mu \otimes s_\nu) &= s_\mu \cdot s_\nu\,, \\ m(s_\mu(x) \otimes s_\nu(y)) &= s_\mu(x)\cdot s_\nu(x) = \sum_\lambda C^\lambda_{\mu,\nu} s_\lambda(x)\,. \end{aligned} \quad (9)$$

Here, and elsewhere if not otherwise specified, tensor products are over $\mathbb{Z}$ (or $\mathbb{Q}$ if power sums are involved).

The **outer coproduct** map is denoted by Δ, and we use the variable or the tensor product notation interchangeably (unary form, Sweedler index form [31] and skew product forms, first without and then with variables, and finally the explicit multilinear form):

$$\Delta(s_\lambda) = s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} = \sum_\nu s_{\lambda/\nu} \otimes s_\nu = \sum_\mu s_\mu \otimes s_{\lambda/\mu} \,,$$
$$\Delta(s_\lambda(x)) = s_\lambda(x,y) = s_{\lambda_{(1)}}(x) s_{\lambda_{(2)}}(y) = \sum_{\mu,\nu} C_\lambda^{\mu,\nu} s_\mu(x) \otimes s_\nu(y) \,. \tag{10}$$

The coefficient $C_{\mu,\nu}^\lambda$ of the multiplication map m in the Schur function basis, and the structure constant $C_\lambda^{\mu,\nu}$ of the coproduct in the same basis turn out to be identical. This equality of coefficients is a consequence of the self-duality condition

$$\langle s_\lambda \mid m(s_\mu \otimes s_\nu) \rangle = \langle \Delta(s_\lambda) \mid s_\mu \otimes s_\nu \rangle \,. \tag{11}$$

In fact, they are both equal to the corresponding Littlewood-Richardson coefficient $c_{\mu,\nu}^\lambda$, which may be evaluated combinatorially using the Littlewood-Richardson rule [22, 24]. Thus we have

$$C_{\mu,\nu}^\lambda = c_{\mu,\nu}^\lambda = C_\lambda^{\mu,\nu} \,. \tag{12}$$

This follows from the well know fact that the dot and skew products of Schur functions are dual with respect to the Schur-Hall scalar product, that is to say [24]

$$\langle s_\lambda \mid s_\mu \cdot s_\nu \rangle = c_{\mu,\nu}^\lambda = \langle s_{\lambda/\nu} \mid s_\mu \rangle \,. \tag{13}$$

Here, in the Schur function basis the operations of outer multiplication and that of skewing are both defined in terms of Littlewood-Richardson coefficients by

$$s_\mu \cdot s_\nu = \sum_\lambda c_{\mu,\nu}^\lambda \, s_\lambda \qquad \text{and} \qquad s_\mu^\perp(s_\lambda) = s_{\lambda/\mu} = \sum_\nu c_{\mu,\nu}^\lambda \, s_\nu \,. \tag{14}$$

The notation $s_\mu^\perp(s_\lambda)$ has been introduced to emphasise the fact that the ring of symmetric functions has a module structure under $^\perp$:

$$f^\perp(g^\perp(h)) = (gf)^\perp(h) \qquad \text{or equivalently} \qquad (h/g)/f = h/(gf). \tag{15}$$

The **unit map** η, **counit map** ϵ, and **antipode** S, are defined by:

$$\eta : 1 \to s_0 \,, \qquad \epsilon : s_\lambda \to \delta_{\lambda,0} \,, \qquad S : s_\lambda \to (-1)^{|\lambda|} s'_\lambda \,. \tag{16}$$

It is important to note that the following antipode identity in the Hopf algebra $\mathsf{Symm}\text{-}\Lambda$:

$$m\,(I \otimes S)\Delta\,(s_\lambda) = \eta\,\epsilon(s_\lambda) \tag{17}$$

yields the result

$$\sum_\nu (-1)^{|\nu|} \, s_{\lambda/\nu} \cdot s_{\nu'} = \delta_{\lambda 0} \, s_0 \,, \tag{18}$$

since

$$m(I \otimes S)\Delta(s_\lambda) = m(I \otimes S)(\sum_\nu s_{\lambda/\nu} \otimes s_\nu)$$
$$= m(\sum_\nu s_{\lambda/\nu} \otimes (-1)^{|\nu|} s_{\nu'}) = \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu} \cdot s_{\nu'}$$

and

$$\eta\, \epsilon(s_\lambda) = \eta(\delta_{\lambda 0}) = \delta_{\lambda 0}\, s_0 \,.$$

Returning to the bases provided by h_λ, e_λ and p_λ in (6), these bases are so-called *multiplicative*, because the outer product is just the *(unordered) concatenation product*. Using self-duality, this means that the coproduct is just deconcatenation of these products, together with the action:

$$\Delta(h_n) = \sum_{r=0}^{n} h_{n-r} \otimes h_r \,, \qquad \Delta(e_n) = \sum_{r=0}^{n} e_{n-r} \otimes e_r \,, \qquad \Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \,. \quad (19)$$

These results all follow immediately from the definitions of (7). The first two of these show that one part complete and elementary symmetric functions are divided powers. The third shows that the one part power sum symmetric functions are the primitive elements of the Hopf algebra $\mathsf{Symm}\text{-}\Lambda$.

3 Characters of the classical groups

3.1 Irreducible representations and their characters

The groups, G, under consideration here are the general linear group GL, the orthogonal group O and the symplectic group Sp. If the classical groups GL, O and Sp act by way of linear transformations in a space V of dimension N, then they are denoted by $\mathsf{GL}(N)$, $\mathsf{O}(N)$ and $\mathsf{Sp}(N)$, respectively. We confine attention to their finite-dimensional irreducible covariant tensor representations V_{G}^{λ}. Each of these is specified by their *highest weight* λ, which in each case is a partition. The corresponding character is denoted by $\mathrm{ch}\, V_{\mathsf{G}}^{\lambda}$. Each of these characters may be expressed by means of Weyl's character formula [33] in terms of the eigenvalues $(x_1, \ldots, x_N)$ of each group element $g \in \mathsf{G}$ realised as a matrix $M \in \mathsf{End}(V)$ of linear transformations in V.

It is well known that in the case of $\mathsf{GL}(N)$ we have

$$\mathrm{ch}\, V_{\mathsf{GL}(N)}^{\lambda} = \{\lambda\}(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N) \,, \quad (20)$$

where the central symbol accords with the notation of Littlewood [22]. This character shares the same stable $N \to \infty$ limit as Schur functions, and in this limit we define the *universal character* [20, 17]

$$\mathrm{ch}\, V_{\mathsf{GL}}^{\lambda} = \{\lambda\} = s_\lambda \,. \quad (21)$$

The orthogonal and symplectic groups leave invariant a symmetric second rank tensor $g_{ij} = g_{ji}$ and an antisymmetric second rank tensor $f_{ij} = -f_{ji}$, respectively. It is necessary to distinguish between the even and odd cases, $N = 2K$ and $N = 2K+1$ with $K \in \mathbb{N}$. The groups

$\mathsf{O}(2K)$, $\mathsf{O}(2K+1)$ and $\mathsf{Sp}(2K)$ are all reductive Lie groups whose finite-dimensional representations are fully reducible. On the other hand $\mathsf{Sp}(2K+1)$, an odd-dimensional symplectic group, is not reductive. This is a consequence of the fact that its invariant bilinear form is singular. It can be realised as an affine extension of $\mathsf{Sp}(2K)\times\mathsf{GL}(1)$, that is the semi-direct product of these groups with a set of translations as explained by Proctor [27]. As a result its finite-dimensional representations are not necessarily fully reducible. Indeed its defining representation V, of dimension $2K+1$ is indecomposable but contains two irreducible constituents of dimensions $2K$ and 1. More genearally, Proctor has established that the representations $V^\lambda_{\mathsf{Sp}(2K+1)}$ are reducible but indecomposable for $\lambda\neq 0$.

Despite these issues associated with the evenness and oddness of N, there still exists a stable $N\to\infty$ limit and associated universal characters [20, 17] denoted here by:

$$\mathrm{ch}\, V^\lambda_{\mathsf{O}} = [\lambda] \qquad \text{and} \qquad \mathrm{ch}\, V^\lambda_{\mathsf{Sp}} = \langle\lambda\rangle\,. \tag{22}$$

Before defining these in terms of Schur functions it is necessary to introduce certain infinite series of Schur functions.

3.2 Schur function series

To describe characters of the orthogonal and symplectic groups effectively, Littlewood [22] introduced a set of infinite series with Schur function coefficients, which we are frequently going to use, consult also [3]. Some of these Schur function series read

$$\begin{array}{lll}
A := \sum_\alpha (-1)^{|\alpha|/2}\{\alpha\} & B := \sum_\beta \{\beta\} & C := \sum_\gamma (-1)^{|\gamma|/2}\{\gamma\} \\
D := \sum_\delta \{\delta\} & E := \sum_\epsilon (-1)^{(|\epsilon|+r)/2}\{\epsilon\} & F := \sum_\zeta \{\zeta\} \\
G := \sum_\epsilon (-1)^{(|\epsilon|-r)/2}\{\epsilon\} & H := \sum_\zeta (-1)^{|\zeta|}\{\zeta\} & L := \sum_m (-1)^m\{1^m\} \\
M := \sum_m \{m\} & P := \sum_m (-1)^m\{m\} & Q := \sum_m \{1^m\}
\end{array} \tag{23}$$

where $m\in\mathbb{Z}_+$ and the following special sets of partitions are in use:

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix}, \qquad (\gamma) = \begin{pmatrix} a_1+1 & a_2+1 & \dots & a_r+1 \\ a_1 & a_2 & \dots & a_r \end{pmatrix}. \tag{24}$$

(δ) is in the set of partitions having even parts only, (β) its conjugate. (ζ) is any partition and (ϵ) are the self conjugate partitions, r is the Frobenius rank of a partition (number of boxes on the diagonal).

Our main usage of these series will be by either using the underlying sets of partitions as index sets for special series, or by skewing with all possible Schur functions indexed by partitions in this set. This results in finite sums, and this is the method applied to the branching problem. A major feature of the Schur function series is that they come in mutually inverse pairs:

$$AB = CD = EF = GH = LM = PQ = 1. \tag{25}$$

To streamline notation, and to simplify the display of the formulae below, we use the convention that Roman indices in sums run over $\mathbb{Z}_+$, Greek indices without further restriction run over all partitions, as in $\sum_\zeta$; and if we use particular series we indicate it with consistent notation for the member partitions, for example $\sum_{\alpha\in A}$, $\sum_{\delta\in D}$. Sometimes we want to indicate summation over Sweedler indices, which we denote by $\sum_{(\lambda)}$. In summations over partitions associated to a Schur function series it is always necessary to include the appropriate sign factors. For example, any summation over the partition α in the A series must involve $\sum_{\alpha\in A}(-1)^{|\alpha|/2}\cdots$. Finally, note that these formal series are implicitly members of a ring $\Lambda[[t]]$ rather than Λ although no formal parameter is included in their definition. The default is grading by partition weight, so that summands $\{\alpha\}$ in the series A should be read as $t^{|\alpha|}\{\alpha\}$.

3.3 Universal characters of the orthogonal and symplectic groups

The Schur function series that we have introduced enable us to write down Schur function expressions for the universal characters of O and Sp in the form [22, 16]

$$\mathrm{ch}\, V^\lambda_{\mathsf{O}} = o_\lambda = [\lambda] = \{\lambda/C\} = s_{\lambda/C} = \sum_{\gamma\in C}(-1)^{|\gamma|/2}s_{\lambda/\gamma}\,, \tag{26}$$

$$\mathrm{ch}\, V^\lambda_{\mathsf{Sp}} = sp_\lambda = \langle\lambda\rangle = \{\lambda/A\} = s_{\lambda/A} = \sum_{\alpha\in A}(-1)^{|\alpha|/2}s_{\lambda/\alpha}\,. \tag{27}$$

These relations are the inverse of the branching rules for the restriction from GL to its subgroups O and Sp:

$$\mathrm{ch}\, V^\lambda_{\mathsf{GL}} = \{\lambda\} \to [\lambda/D] = \sum_{\delta\in D}[\lambda/\delta] = \sum_{\delta\in D,\zeta} C^\lambda_{\delta,\zeta}\,\mathrm{ch}\, V^\zeta_{\mathsf{O}}\,, \tag{28}$$

$$\mathrm{ch}\, V^\lambda_{\mathsf{GL}} = \{\lambda\} \to \langle\lambda/B\rangle = \sum_{\beta\in B}\langle\lambda/\beta\rangle = \sum_{\beta\in B,\zeta} C^\lambda_{\beta,\zeta}\,\mathrm{ch}\, V^\zeta_{\mathsf{Sp}}\,. \tag{29}$$

That the above pairs of relations are mutually inverse is a simple consequence of the identities $AB = CD = 1$.

To recover the irreducible characters of $\mathsf{O}(N)$ and $\mathsf{Sp}(N)$ in the finite N case one merely limits the arguments of the universal characters to the eigenvalues of the relevant group elements g supplemented by zeros. Setting $x_k = e^{i\phi_k}$ and $\overline{x}_k = x_k^{-1} = e^{-i\phi_k}$ with $\phi_k \in \mathbb{R}$ for $k = 1, 2, \ldots, K$ one obtains:

$$\begin{aligned}
\mathrm{ch}\, V^\lambda_{\mathsf{O}(2K)} &= [\lambda](x_1,\ldots,x_K,\overline{x}_1\ldots\overline{x}_K,0,\ldots,0) && \text{for } g\in\mathsf{SO}(2K)\,;\\
\mathrm{ch}\, V^\lambda_{\mathsf{O}(2K)} &= [\lambda](x_1,\ldots,x_{K-1},\overline{x}_1\ldots\overline{x}_{K-1},1,-1,0,\ldots,0) && \text{for } g\notin\mathsf{SO}(2K)\,;\\
\mathrm{ch}\, V^\lambda_{\mathsf{O}(2K+1)} &= [\lambda](x_1,\ldots,x_K,\overline{x}_1\ldots\overline{x}_K,1,0,\ldots,0) && \text{for } g\in\mathsf{SO}(2K+1)\,;\\
\mathrm{ch}\, V^\lambda_{\mathsf{O}(2K+1)} &= [\lambda](x_1,\ldots,x_K,\overline{x}_1\ldots\overline{x}_K,-1,0,\ldots,0) && \text{for } g\notin\mathsf{SO}(2K+1)\,;\\
\mathrm{ch}\, V^\lambda_{\mathsf{Sp}(2K)} &= \langle\lambda\rangle(x_1,\ldots,x_K,\overline{x}_1\ldots\overline{x}_K,0,\ldots,0)\,;\\
\mathrm{ch}\, V^\lambda_{\mathsf{Sp}(2K+1)} &= \langle\lambda\rangle(x_1,\ldots,x_K,\overline{x}_1\ldots\overline{x}_K,x_{2K+1},0,\ldots,0) && \text{with } x_{2K+1} \text{ arbitrary}\,.
\end{aligned} \tag{30}$$

where the final character of $\mathsf{Sp}(2K+1)$ is indecomposable, rather than irreducible, with the first $2K$ eigenvalues being those of an element of $\mathsf{Sp}(2K)$ and the x_{2K+1} being an element of $\mathsf{GL}(1)$.

In describing the Hopf algebras of the character rings of the groups GL, O and Sp we deal only with the universal characters, their restriction to the finite N case necessitates the use of modification rules if the relevant partitions are of too great a length. Further details may be found elsewhere, for example [25, 3, 20].

4 The Hopf algebras of universal character rings

4.1 General remarks

The Hopf algebra of symmetric functions, $\mathsf{Symm}\text{-}\Lambda$, is the archetypical Hopf algebra, in that it is universal, commutative, cocommutative, and self-dual. Its properties have been spelt out in the Schur function basis in Section 2.2. Having identified in Section 3.3 the universal characters of the classical groups and expressed them in terms of Schur functions, the Hopf algebras of their universal character rings may be found as isomorphic copies of $\mathsf{Symm}\text{-}\Lambda$. Despite the fact that the structure maps acting on the character ring Hopf algebra $\mathsf{Char}\text{-}\mathsf{GL}$, $\mathsf{Char}\text{-}\mathsf{O}$ and $\mathsf{Char}\text{-}\mathsf{Sp}$ are ismorphic to those of $\mathsf{Symm}\text{-}\Lambda$, they take different explicit forms in the different canonical bases. These are distinguished by the use of different Littlewood parentheses, $\{\lambda\}$, $[\lambda]$ and $\langle\lambda\rangle$. Furthermore, these Hopf algebras are distinguished by their particular Schur-Hall scalar products.

4.2 The general linear case

By virtue of the identification (21) $\mathsf{Symm}\text{-}\Lambda$ is immediately seen to be isomorphic to the Hopf algebra $\mathsf{Char}\text{-}\mathsf{GL}$ of universal characters of GL. Its structure is well known (for references see [11, 14]) and some of its properties are summarized as follows.

Theorem 4.1. The ring of universal characters of GL is a graded self dual, bicommutative Hopf algebra, which we denote by $\mathsf{Char}\text{-}\mathsf{GL}$. Its structure maps are given by:

$$
\begin{array}{ll}
\text{product} & m(\{\mu\}\otimes\{\nu\}) = \{\mu\}\cdot\{\nu\} = \{\mu\cdot\nu\} = \sum_{\zeta} c^{\lambda}_{\mu,\nu}\{\lambda\} \\
\text{unit} & \eta : 1 \to \{0\} \ \text{with} \ \{0\}\cdot\{\mu\} = \{\mu\} = \{\mu\}\cdot\{0\} \\
\text{coproduct} & \Delta(\{\lambda\}) = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu}\{\mu\}\otimes\{\nu\} \\
\text{counit} & \epsilon(\{\mu\}) = \langle\{0\}\mid\{\mu\}\rangle = \delta_{0,\mu} \\
\text{antipode} & S(\{\lambda\}) = (-1)^{|\lambda|}\{\lambda'\} \\
\text{self-duality} & \langle\Delta(\{\lambda\})\mid\{\mu\}\otimes\{\nu\}\rangle = \langle\{\lambda\}\mid\{\mu\}\cdot\{\nu\}\rangle
\end{array}
\tag{31}
$$

where the coefficients $c^{\lambda}_{\mu,\nu}$ are the Littlewood-Richardson coefficients, λ' is the conjugate (transposed) partition and $\langle\cdot\mid\cdot\rangle : \Lambda\otimes\Lambda\to\mathbb{Z}$ is the usual Schur-Hall scalar product, here expressing the orthogonality of the irreducible characters $\langle\{\lambda\}\mid\{\mu\}\rangle = \delta_{\mu,\nu}$. ■

Because of its importance in what follows we map the antipode identity (18) of $\mathsf{Symm}\text{-}\Lambda$, into the antipode identity of $\mathsf{Char}\text{-}\mathsf{GL}$:

$$\sum_{\nu} (-1)^{|\nu|}\,\{\lambda/\nu\}\cdot\{\nu'\} = \delta_{\lambda 0}\,\{0\}\,. \tag{32}$$

4.3 The orthogonal case

Having shown that the irreducible universal characters $[\lambda]$ of the orthogonal group O can be expressed in terms of universal characters of GL by $[\lambda] = \{\lambda/C\}$, it is possible to exploit infinite Schur function series and the Hopf algebra $\mathsf{Char}\text{-}\mathsf{GL}$ to identify the action of the structure maps on the ring of characters $[\lambda]$ forming the canonical basis of $\mathsf{Char}\text{-}\mathsf{O}$. This action, as will be proved in the following section, takes the following form:

Theorem 4.2. The algebra $\mathsf{Char}\text{-}\mathsf{O}$ generated by the universal characters $[\lambda]$ of the orthogonal group O is a bicommutative Hopf algebra. Its structure maps are given by:

$$\begin{array}{ll}
\text{product} & m([\mu]\cdot[\nu]) = [\mu]\cdot[\nu] = \sum_{\zeta}[\mu/\zeta\cdot\nu/\zeta] \\
\text{unit} & \eta : 1 \to [0] \ \text{ with } \ [0]\cdot[\mu] = [\mu] = [\mu]\cdot[0] \\
\text{coproduct} & \Delta([\lambda]) = \sum_{\zeta}[\lambda/(\zeta D)]\otimes[\zeta] = \sum_{\zeta}[\lambda/\zeta]\otimes[\zeta/D] \\
\text{counit} & \epsilon([\lambda]) = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\delta_{\lambda,\gamma} = \delta_{\lambda,C} = c(\{\lambda\}) \\
\text{antipode} & S([\lambda]) = (-1)^{|\lambda|}[\lambda'/(C'D)]
\end{array} \tag{33}$$

where λ' is the partition conjugate to λ and $C' = A$ is the conjugate of C. ■

4.4 The symplectic case

In the same way, by exploiting the fact that the irreducible (or indecomposable) universal characters $\langle\lambda\rangle$ of the symplectic group Sp can be expressed in terms of universal characters of GL by $\langle\lambda\rangle = \{\lambda/A\}$, we can identify the action of the structure maps on the ring of characters $\langle\lambda\rangle$ forming the canonical basis of $\mathsf{Char}\text{-}\mathsf{Sp}$. This action, as will be proved in the following section, takes the following form:

Theorem 4.3. The algebra $\mathsf{Char}\text{-}\mathsf{Sp}$ generated by the universal characters $\langle\lambda\rangle$ of the symplectic group Sp is a bicommutative Hopf algebra. Its structure maps are given by:

$$\begin{array}{ll}
\text{product} & m(\langle\mu\rangle\cdot\langle\nu\rangle) = \langle\mu\rangle\cdot\langle\nu\rangle = \sum_{\zeta}\langle\mu/\zeta\cdot\nu/\zeta\rangle \\
\text{unit} & \eta : 1 \to \langle 0\rangle \ \text{ with } \ \langle 0\rangle\cdot\langle\mu\rangle = \langle\mu\rangle = \langle\mu\rangle\cdot\langle 0\rangle \\
\text{coproduct} & \Delta(\langle\lambda\rangle) = \sum_{\zeta}\langle\lambda/(\zeta B)\rangle\otimes\langle\zeta\rangle = \sum_{\zeta}\langle\lambda/\zeta\rangle\otimes\langle\zeta/B\rangle \\
\text{counit} & \epsilon(\langle\lambda\rangle) = \sum_{\alpha\in A}(-1)^{|\alpha|/2}\,\delta_{\lambda,\alpha} = \delta_{\lambda,A} = a(\{\mu\}) \\
\text{antipode} & S(\langle\lambda\rangle) = (-1)^{|\lambda|}\langle\lambda'/(A'B)\rangle
\end{array} \tag{34}$$

where λ' is the partition conjugate to λ and $A' = C$ is the conjugate of A. ■

4.5 Directory of results

All the above results, and a considerable amount of additional information, regarding the three Hopf algebras of character rings, $\mathsf{Char}\text{-}\mathsf{GL}$, $\mathsf{Char}\text{-}\mathsf{O}$ and $\mathsf{Char}\text{-}\mathsf{Sp}$, are gathered together in Table 1.

The first column of this directory gives the abstract Hopf algebra notation for bases and morphisms of $\mathsf{Symm}\text{-}\Lambda$. The second column gives the notion for the Hopf algebra of the universal character ring of the general linear group, as studied for example in [11]. The third and fourth columns provide the isomorphic images of the structure maps and bases in the character rings of the orthogonal and symplectic groups.

Remark. Note that the last row does *not* show isomorphic structures. While Λ and $\mathsf{Char}\text{-}\mathsf{GL}$ share the same Schur-Hall scalar product we emphasise that $\mathsf{Char}\text{-}\mathsf{O}$ and $\mathsf{Char}\text{-}\mathsf{Sp}$ come with *new structure maps*, the plethystic Schur-Hall scalar products, indexed by 2 and 11, which are defined so as to ensure that the orthogonal and symplectic Schur functions form orthonormal bases of $\mathsf{Char}\text{-}\mathsf{O}$ and $\mathsf{Char}\text{-}\mathsf{Sp}$, respectively. □

We have used the following notational conventions. $\chi(P)$ is a truth symbol, that is, $\chi(P)$ is one if the proposition P is true, and zero otherwise. For example, $\chi(2|n)$ applied to an expression in n means that only n even are selected (2 is a divisor of n). The sum in the row providing expressions for the power sum symmetric functions is over $a+b+1=n$. Other implicit sums are taken over all integers or over all partitions.

The precise definitions of the bases involved in some of the formulae will be given in the following sections. However, this table makes it clear that there are *unique* instances of symmetric functions, such as power sum symmetric functions, which are tied to the underlying alphabet and are, up to isomorphism, equivalent in all character Hopf algebras under consideration. Despite this, if written in the canonical basis of a specific character Hopf algebra, it can be seen that such objects may look different and may also exhibit combinatorial differences.

5 Orthogonal and symplectic character ring Hopf algebras

5.1 The case of $\mathsf{Char}\text{-}\mathsf{O}$

We consider in turn each of the structure maps listed in Theorem 4.3.

The **product** formula

$$[\mu]\cdot[\nu]=\sum_{\zeta}[\mu/\zeta\cdot\nu/\zeta] \tag{35}$$

will not be rederived here, since it is a classical result of Newell [25] and Littlewood [23] that appears as a special case of the development in [14] for more general subgroups of $\mathsf{GL}(N)$.

Furthermore, since $\{0/\zeta\}=\delta_{\zeta,0}\{0\}$, it is clear that in the special case of the above product with $[\mu]=[0]$ the sum over all ζ's is restricted to just the $\zeta=0$ term. It follows that $[0]\cdot[\nu]=[0\cdot(\nu/0)]$. However, $\{\nu/0\}=\{\nu\}$ and $\{0\cdot\nu\}=\{\nu\}$, so that $[0]\cdot[\nu]=[\nu]$ for all $[\nu]$. Similarly, $[\nu]\cdot[0]=[\nu]$ for all $[\nu]$, so that $[0]$ is the unique **unit** of this product.

To find the **coproduct** we need to find first the ordinary coproduct of a skew Schur function. This can be looked up in Macdonald [24] (Eq. 5.9 and 5.10, p72). The idea is to expand

Table 1: Hopf Algebras of Character Rings, Bases and Morphisms

Λ	Char-GL	Char-O	Char-Sp
1	$\{0\}$	$[0]$	$\langle 0\rangle$
p_n	$\sum(-1)^b\{a+1,1^b\}$	$\sum(-1)^b[a+1,1^b]+\chi(2\mid n)[0]$	$\sum(-1)^b\langle a+1,1^b\rangle+\chi(2\mid n)\langle 0\rangle$
h_n	$\{n\}$	$[n/D]$	$\langle n\rangle$
e_n	$\{1^n\}$	$[1^n]$	$\langle 1^n/B\rangle$
s_λ	$\{\lambda\}$	$[\lambda/D]$	$\langle\lambda/B\rangle$
o_λ	$\{\lambda/C\}$	$[\lambda]$	$\langle\lambda/(BC)\rangle$
sp_λ	$\{\lambda/A\}$	$[\lambda/(AD)]$	$\langle\lambda\rangle$
$M(t)$	$\sum_n\{n\}t^n$	$\frac{1}{1-t^2}\sum_n[n]t^n$	$\sum_n\langle n\rangle t^n$
$L(t)$	$\sum_n(-1)^n\{1^n\}t^n$	$\sum_n(-1)^n[n]t^n$	$\frac{1}{1-t^2}\sum_n(-1^n)\langle n\rangle t^n$
m	$m(\{\mu\}\otimes\{\nu\})=\{\mu\}\cdot\{\nu\}$	$m([\mu]\otimes[\nu])=\sum[\mu/\zeta\cdot\nu/\zeta]$	$m(\langle\mu\rangle\otimes\langle\nu\rangle)=\sum\langle\mu/\zeta\cdot\nu/\zeta\rangle$
Δ	$\Delta(\{\lambda\})=\sum\{\lambda/\zeta\}\otimes\{\zeta\}$	$\Delta([\lambda])=\sum[\lambda/\zeta]\otimes[\zeta/D]$	$\Delta([\lambda])=\sum\langle\lambda/\zeta\rangle\otimes\langle\zeta/B\rangle$
ϵ	$\epsilon(\{\lambda\})=\delta_{\lambda,0}$	$\epsilon([\lambda])=\sum_{\gamma\in C}(-1)^{\vert\gamma\vert/2}\delta_{\lambda,\delta}$	$\epsilon(\langle\lambda\rangle)=\sum_{\alpha\in A}(-1)^{\vert\alpha\vert/2}\delta_{\lambda,\alpha}$
η	$1\to\{0\}$	$1\to[0]$	$1\to\langle 0\rangle$
S	$S(\{\lambda\})=(-1)^{\vert\lambda\vert}\{\lambda'\}$	$S([\lambda])=(-1)^{\vert\lambda\vert}[\lambda'/(C'D)]$	$S(\langle\lambda\rangle)=(-1)^{\vert\lambda\vert}\langle\lambda'/(A'B)\rangle$
$\langle\cdot\mid\cdot\rangle$	$\langle\{\lambda\}\mid\{\mu\}\rangle=\delta_{\lambda,\mu}$	$\langle[\lambda]\mid[\mu]\rangle_2=\delta_{\lambda,\mu}$	$\langle\langle\lambda\rangle\mid\langle\mu\rangle\rangle_{11}=\delta_{\lambda,\mu}$

$s_\lambda(x,y,z)$, a double coproduct, in two different ways:

$$\begin{aligned} s_\lambda(x,y,z) &= \sum_\nu s_{\lambda/\nu}(x,y)s_\nu(z) \\ &= \sum_\mu s_{\lambda/\mu}(x)s_\mu(y,z) = \sum_{\mu,\nu} s_{\lambda/\mu}(x)s_{\mu/\nu}(y)s_\nu(z)\,, \end{aligned} \tag{36}$$

and comparing coefficients of $s_\nu(z)$ gives

$$s_{\lambda/\nu}(x,y) = \sum_\mu s_{\lambda/\mu}(x)s_{\mu/\nu}(y)\,, \qquad \text{that is} \qquad \Delta(s_{\lambda/\nu}) = \sum_\mu s_{\lambda/\mu}\otimes s_{\mu/\nu}\,. \tag{37}$$

Now we can proceed to compute

$$\begin{aligned} \Delta([\lambda]) :&= \Delta\{\lambda/C\} = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\Delta(\{\lambda/\gamma\}) = \sum_{\gamma\in C}\sum_\zeta(-1)^{|\gamma|/2}\{\lambda/\zeta\}\otimes\{\zeta/\gamma\} \\ &= \sum_\zeta\{\lambda/\zeta\}\otimes[\zeta] = \sum_\zeta\{\lambda/(\zeta DC)\}\otimes[\zeta] = \sum_\zeta[\lambda/(\zeta D)]\otimes[\zeta]\,. \end{aligned} \tag{38}$$

This can equally well be rewritten to give a second form of the coproduct derived using a pair of related expansions of a skew Schur function, $s_{\lambda/\zeta} = \sum_\sigma c^\lambda_{\zeta,\sigma}s_\sigma$ and $s_{\lambda/\sigma} = \sum_\zeta c^\lambda_{\zeta,\sigma}s_\zeta$, a move we use below frequently. Here it gives,

$$\begin{aligned} \Delta([\lambda]) &= \sum_\zeta[\lambda/(\zeta D)]\otimes[\zeta] = \sum_\zeta[(\lambda/\zeta)/D)]\otimes[\zeta] \\ &= \sum_{\zeta,\sigma} c^\lambda_{\zeta,\sigma}\,[\sigma/D]\otimes[\zeta] = \sum_\sigma[\sigma/D]\otimes[\lambda/\sigma]\,. \end{aligned} \tag{39}$$

The coproduct $\Delta([\lambda])$ is cocommutative, as can be seen by using in the same way as above the connection between the outer product of Schur functions $s_\zeta\cdot s_\delta = \sum_\eta c^\sigma_{\zeta,\delta}\,s_\sigma$ and the skew Schur functions expansion $s_{\sigma/\delta} = \sum_\zeta c^\sigma_{\zeta,\delta}\,s_\zeta$, to obtain a third form:

$$\begin{aligned} \Delta([\lambda]) &= \sum_\zeta[\lambda/(\zeta D)]\otimes[\zeta] = \sum_{\zeta,\delta\in D}[(\lambda/(\zeta\cdot\delta)]\otimes[\zeta] \\ &= \sum_{\zeta,\sigma,\delta\in D} c^\sigma_{\zeta,\delta}\,[\lambda/\sigma]\otimes[\zeta] = \sum_{\sigma,\delta\in D}[\lambda/\sigma]\otimes[\sigma/\delta] = \sum_\sigma[\lambda/\sigma]\otimes[\sigma/D]\,. \end{aligned} \tag{40}$$

The **counit** is established as follows.

Definition 5.1. The counit ϵ and its inverse ϵ^{-1} for Char-O are defined in terms of linear forms c, d : Char-GL $\to \mathbb{Z}$ as follows:[1]

$$\epsilon([\lambda]) := c(\{\lambda\}) \;\text{ with }\; c(\{\lambda\}) := \langle C \mid \{\lambda\}\rangle = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\langle\{\gamma\}\mid\{\lambda\}\rangle = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\delta_{\gamma,\lambda}\,,$$

(41)

$$\epsilon^{-1}([\lambda]) := d(\{\lambda\}) \;\text{ with }\; d(\{\lambda\}) := \langle D \mid \{\lambda\}\rangle = \sum_{\delta\in D}\langle\{\delta\}\mid\{\lambda\}\rangle = \sum_{\delta\in D}\delta_{\delta,\lambda}\,.$$

■

Corollary 5.2. (see [11]) The linear forms (1-cochains) c and d are convolutive inverses with respect to the Char-GL outer coproduct and product in $\mathbb{Z}$. ■

[1] This definition should be compared with a slightly different point of view developed in the section on adapted normal ordered products in [4], which can be used to define a quantum field theory on an external background.

Proof.

$$
\begin{aligned}
(c \star d)(\{\lambda\}) &= \sum_{(\lambda)} c(\{\lambda_{(1)}\}) d(\{\lambda_{(2)}\}) \\
&= \sum_{(\lambda)} \langle C \mid \{\lambda_{(1)}\}\rangle \langle D \mid \{\lambda_{(2)}\}\rangle = \sum_{(\lambda)} \langle C \otimes D \mid \{\lambda_{(1)}\} \otimes \{\lambda_{(2)}\}\rangle \\
&= \langle CD \mid \{\lambda\}\rangle = \langle 1 \mid \{\lambda\}\rangle = \epsilon(\{\lambda\}) = \delta_{\lambda,0} \,. \qquad (42)
\end{aligned}
$$

□

Remark. Using the Hopf algebra structure of the symmetric functions we can write the *inverse* branching map as

$$[\lambda] = \{\lambda/C\} = \sum_{\gamma \in C} (-1)^{|\gamma|/2} \langle \{\gamma\} \mid \{\lambda_{(1)}\}\rangle \, \{\lambda_{(2)}\} = c(\lambda_{(1)}) \{\lambda_{(2)}\}. \qquad (43)$$

This stresses the similarity with the time-ordering device in quantum field theory (see [7, 4]). More explicitly, the module map can be achieved by using either the original coproduct of Char-GL,

$$
\begin{aligned}
[\lambda] &= \sum_{\gamma \in C} (-1)^{|\gamma|/2} \langle \{\gamma\} \mid \{\lambda_{(1)}\}\rangle \, \{\lambda_{(2)}\} = \sum_{\nu, \gamma \in C} (-1)^{|\gamma|/2} \langle \{\gamma\} \mid \{\nu\}\rangle \, \{\lambda/\nu\} \\
&= \sum_{\nu, \gamma \in C} (-1)^{|\gamma|/2} \delta_{\gamma,\nu} \, \{\lambda/\nu\} = \sum_{\gamma \in C} (-1)^{|\gamma|/2} \{\lambda/\gamma\} = \{\lambda/C\} \,, \qquad (44)
\end{aligned}
$$

or the deformed coproduct of Char-O

$$
\begin{aligned}
[\lambda] &= \sum_{\gamma \in C} (-1)^{|\gamma|/2} \langle \{\gamma\} \mid [\lambda_{[1]}]\rangle \, \{\lambda_{[2]}\} = \sum_{\sigma, \gamma \in C} (-1)^{|\gamma|/2} \langle \{\gamma\} \mid [\sigma/D]\rangle \, \{\lambda/\sigma\} \\
&= \sum_{\sigma, \gamma \in C} (-1)^{|\gamma|/2} \langle \{\gamma\} \mid \{\sigma\}\rangle \, \{\lambda/\sigma\} = \sum_{\sigma, \gamma \in C} (-1)^{|\gamma|/2} \delta_{\gamma,\sigma} \{\lambda/\sigma\} = \{\lambda/C\} \,, \qquad (45)
\end{aligned}
$$

where use has been made of differently parametrized Sweedler indices, in accordance with the Brouder-Schmitt convention [5]. These results show the consistency of the two ways to compute this inverse branching $[\lambda] = \{\lambda/C\}$. The branching itself, $\{\lambda\} = [\lambda/D]$, can also be derived in two ways. □

It remains to show that the counit we have defined satisfies the required Hopf algebra identity

$$(\epsilon \otimes 1) \, \Delta = 1 \,. \qquad (46)$$

Using the fact that $[\lambda/D] = \{\lambda\}$ and $\{\lambda/C\} = [\lambda]$, this is done as follows:

$$(\epsilon \otimes 1)\Delta[\lambda] = \sum_{\zeta} \epsilon([\zeta]) \, [\lambda/(\zeta D)] = \sum_{\lambda, \gamma \in C} (-1)^{|\gamma|/2} \delta_{\gamma,\zeta} \, \{\lambda/\zeta\} = \{\lambda/C\} = [\lambda] \,. \qquad (47)$$

The cocommutativity shows that ϵ is a unique left and right counit.

Furthermore, we have to check that the **antipode** fulfils its defining relation

$$m(1 \otimes S)\Delta = \eta\epsilon = m(S \otimes 1)\Delta \,. \qquad (48)$$

Once more it is enough to compute one equality due to bicommutativity. Using both our techniques for moving from one skew product to another and the fact that the Littlewood-Richardson coefficients satisfy the conjugacy identity $c^{\lambda}_{\mu,\nu} = c^{\lambda'}_{\mu',\nu'}$, as well as the Char-GL antipode condition (32), we find:

$$\begin{aligned}
m(1\otimes S)\Delta[\lambda] &= m\,(1\otimes S)\sum_{\nu}[\lambda/(\nu D)]\otimes[\nu] = m\Big(\sum_{\nu}[\lambda/(\nu D)]\otimes(-1)^{|\nu|}[\nu'/(C'D)]\Big)\\
&= \sum_{\nu,\zeta}(-1)^{|\nu|}[\lambda/(\nu D\zeta)\cdot\nu'/(C'D\zeta)] = \sum_{\nu}(-1)^{|\nu|}[(\lambda/\nu\cdot\nu'/C')/D]\\
&= \sum_{\nu,\mu,\gamma\in C}(-1)^{|\nu|+|\gamma|/2}c^{\nu'}_{\mu',\gamma'}[(\lambda/\nu\cdot\mu')\,/D] = \sum_{\nu,\mu,\gamma\in C}(-1)^{|\mu|+|\gamma|/2}c^{\nu}_{\mu,\gamma}[(\lambda/\nu\cdot\mu')\,/D]\\
&= \sum_{\mu,\gamma\in C}(-1)^{|\mu|+|\gamma|/2}[((\lambda/(\gamma\cdot\mu))\cdot\mu')\,/D] = \sum_{\mu,\gamma\in C}(-1)^{|\mu|+|\gamma|/2}[(((\lambda/\gamma)/\mu)\cdot\mu')\,/D]\\
&= \sum_{\gamma\in C}(-1)^{|\gamma|/2}\delta_{\lambda/\gamma,0}\,[0/D] = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\delta_{\lambda,\gamma}\,[0/D] = \epsilon([\lambda])[0]\,.
\end{aligned} \tag{49}$$

Finally, we need to check that the product and coproduct are mutual **coalgebra** and **algebra homomorphisms**. We establish this fact by direct computation:

$$\begin{aligned}
(\Delta\, m)([\lambda]\otimes[\mu]) &= \sum_{\zeta}\Delta([\lambda/\zeta\cdot\mu/\zeta]) = \sum_{\rho,\zeta}[(\lambda/\zeta\cdot\mu/\zeta)\,/\rho]\otimes[\rho/D]\\
&= \sum_{\sigma,\rho,\zeta}[\lambda/(\zeta\sigma)\cdot\mu/(\zeta(\rho/\sigma))]\otimes[\rho/D] = \sum_{\xi,\sigma,\zeta}[\lambda/(\zeta\sigma)\cdot\mu/(\zeta\xi)]\otimes[(\sigma\xi)/D]\\
&= \sum_{\tau,\xi,\sigma,\zeta}[\lambda/(\zeta\sigma)\cdot\mu/(\zeta\xi)]\otimes[\sigma/(\tau D)\cdot\xi/(\tau D)] = \sum_{\sigma,\xi}([\lambda/\sigma]\cdot[\mu/\xi])\otimes([\sigma/D]\cdot[\xi/D])\\
&= \sum_{\sigma,\xi}([\lambda/\sigma]\otimes[\sigma/D])\cdot([\mu/\xi]\otimes[\xi/D]) = \Delta([\lambda])\cdot\Delta([\mu]) = m(\Delta([\lambda])\otimes\Delta([\mu]))\,,
\end{aligned} \tag{50}$$

showing the claim. □

Remarks. It could be argued that the above proof is unnecessary. We considered just a linear isomorphism on the module underlying the symmetric function Hopf algebra, and the result is in a natural way, a homomorphic image. However, the displayed calculations show explicitly how the structure maps are written in the orthogonal Schur function bases, how the combinatorics alters, and that everything is set up correctly.

Note also the most remarkable fact that the structure of the Hopf algebra Char-O does not distinguish between even and odd orthogonal groups. It does not even rely on the fact that the metric tensor $g_{i,j} = g_{j,i}$ of Schur symmetry type $\{2\}$, which defines the orthogonal group, is invertible. Such degenerate cases are instances of Cayley-Klein groups (see conclusions for further comments). The even or oddness of the underlying group will show up in a subtle way when we define particular bases for these Hopf algebras below.

Orthogonality. It can be readily checked, that Schur functions of orthogonal type $o_\lambda = [\lambda]$ are **not** orthogonal with respect to the Schur-Hall scalar product. As mentioned already, it is hence necessary to define a new, 'orthogonal' Schur-Hall scalar product, accounting for the fact that we consider the orthogonal Schur functions to be characters of irreducible orthogonal group representations.

Definition 5.3. The orthogonal Schur-Hall scalar product expressing the orthonormality of the orthogonal Schur functions is defined by:

$$\langle \cdot \mid \cdot \rangle_2 : \mathsf{Char\text{-}O} \otimes \mathsf{Char\text{-}O} \to \mathbb{Z} \quad \text{with} \quad \langle [\lambda] \mid [\mu] \rangle_2 = \delta_{\lambda,\mu} \,. \tag{51}$$

■

The index 2 is a reminder of the plethystic character of this branching (see [14] and the previous introductory remarks). The relation between the scalar products of $\mathsf{Char\text{-}O}$ and $\mathsf{Char\text{-}GL}$ is

$$\langle [\lambda] \mid [\mu] \rangle_2 = \delta_{\lambda,\mu} = \langle \{\lambda\} \mid \{\mu\} \rangle = \langle [\lambda/D] \mid [\mu/D] \rangle. \tag{52}$$

Corollary 5.4. Neither with respect to the $\mathsf{Char\text{-}GL}$ Schur-Hall scalar product nor with respect to the $\mathsf{Char\text{-}O}$ Schur-Hall scalar product is $\mathsf{Char\text{-}O}$ a self-dual Hopf algebra. ■

Proof. Straightforward verification, or see the Appendix. □

This shows that, the Hopf algebra isomorphism between $\mathsf{Char\text{-}GL}$ and $\mathsf{Char\text{-}O}$ notwithstanding, these Hopf algebras are not identical, by virtue of the latter's non self-duality. Since we will typically consider products such as $\mathsf{H} \otimes \mathsf{H}^*$ of a Hopf algebra and its dual (as in the case of the Drinfeld quantum double, or Schur functors with both multiplication endomorphisms and Foulkes derivatives), we note that the branching process does not provide an isomorphism of this extended structure, and hence the map from one to the other is a nontrivial transformation; see [6] for a further exploration of this fact.

5.2 The case of Char-Sp

The validity of the structure maps of $\mathsf{Char\text{-}Sp}$ given in Theorem 4.4 may be established by copying and pasting the proof for the orthogonal case. One merely changes all orthogonal characters into symplectic ones, $[\lambda] \to \langle \lambda \rangle$, and interchanges Schur function series, $C \to A$ and $D \to B$. All arguments run through as before. In the case of the counit, it is also necessary to interchange the labelling on the linear forms, $c \to a$ and $d \to b$, where by analogy with Definition 5.1 we have:

Definition 5.5. The counit ϵ and its inverse ϵ^{-1} for $\mathsf{Char\text{-}Sp}$ are defined in terms of linear forms $a, b : \mathsf{Char\text{-}GL} \to \mathbb{Z}$ as follows:

$$\begin{aligned} \epsilon(\langle \lambda \rangle) &:= a(\{\lambda\}) \ \text{ with } \ a(\{\lambda\}) := \langle A \mid \{\lambda\} \rangle = \sum_{\alpha \in A} (-1)^{|\alpha|/2} \langle \{\alpha\} \mid \{\lambda\} \rangle = \sum_{\alpha \in A} (-1)^{|\alpha|/2} \delta_{\alpha,\lambda} \,, \\ \epsilon^{-1}(\langle \lambda \rangle) &:= b(\{\lambda\}) \ \text{ with } \ b(\{\lambda\}) := \langle B \mid \{\lambda\} \rangle = \sum_{\beta \in B} \langle \{\beta\} \mid \{\lambda\} \rangle = \sum_{\beta \in B} \delta_{\beta,\lambda} \,. \end{aligned} \tag{53}$$

■

Once again as in Corollary 5.1 we have:

Corollary 5.6. (see [11]) The linear forms (1-cochains) a and b are convolutive inverses with respect to the $\mathsf{Char\text{-}GL}$ outer coproduct and product in $\mathbb{Z}$. ■

Orthogonality. We consider the symplectic Schur functions as irreducible characters and define, analogously to the orthogonal case, a new 'symplectic' Schur-Hall scalar product indexed this time by the plethyistic label 11 rather than 2.

Definition 5.7. The symplectic Schur-Hall scalar product expressing the orthonormality of the symplectic Schur functions is defined by:

$$\langle \cdot \mid \cdot \rangle_{11} : \mathsf{Char}\text{-}\mathsf{Sp} \otimes \mathsf{Char}\text{-}\mathsf{Sp} \rightarrow \mathbb{Z} \quad \text{with} \quad \langle \langle \lambda \rangle \mid \langle \mu \rangle \rangle_{11} = \delta_{\lambda,\mu} \, . \tag{54}$$

■

The relation between the scalar products of $\mathsf{Char}\text{-}\mathsf{Sp}$ and $\mathsf{Char}\text{-}\mathsf{GL}$ is

$$\langle \langle \lambda \rangle \mid \langle \mu \rangle \rangle_{11} = \delta_{\lambda,\mu} = \langle \{\lambda\} \mid \{\mu\} \rangle = \langle \langle \lambda/B \rangle \mid \langle \mu/B \rangle \rangle \, . \tag{55}$$

Corollary 5.8. Neither with respect to the $\mathsf{Char}\text{-}\mathsf{GL}$ Schur-Hall scalar product, nor with respect to the $\mathsf{Char}\text{-}\mathsf{Sp}$ Schur-Hall scalar product, is $\mathsf{Char}\text{-}\mathsf{Sp}$ a self-dual Hopf algebra. ■

Proof. Straightforward verification, or see the Appendix. □

Remarks. Note again that the results do not depend on the invertibility of the bilinear form, so the ring generated by the finite dimensional characters of the odd symplectic groups $\mathsf{Sp}(2K+1)$ in the stable limit as $K \rightarrow \infty$, coincides with that obtained from $\mathsf{Sp}(2K)$ in the same limit. The reverse processes of restriction from Sp to $\mathsf{Sp}(2K+1)$ and $\mathsf{Sp}(2K)$ do however require different treatment and require further more the application of modification rules.

6 Bases for Char-O and Char-Sp

6.1 Power sum symmetric functions

A major issue in setting the above abstract machinery to work in concrete (physical) examples, is a proper identification in the various character rings of the usual canonical bases of the symmetric function ring. In making this identification, we will encounter some familiar and also some surprising results. We start with the power sum symmetric functions on a finite number of variables N. The one part power sum symmetric functions are defined on the variables $(x_1, \ldots, x_N)$ by

$$p_n := \sum_{i=1}^{N} x_i^n \tag{56}$$

which is independent of the meaning of the alphabet.

In the $\mathsf{GL}(N)$ case the x_i are the eigenvalues of a $\mathsf{GL}(N)$ element g within a $\mathsf{GL}(N)$ conjugacy class. There is the constraint $\prod_i x_i \neq 0$ in force to ensure invertibility. We use the well known hook expansion in terms of the Schur functions identified with irreducible $\mathsf{GL}(N)$ characters:

$$p_n(x_1, \ldots, x_n) = \sum_{a+b+1=n} (-1)^b \{a+1, 1^b\}(x_1, \ldots, x_N) \tag{57}$$

This formula is stable with respect to the limit $N \to \infty$ so that we immediately have in the case of $\mathsf{Char}\text{-}\mathsf{GL}$ the identification

$$p_n = \sum_{a+b+1=n} (-1)^b \{a+1, 1^b\} \,. \tag{58}$$

In branching to orthogonal or the symplectic groups, as one can see from (30) the eigenvalues now generally speaking come in pairs x_k and $\overline{x}_k$ and we can split p_n into at least two parts. In the orthogonal $O(N)$ case, there are four possibilities, and in the symplectic $Sp(N)$ case there are two. Confining attention to the unimodular case it follows from (30) that:

$$\begin{aligned}
p_n(x,\overline{x}) &= p_n(x) + p_n(\overline{x}) && \text{for } \mathsf{SO}(2K)\,; \\
p_n(x,\overline{x},1) &= p_n(x) + p_n(\overline{x}) + 1 && \text{for } \mathsf{SO}(2K+1)\,; \\
p_n(x,\overline{x}) &= p_n(x) + p_n(\overline{x}) && \text{for } \mathsf{Sp}(2K)\,; \\
p_n(x,\overline{x},1) &= p_n(x) + p_n(\overline{x}) + 1 && \text{for } \mathsf{Sp}(2K+1)\,,
\end{aligned} \tag{59}$$

where in each case $p_n(x) = \sum_{i=1}^K x_i^n$ and $p_n(\overline{x}) = \sum_{i=1}^K \overline{x}_i^n$.

Clearly, this is the place where the dimensionality $N = 2K$ or $N = 2K+1$ comes into play. However, this does not prevent us from establishing a result stable in the $K \to \infty$ limit. Indeed we find as a corollary to (58) the result appropriate to $\mathsf{Char}\text{-}\mathsf{O}$:

Corollary 6.1.

$$p_0 = [0] \quad \text{and} \quad p_n = \sum_{a+b+1=n} (-1)^b [a+1, 1^b] + \chi(2|n)[0] \qquad \text{for } n \geq 1\,, \tag{60}$$

where χ is the truth function, so that $\chi(2|n)$ is 1 if n is even and 0 if n is odd. Note that the n in this truth function has to do with the index of the one part orthogonal power sums, and not with the number of its variables! The even power sums $p_{2k}, k \geq 1$ pick up an additional term $[0]$. ■

Proof. Note that for $n = 0, 1$ we can directly verify that $p_0 = [0]$ and $p_1 = [1]$ proving the statement in these cases. Henceforth we assume $n \geq 2$. Recall that C contains partitions $(-1)^{|\gamma|/2}\gamma$ with Frobenius representation $\begin{pmatrix} a_1+1 & \ldots & a_r+1 \\ a_1 & \ldots & a_r \end{pmatrix}$, where only those partitions of Frobenius rank 1, that are of hook shape, can skew the hooks $\{a+1, 1^b\}$. Furthermore recall that the D series partitions δ has only even parts, and of these only the partitions of type $\{2k\}$

can fit into a hook. Thus

$$\begin{aligned}
p_n &= \sum_{a+b+1=n} (-1)^b \{a+1, 1^b\} = \sum_{b\geq 0}^{n-1} (-1)^b \{n-b, 1^b\} \\
&= \sum_{b\geq 0}^{n-1} (-1)^b [(n-b, 1^b)/D] = \sum_{k\geq 0} \sum_{b\geq 0}^{n-1} (-1)^b [(n-b, 1^b)/(2k)] \\
&= \sum_{b\geq 0}^{n-1} (-1)^b [(n-b, 1^b)] + \sum_{k\geq 1} \sum_{b\geq 0}^{n-1} (-1)^b [n-b-2k, 1^b] \\
&= \sum_{b\geq 0}^{n-1} (-1)^b [(n-b, 1^b)] + \sum_{k\geq 1} \sum_{\zeta} (-1)^{|\zeta|} [(n-2k)/\zeta \cdot \zeta'] \\
&= \sum_{b\geq 0}^{n-1} (-1)^b [(n-b, 1^b)] + \sum_{k\geq 1} \delta_{n-2k,0} [0] \\
&= \sum_{b\geq 0}^{n-1} (-1)^b [(n-b, 1^b)] + \chi(2|n)[0] \,, \qquad (61)
\end{aligned}$$

where, in the penultimate line, the structure of the second term has resulted from the antipode property (32). □

We have an exactly analogous result for Char-Sp:

Corollary 6.2.

$$p_0 = \langle 0 \rangle \quad \text{and} \quad p_n = \sum_{a+b+1=n} (-1)^b \langle a+1, 1^b \rangle + \chi(2|n)\langle 0 \rangle \qquad \text{for } n \geq 1 \,. \qquad (62)$$

■

Remarks. We recall that the one part power sums are the primitive elements of the symmetric function Hopf algebra. They form a rational basis of this Hopf algebra. This implies that for both Char-O and Char-Sp, as well as Char-GL, we have:

Proposition 6.3. The one part power sums p_n map to the primitive elements of the Hopf algebra of the universal character rings of GL, O and Sp. That is, in each case we have

$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \,. \qquad (63)$$

■

Proof. This is a trivial consequence of (19), since the isomorphism of Hopf algebras which we have established is independent of the underlying alphabet, and hence does not alter the coproduct properties of the power sums. □

6.2 Complete symmetric functions

In this section we investigate the nature of complete symmetric functions in each of our three rings of universal characters, both directly from the maps between Schur functions and the characters, and then using their definition in terms of power sum symmetric functions, which exhibits certain algebraic features most clearly. The second approach involves an explicit treatment in terms of the group element eigenvalues, that is to say the variables $(x, \overline{x})$ and $(x, \overline{x}, 1)$.

First, our maps allow us to see immediately that, in accordance with the formulae of Table 1, we have

$$\begin{aligned} h_n &= s_n = \{n\}\,, \\ h_n &= s_n = \{n\} = [n/D] = \sum_k [n/(2k)] = \sum_{k=0}^{[n/2]} [n-2k]\,, \\ h_n &= s_n = \{n\} = \langle n/B\rangle = \langle n\rangle, \end{aligned} \tag{64}$$

where $[n/2]$ is the integer part of $n/2$. Moreover, we have

Proposition 6.4. The above images of the one part complete symmetric functions h_n under the maps from the Hopf algebra of symmetric functions to the universal character rings of GL, O and Sp are divided powers [26, 2, 32], their coproducts takes the form:

$$\Delta(h_n) = \sum_r \; h_{n-r} \otimes h_r. \tag{65}$$

■

Proof. These results are a direct consequence of (19), since the maps between the Hopf algebras are isomorphisms, but they can also be derived as follows.

$$\begin{aligned} \Delta(\{n\}) &= \sum_\zeta \{n/\zeta\} \otimes \{\zeta\} = \sum_r \{n/r\} \otimes \{r\} = \sum_r \{n-r\} \otimes \{r\}\,, \\ \Delta([n/D]) &= \sum_\zeta [n/(\zeta D)] \otimes [\zeta/D] = \sum_r [(n/r)/D] \otimes [r/D] = \sum_r [(n-r)/D] \otimes [r/D]\,, \\ \Delta(\langle n\rangle) &= \sum_\zeta \langle n/\zeta\rangle \otimes \langle \zeta/B\rangle = \sum_r \langle n/r\rangle \otimes \langle r/B\rangle = \sum_r \langle n-r\rangle \otimes \langle r\rangle\,. \end{aligned} \tag{66}$$

The expression for the image of h_n in $\mathsf{Char\text{-}O}$ given in (64) may also be derived as follows by considering the finite N case $\mathsf{Char\text{-}O}(N)$. The even and odd dimensional cases need separate treatment and we start with the case $N = 2K$. Consider the chain of subgroups $\mathsf{U}(2K) \supset \mathsf{O}(2K) \supset \mathsf{U}(K)$ and expand the characters according to this branching scheme [16].

$$\begin{aligned} H_t &= \exp(\sum_{r\geq 1} \frac{1}{r} p_r t^r) = \exp(\sum_{r\geq 1} \frac{1}{r}(x_1^r + \ldots x_K^r + \overline{x}_1^r + \ldots \overline{x}_K^r) t^r) \\ &= \prod_{i=1}^K \exp(\sum_{r\geq 1} \frac{1}{r}(x_i t)^r) \exp(\sum_{r\geq 1} \frac{1}{r}(\overline{x}_i t)^r) = \prod_{i=1}^K \exp(-\log(1 - x_i t)) \exp(-\log(1 - \overline{x}_i t)) \\ &= \prod_{i=1}^K \frac{1}{1 - x_i t} \frac{1}{1 - \overline{x}_i t} = \sum_{m\geq 0} \{m\}(x) t^m \sum_{n\geq 0} \{n\}(\overline{x}) t^n = \sum_{m\geq 0} \left(\sum_{r=0}^m \{m-r\}(x)\{\overline{r}\}(x) \right) t^m\,. \end{aligned} \tag{67}$$

Note that the last expression is in terms of $\mathsf{U}(K)$ characters [16]. These satisfy the identity

$$\sum_{m_1+m_2=m} \{m_1\}(x)\, \{\overline{m_2}\}(x) = \sum_{r=0}^{\min(m_1,m_2)} \{\overline{m_2 - r}, m_1 - r\}(x)\,. \tag{68}$$

However the restriction from $\mathsf{O}(2K)$ to $\mathsf{U}(K)$ [3] yields the branching

$$[n](x,\overline{x}) = \{n\}(x) + \{\bar{1}, n-1\}(x) + \ldots + \{\bar{n}(x)\}\,. \tag{69}$$

Combining the last two facts reveals that

$$\sum_{n_1+n_2=n} \{n_1\}(x)\{\overline{n_2}\}(x) = \sum_{r\geq 0}[n-2r](x,\overline{x})\,. \tag{70}$$

Reinserting this into the series for H_t results in

$$H_t = \sum_{n\geq 0}\sum_{r=0}^{[n/2]}[n-2r](x,\overline{x})\,t^n\,. \tag{71}$$

In the odd dimensional case, one has to consider the chain of groups $\mathsf{U}(2K+1) \supset \mathsf{O}(2K+1) \supset \mathsf{O}(2K) \supset \mathsf{U}(K)$. This is due to the additional eigenvalue 1 also appearing in the power sum functions. We furthermore distinguish characters in $\mathsf{O}(2K)$ by adding a subscript e for even, and characters in $\mathsf{O}(2K+1)$ by a subscript o for odd, i.e. $[n]_e$, $[n]_o$.

$$\begin{aligned}
H_t &= \exp(\sum_{r\geq 1}\frac{1}{r}p_r t^r) = \exp(\sum_{r\geq 1}\frac{1}{r}(x_1^r + \ldots x_K^r + \overline{x}_1^r + \ldots \overline{x}_K^r + 1)t^r) \\
&= \prod_{i=1}^{K}\exp(\sum_{r\geq 1}\frac{1}{r}(x_i t)^r)\exp(\sum_{r\geq 1}\frac{1}{r}(\overline{x}_i t)^r)\exp(\sum_{r\geq 1}\frac{1}{r}t^r) \\
&= \prod_{i=1}^{K}\frac{1}{1-x_i t}\prod_{i=1}^{K}\frac{1}{1-\overline{x}_i t}\cdot\frac{1}{1-t} = \sum_{m\geq 0}\{m\}(x)t^m\sum_{n\geq 0}\{\bar{n}\}(x)t^n\sum_{r\geq 0}t^r \\
&= \sum_{s\geq 0}\left(\sum_{r\geq 0}[s-2r]_e(x,\overline{x}))\right)t^s\sum_{m\geq 0}t^m \\
&= \sum_{s\geq 0}\left(\sum_{r\geq 0}\sum_{m\geq 0}[s-2r]_e(x,\overline{x})\right)t^{s+m}\,.
\end{aligned} \tag{72}$$

Now, we have

$$\begin{aligned}
H_t &= \sum_{n\geq 0}[n/D]_o(x,\overline{x},1)\,t^n = \sum_{n\geq 0}[n/(DM)]_e(x,\overline{x})\,t^n = \sum_{n\geq 0}\sum_{m\geq 0}[(n-m)/D]_e(x,\overline{x})\,t^n \\
&= \sum_{n,m,r\geq 0}[n-m-2r]_e(x,\overline{x})\,t^n\,.
\end{aligned} \tag{73}$$

Reindexing and comparing the two last results concludes the proof. □

6.3 Elementary symmetric functions

The elementary symmetric functions e_n map as follows to the three character rings of interest:

$$\begin{aligned}
e_n &= s_{1^n} = \{1^n\}\,, \\
e_n &= s_{1^n} = \{1^n\} = [1^n/D] = [1^n]\,, \\
e_n &= s_{1^n} = \{1^n\} = \langle 1^n/B\rangle = \sum_r \langle 1^{n-2r}\rangle = \sum_{r=0}^{[n/2]}\langle 1^{n-2r}\rangle\,.
\end{aligned} \tag{74}$$

Moreover, we have

Proposition 6.5. The above images of the one part elementary symmetric functions e_n under the maps from the Hopf algebra of symmetric functions to the universal character rings of GL, O and Sp are again divided powers since their coproducts all take the form:

$$\Delta(e_n) = \sum_r e_{n-r} \otimes e_r. \tag{75}$$

■

Proof. These results are a direct consequence of (19), since the maps between the Hopf algebras are isomorphisms, but they can also be derived as follows.

$$\begin{aligned}
\Delta(\{1^n\}) &= \sum_\zeta \{1^n/\zeta\} \otimes \{\zeta\} = \sum_r \{1^n/1^r\} \otimes \{1^r\} = \sum_r \{1^{n-r}\} \otimes \{1^r\}\,, \\
\Delta([1^n]) &= \sum_\zeta [1^n/\zeta] \otimes [\zeta/D] = \sum_r [1^n/1^r] \otimes [1^r/D] = \sum_r [1^{n-r}] \otimes [1^r]\,, \\
\Delta(\langle 1^n/B\rangle) &= \sum_\zeta \langle 1^n/(\zeta B)\rangle \otimes \langle \zeta/B\rangle = \sum_r \langle (1^n/1^r)/B\rangle \otimes \langle 1^r/B\rangle = \sum_r \langle 1^{n-r}/B\rangle \otimes \langle 1^r/B\rangle.
\end{aligned} \tag{76}$$

In terms of variables, for $\mathsf{O}(N)$ we have to distinguish once more the even and odd case. For $N = 2K$ we have

$$\begin{aligned}
E_t &= \prod_{i=1}^K (1 + x_i t) \prod_{i=1}^K (1 + \overline{x}_i t) \\
&= \sum_{n\geq 0} \{1^n\}(x)\, t^n \sum_{m\geq 0} \{1^m\}(\overline{x})\, t^m = \sum_{n\geq 0} \left(\sum_{r=0}^n \{1^{n-r}\}(x)\{\overline{1^r}\}(x) \right) t^n \\
&= \sum_{n\geq 0} [1^n]_e(x, \overline{x}) t^n\,.
\end{aligned} \tag{77}$$

where we have used the branching from $\mathsf{O}(2K)$ to $\mathsf{U}(K)$

$$[1^n]_e(x, \overline{x}) = \{1^n\}(x) + \{\overline{1}\}(x)\{1^{n-1}\}(x) + \ldots + \{\overline{1^n}(x)\}\,. \tag{78}$$

In the odd case we have to add an additional $(1 + t)$ term for the additional eigenvalue 1, which once more leads to the relation between odd and even characters

$$[1^n]_o(x, \overline{x}, 1) = [1^n/M]_e(x, \overline{x}) = [1^n]_e(x, \overline{x}) + [1^{n-1}]_e(x, \overline{x})\,. \tag{79}$$

Hence finally, we have from both cases in the limit $K \to \infty$, the universal character result

$$E_t = \sum_{n\geq 0} [1^n]\, t^n\,. \tag{80}$$

A similar derivation applies in the Sp case. □

7 Adjoints and Foulkes derivatives (skewing)

In this section we consider two maps from the ring of symmetric functions Λ into the ring $\mathsf{End}(\Lambda)$ and their orthogonal and symplectic counter part. These are the operators 'multiplying by a Schur function' and its adjoint 'skewing with a Schur function', which we have used frequently above.

$$\begin{aligned} M, \perp \ : \ & \Lambda \to \mathsf{End}(\Lambda) & & \\ & \{\lambda\} \to M_\lambda & M_\lambda(\{\mu\}) &= c^{\nu}_{\lambda,\mu}\{\nu\}\,, \\ & \{\lambda\} \to \{\lambda\}^\perp & \{\lambda\}^\perp(\{\mu\}) &= c^{\mu}_{\lambda,\nu}\{\nu\} = \{\mu/\lambda\}\,. \end{aligned} \tag{81}$$

In the GL case these two operations are related via the Schur-Hall scalar product

$$\langle M_\mu(\{\nu\}) \mid \{\lambda\}\rangle = \langle \{\mu\}\cdot\{\nu\} \mid \{\lambda\}\rangle = \langle \{\nu\} \mid \{\lambda/\mu\}\rangle = \langle \{\nu\} \mid \{\mu\}^\perp(\{\lambda\})\rangle\,. \tag{82}$$

The adjoint of multiplication by a Schur function with respect to the Schur-Hall scalar product, that is the skew or $\perp$, is called the Foulkes derivative. This can be used to introduce differential operators, for example in Macdonald [24] one finds both

$$p_n^\perp = n\frac{\partial}{\partial p_n} \qquad \text{and} \qquad p_n^\perp = \sum_{r\geq 0} h_r \frac{\partial}{\partial h_{n+r}}\,. \tag{83}$$

This leads to the interesting fact, that the coproduct can be written in terms of the adjoint,

$$\Delta(f) = \sum_\mu s_\mu^\perp(f)\otimes s_\mu = \sum_{\mu,(f)} \epsilon(s_\mu^\perp f_{(1)}) f_{(2)} \otimes s_\mu\,, \tag{84}$$

and fulfils a Leibnitz type formula:

$$s_\lambda^\perp(fg) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu}\, s_\mu^\perp(f)\otimes s_\nu^\perp(g)\,, \tag{85}$$

justifying the name derivative. It is furthermore a rather important fact, that using the identification $\pi_0 = 1$, $\pi_n = M_{p_n}$ and $\pi_{-n} = n\partial/\partial_{p_n}$ one easily checks that these operators generate the **Heisenberg Lie algebra**

$$[\pi_n, \pi_m] = n\delta_{n+m,0}\pi_0\,, \tag{86}$$

closely related to vertex operators and the Witt, and Virasoro algebras used in string theory.

The main point we make in this section is to exemplify that in the branched cases, the notion of *adjoint of multiplication* and that of *Foulkes derivative* are no longer identical. Therefore we need new notation, and we choose to write $s_\lambda^\dagger$ for the adjoint, and keep the $s_\lambda^\perp$ for the Foulkes derivative. Since the orthogonal and symplectic case do not differ essentially, we pack them into a single statement.

Theorem 7.1. The adjoint of multiplication in Char-O (Char-Sp) with respect to the orthogonal (symplectic) Schur-Hall scalar product is given by:

$$\begin{aligned} \langle [\nu] \mid [\mu]^\dagger([\lambda])\rangle_2 &= \langle [\mu]\cdot[\nu] \mid [\lambda]\rangle_2\,; \\ \langle \langle\nu\rangle \mid \langle\mu\rangle^\dagger(\langle\lambda\rangle)\rangle_{11} &= \langle \langle\mu\rangle\cdot\langle\nu\rangle \mid \langle\lambda\rangle\rangle_{11}\,, \end{aligned} \tag{87}$$

with explicit form:

$$[\mu]^\dagger([\lambda]) = [\mu]\cdot[\lambda] = \sum_\zeta [\mu/\zeta\cdot\lambda/\zeta]\,;$$
$$\langle\mu\rangle^\dagger(\langle\lambda\rangle) = \langle\mu\rangle\cdot\langle\lambda\rangle = \sum_\zeta \langle\mu/\zeta\cdot\lambda/\zeta\rangle\,. \tag{88}$$

■

Proof. We compute the left and right hand side of the statement separately, but only for the orthogonal case, the symplectic case is identical.

$$\langle[\mu]\cdot[\nu]\mid[\lambda]\rangle = \sum_\rho \langle[\mu/\rho\cdot\nu/\rho]\mid[\lambda]\rangle = \sum_{\rho,\sigma,\tau,\eta} c^\mu_{\rho\sigma}c^\nu_{\rho\tau}c^\eta_{\sigma\tau}\langle[\eta]\mid[\lambda]\rangle = \sum_{\rho,\sigma,\tau} c^\mu_{\rho\sigma}c^\nu_{\rho\tau}c^\lambda_{\sigma\tau}\,,$$
$$\langle[\nu]\mid[\mu]^\star([\lambda])\rangle = \sum_\rho \langle[\nu]\mid[\mu/\rho\cdot\lambda/\rho]\rangle = \sum_{\rho,\sigma,\tau,\eta} c^\mu_{\rho\sigma}c^\lambda_{\rho\tau}c^\eta_{\sigma\tau}\langle[\nu]\mid[\eta]\rangle = \sum_{\rho,\sigma,\tau} c^\mu_{\rho\sigma}c^\nu_{\sigma\tau}c^\lambda_{\rho\tau}\,. \tag{89}$$

Reindexing and the symmetry $c^\mu_{\rho\sigma} = c^\mu_{\sigma\rho}$ yields equality, proving the statement. □

Remark. We are thus left with the fact, that *multiplication is a selfadjoint* operation in Char-O (Char-Sp) with respect to the orthogonal (symplectic) Schur-Hall scalar product. In terms of group representations this amounts to saying that one can use the second order tensor $g_{i,j} = g_{j,i}$ of symmetry type $\{2\}$ ($f_{i,j} = -f_{j,i}$ of symmetry type $\{1,1\}$) to raise or lower indices. Co- and contra-variant representations are hence isomorphic. □

To find the correct Foulkes derivative, we have to see how the primitives act via the comultiplication and the Schur-Hall scalar product. We define the action of any $a^\perp$ as follows

Definition 7.2. The Foulkes derivative is defined in an invariant way as

$$a^\perp(b) = \langle a\mid b_{(1)}\rangle b_{(2)}\,. \tag{90}$$

■

It is easy to check that this definition is equivalent to the skew in the ordinary Symm-Λ case.

$$s^\perp_\lambda(s_\mu) = \sum_\zeta \langle s_\lambda\mid s_\zeta\rangle\, s_{\mu/\zeta} = \sum_\zeta \delta_{\lambda\zeta}\, s_{\mu/\zeta} = s_{\mu/\lambda}\,. \tag{91}$$

Furthermore, this definition can be written down in any character Hopf algebra where we have defined a Schur-Hall scalar product which represents the orthogonality of irreducible (indecomposable) characters.

It is well known that the above definition defines a derivation if the element a is a primitive element in the dual Hopf algebra [10, 9].

Corollary 7.3. The Foulkes derivatives in the case of Char-GL, Char-O and Char-Sp are given in terms of the appropriate Schur-Hall scalar product by:

$$(s_\lambda)^\perp(s_\mu) = \{\lambda\}^\perp(\{\mu\}) = \langle\{\lambda\}\mid\{\mu_{(1)}\}\rangle\,\{\mu_{(2)}\} = \sum_\zeta \langle\{\lambda\}\mid\{\zeta\}\rangle\,\{\mu/\zeta\}\,,$$
$$(o_\lambda)^\perp(o_\mu) = [\lambda]^\perp([\mu]) = \langle[\lambda]\mid[\mu_{[1]}]\rangle_2\,[\mu_{[2]}] = \sum_\zeta \langle[\lambda]\mid[\zeta/D]\rangle_2\,[\mu/\zeta]\,,$$
$$(sp_\lambda)^\perp(sp_\mu) = \langle\lambda\rangle^\perp(\langle\mu\rangle) = \langle\langle\lambda\rangle\mid\langle\mu_{\langle 1\rangle}\rangle\rangle_{11}\,\langle\mu_{\langle 2\rangle}\rangle = \sum_\zeta \langle\langle\lambda\rangle\mid\langle\zeta/B\rangle\rangle_{11}\,\langle\mu/\zeta\rangle\,. \tag{92}$$

In the case of a primitive element ($m \geq 1$) of Char-O we have

$$\begin{aligned} p_n^\perp(p_m) &= \langle p_n \mid p_m\rangle_2 \, [0] + \langle p_n \mid [0]\rangle_2 \, p_m \\ &= n\delta_{n,m} + \chi(2|m)\chi(2|n) + \chi(2|n)p_m , \end{aligned} \tag{93}$$

with a similar expression for Char-Sp. ■

Proof. The Hopf algebra definition for the Foulkes derivative is basis free, but depends on the scalar product, so that the first statement in each of the equations of (92) is a rephrasing in terms of general linear, orthogonal and symplectic characters. The ensuing expressions then follow from the coproduct formulae of Table 1. In the case of (93) the action of a primitive element yields $\langle p_n \mid p_m\rangle_2$ and $\langle p_n \mid [0]\rangle_2$, both of which we have to evaluate, for example using the hook expansion of the power sums also given in Table 1. For Char-O this yields

$$\begin{aligned} \langle p_n \mid p_m\rangle_2 &= \Big\langle \sum_{a+b+1=n} (-1)^b[a+1,1^b] + \chi(2|n)[0] \Big| \sum_{c+d+1=m} (-1)^d[c+1,1^d] + \chi(2|m)[0] \Big\rangle_2 \\ &= n\,\delta_{n,m} \;+\; \chi(2|n)\chi(2|m) \,. \end{aligned} \tag{94}$$

In addition we have:

$$\langle p_n \mid [0]\rangle_2 = \Big\langle \sum_{a+b+1=n} (-1)^b[a+1,1^b] + \chi(2|n)[0] \Big| [0] \Big\rangle = \chi(2|n) \,. \tag{95}$$

Combining these results gives (93). □

Remark. It is easily shown that the power sum basis is not orthogonal with respect to the orthogonal or symplectic Schur-Hall scalar products. Furthermore, due to the different Hopf algebra structures of H and H^*, the power sums p_n are **not** the primitive elements of H^*. Hence the naive identification $p_n^\perp = n\partial/\partial p_n$ fails to hold. The correct way to introduce such (formal) derivatives would be to detect the primitive elements of H^* and to find their dual basis under the relevant Schur-Hall scalar product. After this identifications one could set up an orthogonal (symplectic) Heisenberg Lie algebra. This is, however, beyond the scope of the present paper and will be discussed elsewhere [13]. □

8 Conclusions and discussion

8.1 On the similarity between Char-O and Char-Sp

Our treatment shows that on the Hopf algebraic side the two character ring Hopf algebras Char-O and Char-Sp behave in exactly the same way. They share the same product structure and differ only in the coproduct where the series D and B are involved. This stems from the fact that the deformation of the product actually depends only on the proper cut part Δ' of the coproduct

$$\Delta'(a) = \Delta(a) - 1 \otimes a - a \otimes 1 \,. \tag{96}$$

Incidentally, the proper cut parts of $\Delta'(\{2\})$ and $\Delta'(\{11\})$ are identical (simply the single term $\{1\} \otimes \{1\}$), producing the same deformation. As shown in [14] this is no longer true for deformations based on tensors of higher degree.

The orthogonal and symplectic character of the underlying group finds its counterpart in the proper definition of the various symmetric function bases. While the primitives look similar, complete and orthogonal symmetric functions differ. This is important for applications in physics, since orthogonal, elementary and power sum symmetric functions can be used to encode partition functions of physical systems [29, 30]. Assuming one has a gas of particles, say atoms or even molecules, having an internal orthogonal or symplectic symmetry, one is naturally led to the bases defined in the previous sections.

8.2 Categorical setting and generalisations

A major motivation for studying the orthogonal and symplectic case was to develop a programme for defining character ring Hopf algebras for a wide class of algebraic subgroups H (definable by a finite set of algebraic equations) of the general linear group GL. We are interested in the character ring Hopf algebras of these centralizer subgroups. Let $\mathsf{Char}\text{-}\mathsf{G}$ denote the infinite dimensional character ring (Hopf algebra) of finite dimensional complex representations of G. This is a Tannakian monoidal tensor category. In [14] we studied the character ring Hopf algebras of H_π subgroups, centralizer subgroups of GL, stabilizing a tensor T^π of Young symmetry π. Since we know how to define the subgroup branchings in the stable limit [11, 12] using the Hopf algebra deformation induced by the branching operation $/M_\pi$ (the analogue of $/D = /M_2$ and $/B = /M_{11}$ for the π case), we can use the following commutative diagram as definition for the map $\mathcal{R} : \mathsf{H} \to \mathsf{Char}\text{-}\mathsf{H}$

$$\begin{array}{ccc} \mathsf{GL} & \xrightarrow{\mathcal{R}} & \mathsf{Char}\text{-}\mathsf{GL} \cong \Lambda \\ \Big\downarrow \supset & & \Big\downarrow /M_\pi \\ \mathsf{H} & \xrightarrow{\mathcal{R}} & \mathsf{Char}\text{-}\mathsf{H} \cong \Lambda \end{array} \qquad (97)$$

Note that the left down-arrow is an epimorphism, the right hand side down-arrow is an isomorphism. This raises serious questions about the domain of the functor $\mathcal{R}$. Indeed there is Nagata's counterexample to Hilbert's 14th problem, showing that $\mathsf{GL}(N)$ has subgroups with non-finitely generated invariant rings. Such pathological groups have to be excluded. We conjecture that under such 'technical assumptions' the map $\mathcal{R}$ is a functor from the category of (algebraic) subgroups of $\mathsf{GL}(\infty)$ into the category of Tannakian categories of finite representations. It should be noted that the character rings are isomorphic to the universal ring of symmetric functions, but as Tannakian categories these objects are nonisomorphic, since this structure encodes further information about the representations and the underlying group. However, we exploit the isomorphism between the various (orthogonal and symplectic in this paper) character rings and provide explicit formulae for the structure maps in the realizations obtained by choosing the irreducible (indecomposable) elements as a basis of the underlying (infinite dimensional free) module.

In [11, 12, 14] the focus was on the branchings, that is on the linear isomorphisms of the underlying modules and the ring structure, via the derivation of generalized Newell-Littlewood product formulae. We make here the stronger claim that we actually have an isomorphism between the Hopf algebras. This has important consequences for the interpretation of our results in physics, since the counit was identified with the vacuum of physical theories [7, 8]. Here we see that it changes under reduction or subduction morphisms.

8.3 Further work

We expect the present work to be generalizable along the lines given in [14], but here we have limited ourselves to the case of weight two plethystic branchings (in the notation of that reference; see also the introductory remarks above), namely, the orthogonal and symplectic classical groups. As examples of *non-generic* subgroups we have encountered here the case of odd symplectic groups. Symplectic groups are usually restricted to be defined in even dimensions $\mathsf{Sp}(2K)$, due to the fact that antisymmetric second rank tensors are invertible in $2K$ dimensions only. In the $2K+1$ case the metric is singular. However, from our point of view, that of Hopf algebras of formal universal characters, we were led inexorably to consider the symplectic analogs of the $\mathsf{O}(2K+1)$ groups as defined by Proctor [27], by adding an additional eigenvalue 1 (or arbitrary x_{2K+1}) to the eigenvalues of the even symplectic form.

Further instances of non-generic subgroups are the Cayley-Klein groups, which were studied long ago in the context of projective geometry by Cayley, Klein and Poincaré amongst others. They include groups acting on real, complex and quaternionic homogeneous spaces. In the real case, they can be seen as a 3^N-fold family of generalised orthogonal groups labelled $O_{\omega_1\omega_2\cdots\omega_N}(N+1)$ where up to scaling, the choice $0, \pm 1$ for each ω_i specifies a member of a hierarchy of generalised contraction limits [15, 28]. The situation for quaternionic cases gives a similar classification of generalised symplectic groups. For both Cayley-Klein groups, and odd symplectic groups, the constructs of the present paper which rely on irreducibility of images under character module isomorphisms, and the availability of the modified scalar product, will possibly fail to hold, if applied naively, due to the fact that the underlying modules are at best indecomposable only. See [14] for further examples of non-classical subgroups arising from higher degree plethystic branchings.

Applications of symmetric function techniques are widespread. We have argued in this paper that it is important to pursue the Hopf algebraic machinery behind the character Hopf algebras, to generalize these techniques to form a powerful tool which can deal with more general sub-symmetries than orthogonal and symplectic ones. We restricted our studies here to the classical cases, but even there novel points arose.

We believe that the general branching scenario is quite universal, and have proposed to take it as a blueprint for quantum field calculations [11, 12]. The present work is preparatory to the study of the character ring Hopf algebras at a (conformal) quantum field level using vertex operator techniques. It should allow the construction of vertex operators for orthogonal, symplectic and possibly more general subgroups of the general linear group GL [13].

Additionally, the present work supports our parallel study of knot invariants in the combinatorial Hopf algebra framework [6], obtained from coloring knots by irreducible (indecomposable) representations of $\mathsf{GL}(\infty)$ centralizer subgroups. It was conjectured in [14], appendix, that the deformations which arise from branchings are actually Hopf algebra deformations in the manner of a Drinfeld twist. The work [6] shows that there is yet more subtle information contained in this deformation, as it is clearly rich enough to encode topological invariants.

Finally we reiterate that the presently developed machinery was obtained by literally re-doing the quantum field theory calculations done in [7, 4], in the context of symmetric functions. We hope to show elsewhere, that the insights gained here can in turn be profitably applied in quantum field theory, clarifying algebraic constructions from a group representation point of view. In particular the non-classical subgroup branchings may lead to new methods allowing the computation of nontrivial, that is non-quadratic, invariants, and offering the possibility of new frameworks for general interacting quantum field theories.

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A The dual Hopf algebra

In this appendix we give explicit formulae for the dual Hopf algebras of the orthogonal and symplectic character Hopf algebras. Since once more the orthogonal and symplectic cases work out similarly, we give only the orthogonal versions. Symplectic versions can be easily obtained by the recipe to change the character brackets $[\,] \rightarrow \langle\,\rangle$ and the series $C, D \rightarrow A, B$.

We had come across the necessity to define a new Schur-Hall scalar product in the deformed cases, and we use here these new scalar products to impose duality.

Definition A.1. Let $\mathrm{mod}-_{[]}$ be the module underlying Char-O. Define the linear dual module $\mathrm{mod}-_{[]^*}$ as

$$\begin{aligned} &[\lambda]^* \in \mathrm{mod}-_{[]^*} \\ &[\lambda]^*([\mu]) = \langle [\lambda] \mid [\mu] \rangle_2 = \delta_{\lambda,\mu} \,. \end{aligned} \tag{98}$$

■

Proposition A.2. The dual Hopf algebra Char-O* $=(\mathrm{mod}-_{[]^*}, \mu, \delta)$ is given by the following structure maps:

$$\begin{array}{lll} \text{product} & & \mu([\mu]^* \otimes [\nu]^*) = [\mu \cdot \nu D]^* \\ \text{unit} & \eta^* : \mathbb{Z} \rightarrow \mathrm{mod}-_{[]^*} & \mu([C]^* \otimes [\mu]^*) = [\mu]^* \\ \text{coproduct} & & \delta([\lambda]^*) = \sum_{\sigma,\zeta} [(\lambda/\sigma) \cdot \zeta]^* \otimes [\sigma \cdot \zeta]^* \\ \text{counit} & \epsilon^* : \mathrm{mod}-_{[]^*} \rightarrow \mathbb{Z} & \epsilon^*([\lambda]^*) = \delta_{\lambda,0} \\ \text{antipode} & & {}^tS([\lambda]^*) = (-1)^{|\lambda|}[\lambda' C D']^* \end{array} \tag{99}$$

■

Proof. We have different opportunities to check these results and decide to use the duality directly. For the **product** we compute

$$\begin{aligned} \mu([\mu]^* \otimes [\nu]^*)([\lambda]) &= \langle [\mu] \otimes [\nu] \mid \Delta([\lambda]) \rangle_2 = \sum_\sigma \langle [\mu] \otimes [\nu] \mid [\lambda/\sigma] \otimes [\sigma/D] \rangle_2 \\ &= \sum_\sigma \langle [\mu] \mid [\lambda/\sigma] \rangle_2 \, \langle [\nu] \mid [\sigma/D] \rangle_2 = \sum_\sigma \langle [\mu \cdot \sigma] \mid [\lambda] \rangle_2 \, \langle [\nu D] \mid [\sigma] \rangle_2 \\ &= \langle [\mu \cdot \nu D] \mid [\lambda] \rangle_2 = [\mu \cdot \nu D]^*([\lambda]) \,. \end{aligned} \tag{100}$$

The **unit** map $\eta^* : \mathbb{Z} \rightarrow \mathrm{mod}-_{[]^*}$ is easily seen to be $\eta^*(1) = [C]^*$ due to

$$\mu([\lambda]^* \otimes [C]^*) = [\lambda C D]^* = [\lambda]^* \tag{101}$$

The **coproduct** is obtained from

$$\begin{aligned}
&(\delta([\lambda]^*))([\mu]\otimes[\nu]) = \langle[\lambda]\mid[\mu]\cdot[\nu]\rangle_2 = \sum_{\zeta}\langle[\lambda]\mid[\mu/\zeta\cdot\nu/\zeta]\rangle_2 \\
&= \sum_{\zeta,\sigma}\langle[\lambda/\sigma]\otimes[\sigma]\mid[\mu/\zeta]\otimes[\nu/\zeta]\rangle_2 = \sum_{\zeta,\sigma}\langle[(\lambda/\sigma)\cdot\zeta]\otimes[\sigma\cdot\zeta]\mid[\mu]\otimes[\nu]\rangle_2 \\
&= \left(\sum_{\zeta,\sigma}[(\lambda/\sigma)\cdot\zeta]^*\otimes[\sigma\cdot\zeta]^*\right)([\mu]\otimes[\nu])\,. \qquad (102)
\end{aligned}$$

The **counit** map $\epsilon^* : \mathrm{mod}-_{[]^*} \to \mathbb{Z}$ is obtained as

$$(\epsilon^*\otimes 1)\delta([\lambda]^*) = \sum_{\zeta,\sigma}\epsilon^*([(\lambda/\sigma)\cdot\zeta]^*)\,[\sigma\cdot\zeta]^* = [\lambda]^*\,. \qquad (103)$$

In order that the single term $[\lambda]^*$ survives, as required, we must have $\epsilon^*([0]^*) = 1$ in the contribution arising from the case $\zeta = 0$ and $\sigma = \lambda$, and $\epsilon^*([(\lambda/\sigma)\cdot\zeta]^*) = 0$ in all other cases, with no term $[\tau]^*$ of $[(\lambda/\sigma)\cdot\zeta]^*$ being zero. This implies that for all ν we must have:

$$\epsilon^*([\nu]^*) = \delta_{\nu,0}\,. \qquad (104)$$

Along the same lines we obtain the explicit form of the **antipode** for the dual Hopf algebra:

$$\begin{aligned}
&{}^tS([\lambda]^*)([\nu]) = [\lambda]^*(S[\nu]) = \langle[\lambda]\mid S([\nu])\rangle_2 \\
&= (-1)^{|\nu|}\langle[\lambda]\mid[\nu'/(C'D)]\rangle_2 = (-1)^{|\nu|}\langle[\lambda DC']\mid[\nu']\rangle_2 = (-1)^{|\lambda|}\langle[\lambda' D' C]\mid[\nu]\rangle_2\,. \qquad (105)
\end{aligned}$$

□

Remark. A dramatic difference between the character ring Hopf algebras for orthogonal and symplectic groups and their dual Hopf algebras is that the product maps of the former are filtered and hence contain only finitely many terms. The dual character ring Hopf algebras, however, have products based on the infinite Schur function series and acquire thereby an infinite number of terms. This parallels the branchings of noncompact real forms of the symplectic groups as studied in [18, 19]. As long as we consider evaluation, for example, of pairings between a character ring Hopf algebra and its dual, the finiteness of the former ensures that only finitely many terms of the dual Hopf algebra product and coproduct formulae are needed and one can study a truncated version.

We add a few (more or less obvious) statements about this structure without explicit proof.

Corollary A.3. The dual Hopf algebra $\mathsf{Char}\text{-}\mathsf{GL}^*$ is connected, that is we have:

$$\delta(\eta^*(1)) = \eta^*(1)\otimes\eta^*(1) \qquad \text{and} \qquad \epsilon^*(\mu([\lambda]^*\otimes[\nu]^*)) = \epsilon^*([\lambda]^*)\,\epsilon^*([\nu]^*)\,. \qquad (106)$$

■

However, note that neither the product μ nor the coproduct δ is graded. The connectedness property allows to conclude that the antipode still is an antialgebra homomorphism (though we are bicommutaive here), that is

$${}^tS(\mu([\lambda]^*\otimes[\rho]^*)) = {}^tS([\rho]^*)\,{}^tS([\lambda]^*)\,. \qquad (107)$$

The fact that the antipode fulfils its defining relation is established by

$$
\begin{aligned}
\mu(1\otimes {}^tS)\delta([\lambda]^*) &= \mu(1\otimes {}^tS)\sum_{\zeta,\sigma}[(\lambda/\sigma)\cdot\zeta]^*\otimes[\sigma\cdot\zeta]^*) \\
&= \mu\ \sum_{\zeta,\sigma}[(\lambda/\sigma)\cdot\zeta]^*\otimes(-1)^{|\sigma|+|\zeta|}[\sigma'\cdot\zeta'\,C\,D']^*) \\
&= \sum_{\zeta,\sigma}(-1)^{|\sigma|+|\zeta|}[(\lambda/\sigma)\cdot\zeta\cdot\sigma'\cdot\zeta'\,C\,D'\,D]^* \\
&= \delta_{\lambda,0}\ \sum_{\zeta}(-1)^{|\zeta|}[\zeta\cdot\zeta'\,B]^* = \delta_{\lambda,0}\,[A\,C\,B]^* = \delta_{\lambda,0}[C]^* = \epsilon^*([\lambda]^*)\eta^*(1)\,,
\end{aligned}
\tag{108}
$$

as required. In the last line use has been made of the antipode identity (32), $CD=1$, $D'=B$, $AB=1$ and the fact that $\sum_\zeta(-1)^{|\zeta|}\,\{\zeta\}\cdot\{\zeta'\}=A\,C$. This last identity can be established by comparing the inverse Cauchy identity with the product of the generating functions for the Schur function series A and C [14].

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