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Convolution of  $hp$ -functions on locally refined  
grids - extended version

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# Convolution of $hp$ -functions on locally refined grids

## Extended version

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### Abstract

Usually, the fast evaluation of a convolution integral  $\int_{\mathbb{R}} f(y)g(x-y)dy$  requires that the functions  $f, g$  have a simple structure based on an equidistant grid in order to apply the fast Fourier transform. Here we discuss the efficient performance of the convolution of  $hp$ -functions in certain locally refined grids. More precisely, the convolution result is projected into some given  $hp$ -space (Galerkin approximation). The overall cost is  $\mathcal{O}(p^2 N \log N)$ , where  $N$  is the sum of the dimensions of the subspaces containing  $f, g$  and the resulting function, while  $p$  is the maximal polynomial degree.

*AMS Subject Classifications:* 44A35, 42A55

*Key words:* convolution integral,  $hp$ -finite elements, non-uniform grids, local refinement

## 1 Introduction

We consider the convolution integral

$$\omega_{\text{exact}}(x) := (f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)dy \quad (1.1)$$

for  $hp$ -functions  $f, g$  of bounded support. We do not compute the exact result  $\omega_{\text{exact}}$ , but its  $L^2$ -orthogonal projection  $\omega := P\omega_{\text{exact}}$  into a certain subspace of  $hp$ -functions.

Convolutions involving a kernel function  $f = k$  occur for instance when integral operators  $Kg(x) = \int_{\mathbb{R}} k(x-y)g(y)dy$  are to be evaluated. In [1] and [2] one finds applications where the convolutions are not derived from integral operators. In the case of integral operators, the kernel function  $k$  is often assumed to satisfy special (smoothness) conditions. This allows various approximations and various methods for its efficient numerical treatment (e.g., [6], [7]). In the rest of the paper we make no assumptions about  $f, g$  except that they belong to certain  $hp$ -subspaces.

The convolution of piecewise constant functions has been considered in [3]. There one finds further comments on the nature of the problem which are not repeated here but also apply in the present case. A variant of the method with “mass conservation” in the piecewise constant case can be found in [5]. The mass conservation holds in general for approaches with polynomial degrees  $p \geq 1$ . The particular case of  $p = 1$  is discussed in [4]. The present article concentrates on the algorithmic aspects when large polynomial degrees appear as it is generally assumed in the  $hp$ -case, where coarser grid sizes are compensated by higher polynomial degrees.

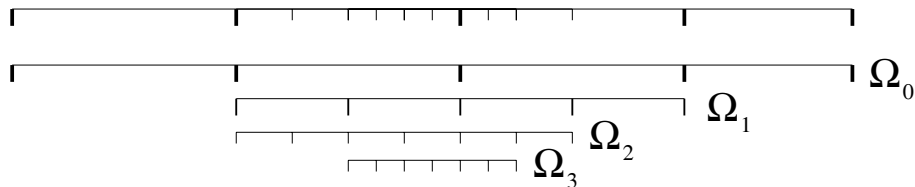


Figure 1.1: Refined grid (first line) composed by local refinements at the levels 0-3 in the zones  $\Omega_\ell$

The  $hp$ -structure is defined as follows. There are nested refinement zones

$$\mathbb{R} \supset \Omega_0 \supset \dots \supset \Omega_{\ell-1} \supset \Omega_\ell \supset \dots \supset \Omega_L \quad (1.2)$$

(cf. Figure 1.1) corresponding to step sizes

$$h_\ell = 2^{-\ell}h \quad (1.3)$$

with a fixed (coarsest) step size  $h = h_0$ . More precisely,  $\Omega_\ell$  must be a nonempty interval consistent with the  $h_\ell$ -mesh, i.e.,

$$\Omega_\ell = [i_{a,\ell}h_\ell, i_{b,\ell}h_\ell] \quad \text{for some } i_{a,\ell}, i_{b,\ell} \in \mathbb{Z}.$$

The nestedness (1.2) can be rewritten as  $i_{a,0}h_0 \leq i_{a,1}h_1 \leq \dots \leq i_{a,L}h_L < i_{b,L}h_L \leq \dots \leq i_{b,1}h_1 \leq i_{b,0}h_0$ . The following definitions can be formulated more easily when we formally introduce the empty refinement zone

$$\Omega_{L+1} := \emptyset$$

of a fictitious level  $L+1$ , since then  $\Omega_L \setminus \Omega_{L+1} = \Omega_L$ .

Let  $\mathcal{M}_\ell$  be the infinite grid of level  $\ell$ ,

$$\mathcal{M}_\ell := \{I_\nu^\ell : \nu \in \mathbb{Z}\} \quad \text{for } \ell \in \mathbb{N}_0, \quad (1.4)$$

containing the intervals

$$I_\nu^\ell := [\nu h_\ell, (\nu+1)h_\ell] \quad \text{for } \nu \in \mathbb{Z}, \ell \in \mathbb{N}_0.$$

The geometric mesh for the  $hp$ -functions consists of the intervals in the set

$$\mathcal{M} := \left\{ I_\nu^\ell \in \mathcal{M}_\ell : I_\nu^\ell \subset \overline{\Omega_\ell \setminus \Omega_{\ell+1}}, 0 \leq \ell \leq L \right\},$$

i.e., inside of  $\Omega_0 \setminus \Omega_1$  the  $h_0$ -mesh is used, inside of  $\Omega_1 \setminus \Omega_2$  the  $h_1$ -mesh,  $\dots$  and finally  $\Omega_L \setminus \Omega_{L+1} = \Omega_L$  is filled with the  $h_L$ -mesh.

Furthermore, we associate to each interval  $I_\nu^\ell \in \mathcal{M}$  with a polynomial degree  $p_\nu^\ell \in \mathbb{N}_0$  and define the space  $\mathcal{S} = \mathcal{S}(\mathcal{M})$  of  $hp$ -functions by all functions  $\psi$  with

$$\psi|_{I_\nu^\ell} \text{ is a polynomial of degree } \leq p_\nu^\ell \quad \text{for all } I_\nu^\ell \in \mathcal{M} \quad (1.5)$$

and  $\psi = 0$  outside of  $\Omega_0$ . Note that no continuity of  $\psi$  is required.

In the standard  $hp$ -version smaller step sizes  $h_\ell$  correlate with lower degree  $p_\nu^\ell$ . Hence the maximal degree  $p_\ell := \max \{p_\nu^\ell : I_\nu^\ell \in \mathcal{M} \cap \mathcal{M}_\ell\}$  per level should decrease:  $p_0 \geq p_1 \geq \dots \geq p_L$ . For the sake of easier notation we introduce

$$p := \max \{p_\nu^\ell : I_\nu^\ell \in \mathcal{M}\} = \max_{0 \leq \ell \leq L} p_\ell \quad (1.6)$$

and allow generally that  $\psi|_{I_\nu^\ell}$  is a polynomial of degree  $\leq p$ .

We allow that the two factors  $f$  and  $g$  of the convolution belong to *different*  $hp$ -spaces characterised by different refinement zones. For the resulting projection  $\omega := P\omega_{\text{exact}}$  ( $\omega_{\text{exact}}$  from (1.1)) a third  $hp$ -space may be defined. We denote these three spaces by the superscripts “ $f, g, \omega$ ”. Therefore, we have to replace the sets  $\Omega_\ell, \mathcal{M}, \mathcal{S}$  by

$$\Omega_\ell^f, \Omega_\ell^g, \Omega_\ell^\omega, \mathcal{M}^f, \mathcal{M}^g, \mathcal{M}^\omega, \mathcal{S}^f, \mathcal{S}^g, \mathcal{S}^\omega, \text{ and } p_i^{\ell,f}, p_i^{\ell,g}, p_i^{\ell,\omega}.$$

Now, the problem can be formulated.

**Problem 1.1** *Given  $f \in \mathcal{S}^f$  and  $g \in \mathcal{S}^g$ , we want to compute the (exact) projection  $\omega = P(f * g)$ , where  $P$  is the  $L^2$ -orthogonal projection onto the subspace  $\mathcal{S}^\omega \subset L^2(\mathbb{R})$ .*

The outline of the paper is as follows.

The  $hp$ -subspaces are defined in Section 2. For this purpose we introduce the basis functions in §2.1. An essential detail is their refinement rule (2.3). §2.2 defines the decomposition of  $f, g$  into contributions  $f_\ell, g_\ell$  at the various refinement levels  $\ell$ . After introducing notations for the projections onto certain subspaces and for the coefficients, the terms  $f_\ell, g_\ell$  are used to represent the convolution  $f * g$  in §2.5. The section is finished by considerations about the discrete convolution of sequences in §2.6.

Section 3 defines the auxiliary coefficients  $\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}, \gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell}, \Gamma_{i,(\alpha,\beta)}^{\ell',\ell}$ , which play a key role in the later algorithms. Also the prolongation operator  $\mathcal{P}$  and the restriction operator  $\mathcal{R}$  from §3.4 will appear in the algorithm.

The convolution algorithm is given in Section 4. Here three different cases appear which are to be treated in different ways.

Section 5 describes how the discrete convolution of infinite sequences has to be handled to obtain an efficient performance. The computational cost is only briefly discussed. The corresponding details can be found in [3].

Section 6 discusses some modifications and extensions.

Appendices A-D contain details about how to compute the involved coefficients efficiently. Furthermore, some identities are proved which are used in the algorithms.

## 2 Spaces

### 2.1 Basis functions

Functions from  $\mathcal{S}$  may be discontinuous at the grid points of the mesh. This fact has the advantage that the basis functions spanning  $\mathcal{S}$  have minimal support (the support is just one interval of  $\mathcal{M}$ ). In the following we discuss the basis derived from Legendre polynomials. Let  $L_\alpha$  be the Legendre polynomial of degree  $\alpha \in \mathbb{N}_0$  defined in  $(-1, 1)$  and normalised such that  $\int_{-1}^1 (L_\alpha(x))^2 dx = 1$  (for details about Legendre polynomials see Appendix A). Then the affine transformation from  $(-1, 1)$  onto  $(0, h)$  leads to

$$\Phi_{0,\alpha}^0(x) := \begin{cases} \sqrt{2/h} L_\alpha(-1 + 2x/h) & \text{if } x \in (0, h), \\ 0 & \text{otherwise.} \end{cases} \quad (2.1a)$$

Translation of  $\Phi_{0,\alpha}^0$  yields the basis functions of level  $\ell = 0$ :

$$\Phi_{i,\alpha}^0(x) := \Phi_{0,\alpha}^0(x - ih) \quad (i \in \mathbb{Z}). \quad (2.1b)$$

For levels  $\ell > 0$  we define

$$\Phi_{0,\alpha}^\ell(x) := 2^{\ell/2} \Phi_{0,\alpha}^0(2^\ell x), \quad \Phi_{i,\alpha}^\ell(x) := \Phi_{0,\alpha}^\ell(x - ih_\ell) = 2^{\ell/2} \Phi_{i,\alpha}^0(2^\ell x) \quad (\alpha, \ell \in \mathbb{N}_0, i \in \mathbb{Z}). \quad (2.1c)$$

Note that  $\text{supp}(\Phi_{i,\alpha}^\ell) = I_i^\ell$ .

The space  $\mathcal{S} = \mathcal{S}(\mathcal{M})$  introduced above has the representation

$$\mathcal{S}(\mathcal{M}) = \text{span} \{ \Phi_{i,\alpha}^\ell : I_i^\ell \in \mathcal{M}, 0 \leq \alpha \leq p_i^\ell \}.$$

With  $p$  from (1.6) we define  $\mathcal{S}_\ell$  as the space of piecewise polynomials of degree  $\leq p$  of level  $\ell$  (on the infinite mesh  $\mathcal{M}_\ell$  from (1.4)):

$$\mathcal{S}_\ell := \text{span} \{ \Phi_{i,\alpha}^\ell : i \in \mathbb{Z}, 0 \leq \alpha \leq p \} \quad (\ell \in \mathbb{N}_0). \quad (2.2)$$

**Remark 2.1** a) For each level  $\ell$ ,  $\{ \Phi_{i,\alpha}^\ell : i \in \mathbb{Z}, \alpha \in \mathbb{N}_0 \}$  is an orthonormal system of functions. Basis functions of different level are not necessarily orthogonal as can be seen from (2.3) below.

b) The functions  $\{ \Phi_{i,\alpha}^\ell : I_i^\ell \in \mathcal{M}, 0 \leq \alpha \leq p_i^\ell \}$  spanning  $\mathcal{S}(\mathcal{M})$  are orthonormal. In this case, basis functions from different levels are orthogonal, since their supports are disjoint<sup>1</sup>:  $I_i^\ell, I_{i'}^{\ell'} \in \mathcal{M}$  with  $\ell \neq \ell'$  implies  $I_i^\ell \cap I_{i'}^{\ell'} = \emptyset$ .

The spaces  $\mathcal{S}_\ell$  are nested, i.e.,

$$\mathcal{S}_\ell \subset \mathcal{S}_{\ell+1}.$$

In particular,  $\Phi_{i,\varkappa}^\ell$  can be represented by means of  $\Phi_{j,\alpha}^{\ell+1}$ . Appendix B contains the representation formula (B.3), which is

$$\Phi_{i,\varkappa}^\ell = \sum_{q=0}^{\varkappa} \xi_{\varkappa,q} \left( (-1)^{\varkappa+q} \Phi_{2i,q}^{\ell+1} + \Phi_{2i+1,q}^{\ell+1} \right). \quad (2.3)$$

The coefficients  $\xi_{\varkappa,\alpha}$  are independent of  $i$  and  $\ell$  and can easily be computed as shown in the Appendix B. There also concrete values are listed.

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<sup>1</sup>The true supports are the closed intervals  $\overline{I_i^\ell}, \overline{I_{i'}^{\ell'}}$  which are not necessarily disjoint. What we mean by “disjoint supports” is that the intersection of the supports has measure zero.

## 2.2 Representations of $f \in \mathcal{S}^f$ and $g \in \mathcal{S}^g$

Following the definition of  $\mathcal{S}^f$ , we have  $\mathcal{S}^f = \text{span}\{\Phi_{i,\varkappa}^\ell : I_i^\ell \in \mathcal{M}^f, 0 \leq \varkappa \leq p_i^\ell\}$ . We can decompose the set  $\mathcal{M}^f$  into different levels:  $\mathcal{M}^f = \bigcup_{\ell=0}^{L^f} \mathcal{M}_\ell^f$ , where  $\mathcal{M}_\ell^f := \mathcal{M}^f \cap \mathcal{M}_\ell$ . This gives rise to the related index set

$$\mathcal{I}_\ell^f := \left\{ i \in \mathbb{Z} : I_i^\ell \in \mathcal{M}_\ell^f \right\} = \left\{ i \in \mathbb{Z} : I_i^\ell \subset \overline{\Omega_\ell \setminus \Omega_{\ell+1}} \right\}$$

and to the corresponding decomposition

$$\mathcal{S}^f = \bigcup_{\ell=0}^{L^f} \mathcal{S}_\ell^f \quad \text{with } \mathcal{S}_\ell^f = \text{span} \left\{ \Phi_{i,\varkappa}^\ell : i \in \mathcal{I}_\ell^f, 0 \leq \varkappa \leq p_i^\ell \right\}. \quad (2.4)$$

Here,  $L^f$  is the largest level  $\ell$  with  $\mathcal{M}_\ell^f \neq \emptyset$ .

The following computation uses the representation<sup>2</sup>

$$f = \sum_{\ell=0}^{L^f} f_\ell, \quad f_\ell = \sum_{i \in \mathcal{I}_\ell^f} \sum_{\varkappa=0}^p f_{i,\varkappa}^\ell \Phi_{i,\varkappa}^\ell \in \mathcal{S}_\ell^f \subset \mathcal{S}_\ell, \quad (2.5)$$

( $p$  from (1.6)) and, similarly,

$$g = \sum_{\ell=0}^{L^g} g_\ell, \quad g_\ell = \sum_{i \in \mathcal{I}_\ell^g} \sum_{\varkappa=0}^p g_{i,\varkappa}^\ell \Phi_{i,\varkappa}^\ell \in \mathcal{S}_\ell^g \subset \mathcal{S}_\ell, \quad (2.6)$$

for the factors  $f, g$  of the convolution.

## 2.3 Projection $P_\ell$

The  $L^2$ -orthogonal projection  $P_\ell$  onto  $\mathcal{S}_\ell$  from (2.2) is defined by

$$P_\ell \psi := \sum_{i \in \mathbb{Z}} \sum_{\varkappa=0}^p \langle \psi, \Phi_{i,\varkappa}^\ell \rangle \Phi_{i,\varkappa}^\ell \quad (2.7)$$

where<sup>3</sup>

$$\langle \varphi, \psi \rangle := \int_{\mathbb{R}} \varphi(x) \psi(x) dx.$$

Here, the orthonormality of the system  $\{\Phi_{i,\varkappa}^\ell\}$  is used. Hence, the projected function  $\psi_\ell := P_\ell \psi \in \mathcal{S}_\ell$  has the representation  $\sum_{i \in \mathbb{Z}} \sum_{\varkappa=0}^p \psi_{i,\varkappa}^\ell \Phi_{i,\varkappa}^\ell$  with the coefficients  $\psi_{i,\varkappa}^\ell = \langle \Phi_{i,\varkappa}^\ell, \psi \rangle$ .

The relation between the projections  $P_\ell$  and  $P$  is given by

$$(P\varphi)|_{\Omega_\ell^\omega \setminus \Omega_{\ell+1}^\omega} = (P_\ell \varphi)|_{\Omega_\ell^\omega \setminus \Omega_{\ell+1}^\omega} \quad \text{for } 0 \leq \ell \leq L^\omega.$$

## 2.4 Compact notation of the coefficients

Functions  $\psi_\ell \in \mathcal{S}_\ell$  are either explicitly given as functions in  $\mathcal{S}_\ell$  or are produced as projection  $\psi_\ell = P_\ell \psi \in \mathcal{S}_\ell$  of a general function  $\psi \in L^2(\mathbb{R})$ . The coefficients  $\psi_{i,\varkappa}^\ell$  of  $\psi_\ell = \sum_{i \in \mathbb{Z}} \sum_{\varkappa=0}^p \psi_{i,\varkappa}^\ell \Phi_{i,\varkappa}^\ell$  have position indices  $i \in \mathbb{Z}$  and degree indices  $\varkappa = 0, \dots, p$ . First we form the infinite sequences

$$\boldsymbol{\psi}_{\ell,\varkappa} = (\psi_{i,\varkappa}^\ell)_{i \in \mathbb{Z}} \quad (0 \leq \varkappa \leq p) \quad (2.8a)$$

and next we build the  $(p+1)$ -tuples

$$\boldsymbol{\psi}[\ell] := (\boldsymbol{\psi}_{\ell,\alpha})_{\alpha=0}^p = \left( (\psi_{i,\alpha}^\ell)_{i \in \mathbb{Z}} \right)_{\alpha=0}^p. \quad (2.8b)$$

<sup>2</sup>For the sake of a simple notation, the summation over the polynomial degree is always from 0 to  $p$ . The definition of  $\mathcal{S}_\ell^f$  implies that  $f_{i,\varkappa}^\ell = 0$  for  $\varkappa > p_i^\ell$ .

<sup>3</sup>Here, real-valued functions are assumed. Extensions to complex-valued ones are obvious.

Depending on the name of the function, the letter  $\psi$  may change, but the collected data  $\boldsymbol{\psi}_{\ell, \varkappa}$  and  $\boldsymbol{\psi}[\ell]$  are always written in bold face.

Particular examples of  $\boldsymbol{\psi}_\ell \in \mathcal{S}_\ell$  are the functions  $f_\ell, g_\ell$  from (2.5) and (2.6). They give rise to the data

$$\mathbf{f}_{\ell, \varkappa}, \mathbf{f}[\ell], \mathbf{g}_{\ell, \varkappa}, \mathbf{g}[\ell].$$

The representation (2.7) defines the mapping

$$\mathbf{P}_\ell : \psi \in L^2(\mathbb{R}) \mapsto \mathbf{P}_\ell(\psi) := \boldsymbol{\psi}[\ell] = \left( (\psi_{i, \varkappa}^\ell)_{i \in \mathbb{Z}} \right)_{\varkappa=0}^p \quad \text{with } \psi_{i, \varkappa}^\ell = \langle \psi, \Phi_{i, \varkappa}^\ell \rangle, \quad (2.8c)$$

i.e.,  $\mathbf{P}_\ell(\psi)$  are the coefficients of  $P_\ell \psi \in \mathcal{S}_\ell$  in the basis at level  $\ell$ . Note that  $\mathbf{P}_\ell : \psi \mapsto \boldsymbol{\psi}[\ell]$  is bijective for all  $\psi \in \mathcal{S}_\ell$ .

## 2.5 Decomposition into levels

We use the decomposition into scales expressed by  $f = \sum_{\ell'=0}^{L^f} f_{\ell'}$  and  $g = \sum_{\ell=0}^{L^g} g_\ell$  (see (2.5) and (2.6)). The convolution  $f * g$  can be written as

$$f * g = \sum_{\ell'=0}^{L^f} \sum_{\ell=0}^{L^g} f_{\ell'} * g_\ell.$$

Since the convolution is symmetric, we can rewrite the sum as

$$f * g = \sum_{\ell' \leq \ell} f_{\ell'} * g_\ell + \sum_{\ell < \ell'} g_\ell * f_{\ell'}, \quad (2.9)$$

where  $\ell', \ell$  are restricted to the level intervals  $0 \leq \ell' \leq L^f, 0 \leq \ell \leq L^g$ .

The projection  $P(f * g)$  onto the  $hp$ -space  $\mathcal{S}^\omega$  has the representation

$$P(f * g) = \sum_{\ell''=0}^{L^\omega} \sum_{i \in \mathcal{I}_{\ell''}^\omega} \sum_{\alpha=0}^p \omega_{i, \alpha}^{\ell''} \Phi_{i, \alpha}^{\ell''}. \quad (2.10)$$

Hence, we have to compute

$$\omega_{i, \alpha}^{\ell''} = \left\langle f * g, \Phi_{i, \alpha}^{\ell''} \right\rangle \quad \text{for all } i \in \mathcal{I}_{\ell''}^\omega, 0 \leq \ell'' \leq L^\omega, 0 \leq \alpha \leq p. \quad (2.11)$$

In the following, we do not compute  $\left\langle f * g, \Phi_{i, \alpha}^{\ell''} \right\rangle$  directly, but  $\left\langle \sum_{\ell' \leq \ell} f_{\ell'} * g_\ell, \Phi_{i, \alpha}^{\ell''} \right\rangle$  involving the first term in (2.9). Analogously, we can compute the second part  $\left\langle \sum_{\ell < \ell'} g_\ell * f_{\ell'}, \Phi_{i, \alpha}^{\ell''} \right\rangle$ . Together we get the values from (2.11). The reason for the decomposition is that we make use of the fact that the level index of the first factor in the convolution is not exceeding the level index of the second factor ( $\ell' \leq \ell$  in the first term and  $\ell < \ell'$  in the second one). Note that for the treatment of  $\left\langle \sum_{\ell < \ell'} g_\ell * f_{\ell'}, \Phi_{i, \alpha}^{\ell''} \right\rangle$  we only have to interchange the roles of the symbols  $f$  and  $g$  and to omit the cases  $\ell = \ell'$ .

## 2.6 Discrete convolution

Let  $\mathbf{a} = (a_i)_{i \in \mathbb{Z}}$  and  $\mathbf{b} = (b_i)_{i \in \mathbb{Z}}$  be two infinite sequences. Then the *discrete convolution* is defined by

$$\mathbf{c} := \mathbf{a} * \mathbf{b} \quad \text{with } c_i = \sum_{j \in \mathbb{Z}} a_j b_{i-j}. \quad (2.12)$$

In our applications, the supports<sup>4</sup> of  $\mathbf{a}, \mathbf{b}$  are bounded. Furthermore, the resulting coefficients  $c_i$  are only of interest for  $i$  in a certain interval. Then the computation of (the interesting part of)  $\mathbf{c}$  can be performed by the Fast Fourier Transform (see the extensive discussion in [3]). The truncation of  $\mathbf{a} * \mathbf{b}$  to the desired range of indices will be discussed again in §5.

In (3.6) we will define an extended version of the discrete convolution.

Before we present the solution algorithm in Section 4, we introduce some required quantities in the next Section.

<sup>4</sup>The support of a sequence  $a = (a_i)_{i \in \mathbb{Z}}$  is the index subset  $\{i \in \mathbb{Z} : a_i \neq 0\}$ .

### 3 Auxiliary coefficients

#### 3.1 $\gamma$ -Coefficients

For level numbers  $\ell'', \ell', \ell \in \mathbb{N}_0$ , position indices  $i, j, k \in \mathbb{Z}$ , and degrees  $0 \leq \alpha, \beta, \varkappa \leq p$  we define

$$\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell} := \iint \Phi_{i,\alpha}^{\ell''}(x) \Phi_{j,\beta}^{\ell'}(y) \Phi_{k,\varkappa}^{\ell}(x-y) dx dy = \left\langle \Phi_{i,\alpha}^{\ell''}, \Phi_{j,\beta}^{\ell'} * \Phi_{k,\varkappa}^{\ell} \right\rangle$$

(all integrations over  $\mathbb{R}$ ). The fundamental role of these coefficients can be seen from the next statement.

The coefficients  $\omega_{i,\alpha}^{\ell''}$  of the projected function

$$\omega_{\ell''} = P_{\ell''}(f_{\ell'} * g_{\ell}) = \sum_{i \in \mathbb{Z}} \sum_{\alpha=1}^p \omega_{i,\alpha}^{\ell''} \Phi_{i,\alpha}^{\ell''} \quad (3.1a)$$

result from

$$\omega_{i,\alpha}^{\ell''} = \sum_{j,k \in \mathbb{Z}} \sum_{\beta=0}^p f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}. \quad (3.1b)$$

The *proof* of the basic identity (3.1a,b) follows by inserting the representations (2.5) and (2.6) of  $f_{\ell'}$  and  $g_{\ell}$  into  $\omega_{\text{exact}} := f_{\ell'} * g_{\ell}$  which yields

$$\omega_{\text{exact}} = \left( \sum_{j \in \mathcal{I}_{\ell'}^f} \sum_{\beta=0}^p f_{j,\beta}^{\ell'} \Phi_{j,\beta}^{\ell'} \right) * \left( \sum_{k \in \mathcal{I}_{\ell}^g} \sum_{\varkappa=0}^p g_{k,\varkappa}^{\ell} \Phi_{k,\varkappa}^{\ell} \right) = \sum_{j,k \in \mathbb{Z}} \sum_{\beta,\varkappa=0}^p f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \Phi_{j,\beta}^{\ell'} * \Phi_{k,\varkappa}^{\ell}.$$

Here,  $f_{j,\beta}^{\ell'} = g_{k,\varkappa}^{\ell} = 0$  holds for all  $j, k$  not belonging to  $\mathcal{I}_{\ell'}^f$  or  $\mathcal{I}_{\ell}^g$  respectively. Since the coefficients  $\omega_{i,\alpha}^{\ell''}$  of  $P_{\ell''}(\omega_{\text{exact}})$  are given by  $\omega_{i,\alpha}^{\ell''} = \left\langle \Phi_{i,\alpha}^{\ell''}, \omega_{\text{exact}} \right\rangle$ , we obtain the result (3.1b).

#### 3.2 Simplified $\gamma$ -coefficients

For levels  $\ell, \ell', \ell''$  with  $\ell \geq \max\{\ell', \ell''\}$  we define

$$\gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell} := \iint \Phi_{0,\alpha}^{\ell''}(x) \Phi_{0,\beta}^{\ell'}(y) \Phi_{\nu,\varkappa}^{\ell}(x-y) dx dy \quad (\nu \in \mathbb{Z}, 0 \leq \alpha, \beta, \varkappa \leq p). \quad (3.2)$$

Under the condition  $\ell \geq \max\{\ell', \ell''\}$ , it suffices to use the quantities  $\gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell}$  from (3.2) as can be seen from Lemma C.2:

$$\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell} = \gamma_{k-i2^{\ell-\ell'}+j2^{\ell-\ell''},(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell} \quad \text{for } \ell \geq \max\{\ell', \ell''\}. \quad (3.3)$$

We will see that we need to determine the coefficients  $\gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell}$  only for the cases  $\ell'' = \ell' = \ell = 0$ ,  $-1 \leq \nu \leq 0$ ,  $0 \leq \alpha \leq \beta \leq \varkappa \leq p$ . All other index combinations can be derived from these data. Appendix C shows how these coefficients can be determined. Furthermore, concrete values are given there.

#### 3.3 $\Gamma$ -coefficients

Consider the convolution  $f_{\ell'} * g_{\ell}$  for  $\ell' < \ell$ . As long as the two factors belong to different step sizes, the Fast Fourier Transform cannot be applied. The purpose of the following  $\Gamma$ -coefficients is to transport the  $\mathbf{g}[\ell]$  data into data at level  $\ell' < \ell$  corresponding to a coarser step size in order to perform the discrete convolution with  $\mathbf{f}[\ell']$  in  $\Omega_{\ell'} \setminus \Omega_{\ell}$ . The key result will be given in Lemma 3.1.

The alternative would be to use the prolongation from §3.4 for transferring the  $\mathbf{f}[\ell']$  data to level  $\ell$  and convolve with  $\mathbf{g}[\ell]$ . This, however, would increase the data size of  $\mathbf{f}[\ell']$  (and the computational cost) by the factor  $2^{\ell-\ell'}$ , although subsequently the results at level  $\ell$  must be restricted to much fewer data at level  $\ell'$ .

We define

$$\Gamma_{i,(\alpha,\beta)}^{\ell',\ell} := \sum_{k \in \mathbb{Z}} \sum_{\varkappa=0}^p g_{k,\varkappa}^{\ell} \gamma_{k-i2^{\ell-\ell'},(\alpha,\beta,\varkappa)}^{\ell',\ell',\ell}. \quad (3.4)$$



Analogously to (2.8a,b) we collect the single coefficients  $\Gamma_{i,(\alpha,\beta)}^{\ell',\ell}$  in sequences  $\mathbf{\Gamma}_{\ell',\ell,(\alpha,\beta)}$  and tuples  $\mathbf{\Gamma}_\ell[\ell']$ :

$$\mathbf{\Gamma}_{\ell',\ell,(\alpha,\beta)} := \left( \Gamma_{i,(\alpha,\beta)}^{\ell',\ell} \right)_{i \in \mathbb{Z}}, \quad \mathbf{\Gamma}_\ell[\ell'] := \left( \mathbf{\Gamma}_{\ell',\ell,(\alpha,\beta)} \right)_{\alpha,\beta=0}^p \quad \text{for } \ell \geq \ell'. \quad (3.4^*)$$

The contributions of  $g_\ell$  in  $\Gamma_{i,(\alpha,\beta)}^{\ell',\ell}$  for  $\ell' \leq \ell \leq L^g$  are summed up in

$$\mathbf{\Gamma}_{i,(\alpha,\beta)}^{\ell'} := \sum_{\ell=\ell'}^{L^g} \Gamma_{i,(\alpha,\beta)}^{\ell',\ell} \quad \text{for all } 0 \leq \ell' \leq L^g \text{ and } 0 \leq \alpha, \beta \leq p. \quad (3.5)$$

Again these data are collected in

$$\mathbf{\Gamma}_{\ell',(\alpha,\beta)} := \left( \mathbf{\Gamma}_{i,(\alpha,\beta)}^{\ell'} \right)_{i \in \mathbb{Z}}, \quad \mathbf{\Gamma}[\ell'] := \left( \mathbf{\Gamma}_{\ell,(\alpha,\beta)} \right)_{\alpha,\beta=0}^p = \sum_{\ell=\ell'}^{L^g} \mathbf{\Gamma}_\ell[\ell']. \quad (3.5^*)$$

Let the coefficients  $\boldsymbol{\psi}[\ell'] := \left( \boldsymbol{\psi}_{\ell',\beta} \right)_{\beta=0}^p$  belong to a function  $\psi_{\ell'} \in \mathcal{S}_{\ell'}$ , i.e.,  $\boldsymbol{\psi}[\ell'] = P_{\ell'}(\psi_{\ell'})$ . Then we define an extended version of the discrete convolution by

$$\boldsymbol{\psi}[\ell'] * \mathbf{\Gamma}[\ell'] := \left( \sum_{\beta=0}^p \boldsymbol{\psi}_{\ell',\beta} * \mathbf{\Gamma}_{\ell',(\alpha,\beta)} \right)_{\alpha=0}^p \quad \text{and} \quad \boldsymbol{\psi}[\ell'] * \mathbf{\Gamma}_\ell[\ell'] := \left( \sum_{\beta=0}^p \boldsymbol{\psi}_{\ell',\beta} * \mathbf{\Gamma}_{\ell,(\alpha,\beta)} \right)_{\alpha=0}^p. \quad (3.6)$$

**Lemma 3.1** *Let  $\psi_{\ell'} \in \mathcal{S}_{\ell'}$  and  $\boldsymbol{\psi}[\ell'] = P_{\ell'}(\psi_{\ell'})$ , i.e.,  $\psi_{\ell'} = \sum_{i \in \mathbb{Z}} \sum_{\beta=0}^p \psi_{i,\beta}^{\ell'} \Phi_{i,\beta}^{\ell'}$ . Then<sup>5</sup>*

$$\boldsymbol{\psi}[\ell'] * \mathbf{\Gamma}_\ell[\ell'] = \mathbf{P}_{\ell'}(\psi_{\ell'} * g_\ell) \quad \text{for } \ell \geq \ell',$$

while the convolution with  $\mathbf{\Gamma}[\ell']$  from (3.5\*) yields

$$\boldsymbol{\psi}[\ell'] * \mathbf{\Gamma}[\ell'] = \mathbf{P}_{\ell'} \left( \psi_{\ell'} * \sum_{\ell=\ell'}^{L^g} g_\ell \right).$$

*Proof.* a) The second statement follows from the first one together with the last identity in (3.5\*).

b) Let  $\mathbf{c}[\ell'] = \boldsymbol{\psi}[\ell'] * \mathbf{\Gamma}_\ell[\ell']$ . Following definition (3.6), its coefficients are

$$\begin{aligned} c_{i,\alpha}^{\ell'} &= \sum_{j \in \mathbb{Z}} \sum_{\beta=0}^p \psi_{j,\beta}^{\ell'} \Gamma_{i-j,(\alpha,\beta)}^{\ell',\ell} \stackrel{(3.4)}{=} \sum_{j \in \mathbb{Z}} \sum_{\beta=0}^p \psi_{j,\beta}^{\ell'} \sum_{k \in \mathbb{Z}} \sum_{\varkappa=0}^p g_{k,\varkappa}^\ell \gamma_{k-(i-j)2^{\ell-\ell'},(\alpha,\beta,\varkappa)}^{\ell',\ell',\ell} \stackrel{(3.3)}{=} \\ &= \sum_{j,k \in \mathbb{Z}} \sum_{\beta,\varkappa=0}^p \psi_{j,\beta}^{\ell'} g_{k,\varkappa}^\ell \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell',\ell',\ell}. \end{aligned}$$

(3.1a,b) shows that  $c_{i,\alpha}^{\ell'}$  are the coefficients of the function  $P_{\ell'}(\psi_{\ell'} * g_\ell)$  at level  $\ell'$  ( $\ell'' = \ell'$  in (3.1a,b)), i.e.,  $\mathbf{c}[\ell'] = \mathbf{P}_{\ell'}(\psi_{\ell'} * g_\ell)$ .  $\blacksquare$

The computation of the  $\Gamma$ -quantities is based on the following two identities which are proved in Appendix D (Lemma D.2):

$$\Gamma_{i,(\alpha,\beta)}^{\ell,\ell} = \sum_{\varkappa=0}^p \left( g_{i,\varkappa}^\ell + (-1)^{\alpha+\beta+\varkappa} g_{i-1,\varkappa}^\ell \right) \gamma_{0,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} \quad \text{for all } i \in \mathbb{Z}, 0 \leq \alpha, \beta \leq p, \quad (3.7a)$$

$$\begin{aligned} \Gamma_{i,(\alpha,\beta)}^{\ell',\ell} &= \sum_{p=0}^{\alpha} \sum_{q=0}^{\beta} \xi_{\alpha,p} \xi_{\beta,q} \left( (-1)^{\alpha+p} \Gamma_{2i-1,(p,q)}^{\ell'+1,\ell} + \left( 1 + (-1)^{\alpha+\beta+p+q} \right) \Gamma_{2i,(p,q)}^{\ell'+1,\ell} + (-1)^{\beta+q} \Gamma_{2i+1,(p,q)}^{\ell'+1,\ell} \right) \\ &\quad \text{for } 0 \leq \ell' < \ell. \end{aligned} \quad (3.7b)$$

<sup>5</sup>On the left-hand side  $*$  is the discrete convolution of sequences, whereas on the right-hand side  $*$  is the convolution of functions.

(3.7a,b) can be reformulated as

$$\mathbf{\Gamma}_\ell[\ell] = \Lambda_\ell(\mathbf{g}[\ell]), \quad \mathbf{\Gamma}_\ell[\ell'] = \Lambda(\mathbf{\Gamma}_\ell[\ell' + 1]), \quad (3.7c)$$

with  $\Lambda_\ell$  and  $\Lambda$  defined by

$$\Lambda_\ell(g[\ell]) := \left( \left( \sum_{\varkappa=0}^p \left( g_{i,\varkappa}^\ell + (-1)^{\alpha+\beta+\varkappa} g_{i-1,\varkappa}^\ell \right) \gamma_{0,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} \right)_{i \in \mathbb{Z}} \right)_{\alpha,\beta=0}^p \quad (3.7a^*)$$

and

$$\Lambda(\Xi[\lambda]) := \left( \left( \sum_{p=0}^{\alpha} \sum_{q=0}^{\beta} \xi_{\alpha,p} \xi_{\beta,q} \left( (-1)^{\alpha+p} \Xi_{2i-1,(p,q)}^\lambda + \left( 1 + (-1)^{\alpha+\beta+p+q} \right) \Xi_{2i,(p,q)}^\lambda + (-1)^{\beta+q} \Xi_{2i+1,(p,q)}^\lambda \right) \right)_{i \in \mathbb{Z}} \right)_{\alpha,\beta=0}^p \quad (3.7b^*)$$

for an argument  $\Xi[\lambda] = \left( (\Xi_{i,(\alpha,\beta)}^\lambda)_{i \in \mathbb{Z}} \right)_{\alpha,\beta=0}^p$ .

### 3.4 Prolongation $\mathcal{P}$ and restriction $\mathcal{R}$

The mapping  $\mathcal{P} : \psi[\ell] \mapsto \psi[\ell + 1]$  is defined as follows. The coefficients  $\psi[\ell]$  define the function  $\psi_\ell = \sum_{j \in \mathbb{Z}} \sum_{\beta=0}^p \psi_{j,\beta}^\ell \Phi_{j,\beta}^\ell \in \mathcal{S}_\ell$ . Since  $\mathcal{S}_\ell \subset \mathcal{S}_{\ell+1}$ , the function  $\psi_\ell$  belongs also to  $\mathcal{S}_{\ell+1}$ . The corresponding coefficients at level  $\ell + 1$  are  $\psi[\ell + 1] := \mathbf{P}_{\ell+1}(\psi_\ell)$ . A compact characterisation is given by

$$\mathcal{P}(\mathbf{P}_\ell(\psi)) = \mathbf{P}_{\ell+1}(\psi) \quad \text{for all functions } \psi \in \mathcal{S}_\ell. \quad (3.8a)$$

For the concrete description one has to study the action of  $\mathcal{P}(\mathbf{P}_\ell(\psi))$  for the basis functions  $\psi = \Phi_{j,\beta}^\ell \in \mathcal{S}_\ell$ . The answer is given by (2.3), from which one concludes that

$$\mathcal{P}(\psi[\ell]) = \psi[\ell + 1] \quad \text{with } \psi_{2j,\beta}^{\ell+1} := \sum_{\alpha=\beta}^p \psi_{j,\alpha}^\ell \xi_{\alpha,\beta} (-1)^{\alpha+\beta} \quad \text{and } \psi_{2j+1,\beta}^{\ell+1} := \sum_{\alpha=\beta}^p \psi_{j,\alpha}^\ell \xi_{\alpha,\beta}. \quad (3.8b)$$

The adjoint operation is the coarsening operator  $\mathcal{R}$  which maps the coefficients  $\omega[\ell] = \mathbf{P}_\ell(\sigma)$  of a projected function  $\sigma$  into the coefficients  $\omega[\ell - 1] = \mathbf{P}_{\ell-1}(\sigma)$  of the projection at the coarser level  $\ell - 1$ :

$$\mathcal{R}(\mathbf{P}_\ell(\sigma)) = \mathbf{P}_{\ell-1}(\sigma) \quad \text{for any function } \sigma \in L^2(\mathbb{R}). \quad (3.9a)$$

Since  $\omega_{i,\alpha}^{\ell-1} = \langle \Phi_{i,\alpha}^{\ell-1}, \sigma \rangle$ , the identity (2.3) proves

$$\mathcal{R}(\omega[\ell]) = \omega[\ell - 1] \quad \text{with } \omega_{i,\alpha}^{\ell-1} := \sum_{\varkappa=0}^{\alpha} \xi_{\alpha,\varkappa} \left( (-1)^{\varkappa+\alpha} \omega_{2i,\varkappa}^\ell + \omega_{2i+1,\varkappa}^\ell \right). \quad (3.9b)$$

## 4 Algorithm

In (3.1a) three level numbers  $\ell''$ ,  $\ell'$ ,  $\ell$  appear. Without loss of generality  $\ell' \leq \ell$  holds. In the following we have to distinguish the following three cases:

- (A)  $\ell'' \leq \ell' \leq \ell$ ,
- (B)  $\ell' < \ell'' \leq \ell$ ,
- (C)  $\ell' \leq \ell < \ell''$ .

The reason is that the projection  $P_{\ell''} \left( \sum_{\ell' \leq \ell} f_{\ell'} * g_\ell \right)$  (see first term in (2.9)) is split into three terms

$$P_{\ell''} \left( \sum_{\ell', \ell \text{ with } \ell'' \leq \ell' \leq \ell} f_{\ell'} * g_\ell \right) + P_{\ell''} \left( \sum_{\ell', \ell \text{ with } \ell' < \ell'' \leq \ell} f_{\ell'} * g_\ell \right) + P_{\ell''} \left( \sum_{\ell', \ell \text{ with } \ell' \leq \ell < \ell''} f_{\ell'} * g_\ell \right), \quad (4.1)$$

which are computed separately as described below.

Formally, the coefficients  $\omega_{i,\alpha}^{\ell''} = P_{\ell''}(\sum_{\ell' \leq \ell} f_{\ell'} * g_{\ell'})$  are computed for all levels  $0 \leq \ell'' \leq L^\omega$  and all  $i \in \mathbb{Z}$ . In particular, these data contain the coefficients  $\{\omega_{i,\alpha}^{\ell''} : 0 \leq \ell'' \leq L^\omega, i \in \mathcal{I}_{\ell''}^\omega, 0 \leq \alpha \leq p_i^{\ell'',\omega}\}$  which are the only data required for the final result (2.10). The restriction to the necessary data will be discussed in §5.

In §§4.1-4.3 we describe the computation of the coefficients of each of the three terms in (4.1). The sum yields  $\omega[\ell''] = \mathbf{P}_{\ell''}(\sum_{\ell' \leq \ell} f_{\ell'} * g_{\ell'})$ . The treatment of the second term  $\sum_{\ell < \ell'} g_{\ell'} * f_{\ell'}$  in (2.9) is very similar as explained in §4.4.

#### 4.1 Case A: $\ell'' \leq \ell' \leq \ell$

**Proposition 4.1** Algorithm (4.2) computes the quantities  $\Gamma_{i,(\alpha,\beta)}^\ell$  ( $0 \leq \ell \leq L^g$ ) defined in (3.5):

$$\boxed{\text{for } \ell := L^g \text{ downto } 0 \text{ do } \Gamma[\ell] := \left\{ \begin{array}{ll} 0 & \text{if } \ell = L^g \\ \Lambda(\Gamma[\ell+1]) & \text{if } \ell < L^g \end{array} \right\} + \Lambda_\ell(\mathbf{g}[\ell]);} \quad (4.2)$$

*Proof.* The first step  $\ell = L^g$  in (4.2) produces  $\Lambda_\ell(\mathbf{g}[\ell]) = \Gamma_\ell[\ell] \stackrel{(3.5*)}{=} \sum_{\lambda=\ell}^{L^g} \Gamma_\lambda[\ell] = \Gamma[\ell]$ .

By induction hypothesis,  $\Gamma[\ell']$  are determined for  $\ell' \geq \ell + 1$ . By definition,  $\Lambda(\Gamma_\lambda[\ell+1]) = \Gamma_\lambda[\ell]$  holds (cf. (3.7c)). This proves  $\Lambda(\Gamma[\ell+1]) = \Lambda(\sum_{\lambda=\ell+1}^{L^g} \Gamma_\lambda[\ell+1]) = \sum_{\lambda=\ell+1}^{L^g} \Lambda(\Gamma_\lambda[\ell+1]) = \sum_{\lambda=\ell+1}^{L^g} \Gamma_\lambda[\ell]$ . Together with  $\Lambda_\ell(\mathbf{g}[\ell]) = \Gamma_\ell[\ell]$  the data  $\Gamma[\ell]$  produced in (4.2) contains the coefficients  $\Gamma_{i,(\alpha,\beta)}^\ell$  from (3.5). ■

The algorithm for the coefficients  $\omega_{i,\alpha}^\ell$  (encoded by  $\omega[\ell]$ ) is

$$\boxed{\text{for } \ell := \min\{L^f, L^g\} \text{ downto } 0 \text{ do } \omega[\ell] := \left\{ \begin{array}{ll} 0 & \text{if } \ell = \min\{L^f, L^g\} \\ \mathcal{R}(\omega[\ell+1]) & \text{if } \ell < \min\{L^f, L^g\} \end{array} \right\} + \mathbf{f}[\ell] * \Gamma[\ell]} \quad (4.3)$$

**Proposition 4.2** Algorithm (4.3) yields the coefficients

$$\omega[\ell''] = \mathbf{P}_{\ell''} \left( \sum_{\ell'=\ell''}^{L^f} \sum_{\ell=\ell'}^{L^g} f_{\ell'} * g_{\ell} \right) = \mathbf{P}_{\ell''} \left( \sum_{0 \leq \ell'' \leq \ell' \leq \ell} f_{\ell'} * g_{\ell} \right) \quad (0 \leq \ell'' \leq \min\{L^f, L^g\}) \quad (4.4)$$

involving all combinations of indices with  $\ell'' \leq \ell' \leq \ell$ . An equivalent notation of (4.4) is

$$\omega[\ell''] = \sum_{\ell'=\ell''}^{L^f} \omega_{\ell'}[\ell''] \quad \text{with} \quad \omega_{\ell'}[\ell''] := \mathbf{P}_{\ell''} \left( f_{\ell'} * \sum_{\ell=\ell'}^{L^g} g_{\ell} \right).$$

If  $L^\omega > \min\{L^f, L^g\}$ ,  $\omega_{i,\alpha}^\ell = 0$  holds for all  $\min\{L^f, L^g\} < \ell \leq L^\omega$ .

*Proof.* a) By Lemma 3.1 we have  $\omega_\ell[\ell] = \mathbf{P}_\ell(f_\ell * \sum_{\lambda=\ell}^{L^g} g_\lambda) = \mathbf{f}[\ell] * \Gamma[\ell]$ . Note that  $\omega_{\ell''}[\ell''] = 0$  for  $\ell'' > \min\{L^f, L^g\}$ .

b) The first step of the loop yields  $\mathbf{f}[\ell''] * \Gamma[\ell''] \stackrel{a)}{=} \omega_{\ell''}[\ell''] \stackrel{a)}{=} \sum_{\ell'=\ell''}^{L^f} \omega_{\ell'}[\ell''] = \omega[\ell'']$  for  $\ell'' = \min\{L^f, L^g\}$ .

Hence,  $\omega[\ell'']$  is correctly computed.

c) By induction assume that (4.4) holds for  $\ell + 1$ , i.e.,  $\omega[\ell+1] = \sum_{\ell'=\ell+1}^{L^f} \omega_{\ell'}[\ell+1]$ . Then

$$\mathcal{R}(\omega[\ell+1]) = \sum_{\ell'=\ell+1}^{L^f} \mathcal{R}(\omega_{\ell'}[\ell+1]) \stackrel{(3.9b)}{=} \sum_{\ell'=\ell+1}^{L^f} \omega_{\ell'}[\ell]$$

is the first term in step  $\ell$  of (4.3). The second term is  $\mathbf{f}[\ell] * \Gamma[\ell] = \omega_\ell[\ell]$ . Together we obtain the correct value  $\sum_{\ell'=\ell}^{L^f} \omega_{\ell'}[\ell] = \omega[\ell]$ . ■

The coefficients  $\omega_{i,\alpha}^\ell$  in (4.3) contain formally all  $i \in \mathbb{Z}$ . In the practical performance, the index  $i$  is restricted to a finite subset. Furthermore, the upper bound  $p$  from (1.6) can be replaced by a possibly smaller degree. Both aspects are explained in the next remark.

**Remark 4.3** a) We recall that all intervals  $I_i^\ell \in \mathcal{M}^\omega$  are associated with a polynomial degree  $p_i^\ell$  (cf. (1.5)). For all intervals  $I_j^{\ell'} \subset I_i^\ell$  ( $\ell' > \ell$ ) define an associated degree  $p_j^{\ell'} := p_i^\ell$ .

b) Consider the right-hand side of (4.3) for fixed value of the loop variable  $\ell$ . Then for  $i \in \mathbb{Z}$  the following disjoint cases appear:

1.  $I_i^\ell \subset \overline{\mathbb{R} \setminus \Omega_0^\omega}$ . Then no action is needed, since  $I_i^\ell$  is outside of the support of  $\omega = P\omega_{\text{exact}}$ .
2.  $I_i^\ell \subset \Omega_{\ell'}^\omega$ , for some  $0 \leq \ell' \leq \ell$ . Then the computations have to be performed with the polynomial degree  $p_i^\ell$  associated to  $I_i^\ell$  by part a) of the Remark. If  $\ell'' = \ell$ , the values  $\omega_{i,\alpha}^\ell$  ( $0 \leq \alpha \leq p_i^\ell$ ) defined in (4.3) are part of the projected result  $P(f * g)|_{I_i^\ell}$  since  $I_i^\ell \in \mathcal{M}^\omega$ .
3.  $I_i^\ell \subset \Omega_{\ell+1}^\omega$ . No action needed, since the result is already determined in  $\Omega_{\ell+1}^\omega$ .

c) The convolution  $\mathbf{f}[\ell] * \mathbf{\Gamma}[\ell]$  in (4.3) must be evaluated for all  $i$  with  $I_i^\ell \subset \Omega_0^\omega$ . The required degree indices are  $0 \leq \alpha \leq \max\{p_i^\ell : I_i^\ell \subset \Omega_0^\omega\}$ .

## 4.2 Case B: $\ell' < \ell'' \leq \ell$

In the following we need the coefficients  $F_{j,\alpha}^\ell$  of the functions

$$F_\ell := \sum_{\ell'=0}^{\ell-1} f_{\ell'} \in S_{\ell-1} \subset S_\ell \quad \text{for } \ell = 1, \dots, L^\omega \quad (4.5)$$

at level  $\ell$ , i.e.  $\sum_{j \in \mathbb{Z}} \sum_{\alpha=0}^p F_{j,\alpha}^\ell \Phi_{j,\alpha}^\ell = F_\ell$ . All components  $F_{j,\alpha}^\ell$  are collected in  $\mathbf{F}[\ell] = \mathbf{P}_\ell(F_\ell)$  and computed by

$$\boxed{\begin{array}{l} \mathbf{F}[0] := 0; \\ \text{for } \ell := 1 \text{ to } L^\omega \text{ do } \mathbf{F}[\ell] := \mathcal{P} \left( \begin{array}{l} \mathbf{F}[\ell-1] \quad \text{if } \ell > L^f + 1 \\ \mathbf{F}[\ell-1] + \mathbf{f}[\ell-1] \quad \text{if } \ell \leq L^f + 1 \end{array} \right); \end{array}} \quad (4.6)$$

with  $\mathcal{P}$  from (3.8b).

*Proof.* The definition of  $\mathcal{P}$  yields  $\mathbf{F}[\ell] = \mathcal{P}(\mathbf{F}[\ell-1] + \mathbf{f}[\ell-1])$ . If  $\ell > L^f + 1$ ,  $\mathbf{f}[\ell-1] = 0$  allows to avoid the addition of  $\mathbf{f}[\ell-1]$ . ■

**Proposition 4.4** *The algorithm*

$$\boxed{\text{for } \ell := 1 \text{ to } \min\{L^\omega, L^g\} \text{ do } \omega[\ell] := \mathbf{F}[\ell] * \mathbf{\Gamma}[\ell];} \quad (4.7)$$

computes  $\omega[\ell'']$  containing the coefficients  $\omega_{i,\alpha}^{\ell''}$  of

$$\omega[\ell''] = \mathbf{P}_{\ell''} \left( \sum_{\ell=\ell''}^{L^g} \sum_{\ell'=0}^{\ell''-1} f_{\ell'} * g_\ell \right) = \mathbf{P}_{\ell''} \left( \sum_{\ell', \ell \text{ with } \ell' < \ell'' \leq \ell} f_{\ell'} * g_\ell \right) \quad (1 \leq \ell'' \leq \min\{L^\omega, L^g\}) \quad (4.8)$$

involving all combinations of indices with  $\ell' < \ell'' \leq \ell$ . For  $\ell = 0$  and  $L^g < \ell \leq L^\omega$ ,  $\omega[\ell] = 0$  holds, while for  $\ell > L^\omega$  the values  $\omega[\ell]$  are not needed.

*Proof.* The convolution  $\omega[\ell''] := \mathbf{F}[\ell''] * \mathbf{\Gamma}[\ell'']$  produces the coefficients

$$\omega[\ell''] = \mathbf{P}_{\ell''} \left( F_{\ell''} * \sum_{\ell=\ell''}^{L^g} g_\ell \right) = \mathbf{P}_{\ell''} \left( \sum_{\ell'=0}^{\ell''-1} f_{\ell'} * \sum_{\ell=\ell''}^{L^g} g_\ell \right)$$

(cf. Lemma 3.1). This proves the assertion (4.8). ■

### 4.3 Case C: $\ell' \leq \ell < \ell''$

We recall that  $\mathbf{\Gamma}_\ell[\ell] = \Lambda_\ell(g[\ell])$  is already evaluated in (4.2). Furthermore, we use the data  $\mathbf{F}[\ell]$  determined by (4.6). Different from the previous setting, we define  $\hat{\omega}[\ell'']$  as  $(2p+1)$ -tuple  $(\hat{\omega}_{i,\alpha}^{\ell''})_{\alpha=0}^{2p+1}$  containing the coefficients  $\hat{\omega}_{i,\alpha}^{\ell''}$  of the function

$$\omega_{\ell'',\text{exact}} := \sum_{0 \leq \ell' \leq \ell < \ell''} f_{\ell'} * g_\ell = \sum_{i \in \mathbb{Z}} \sum_{\alpha=0}^{2p+1} \hat{\omega}_{i,\alpha}^{\ell''} \Phi_{i,\alpha}^{\ell''}.$$

Since for any interval  $I_i^{\ell''}$  the restriction  $\omega_{\ell'',\text{exact}}|_{I_i^{\ell''}}$  is a polynomial of degree  $\leq 2p+1$ ,  $\sum_{i \in \mathbb{Z}} \sum_{\alpha=0}^{2p+1} \hat{\omega}_{i,\alpha}^{\ell''} \Phi_{i,\alpha}^{\ell''}$  is an *exact* representation of  $\omega_{\ell'',\text{exact}}$ . The following algorithm for Case C uses the extended data  $\hat{\omega}[\ell'']$  and contains the standard data  $\omega[\ell''] = (\hat{\omega}_{\ell'',\alpha})_{\alpha=0}^p$  as a subset of  $\hat{\omega}[\ell'']$ .

$$\begin{aligned} &\hat{\omega}[0] := 0; \\ &\text{for } \ell := 1 \text{ to } L^\omega \text{ do} \\ &\quad \text{begin if } \ell \leq L^g + 1 \text{ then } \hat{\omega}[\ell - 1] := \hat{\omega}[\ell - 1] + (\mathbf{F}[\ell - 1] + \mathbf{f}[\ell - 1]) * \hat{\mathbf{\Gamma}}_{\ell-1}[\ell - 1]; \\ &\quad \quad \hat{\omega}[\ell] := \hat{\mathcal{P}}(\hat{\omega}[\ell - 1]) \\ &\quad \text{end;} \end{aligned} \tag{4.9}$$

The coefficient vectors  $\hat{\mathbf{\Gamma}}_\ell$  and the mapping  $\hat{\mathcal{P}}$  is defined as  $\mathbf{\Gamma}_\ell$ ,  $\mathcal{P}$  but with degree indices in the range<sup>6</sup>  $0, 1, \dots, 2p+1$ .

**Proposition 4.5** *Restricting the data  $\hat{\omega}[\ell''] = (\hat{\omega}_{\ell'',\alpha})_{\alpha=0}^{2p+1}$  produced by (4.9) to  $0 \leq \alpha \leq p$ , we obtain the desired coefficients  $\omega[\ell''] = \mathbf{P}_{\ell''} \left( \sum_{0 \leq \ell' \leq \ell < \ell''} f_{\ell'} * g_\ell \right)$ .*

*Proof.* a) The first step yields  $\hat{\omega}[1] = \hat{\mathcal{P}}(\mathbf{f}[0] * \hat{\mathbf{\Gamma}}_0[0]) \stackrel{\text{Lemma 3.1}}{=} \hat{\mathcal{P}}(\hat{\mathbf{P}}_0(f_0 * g_0)) \stackrel{(3.8a)}{=} \hat{\mathbf{P}}_1(f_0 * g_0)$ , since  $\hat{\omega}[0] = \mathbf{F}[0] = 0$ . Again,  $\hat{\mathbf{P}}_\ell$  is defined as  $\mathbf{P}_\ell$  but with extended range of the degree indices. Since  $\omega_{1,\text{exact}} = f_0 * g_0$ , the assertion holds for  $\ell = 1$ .

b) Let  $\ell > 1$ . Note that  $\omega_{\ell,\text{exact}} = \omega_{\ell-1,\text{exact}} + \sum_{\ell'=0}^{\ell-1} f_{\ell'} * g_{\ell-1} = \omega_{\ell-1,\text{exact}} + (F_{\ell-1} + f_{\ell-1}) * g_{\ell-1}$  with  $F_{\ell-1}$  from (4.5). Assume by induction hypothesis that (4.9) has produced  $\hat{\omega}[\ell - 1] = \hat{\mathbf{P}}_{\ell-1}(\omega_{\ell-1,\text{exact}})$ . Then (4.9) produces in step  $\ell$

$$\begin{aligned} &\hat{\mathcal{P}} \left( \hat{\omega}[\ell - 1] + (\mathbf{F}_{\ell-1} + \mathbf{f}_{\ell-1}) * \hat{\mathbf{\Gamma}}_{\ell-1} \right) \stackrel{\text{Lemma 3.1}}{=} \hat{\mathcal{P}} \left( \hat{\mathbf{P}}_{\ell-1}(\omega_{\ell-1,\text{exact}}) + \hat{\mathbf{P}}_{\ell-1}((F_{\ell-1} + f_{\ell-1}) * g_{\ell-1}) \right) \stackrel{(3.8a)}{=} \\ &= \hat{\mathbf{P}}_\ell(\omega_{\ell-1,\text{exact}} + (F_{\ell-1} + f_{\ell-1}) * g_{\ell-1}) = \hat{\mathbf{P}}_\ell(\omega_{\ell,\text{exact}}) \end{aligned}$$

the correct value  $\hat{\omega}[\ell] = \hat{\mathbf{P}}_\ell(\omega_{\ell,\text{exact}})$ . If  $\ell > L^g + 1$ , the factor  $\hat{\mathbf{\Gamma}}_{\ell-1}$  vanishes and the update of  $\hat{\omega}[\ell - 1]$  is not necessary.  $\blacksquare$

**Remark 4.6** *For fixed loop index  $\ell$ , the computations in (4.9) are to be performed for all  $i$  with  $I_i^\ell \subset \Omega_\ell^\omega$ .*

### 4.4 Treatment of the second sum in (2.9)

The treatment of the second sum

$$\sum_{\ell < \ell'} g_\ell * f_{\ell'} \stackrel{\text{notations of } \ell, \ell' \text{ interchanged}}{=} \sum_{\ell' < \ell} g_{\ell'} * f_\ell \tag{4.10}$$

in (2.9) is almost analogous to the first sum  $\sum_{\ell' \leq \ell} f_{\ell'} * g_\ell$ . Besides the trivial permutation of the letters  $f$  and  $g$  one has to notice that the sum in (4.10) excludes the cases of  $\ell', \ell$  with  $\ell' = \ell$ .

Below we describe the changes in the algorithmic steps (4.2), (4.3), (4.6), (4.7), (4.9) from above.

<sup>6</sup>More precisely, the indices  $\beta, \varkappa$  of  $\gamma_{k-i2^{\ell-\ell'}, (\alpha, \beta, \varkappa)}^{\ell', \ell', \ell}$  and  $\Gamma_{i, (\alpha, \beta)}^{\ell', \ell}$  are still from 0 to  $p$ , whereas  $\alpha$  varies from 0 to  $2p+1$ .

**Case A.** First,  $g$  and  $L^g$  are replaced by  $f$  and  $L^f$ . Besides  $\mathbf{\Gamma}[\ell] = \sum_{\lambda=\ell}^{L^f} \mathbf{\Gamma}_\lambda[\ell]$  (cf. (3.5\*)) we also define the sum

$$\mathbf{\Gamma}'[\ell] = \sum_{\lambda=\ell+1}^{L^f} \mathbf{\Gamma}_\lambda[\ell].$$

Then (4.2), (4.3) become

```

 $\mathbf{\Gamma}'[L^f] := 0;$ 
for  $\ell := L^f$  downto 0 do
begin  $\mathbf{\Gamma}_\ell[\ell] := \Lambda_\ell(\mathbf{f}[\ell]);$ 
  if  $\ell < L^f$  then  $\mathbf{\Gamma}'[\ell] := \Lambda(\mathbf{\Gamma}'[\ell - 1] + \mathbf{\Gamma}_{\ell-1}[\ell - 1]);$ 
   $\mathbf{\Gamma}[\ell] := \mathbf{\Gamma}'[\ell] + \mathbf{\Gamma}_\ell[\ell]$ 
end;
```

```

 $L := \min\{L^f - 1, L^g\};$ 
for  $\ell := L$  downto 0 do  $\omega[\ell] := \left\{ \begin{array}{ll} 0 & \text{if } \ell = L \\ \mathcal{R}(\omega[\ell + 1]) & \text{if } \ell < L \end{array} \right\} + \mathbf{g}[\ell] * \mathbf{\Gamma}'[\ell]$ 
```

**Case B.** In this case  $\ell' = \ell$  cannot happen. There only the name  $f, g$  are to be interchanged. For a consistent naming we also change  $\mathbf{F}$  into  $\mathbf{G}$ :

```

 $\mathbf{G}[0] := 0;$ 
for  $\ell := 1$  to  $L^\omega$  do  $\mathbf{G}[\ell] := \mathcal{P} \left( \left\{ \begin{array}{ll} \mathbf{G}[\ell - 1] & \text{if } \ell > L^g + 1 \\ \mathbf{G}[\ell - 1] + \mathbf{g}[\ell - 1] & \text{if } \ell \leq L^g + 1 \end{array} \right\} \right);$ 
for  $\ell := 1$  to  $\min\{L^\omega, L^f\}$  do  $\omega[\ell] := \mathbf{G}[\ell] * \mathbf{\Gamma}[\ell];$ 
```

**Case C.** Again  $\ell' = \ell$  must be excluded:

```

 $\hat{\omega}[0] := \hat{\omega}[1] := 0;$ 
for  $\ell := 2$  to  $L^\omega$  do
begin if  $\ell \leq L^f + 1$  then  $\hat{\omega}[\ell - 1] := \hat{\omega}[\ell - 1] + \mathbf{G}[\ell - 1] * \hat{\mathbf{\Gamma}}_{\ell-1}[\ell - 1];$ 
   $\hat{\omega}[\ell] := \hat{\mathcal{P}}(\hat{\omega}[\ell - 1])$ 
end;
```

## 5 Discrete convolution revisited and cost of the algorithm

The algorithms in §§4.1-4.4 involve the operation  $*$  from (3.6) which is a sum of usual discrete convolutions of infinite sequences as defined in (2.12). The sequences appearing in our applications have a bounded support, i.e., almost all components are zero. To measure the size of the support, we say that a sequence  $a$  has length  $N(a) \in \mathbb{N}_0$  if there is an index  $i_a$  so that  $a_i = 0$  for all  $i \notin [i_a, i_a + N(a) - 1] \cap \mathbb{Z}$ . Obviously,  $N(a)$  is a measure for the storage needed for the coefficients of  $a$ . The convolution of two sequences of bounded support has again bounded support with the length  $N(a * b) = N(a) + N(b) - 1$ . The computational cost of the Fast Fourier Transform is<sup>7</sup>  $\mathcal{O}((N(a) + N(b)) \log(N(a) + N(b)))$ .

With the foregoing comments the given algorithms have obviously a finite runtime. However, the restriction to the essential support of the sequences is still insufficient. A critical example is algorithm (4.6), where the quantities  $\mathbf{F}[\ell]$  are computed. The difficulty arises from the fact that the prolongation mapping  $\mathcal{P}$  has an output length which is about twice the input length. This causes an exponential growth with respect to the level number and makes the algorithm inefficient. The remedy is given by the following trivial observation.

Let  $c := a * b$ , where  $N(b) < \infty$ , whereas  $a$  may have even an unbounded support. Assume that the resulting coefficients  $c_i$  are only of interest for  $i \in [i_c, i_c + L_c - 1] \cap \mathbb{Z}$  ( $L_c$ : output length). Then there is a truncated version  $a'$  of  $a$  with  $N(a') = N(b) + L_c - 1$  such that  $c' := a' * b$  yields the same components  $c'_i = c_i$  for  $i \in [i_c, i_c + L_c - 1]$ . By Remark 4.6, the length  $L_c$  is given by  $\text{length}(\Omega_\ell^\omega)/h_\ell$ .

The application to algorithm (4.6) is as follows. In (4.7) the convolution  $\omega[\ell] := \mathbf{F}[\ell] * \mathbf{\Gamma}[\ell]$  appears. The size of  $\mathbf{\Gamma}[\ell]$  is bounded by the size of the input data  $g_\ell$  from (2.6), while the necessary coefficients of

<sup>7</sup>In [3] improvements are discussed if  $N(a)$  and  $N(b)$  are of quite different size.

$\omega[\ell]$  are characterised by  $i \in \mathcal{I}_\ell^\omega$  (cf. (2.10)). This allows to shorten the sequences in  $\mathbf{F}[\ell]$ . The remaining size is bounded by the input sizes<sup>8</sup> of the data  $\{f_\ell : 0 \leq \ell \leq L^f\}$ ,  $\{g_\ell : 0 \leq \ell \leq L^g\}$ , and the output sizes  $N(\mathcal{I}_\ell^\omega) := 1 + \max_{i \in \mathcal{I}_\ell^\omega} i - \min_{i \in \mathcal{I}_\ell^\omega} i$ . Let  $N$  be the sum of these input and output sizes. The storage size of these data is  $N \cdot p$ . The data needed in the algorithms of §§4.1-4.4 require a storage of  $\mathcal{O}(N \cdot p^2)$ . The number of operations is  $\mathcal{O}(N \log(N) \cdot p^2)$ .

A detailed complexity analysis can be found in [3].

## 6 Modifications and Extensions

### 6.1 Nondisjoint supports of $f_\ell$ or $g_\ell$

By definition of the index set  $\mathcal{I}_\ell^f$  and the space  $\mathcal{S}_\ell^f$ , the functions  $f_0, f_1, \dots$  arising in the decomposition (2.5) have disjoint support (see footnote 1). In fact,  $\text{supp}(f_\ell) \subset \overline{\Omega_\ell} \setminus \Omega_{\ell+1}$  holds. This assumption makes the decomposition (2.5) unique. However, neither the algorithms nor the discussion in §5 make use of the fact that the support of  $\text{supp}(f_\ell)$  has a “hole” in  $\Omega_{\ell+1}$ . Therefore, nondisjoint supports of  $f_\ell$  can be allowed without any change of the algorithm.

### 6.2 Other systems of basis functions

We started from the condition that the basis functions should be orthonormal (cf. Remark 2.1b) with  $\text{supp}(\Phi_{i,\alpha}^\ell) = I_i^\ell$ . Then the described system of transformed Legendre polynomial is the optimal one.

Weakening the condition  $\text{supp}(\Phi_{i,\alpha}^\ell) = I_i^\ell$  we can ask for an orthonormal system with a support of  $\mathcal{O}(1)$  subintervals. An obvious candidate are orthogonal wavelet representations. In this case we make use of nondisjoint supports as discussed above. Then the set of basis functions  $\Phi_{i,\alpha}^\ell$  consists of generating functions and wavelets at various levels.

For any orthonormal system one has to describe the refinement rule (2.3) which is obvious in the wavelet case. Another requirement is that the values of  $\gamma_{v,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$  can be determined (see (3.2) and Appendix C), since these quantities are used in (3.7a), i.e., in the definition of the mapping  $\Lambda_\ell$  from (3.7c).

Even a generalisation to non-orthonormal systems of basis functions  $\Phi_{i,\alpha}^\ell$  is possible, if there is an associated bi-orthonormal system of functions  $\Psi_{i,\alpha}^\ell$  so that  $\langle \Phi_{i,\alpha}^\ell, \Psi_{j,\beta}^\ell \rangle = \delta_{ij} \delta_{\alpha\beta}$ .

### 6.3 Multidimensional convolution

Generalisations to  $d$ -variate functions are straightforward, provided that the basis functions are tensor products of the one-dimensional basis functions  $\Phi_{i,\alpha}^\ell$  used here.

### 6.4 Local refinements towards multiple subregions

The structure of the refinement is defined by means of the refinement zones  $\Omega_\ell = [i_{a,\ell} h_\ell, i_{b,\ell} h_\ell]$ . Note that the description of an  $hp$ -function  $g$  in such a grid requires a storage<sup>9</sup>  $N(g) = \mathcal{O}(\sum_{\ell=0}^{L^g} \text{length}(\Omega_\ell)/h_\ell) = h^{-1} \mathcal{O}(\sum_{\ell=0}^{L^g} 2^\ell \text{length}(\Omega_\ell))$ . Therefore, the size of the refinement zones should decrease with  $\ell$ .

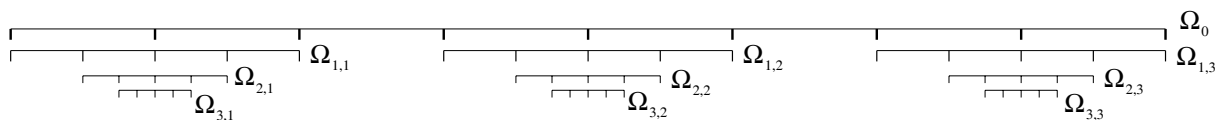


Figure 6.1: Local grid refinements in three disjoint subregions

<sup>8</sup>For a function  $f_\ell$  with support in  $\Omega_\ell^f = [i_{a,\ell}^f h_\ell, i_{b,\ell}^f h_\ell]$  we define the size  $N(f_\ell) := i_{b,\ell}^f - i_{a,\ell}^f + 1$  (number of involved subintervals).

<sup>9</sup>Here, the maximal polynomial degree  $p$  is considered as a constant.

Now we consider the situation of Figure 6.1, where the starting grid in  $\Omega_0$  is refined in three disjoint subregions  $\Omega_{\ell,1}, \Omega_{\ell,2}, \Omega_{\ell,3}$ . In principle, we can form the convex hulls  $\Omega_\ell := [a_\ell, b_\ell]$  with  $a_\ell := \min_{x \in \Omega_{\ell,1}} x$  and  $b_\ell := \max_{x \in \Omega_{\ell,3}} x$  to reach the situation assumed in the introduction (as in Figure 1.1). Then, however,  $\Omega_\ell$  has the size  $\mathcal{O}(1)$  and the storage of an  $hp$ -function  $g$  increases as  $\mathcal{O}(2^{L^g})$ . Using the ideas of §6.1, the storage can be reduced, since the functions  $g_\ell$  ( $\ell \geq 1$ ) from (2.6) have a sparse representation: nonzero components appear only for subintervals in  $\Omega_{\ell,1} \cup \Omega_{\ell,2} \cup \Omega_{\ell,3}$ , while zero components belongs to the holes in between. However, this does not help for the convolutions at the levels  $\ell \geq 1$ , since for this purpose the discrete Fourier transform is to be applied and, unfortunately, the Fourier image of a sparse vector is not sparse again.

A possible remedy is the division of all  $g_\ell$  ( $0 \leq \ell \leq L^g$ ) into three terms  $g_\ell^{(1)} + g_\ell^{(2)} + g_\ell^{(3)}$  so that each term is associated with only one refinement subregion and therefore satisfies the assumptions of §1. Provided that also  $f$  satisfies these conditions, the convolutions  $f * g^{(i)}$  ( $1 \leq i \leq 3$ ,  $g^{(i)} := \sum_\ell g_\ell^{(i)}$ ) can be computed as usual. More or less, the computational cost is increased by the factor 3. This fact makes the proposed remedy less attractive if  $f$  and/or  $g$  is refined in many subregions.

So far, the aim was to compute the projected convolution *exactly*. Finally, we discuss a case which is quite typical. Let  $f$  be a function which is refined in the neighbourhood of 0 and increasingly smooth<sup>10</sup> away from 0, while  $g$  may be of the form like in Figure 6.1. Furthermore, we assume that the  $hp$ -structure of the desired result  $\omega$  is identical to that of  $g$ , i.e.,  $\mathcal{S}^\omega = \mathcal{S}^g$ . Again we split  $g$  into  $\sum_i g^{(i)}$  so that each  $g^{(i)}$  uses a grid refinement as in Figure 1.1. Define  $\mathcal{S}^{\omega,i}$  ( $i = 1, 2, 3$ ) for instance as follows:  $\mathcal{S}^{\omega,i} := \mathcal{S}_0^g + \sum_\ell \mathcal{S}_\ell^g|_{\Omega_{\ell,i}}$ , where  $\mathcal{S}_\ell^g = \mathcal{S}^g \cap \mathcal{S}_\ell$  (cf. (2.4)) is restricted to the  $i$ th subregion  $\Omega_{\ell,i}$ . Note that  $\mathcal{S}^\omega = \sum_i \mathcal{S}^{\omega,i}$ . The subspace  $\mathcal{S}^{\omega,i}$  is of the standard form and offers a good approximation of  $f * g^{(i)}$ , since  $f * g^{(i)}$  is smoother than  $g^{(i)}$ . The projection  $\omega^{(i)}$  of the convolution  $f * g^{(i)}$  to  $\mathcal{S}^{\omega,i}$  is cheaper because  $\mathcal{S}^{\omega,i}$  has a simpler structure than  $\mathcal{S}^\omega$ . In this case, the sum  $\sum_i \omega^{(i)} \in \mathcal{S}^\omega$  is not the exact projection of  $f * g$  to  $\mathcal{S}^\omega$  but a perfect approximation.

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<sup>10</sup>A typical characterisation of such smooth functions is by the “asymptotically smoothness” (cf. [6]).



## A Legendre polynomials

The standard definition of Legendre polynomials  $P_n$  of degree  $n$  is by the recursion formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad P_0(x) = 1, \quad (\text{A.1a})$$

which leads to the normalisation condition

$$P_n(1) = 1. \quad (\text{A.1b})$$

We recall some properties of  $P_n$ .

**Lemma A.1** *a)  $P_n$  is an even [odd] function, if  $n$  is even [odd].*

*b) The Legendre polynomials  $P_n$  are orthogonal, i.e.,  $\int_{-1}^1 P_n(x)P_m(x)dx = 0$  for  $n \neq m$ .*

*c) The squared  $L^2$ -norm is*

$$\int_{-1}^1 P_n^2(x)dx = \frac{2}{2n+1}. \quad (\text{A.1c})$$

In the proof of Lemma C.4 we will need the following representation of the antiderivative of  $P_n$ .

**Lemma A.2** *For any  $n \in \mathbb{N}$  we have*

$$\int_{-1}^x P_n(t)dt = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}. \quad (\text{A.1d})$$

*Proof.* a) Since  $Q_{n+1}(x) := \int_{-1}^x P_n(t)dt$  is a polynomial of degree  $n+1$ , it is of the form

$$Q_{n+1}(x) = \sum_{\nu=0}^{n+1} \alpha_\nu P_\nu(x) \quad \text{with } \alpha_\nu = \frac{2\nu+1}{2} \int_{-1}^1 Q_{n+1}(x)P_\nu(x)dx \text{ (cf. (A.1b)).}$$

b) Let  $\nu \leq n-2$ . We want to show that  $\alpha_\nu = 0$ . Partial integration yields

$$\begin{aligned} \int_{-1}^1 Q_{n+1}(x)P_\nu(x)dx &= \int_{-1}^1 \left( \int_{-1}^x P_n(t)dt \right) P_\nu(x)dx \\ &= \left( \int_{-1}^x P_n(t)dt \right) \left( \int_{-1}^x P_\nu(t)dt \right) \Big|_{x=-1}^1 - \int_{-1}^1 P_n(x) \left( \int_{-1}^x P_\nu(t)dt \right) dx \\ &= Q_{n+1}(1)Q_{\nu+1}(1) - \int_{-1}^1 P_n(x)Q_{\nu+1}(x)dx \end{aligned}$$

since obviously  $Q_{n+1}(0) = 0$ . By orthogonality,  $Q_{n+1}(1) = \int_{-1}^1 P_n(x)P_0(x)dx = 0$  holds. Since the degree of  $Q_{\nu+1}$  is  $\nu+1 \leq n-1$ , this polynomial is orthogonal to  $P_n$  proving  $\int_{-1}^1 Q_{n+1}(x)P_\nu(x)dx = 0$ , i.e.,  $\alpha_\nu = 0$ .

c) Next we prove  $\alpha_n = 0$ . Let  $n$  be even [odd]. Lemma A.1a implies that  $\tilde{Q}_{n+1}(x) = \int_0^x P_n(t)dt$  is odd [even].  $Q_{n+1}$  equals  $c + \tilde{Q}_{n+1} = cP_0 + \tilde{Q}_{n+1}$  with  $c := \int_0^1 P_n(t)dt$ . The integral  $\int_{-1}^1 \tilde{Q}_{n+1}P_n(x)dx$  vanishes since the integrand is even (product of an even and an odd function). Finally,  $\int_{-1}^1 P_0P_n(x)dx = 0$  shows that  $\int_{-1}^1 Q_{n+1}(x)P_n(x)dx = 0$ , i.e.,  $\alpha_n = 0$ .

d) Parts a-c yield the representation  $Q_{n+1}(x) = \alpha_{n+1}P_{n+1}(x) + \alpha_{n-1}P_{n-1}(x)$ . Evaluation at  $x = -1$  shows  $0 = \alpha_{n+1}(-1)^{n+1} + \alpha_{n-1}(-1)^{n-1} = \pm(\alpha_{n+1} + \alpha_{n-1})$ , proving  $\alpha_{n-1} = -\alpha_{n+1}$ , i.e.

$$Q_{n+1}(x) = \alpha_{n+1}(P_{n+1}(x) - P_{n-1}(x)). \quad (\text{A.2})$$

e) To fix the value of  $\alpha_{n+1}$ , we consider the leading term.  $P_n(x) = c_n x^n + \dots$  yields  $Q_{n+1}(x) = \frac{1}{n+1}c_n x^{n+1} + \dots$ , while the recursion formula (A.1a) implies that  $(n+1)c_{n+1} = (2n+1)c_n$ . Equating the leading coefficients on both sides in (A.2) yields  $\frac{1}{n+1}c_n = \alpha_{n+1} \frac{2n+1}{n+1}c_n$ , i.e.,  $\alpha_{n+1} = \frac{1}{2n+1}$  proving the assertion (A.1c).  $\blacksquare$

In order to obtain an orthonormal system of polynomials (i.e.,  $\int_{-1}^1 L_n(x)L_m(x)dx = \delta_{nm}$ ), we introduce the scaled Legendre polynomials

$$L_n(x) := \sqrt{\frac{2n+1}{2}} P_n(x).$$

(A.1a) implies the recursion formula

$$L_{n+1}(x) = a_n x L_n(x) - b_n L_{n-1}(x), \quad a_n = \frac{\sqrt{2n+3}\sqrt{2n+1}}{n+1}, \quad b_n = \frac{\sqrt{2n+3}}{\sqrt{2n-1}} \frac{n}{n+1}. \quad (\text{A.3})$$

**Remark A.3** The recursion formula (A.3) can also be used for  $n = 0$ . Then the term  $b_0 L_{-1}(x)$  must be ignored ( $L_{-1}$  undefined, but  $b_0 = 0$ ). Therefore, one needs only one starting value  $L_0(x) = 1/\sqrt{2}$  to obtain all  $L_n$  from (A.3).

The basis functions  $\Phi_{i,\alpha}^0$  defined in (2.1a-c) satisfy the related recursion formula

$$\Phi_{0,n+1}^0(x) = a_n (-1 + 2x/h) \Phi_{0,n}^0(x) - b_n \Phi_{0,n-1}^0(x), \quad (\text{A.4a})$$

$$\Phi_{i,n+1}^\ell(x) = a_n (-1 - 2i + 2x/h_\ell) \Phi_{i,n}^\ell(x) - b_n \Phi_{i,n-1}^\ell(x). \quad (\text{A.4b})$$

The first  $\Phi_{0,\alpha}^0$  are  $\Phi_{0,0}^0(x) = 1/\sqrt{h}$ ,  $\Phi_{0,1}^0(x) = \sqrt{3/h}(2x/h - 1)$ ,  $\Phi_{0,2}^0(x) = \sqrt{5/h}(6(x/h)^2 - 6x/h + 1)$  for  $0 \leq x \leq h$  and zero outside of  $[0, h]$ .

**Remark A.4** The symmetry property  $\Phi_{0,\alpha}^0(\frac{h}{2} + x) = (-1)^\alpha \Phi_{0,\alpha}^0(\frac{h}{2} - x)$  holds for all  $\alpha \in \mathbb{N}_0$ .

## B Coefficients $\xi_{\varkappa,\alpha}$

We make the ansatz

$$\Phi_{0,\varkappa}^0 = \sum_{\alpha=0}^{\varkappa} (\xi_{\varkappa,\alpha,0} \Phi_{0,\alpha}^1 + \xi_{\varkappa,\alpha,1} \Phi_{1,\alpha}^1). \quad (\text{B.1})$$

Note that (B.1) restricted to  $I_0^1 = (0, h_1) = (0, h_0/2)$  reads  $\Phi_{0,\varkappa}^0|_{(0,h_1)} = \sum_{\alpha=0}^{\varkappa} \xi_{\varkappa,\alpha,0} \Phi_{0,\alpha}^1|_{(0,h_1)}$ , while  $\Phi_{0,\varkappa}^0|_{(h_1,h_0)} = \sum_{\alpha=0}^{\varkappa} \xi_{\varkappa,\alpha,1} \Phi_{1,\alpha}^1|_{(h_1,h_0)}$  on the second half of  $I_0^0$ . Outside of  $I_0^0$  both sides in (B.1) are zero.

**Remark B.1** The coefficients from (B.1) also satisfy

$$\Phi_{i,\varkappa}^\ell = \sum_{\alpha=0}^{\varkappa} (\xi_{\varkappa,\alpha,0} \Phi_{2i,\alpha}^{\ell+1} + \xi_{\varkappa,\alpha,1} \Phi_{2i+1,\alpha}^{\ell+1}).$$

Since  $\{\Phi_{0,\alpha}^1\}$  is an orthonormal system, the coefficients  $\xi_{\varkappa,\alpha,0}$ ,  $\xi_{\varkappa,\alpha,1}$  are characterised by the scalar products

$$\xi_{\varkappa,\alpha,0} = \int_0^{h_1} \Phi_{0,\varkappa}^0(x) \Phi_{0,\alpha}^1(x) dx, \quad \xi_{\varkappa,\alpha,1} = \int_{h_1}^{2h_1} \Phi_{0,\varkappa}^0(x) \Phi_{1,\alpha}^1(x) dx. \quad (\text{B.2})$$

The symmetry property from Remark A.4 implies the following result.

**Remark B.2**  $\xi_{\varkappa,\alpha,0} = (-1)^{\varkappa+\alpha} \xi_{\varkappa,\alpha,1}$  for all  $\varkappa, \alpha \in \mathbb{N}_0$ .

We rename  $\xi_{\varkappa,\alpha,1}$  by  $\xi_{\varkappa,\alpha}$  and get

$$\Phi_{i,\varkappa}^\ell = \sum_{\alpha=0}^{\varkappa} \xi_{\varkappa,\alpha} \left( (-1)^{\varkappa+\alpha} \Phi_{2i,\alpha}^{\ell+1} + \Phi_{2i+1,\alpha}^{\ell+1} \right). \quad (\text{B.3})$$

It remains to determine the coefficients  $\xi_{\varkappa,\alpha}$ . The answer is given in the next lemma.

**Lemma B.3** a)  $\xi_{n,m} = 0$  for  $n < m$ .

b) The coefficients  $\xi_{n,m}$  can be computed from the recursion formula

$$\xi_{n,m} = \frac{a_{n-1}}{2a_m} \xi_{n-1,m+1} + \frac{a_{n-1}}{2} \frac{b_m}{a_m} \xi_{n-1,m-1} + \frac{a_{n-1}}{2} \xi_{n-1,m} - b_{n-1} \xi_{n-2,m}$$

with  $a_\nu, b_\mu$  from (A.3). The recursion leads to the following algorithm<sup>11</sup>:

```

 $\xi_{0,0} := 1/\sqrt{2};$ 
for  $q := 1$  to  $2p$  do for  $n := 0$  to  $q$  do
begin  $m := q - n;$ 
  if  $n < m$  then  $\xi_{n,m} := 0$  else
  begin  $\xi_{n,m} := \frac{a_{n-1}}{2} (\xi_{n-1,m+1}/a_m + \xi_{n-1,m});$ 
    if  $m > 0$  then  $\xi_{n,m} := \xi_{n,m} + \frac{a_{n-1}}{2} \frac{b_m}{a_m} \xi_{n-1,m-1};$ 
    if  $n \geq 2$  then  $\xi_{n,m} := \xi_{n,m} - b_{n-1} \xi_{n-2,m}$ 
  end
end end;
```

*Proof.* a) The Legendre polynomial is orthogonal to any polynomial of lower degree. This shows part a).

b) We make use of the recursion formula (A.4). Remark B.1 shows that the scalar products (B.2) do not depend on the choice of the step size  $h_\ell$ . For convenience we take  $h = h_0 = 2, h_1 = 1$  and obtain from (B.2) that

$$\begin{aligned} \xi_{n,m} &= \int_1^2 \Phi_{0,n}^0(x) \Phi_{1,m}^1(x) dx = \int_1^2 (a_{n-1}(x-1) \Phi_{0,n-1}^0(x) - b_{n-1} \Phi_{0,n-2}^0(x)) \Phi_{1,m}^1(x) dx \\ &= a_{n-1} \int_1^2 (x-1) \Phi_{0,n-1}^0(x) \Phi_{1,m}^1(x) dx - b_{n-1} \xi_{n-2,m} \\ &= \frac{a_{n-1}}{2} \left( \int_1^2 (2x-3) \Phi_{0,n-1}^0(x) \Phi_{1,m}^1(x) dx + \int_1^2 \Phi_{0,n-1}^0(x) \Phi_{1,m}^1(x) dx \right) - b_{n-1} \xi_{n-2,m} \\ &= \frac{a_{n-1}}{2} \int_1^2 \Phi_{0,n-1}^0(x) [(2x-3) \Phi_{1,m}^1(x)] dx + \frac{a_{n-1}}{2} \xi_{n-1,m} - b_{n-1} \xi_{n-2,m} \\ &= \frac{a_{n-1}}{2} \int_1^2 \Phi_{0,n-1}^0(x) \frac{\Phi_{1,m+1}^1(x) + b_m \Phi_{1,m-1}^1(x)}{a_m} dx + \frac{a_{n-1}}{2} \xi_{n-1,m} - b_{n-1} \xi_{n-2,m} \\ &= \frac{a_{n-1}}{2a_m} (\xi_{n-1,m+1} + b_m \xi_{n-1,m-1}) + \frac{a_{n-1}}{2} \xi_{n-1,m} - b_{n-1} \xi_{n-2,m}. \end{aligned}$$

From line 4 to 5 we used  $\Phi_{1,m+1}^1(x) = a_m(2x-3)\Phi_{1,m}^1(x) - b_m\Phi_{1,m-1}^1(x)$ , which is (A.4b) for  $\ell = i = 1$  and  $n$  replaced by  $m$ . ■

Table B.1 contains the scaled values

$$\xi_{n,m}^* := 2^{n+1/2} \xi_{n,m} / \sqrt{(2n+1)(2m+1)},$$

since they have a rather simple form (mostly small integers).

## C Coefficients $\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$

We recall the general  $\gamma$ -coefficients

$$\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell} = \iint \Phi_{i,\alpha}^{\ell''}(x) \Phi_{j,\beta}^{\ell'}(y) \Phi_{k,\varkappa}^{\ell}(x-y) dx dy. \quad (\text{C.1})$$

and the simpler version

$$\gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell} = \iint \Phi_{(0,\alpha)}^{\ell''}(x) \Phi_{(0,\beta)}^{\ell'}(y) \Phi_{(\nu,\varkappa)}^{\ell}(x-y) dx dy.$$

Recursion formula (B.3) can be applied to all three basis functions in the integrand in (C.1). The resulting formulae for  $\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}$  are given in the next Remark.

<sup>11</sup>This loop defines all  $\xi_{n,m}$  with  $n+m \leq 2p$ , hence, in particular, all  $\xi_{n,m}$  for  $0 \leq n, m \leq p$ .

$m \setminus n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	0	-1	0	2	0	-5	0	14	0	-42	0	132	0	-429	0
1	0	$\frac{1}{3}$	1	1	$-\frac{2}{3}$	-2	1	5	-2	-14	$\frac{14}{3}$	42	-12	-132	33	429	$-\frac{286}{3}$
2	0	0	$\frac{1}{5}$	1	2	1	-3	$-\frac{19}{5}$	6	12	-14	-38	36	123	-99	-407	286
3	0	0	0	$\frac{1}{7}$	1	3	4	-1	-10	-4	$\frac{176}{7}$	22	-67	-87	188	319	-550
4	0	0	0	0	$\frac{1}{9}$	1	4	$\frac{25}{3}$	6	-12	$-\frac{80}{3}$	13	85	$\frac{43}{9}$	-260	-113	$\frac{2398}{9}$
5	0	0	0	0	0	$\frac{1}{11}$	1	5	14	20	-1	-53	-50	113	229	-215	$-\frac{9206}{11}$
6	0	0	0	0	0	0	$\frac{1}{13}$	1	6	21	43	35	-58	-170	-7	509	406
7	0	0	0	0	0	0	0	$\frac{1}{15}$	1	7	$\frac{88}{3}$	77	$\frac{556}{5}$	$\frac{14}{3}$	-304	-397	$\frac{1498}{3}$
8	0	0	0	0	0	0	0	0	$\frac{1}{17}$	1	8	39	124	246	208	-321	-1090
9	0	0	0	0	0	0	0	0	0	$\frac{1}{19}$	1	9	50	186	461	657	46
10	0	0	0	0	0	0	0	0	0	0	$\frac{1}{21}$	1	10	$\frac{187}{3}$	265	781	1494
11	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{23}$	1	11	76	363	1234
12	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{25}$	1	12	91	482
13	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{27}$	1	13	$\frac{322}{3}$
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{29}$	1	14
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{31}$	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{33}$

Table B.1: Exact values of  $\xi_{n,m}^*$  from which  $\xi_{n,m} = 2^{-n-1/2} \xi_{n,m}^* \sqrt{(2n+1)(2m+1)}$  can be derived.

**Remark C.1** For all  $\ell'', \ell', \ell \in \mathbb{N}_0$ , all  $i, j, k \in \mathbb{Z}$  and all  $0 \leq \alpha, \beta, \varkappa \leq p$  we have

$$\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell} = \sum_{q=0}^{\alpha} \xi_{\alpha,q} \left( (-1)^{\alpha+q} \gamma_{(2i,q),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} + \gamma_{(2i+1,q),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} \right) \quad (\text{C.2a})$$

$$= \sum_{q=0}^{\alpha} \xi_{\beta,q} \left( (-1)^{\beta+q} \gamma_{(i,\alpha),(2j,q),(k,\varkappa)}^{\ell'',\ell'+1,\ell} + \gamma_{(i,\alpha),(2j+1,q),(k,\varkappa)}^{\ell'',\ell'+1,\ell} \right) \quad (\text{C.2b})$$

$$= \sum_{q=0}^{\alpha} \xi_{\varkappa,q} \left( (-1)^{\varkappa+q} \gamma_{(i,\alpha),(j,\beta),(2k,q)}^{\ell'',\ell',\ell+1} + \gamma_{(i,\alpha),(j,\beta),(2k+1,q)}^{\ell'',\ell',\ell+1} \right). \quad (\text{C.2c})$$

The basic relations of the  $\gamma$ -coefficients are gathered below.

**Lemma C.2** a) *Shift properties:* Let  $\ell \geq \max\{\ell', \ell''\}$ . Then

$$\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell} = \gamma_{k-i2^{\ell-\ell''}+j2^{\ell-\ell'},(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell} \quad (i, j, k \in \mathbb{Z}, \alpha, \beta, \varkappa \in \mathbb{N}_0). \quad (\text{C.3})$$

b) *Symmetries properties:*

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = \gamma_{k,(\alpha,\varkappa,\beta)}^{\ell,\ell,\ell}, \quad (\text{C.4a})$$

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = (-1)^{\alpha+\beta} \gamma_{k,(\beta,\alpha,\varkappa)}^{\ell,\ell,\ell}, \quad (\text{C.4b})$$

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = (-1)^{\alpha+\beta+\varkappa} \gamma_{-k-1,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}. \quad (\text{C.4c})$$

c) *Level dependence:*

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = 2^{-\ell/2} \gamma_{k,(\alpha,\beta,\varkappa)}^{0,0,0}. \quad (\text{C.5})$$

d) *Support:*

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = 0 \quad \text{for all } k \notin \{-1, 0\}. \quad (\text{C.6})$$

*Proof.* a) (C.3) follows by substituting the integration variables  $x, y$  by shifted ones.

ba) Another writing of  $\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$  is  $\gamma_{(-k,\alpha),(0,\beta),(0,\varkappa)}^{\ell,\ell,\ell} = \langle \Phi_{-k,\alpha}^{\ell} * \Phi_{0,\beta}^{\ell} * \Phi_{0,\varkappa}^{\ell} \rangle$ . Since the convolution is symmetric:  $\Phi_{0,\beta}^{\ell} * \Phi_{0,\varkappa}^{\ell} = \Phi_{0,\varkappa}^{\ell} * \Phi_{0,\beta}^{\ell}$ , (C.4a) follows.

bb) Interchanging the notations of  $x$  and  $y$  yields

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = \iint \Phi_{0,\alpha}^\ell(x) \Phi_{0,\beta}^\ell(y) \Phi_{k,\varkappa}^\ell(x-y) dx dy = \iint \Phi_{0,\beta}^\ell(x) \Phi_{0,\alpha}^\ell(y) \Phi_{k,\varkappa}^\ell(y-x) dx dy.$$

Due to Remark A.4, we have  $\Phi_{0,\beta}^\ell(x) = (-1)^\beta \Phi_{0,\beta}^\ell(h_\ell - x)$  and  $\Phi_{0,\alpha}^\ell(y) = (-1)^\alpha \Phi_{0,\alpha}^\ell(h_\ell - y)$ . Substituting new variables  $x' = h_\ell - x$ ,  $y' = h_\ell - y$ , we obtain the statement (C.4b):

$$\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = (-1)^{\alpha+\beta} \iint \Phi_{0,\beta}^\ell(x') \Phi_{0,\alpha}^\ell(y') \Phi_{k,\varkappa}^\ell(x'-y') dx' dy' = (-1)^{\alpha+\beta} \gamma_{k,(\beta,\alpha,\varkappa)}^{\ell,\ell,\ell}.$$

bc) Part ba) shows  $\gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = \iint \Phi_{0,\alpha}^\ell(x) \Phi_{0,\beta}^\ell(x-y) \Phi_{\nu,\varkappa}^\ell(y) dx dy$ . The symmetry  $\Phi_{0,\beta}^\ell(t) = (-1)^\beta \Phi_{0,\beta}^\ell(h_\ell - t)$  from Remark A.4 implies

$$\begin{aligned} \gamma_{\nu,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} &= \iint \Phi_{0,\alpha}^\ell(x) \Phi_{0,\beta}^\ell(x-y) \Phi_{\nu,\varkappa}^\ell(y) dx dy \\ &= (-1)^\beta \iint \Phi_{0,\alpha}^\ell(x) \Phi_{0,\beta}^\ell(h_\ell - x + y) \Phi_{0,\varkappa}^\ell(y - \nu h_\ell) dx dy \quad \text{substitute } y' = y - \nu h_\ell \\ &= (-1)^\beta \iint \Phi_{0,\alpha}^\ell(x) \Phi_{0,\beta}^\ell((\nu+1)h_\ell - x + y') \Phi_{0,\varkappa}^\ell(y') dx dy' \quad \text{rename } y' \text{ by } y \\ &= (-1)^\beta \iint \Phi_{0,\alpha}^\ell(x) \Phi_{-\nu-1,\beta}^\ell(-x+y) \Phi_{0,\varkappa}^\ell(y) dx dy. \end{aligned}$$

Interchanging the notations of  $x$  and  $y$  yields

$$\begin{aligned} \gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} &= (-1)^\beta \iint \Phi_{0,\alpha}^\ell(y) \Phi_{-k-1,\beta}^\ell(x-y) \Phi_{0,\varkappa}^\ell(x) dx dy \\ &= (-1)^\beta \iint \Phi_{0,\varkappa}^\ell(x) \Phi_{0,\alpha}^\ell(y) \Phi_{-k-1,\beta}^\ell(x-y) dx dy = (-1)^\beta \gamma_{-k-1,(\varkappa,\alpha,\beta)}^{\ell,\ell,\ell}. \end{aligned}$$

Together,  $\gamma_{-k-1,(\varkappa,\alpha,\beta)}^{\ell,\ell,\ell} \stackrel{(C.4b)}{=} (-1)^{\varkappa+\alpha} \gamma_{-k-1,(\alpha,\varkappa,\beta)}^{\ell,\ell,\ell} \stackrel{(C.4a)}{=} (-1)^{\varkappa+\alpha} \gamma_{-k-1,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$  holds proving (C.4c).

c)  $\Phi_{0,\alpha}^\ell(x) = 2^{\ell/2} \Phi_{0,\alpha}^\ell(2^\ell x)$  yields  $\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = 2^{3\ell/2} \iint \Phi_{0,\alpha}^0(2^\ell x) \Phi_{0,\beta}^0(2^\ell y) \Phi_{k,\varkappa}^0(2^\ell(x-y)) dx dy$ . Substitution by  $x' = 2^\ell x$ ,  $y' = 2^\ell y$  leads to the (C.5).

d) The supports of  $\Phi_{0,\beta}^\ell$  and  $\Phi_{k,\varkappa}^\ell$  are  $[0, h_\ell]$  and  $[kh_\ell, (k+1)h_\ell]$ . Hence,  $\Phi_{0,\beta}^\ell * \Phi_{k,\varkappa}^\ell$  has its support in  $[kh_\ell, (k+2)h_\ell]$ . Since  $\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = \langle \Phi_{0,\alpha}^\ell, \Phi_{0,\beta}^\ell * \Phi_{k,\varkappa}^\ell \rangle$  and  $\text{supp}(\Phi_{0,\alpha}^\ell) = [0, h_\ell]$ , nonzero values can only result if  $[0, h_\ell] \subset [kh_\ell, (k+2)h_\ell]$ , i.e.,  $k = -1$  or  $k = 0$ .  $\blacksquare$

Note that (C.4b) implies  $\gamma_{-1,(\alpha,\beta,\varkappa)}^{0,0,0} = (-1)^{\alpha+\beta+\varkappa} \gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$ . Because of this identity, (C.5), and (C.6), the values of  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  suffice to determine all  $\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$ . Due to the symmetry properties (C.4a,c), it is even sufficient to know  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  for  $0 \leq \alpha \leq \beta \leq \varkappa \leq p$ .

**Lemma C.3** a)  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} = 0$  for  $\alpha > \beta + \varkappa + 1$  or  $\varkappa > \alpha + \beta + 1$  or  $\beta > \alpha + \varkappa + 1$ .

b) For all  $\alpha \geq 1, \beta, \varkappa \geq 0$ , the  $\gamma$ -coefficients satisfy the relation

$$\begin{aligned} \gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} &= \frac{a_{\alpha-1}}{a_\varkappa} \gamma_{0,(\alpha-1,\beta,\varkappa+1)}^{0,0,0} + \frac{a_{\alpha-1}}{a_\beta} \gamma_{0,(\alpha-1,\beta+1,\varkappa)}^{0,0,0} \\ &+ a_{\alpha-1} \gamma_{0,(\alpha-1,\beta,\varkappa)}^{0,0,0} + a_{\alpha-1} \frac{b_\varkappa}{a_\varkappa} \gamma_{0,(\alpha-1,\beta,\varkappa-1)}^{0,0,0} + a_{\alpha-1} \frac{b_\beta}{a_\beta} \gamma_{0,(\alpha-1,\beta-1,\varkappa)}^{0,0,0} - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0}. \end{aligned} \quad (C.7)$$

*Proof.* a) For instance for  $\alpha > \beta + \varkappa + 1$  use that  $\Phi_{0,\beta}^\ell * \Phi_{0,\varkappa}^\ell \Big|_{[0,h]}$  is a polynomial of degree  $\leq \beta + \varkappa + 1$  and therefore orthogonal to  $\Phi_{0,\alpha}^\ell$ . Hence,  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} = \langle \Phi_{0,\alpha}^\ell, \Phi_{0,\beta}^\ell * \Phi_{0,\varkappa}^\ell \rangle = 0$ .

b) For the proof of (C.7) we start from the definition

$$\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} = \iint \Phi_{0,\alpha}^0(x) \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy.$$

For  $\alpha \geq 1$  we can make use of  $\Phi_{0,\alpha}^0(x) = a_{\alpha-1}(-1 + 2x/h)\Phi_{0,\alpha-1}^0(x) - b_{\alpha-1}\Phi_{0,\alpha-2}^0(x)$  (use Remark A.3 for  $b_{\alpha-1}\Phi_{0,\alpha-2}^0$  if  $\alpha = 1$ ):

$$\begin{aligned}\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} &= a_{\alpha-1} \iint \left(\frac{2x}{h} - 1\right) \Phi_{0,\alpha-1}^0(x) \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy \\ &\quad - b_{\alpha-1} \iint \Phi_{0,\alpha-2}^0(x) \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy \\ &= a_{\alpha-1} \iint \left(\frac{2x}{h} - 1\right) \Phi_{0,\alpha-1}^0(x) \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0}.\end{aligned}$$

Split  $\frac{2x}{h} - 1$  into  $\frac{2(x-y)}{h} - 1$  and  $\frac{2y}{h}$ :

$$\begin{aligned}\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} &= a_{\alpha-1} \iint \Phi_{0,\alpha-1}^0(x) \Phi_{0,\beta}^0(y) \left(\frac{2(x-y)}{h} - 1\right) \Phi_{0,\varkappa}^0(x-y) dx dy \\ &\quad + a_{\alpha-1} \iint \Phi_{0,\alpha-1}^0(x) \frac{2y}{h} \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0}.\end{aligned}$$

The recursion  $\Phi_{0,\varkappa+1}^0(t) = a_{\varkappa}(-1 + 2t/h)\Phi_{0,\varkappa}^0(t) - b_{\varkappa}\Phi_{0,\varkappa-1}^0(t)$  from (A.4a) yields

$$\left(\frac{2(x-y)}{h} - 1\right) \Phi_{0,\varkappa}^0(x-y) = \frac{\Phi_{0,\varkappa+1}^0(x-y) + b_{\varkappa}\Phi_{0,\varkappa-1}^0(x-y)}{a_{\varkappa}},$$

so that

$$\begin{aligned}\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} &= \frac{a_{\alpha-1}}{a_{\varkappa}} \iint \Phi_{0,\alpha-1}^0(x) \Phi_{0,\beta}^0(y) [\Phi_{0,\varkappa+1}^0(x-y) + b_{\varkappa}\Phi_{0,\varkappa-1}^0(x-y)] dx dy \\ &\quad + a_{\alpha-1} \iint \Phi_{0,\alpha-1}^0(x) \frac{2y}{h} \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0} \\ &= \frac{a_{\alpha-1}}{a_{\varkappa}} \gamma_{0,(\alpha-1,\beta,\varkappa+1)}^{0,0,0} + \frac{a_{\alpha-1}b_{\varkappa}}{a_{\varkappa}} \gamma_{0,(\alpha-1,\beta,\varkappa-1)}^{0,0,0} \\ &\quad + a_{\alpha-1} \iint \Phi_{0,\alpha-1}^0(x) \frac{2y}{h} \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0}.\end{aligned}$$

Similarly, we split  $\frac{2y}{h}$  in  $\frac{2y}{h} - 1$  plus 1 and use  $(\frac{2y}{h} - 1)\Phi_{0,\beta}^0(y) = \frac{\Phi_{0,\beta+1}^0(y) + b_{\beta}\Phi_{0,\beta-1}^0(y)}{a_{\beta}}$ :

$$\begin{aligned}\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} &= a_{\alpha-1} \iint \Phi_{0,\alpha-1}^0(x) \frac{2y}{h} \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy \\ &\quad + a_{\alpha-1} \gamma_{0,(\alpha-1,\beta,\varkappa+1)}^{0,0,0} + b_{\varkappa} \gamma_{0,(\alpha-1,\beta,\varkappa-1)}^{0,0,0} - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0} \\ &= \frac{a_{\alpha-1}}{a_{\beta}} \iint \Phi_{0,\alpha-1}^0(x) [\Phi_{0,\beta+1}^0(y) + b_{\beta}\Phi_{0,\beta-1}^0(y)] \Phi_{0,\varkappa}^0(x-y) dx dy \\ &\quad + a_{\alpha-1} \iint \Phi_{0,\alpha-1}^0(x) \Phi_{0,\beta}^0(y) \Phi_{0,\varkappa}^0(x-y) dx dy \\ &\quad + \frac{a_{\alpha-1}}{a_{\varkappa}} \gamma_{0,(\alpha-1,\beta,\varkappa+1)}^{0,0,0} + \frac{a_{\alpha-1}b_{\varkappa}}{a_{\varkappa}} \gamma_{0,(\alpha-1,\beta,\varkappa-1)}^{0,0,0} - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0} \\ &= \frac{a_{\alpha-1}}{a_{\beta}} \left( \gamma_{0,(\alpha-1,\beta+1,\varkappa)}^{0,0,0} + b_{\beta} \gamma_{0,(\alpha-1,\beta-1,\varkappa)}^{0,0,0} \right) + a_{\alpha-1} \gamma_{0,(\alpha-1,\beta,\varkappa)}^{0,0,0} \\ &\quad + \frac{a_{\alpha-1}}{a_{\varkappa}} \gamma_{0,(\alpha-1,\beta,\varkappa+1)}^{0,0,0} + \frac{a_{\alpha-1}b_{\varkappa}}{a_{\varkappa}} \gamma_{0,(\alpha-1,\beta,\varkappa-1)}^{0,0,0} - b_{\alpha-1} \gamma_{0,(\alpha-2,\beta,\varkappa)}^{0,0,0}.\end{aligned}$$

This proves (C.7). ■

To make use of the recursion (C.7), one need to know  $\gamma_{0,(0,\beta,\varkappa)}^{0,0,0}$  for  $\alpha = 0$ . These values are easy to describe.

**Lemma C.4** All coefficients  $\gamma_{0,(0,\beta,\varkappa)}^{0,0,0}$ ,  $0 \leq \beta \leq \varkappa$ , are zero except the following ones:

$$\begin{aligned} \gamma_{0,(0,0,0)}^{0,0,0} &= \frac{1}{2}\sqrt{h}, \\ \gamma_{0,(0,\varkappa-1,\varkappa)}^{0,0,0} &= (-1)^\varkappa \frac{1}{2\sqrt{(2\varkappa-1)(2\varkappa+1)}}\sqrt{h} \quad \text{for } \varkappa \in \mathbb{N}. \end{aligned} \tag{C.8}$$

*Proof.* a)  $\gamma_{0,(0,0,0)}^{0,0,0} = \frac{1}{2}\sqrt{h}$  and  $\gamma_{0,(0,0,1)}^{0,0,0} = -\frac{1}{2\sqrt{3}}\sqrt{h}$  follow by direct calculation. In the following we consider the case  $\varkappa \geq 2$ .

b) The symmetry properties (C.4a,b) show that the second statement is equivalent to

$$\gamma_{0,(\varkappa,\varkappa-1,0)}^{0,0,0} = \frac{1}{2\sqrt{(2\varkappa-1)(2\varkappa+1)}}\sqrt{h}.$$

The definition of  $\gamma_{0,(\varkappa,\varkappa-1,0)}^{0,0,0}$  is  $\int_0^h \Phi_{0,\varkappa}^0(x) \left( \int_0^h \Phi_{0,\varkappa-1}^0(y) \Phi_{0,0}^0(x-y) dy \right) dx$ . By  $\Phi_{0,0}^0(x) = 1/\sqrt{h}$  in  $[0, h]$  and  $\Phi_{0,0}^0(x) = 0$  outside, we have

$$(\Phi_{0,\varkappa-1}^0 * \Phi_{0,0}^0)(x) = \int_0^h \Phi_{0,\varkappa-1}^0(y) \Phi_{0,0}^0(x-y) dy = \frac{1}{\sqrt{h}} \int_0^x \Phi_{0,\varkappa-1}^0(y) dy \quad \text{for } 0 \leq x \leq h.$$

Hence,  $\Phi_{0,\varkappa-1}^0 * \Phi_{0,0}^0$  restricted to  $[0, h]$  is the antiderivative of  $\Phi_{0,\varkappa-1}^0$ . The relation between  $\Phi_{0,\varkappa-1}^0$  and the Legendre polynomial  $P_{\varkappa-1}$  allows to translate the formula (A.1c) into

$$\frac{1}{\sqrt{h}} \int_0^x \Phi_{0,\varkappa-1}^0(y) dy = \frac{\sqrt{h}}{2\sqrt{(2\varkappa-1)(2\varkappa+1)}} \Phi_{0,\varkappa}^0 - \frac{\sqrt{h}}{2\sqrt{(2\varkappa-3)(2\varkappa-1)}} \Phi_{0,\varkappa-2}^0 \quad \text{for } 0 \leq x \leq h.$$

Hence, the scalar product with  $\Phi_{0,\nu}^0$  yields

$$\gamma_{0,(\nu,\varkappa-1,0)}^{0,0,0} = \int_0^h \Phi_{0,\nu}^0(x) \frac{1}{\sqrt{h}} \int_0^x \Phi_{0,\varkappa-1}^0(y) dy dx = \begin{cases} 0 & \text{for } \nu \notin \{\varkappa, \varkappa-2\}, \\ \frac{\sqrt{h}}{2\sqrt{(2\varkappa-1)(2\varkappa+1)}} & \text{for } \nu = \varkappa, \\ \frac{-\sqrt{h}}{2\sqrt{(2\varkappa-3)(2\varkappa-1)}} & \text{for } \nu = \varkappa-2. \end{cases}$$

Together with (C.4c) the assertion follows. ■

**Remark C.5** The recursion (C.7) for computing  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  proceeds with increasing  $k := \alpha + \beta + \varkappa = 0, 1, 2, \dots$ . For fixed  $k$ , the loop runs from  $\alpha = 1$  to  $k$  using the values (C.8) for  $\alpha = 0$ .

We remark that  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  can be represented by  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0} = \gamma_{\alpha,\beta,\varkappa}^* \sqrt{\frac{h}{(2\alpha+1)(2\beta+1)(2\varkappa+1)}}$  with rational numbers  $\gamma_{\alpha,\beta,\varkappa}^*$ . However, the involved integers in the nominators and denominators are not as small as in Table B.1.

In the following we give the first values of  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$ . Here, the underlying step size  $h = h_0$  is chosen as  $h = 1$ . For other step sizes the given values  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  must be multiplied by  $\sqrt{h}$ . The values in Table C.1 belong to the indices  $0 \leq \alpha, \beta, \varkappa \leq 5$  and can be used in the Cases A,B if  $p \leq 4$ . In Case C, the convolution yields polynomial up to degree  $2p + 1$ . There  $p \leq 4$  requires coefficients  $\gamma_{\alpha,\beta,\varkappa}^*$  in the range  $0 \leq \alpha \leq 11$ ,  $0 \leq \beta, \varkappa \leq 5$ . Table C.2 shows the missing coefficients for  $6 \leq \alpha \leq 11$ .

values $\gamma_{0,(0,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 0$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	5.000000000E-1	-2.886751346E-1	0	0	0	0
1	-2.886751346E-1	0	1.290994449E-1	0	0	0
2	0	1.290994449E-1	0	-8.451542547E-2	0	0
3	0	0	-8.451542547E-2	0	6.299407883E-2	0
4	0	0	0	6.299407883E-2	0	-5.025189076E-2
5	(see Lemma C.4)	0	0	0	-5.025189076E-2	0

values $\gamma_{0,(1,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 1$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	2.886751346E-1	0	-1.290994449E-1	0	0	0
1	0	-3.464101615E-1	2.236067977E-1	3.779644730E-2	0	0
2	-1.290994449E-1	2.236067977E-1	8.247860988E-2	-1.463850109E-1	-1.844277784E-2	0
3	0	3.779644730E-2	-1.463850109E-1	-3.849001795E-2	1.091089451E-1	1.096586158E-2
4	0	0	-1.844277784E-2	1.091089451E-1	2.249416633E-2	-8.703882798E-2
5	0	0	0	1.096586158E-2	-8.703882798E-2	-1.480385306E-2

values $\gamma_{0,(2,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 2$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	0	1.290994449E-1	0	-8.451542547E-2	0	0
1	1.290994449E-1	-2.236067977E-1	-8.247860988E-2	1.463850109E-1	1.844277784E-2	0
2	0	-8.247860988E-2	3.194382825E-1	-1.259881577E-1	-7.142857143E-2	-7.178841538E-3
3	-8.451542547E-2	1.463850109E-1	-1.259881577E-1	-1.490711985E-1	1.178093810E-1	4.247059929E-2
4	0	1.844277784E-2	-7.142857143E-2	1.178093810E-1	8.711953159E-2	-1.012534592E-1
5	0	0	-7.178841538E-3	4.247059929E-2	-1.012534592E-1	-5.733507635E-2

values $\gamma_{0,(3,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 3$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	0	0	8.451542547E-2	0	-6.299407883E-2	0
1	0	3.779644730E-2	-1.463850109E-1	-3.849001795E-2	1.091089451E-1	1.096586158E-2
2	8.451542547E-2	-1.463850109E-1	1.259881577E-1	1.490711985E-1	-1.178093810E-1	-4.247059929E-2
3	0	-3.849001795E-2	1.490711985E-1	-2.725925593E-1	3.030303030E-2	8.504166129E-2
4	-6.299407883E-2	1.091089451E-1	-1.178093810E-1	3.030303030E-2	1.823744660E-1	-6.720751945E-2
5	0	1.096586158E-2	-4.247059929E-2	8.504166129E-2	-6.720751945E-2	-1.259881577E-1

values $\gamma_{0,(4,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 4$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	0	0	0	6.299407883E-2	0	-5.025189076E-2
1	0	0	1.844277784E-2	-1.091089451E-1	-2.249416633E-2	8.703882798E-2
2	0	1.844277784E-2	-7.142857143E-2	1.178093810E-1	8.711953159E-2	-1.012534592E-1
3	6.299407883E-2	-1.091089451E-1	1.178093810E-1	-3.030303030E-2	-1.823744660E-1	6.720751945E-2
4	0	-2.249416633E-2	8.711953159E-2	-1.823744660E-1	2.007992008E-1	4.969967218E-2
5	-5.025189076E-2	8.703882798E-2	-1.012534592E-1	6.720751945E-2	4.969967218E-2	-1.794871795E-1

values $\gamma_{0,(5,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 5$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	0	0	0	0	5.025189076E-2	0
1	0	0	0	1.096586158E-2	-8.703882798E-2	-1.480385306E-2
2	0	0	7.178841538E-3	-4.247059929E-2	1.012534592E-1	5.733507635E-2
3	0	1.096586158E-2	-4.247059929E-2	8.504166129E-2	-6.720751945E-2	-1.259881577E-1
4	5.025189076E-2	-8.703882798E-2	1.012534592E-1	-6.720751945E-2	-4.969967218E-2	1.794871795E-1
5	0	-1.480385306E-2	5.733507635E-2	-1.259881577E-1	1.794871795E-1	-1.150563653E-1

Table C.1: Values of  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  for  $0 \leq \alpha, \beta, \varkappa \leq 5$



values $\gamma_{0,(6,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 6$						
$\beta$	$\varkappa = 0$	1	2	3	4	5
0	0	0	0	0	0	4.181210050E-2
1	0	0	0	0	7.278552446E-3	-7.242068244E-2
2	0	0	0	3.551569931E-3	-2.818971241E-2	8.726171907E-2
3	0	0	3.551569931E-3	-2.101137107E-2	6.035579363E-2	-7.374961314E-2
4	0	7.278552446E-3	-2.818971241E-2	6.035579363E-2	-7.564093585E-2	8.198451078E-3
5	4.181210050E-2	-7.242068244E-2	8.726171907E-2	-7.374961314E-2	8.198451078E-3	1.033265071E-1

values $\gamma_{0,(7,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 7$						
$\beta$	$\varkappa = 1$	2	3	4	5	
1	( $\gamma = 0$ in rows/columns $\beta, \varkappa = 0$ )			0	0	5.186152788E-3
2	0	0	0	2.018707235E-3	-2.008588338E-2	
3	0	0	1.504655534E-3	-1.194283306E-2	4.442521691E-2	
4	0	2.018707235E-3	-1.194283306E-2	3.664278773E-2	-6.516846365E-2	
5	5.186152788E-3	-2.008588338E-2	4.442521691E-2	-6.516846365E-2	4.683531500E-2	

values $\gamma_{0,(8,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 8$					
$\beta$	$\varkappa = 2$	3	4	5	
2	( $\gamma = 0$ in rows/columns $\beta, \varkappa \leq 1$ )			0	1.257826807E-3
3	0	0	7.478892157E-4	-7.441403750E-3	
4	0	7.478892157E-4	-5.936186627E-3	2.358624958E-2	
5	1.257826807E-3	-7.441403750E-3	2.358624958E-2	-4.811435476E-2	

values $\gamma_{0,(9,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 9$					
$\beta$	$\varkappa = 0, 1, 2$	3	4	5	
3	( $\gamma = 0$ in rows/columns $\beta, \varkappa \leq 2$ )			0	4.140507769E-4
4	0	0	3.302982550E-4	-3.286426189E-3	
5	0	4.140507769E-4	-3.286426189E-3	1.349504317E-2	

values $\gamma_{0,(10,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 10$				
$\beta$	$\varkappa = 0, 1, 2, 3$	4	5	
4	( $\gamma = 0$ in rows/columns $\beta, \varkappa \leq 3$ )		0	1.645270785E-4
5	0	1.645270785E-4	-1.637023960E-3	

values $\gamma_{0,(11,\beta,\varkappa)}^{0,0,0}$ for $\alpha = 11$			
$\beta$	$\varkappa = 0, 1, 2, 3, 4$	5	
5	( $\gamma = 0$ in rows/columns $\beta, \varkappa \leq 4$ )		0
			7.448707785E-5

Table C.2: Values of  $\gamma_{0,(\alpha,\beta,\varkappa)}^{0,0,0}$  for  $6 \leq \alpha \leq 11$ ,  $0 \leq \beta, \varkappa \leq 5$

## D Coefficients $G$ and $\Gamma$

The coefficients

$$G_{(i,\alpha),(j,\beta)}^{\ell'',\ell',\ell} := \sum_{k \in \mathbb{Z}} \sum_{\varkappa=0}^p g_{k,\varkappa}^\ell \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell} = \sum_{k \in \mathbb{Z}} \sum_{\varkappa=0}^p g_{k,\varkappa}^\ell \gamma_{k-i, 2^{\ell-\ell''}+j, 2^{\ell-\ell'}+(\alpha,\beta,\varkappa)}^{\ell'',\ell',\ell} \quad (\text{D.1})$$

(last equality only for  $\ell \geq \max\{\ell', \ell''\}$ ) allow to rewrite (3.1b) in the form

$$\omega_{i,\alpha}^{\ell''} = \sum_{j,k \in \mathbb{Z}} \sum_{\beta,\varkappa=0}^p f_{j,\beta}^{\ell'} G_{(i,\alpha),(j,\beta)}^{\ell'',\ell',\ell}.$$

Combining the definition (D.1) of  $G_{(i,\alpha),(j,\beta)}^{\ell'',\ell',\ell}$  with the recursion formulae (C.2a,b) for  $\gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}$ , one obtains the following recursion formulae for  $G_{(i,\alpha),(j,\beta)}^{\ell'',\ell',\ell}$ .

**Remark D.1** For all  $\ell'', \ell', \ell \in \mathbb{N}_0$  and all  $i, j \in \mathbb{Z}$  we have

$$G_{(i,\alpha),(j,\beta)}^{\ell'',\ell',\ell} = \sum_{q=0}^{\alpha} \xi_{\alpha,q} \left( (-1)^{\alpha+q} G_{(2i,q),(j,\beta)}^{\ell''+1,\ell',\ell} + G_{(2i+1,q),(j,\beta)}^{\ell''+1,\ell',\ell} \right) \quad (\text{D.2a})$$

$$= \sum_{q=0}^{\alpha} \xi_{\beta,q} \left( (-1)^{\beta+q} G_{(i,\alpha),(2j,q)}^{\ell'',\ell'+1,\ell} + G_{(i,\alpha),(2j+1,q)}^{\ell'',\ell'+1,\ell} \right). \quad (\text{D.2b})$$

For the case  $\ell'' = \ell' \leq \ell$ , one concludes from the definition (D.1) that  $G_{(i,\alpha),(j,\beta)}^{\ell',\ell',\ell}$  depends only on the difference  $i - j$ , i.e.,  $G_{(i,\alpha),(j,\beta)}^{\ell',\ell',\ell} = G_{(i-j,\alpha),(0,\beta)}^{\ell',\ell',\ell} = \Gamma_{i-j,(\alpha,\beta)}^{\ell',\ell}$  with  $\Gamma_{i-j,(\alpha,\beta)}^{\ell',\ell}$  defined in (3.4).

The essential identities (3.7a,b) of the  $\Gamma$ -coefficients  $\Gamma_{i,(\alpha,\beta)}^{\ell',\ell}$  are repeated in the next lemma.

**Lemma D.2** a) For  $\ell' = \ell$  we have

$$\Gamma_{i,(\alpha,\beta)}^{\ell,\ell} = \sum_{\varkappa=0}^p \left( g_{i,\varkappa}^\ell + (-1)^{\alpha+\beta+\varkappa} g_{i-1,\varkappa}^\ell \right) \gamma_{0,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} \quad \text{for all } i \in \mathbb{Z}, 0 \leq \alpha, \beta \leq p.$$

b) For  $0 \leq \ell' < \ell$ , the following relation holds for all  $i \in \mathbb{Z}$  and all  $0 \leq \alpha, \beta \leq p$ :

$$\Gamma_{i,(\alpha,\beta)}^{\ell',\ell} = \sum_{p=0}^{\alpha} \sum_{q=0}^{\beta} \xi_{\alpha,p} \xi_{\beta,q} \left( (-1)^{\alpha+p} \Gamma_{2i-1,(p,q)}^{\ell'+1,\ell} + \left( 1 + (-1)^{\alpha+\beta+p+q} \right) \Gamma_{2i,(p,q)}^{\ell'+1,\ell} + (-1)^{\beta+q} \Gamma_{2i+1,(p,q)}^{\ell'+1,\ell} \right).$$

*Proof.* a) Since  $\gamma_{k,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} = 0$  for  $k \notin \{-1, 0\}$ , the definition (3.4) yields

$$\Gamma_{i,(\alpha,\beta)}^{\ell,\ell} = \sum_{\varkappa=0}^p \left( g_{i,\varkappa}^\ell \gamma_{0,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} + g_{i-1,\varkappa}^\ell \gamma_{-1,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} \right).$$

The symmetry property (C.4c) allows to express  $\gamma_{-1,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$  as  $(-1)^{\alpha+\beta+\varkappa} \gamma_{0,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}$ .

b) Application of (D.2a) to  $\Gamma_{i,(\alpha,\beta)}^{\ell',\ell} = G_{(i,\alpha),(0,\beta)}^{\ell',\ell',\ell}$  yields

$$\Gamma_{i,(\alpha,\beta)}^{\ell',\ell} = \sum_{q=0}^{\alpha} \xi_{\alpha,q} \left( (-1)^{\alpha+q} G_{(2i,q),(j,\beta)}^{\ell'+1,\ell',\ell} + G_{(2i+1,q),(j,\beta)}^{\ell'+1,\ell',\ell} \right).$$

Now (D.2b) can be applied to both  $G_{(*,q),(*,\beta)}^{\ell'+1,\ell',\ell}$ -terms. The resulting  $G_{(\nu,q),(\mu,\beta)}^{\ell'+1,\ell'+1,\ell}$ -terms equal  $\Gamma_{\nu-\mu,(q,\beta)}^{\ell',\ell}$  and yield the expression in part b).  $\blacksquare$