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by

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Abstract

In this paper, we introduce a model of adaptive dynamical network by integrating the complex network model and adaptive technique. This model is characterized by that the adaptive updating laws for each vertex in the network depend only on the state information of its neighborhood besides itself and external controllers. This suggests that adaptive technique be added to a complex network without breaking its intrinsic existing network topology. The core of adaptive dynamical networks is to design suitable adaptive updating laws to attain certain aims. Here, we propose two series of adaptive laws to synchronize and pin a complex network respectively. Based on the Lyapunov function method, we can prove that under several mild conditions, with the adaptive technique, a connected network topology is sufficient to synchronize or stabilize any chaotic dynamics of the uncoupled system. This implies that these adaptive updating laws actually enhance synchronizability and stabilizability respectively. We find out that even though these adaptive methods can success for all networks with connectivity, the underlying network topology can affect the convergent rate and the terminal average coupling and pinning strength. And, this influence can be measured by the smallest nonzero eigenvalue of the corresponding Laplacian. Moreover, we detailed study the influence of the prior parameters in this adaptive laws and present several numerical examples to verify our theoretical results and further discussions.

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Synchronization and control of complex networks has been one of the focal points in many research and application fields. Most of such research has been focused so far on networks with prior given coupling and controlling parameters (for example, coupling and pinning strengths). The choice of such parameters is determined by the network synchronizability, which can be characterized by the eigenvalue information of the corresponding Laplacian. In this paper, we propose a model of adaptive dynamical network for which the coupling and pinning strengths as well as other parameters of each vertex are adapted only dependently on the state information of its neighborhood and itself as well as external controllers. By this way, synchronization and pinning control can be achieved with any initial coupling strengths, mismatched parameters, and network topology of connectivity. That is to say, this adaptive technique actually enhances synchronizability or stabilizability of the network. However, we also find out that the underlying network topology can affect the convergent rate and the terminal average of coupling and pinning strengths.

I. INTRODUCTION AND MODEL DESCRIPTION

Models of complex networks have been widely used to describe systems in science, engineering, and nature. Typical examples of complex networks include the Internet, WWW, food webs, cellular and metabolic networks [1–4]. As an implicit assumption, these networks are described by the mathematical term of graph. In such graphs, each vertex represents an individual element in the system, while edges represent the relations between them. That is, each vertex can receive the state information from its neighbors and send its state information to all its neighbors simultaneously. Thus, the evolution of the dynamics of each vertex is derived by two factors: its own dynamical property and its neighborhood's. Generally, complex networks can be formalized as:

$$\dot{x}^i(t) = f^i(x^i, c_i\phi^i(\{x^j:\ j \in \mathcal{N}(i)\}), t, q^i),\ i = 1, \cdots, m, \tag{1}$$

where x^i is the state variable vector of vertex i, $t \in \mathbb{R}^+$ is the continuous time, ϕ^i is the coupling function which indicates the interactions from the neighborhood to vertex i, $c_i \in \mathbb{R}$ is the coupling strength of vertex i, $\mathcal{N}(i)$ denotes the neighborhood of the vertex i, and q^i is the parameter vector characterizing the difference between vertices, $i = 1, \cdots, m$. A prime example, linearly coupled identical oscillators formulated as:

$$\dot{x}^i = g(x^i, t) + c \sum_{j \in \mathcal{N}(i)} [H(x^j) - H(x^i)],\ i = 1, \cdots, m, \tag{2}$$

where $g(\cdot,\cdot)$ describes the dynamics of each individual oscillator, c is the common coupling strength, and $H(\cdot)$ is the output function, were widely studied in the recent last few years [5–7]. If $H(\cdot)$ is a linear function, i.e., $H(u) = \Gamma u$, where $\Gamma \in \mathbb{R}^{n,n}$, then the coupled system (2) has a special form named by linearly coupled ordinary differential systems which can be written as:

$$\dot{x}^i = g(x^i, t) + c \sum_{j\in\mathcal{N}(i)} \Gamma(x^j - x^i),\ i = 1, \cdots, m. \tag{3}$$

See [8–11] for references.

Besides many existing studies of this issue which have been only focused on networks with a regular structure, such as chains, lattices, and grids, the new discoveries of small-world phenomena [12] and scale-free features [13] in many natural and artificial complex networking systems have recently arose very wide concentrations on the complexity of graph topology by scientists from various fields [2, 4], yielding fruitful studies which enrich and deepen the understanding of real-world complex networks, an important step forward from the random graph theory built in [14].

From then on, control and synchronization of complex networks have been one of the focal points in many research and application fields [5–7, 15–22]. Most of all, how the collective dynamics of a large ensemble of dynamical systems depend on the complex wiring topology is the most important question [4]. In [6, 7, 16], the master stability function based on the transverse Lyapunov exponents is used to study local synchronization. Global synchronization of coupled nonlinear dynamical systems are investigated by introducing a distance between the collective states in [10, 11, 23, 24]. In [25, 26], the authors have stabilized a complex dynamical network to its homogenous static state with a small fraction of vertices placing local feedback controllers, i.e., the so-called pinning control algorithm in [20, 21]. The results of previous literature indicate that the region of parameters such as coupling or pinning strengths for which the complex networks can be synchronized, stabilized, or achieve other dynamical aims is significantly affected by the nontrivial topological patterns of complex networks, for example, the eigenvalues of the corresponding Laplacian [4, 6, 9, 26]. For some network topology which has weak synchronizability (or stabilizability), the lower-bound of the region is relatively large in the case of linear output function (3) or even null in the case of general nonlinear output function (2). Then, a question arises how to achieve synchronization, stability, or other aims for most network whether they have "good" topologies or not, and without knowing global information of the coupling topology.

One method to solve this problem introduced recently is adding weights to vertices and edges. Different from the model (2), which has a common coupling strength for all vertices and edges, in

[27–30], the vertices or edges are set with different strengths. As evidences, certain weighting procedures which break the symmetry of the coupling matrix can actually enhance synchronization in scale-free networks [28, 29]. Furthermore, [27, 30] show that for random complex networks, synchronizability is drastically enhanced when the heterogeneity of the distribution of vertex's intensity is reduced. As a basic condition, [31] points out that under several mild conditions, a directed and weighted network can completely synchronize the coupled system (3) for sufficiently large coupling strengths if and only if the underlying graph has a spanning tree. Another method to enhance synchronizability is evolving the network topology through the same time scale as its dynamics. [32, 33] prove that if the average networks topology is able to synchronize the coupled system, then the coupled system can be synchronized when the switch period is sufficiently small. [34] evolves the network graph through time providing the coupling matrices commute and finds out that the synchronizability of the graph process can be significantly improved. All these temporal variations of topology are independent on the dynamics.

During the last several decades, adaptive technique has emerged as an exciting research area for nonlinear system control. Rapid and impressive developments have been witnessed to utilize adaptive technique to stabilize [35–37], synchronize [38–41] nonlinear systems, and identifying unknown parameters [40–43]. For more details, the interested readers can be referred to [44]. This technique suggests a way to realize such aims without knowing much information about the object systems. Besides the controllers, the core of this technique is to design suitable adaptive update laws for the parameters according to the control problem. An idea arises to utilize the adaptive technique to complex dynamical networks to achieve the given results such as stability and synchronization without knowing much information of the network topology. In [45, 46], the authors presented methods to synchronize or stabilize a dynamical network by setting adaptive controller to all or most vertices. In [47], the adaption is realized by knowing the global dynamical information of the network, i.e., all states of vertices. However, when considering complex networks, it seems very expensive to be realized.

Therefore, it is natural to conceive a dynamical network model of which the adaption is carried out following the intrinsic network topology. In such adaptive dynamical networks, on each vertex the weight evolves only by the signals received from its neighborhood. This implies that the added adaption shall not destroy the existing network topology. For the model (1), adaptive dynamical network is characterized by that the coupling strength and parameter vector of each vertex is adapted via the feedback information of its neighborhood and itself. Generally, design certain adaptive updating laws $\xi^i(\cdot)$ and $\zeta^i(\cdot)$ by

$$\begin{cases} \dot{q}^i = \xi^i(x^i, \{x^j,\ j \in \mathcal{N}(i)\}) \\ \dot{c}_i = \zeta^i(x^i, \{x^j,\ j \in \mathcal{N}(i)\}) \end{cases},\ i = 1, \cdots, m, \tag{4}$$

to realize the given task. [48, 49] proposed adaptive schemes for the coupling weights of vertices or edges according to the local synchronization property and indicate that these methods can enhance synchronizability. These adaptive schemes seem intuitionistic, simple, generic, and available, but can not be rigorously proved to be universally available [55].

In this paper, we consider the model of a linearly coupled array of nonidentical systems of which the nonidentity is characterized by mismatched parameters. Thus, the linearly coupled systems can be formulated as:

$$\dot{x}^i = g(x^i, t) + f(x^i, t)q^i + c_i \sum_{j \in \mathcal{N}(i)} \Gamma(x^j - x^i),\ i = 1, \cdots, m, \tag{5}$$

where $g(z,t) = [g_1(z,t), \cdots, g_n(z,t)]^\top : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a common continuous vector function, $f(z,t) = [f_1(z,t), \cdots, f_d(z,t)] : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n,d}$ is a continuous matrix function involved by mismatched parameter vector, $q^i = [q^i_1, \cdots, q^i_d]^\top \in \mathbb{R}^d$ is the parameter vector of vertex i, c_i is the coupling strength of vertex i, and $\Gamma = [\gamma_{kl}]_{k,l=1}^n \in \mathbb{R}^{n,n}$ denotes the inner connection matrix of which the element $\gamma_{kl} \neq 0$ implies that the l-th component of vertex j is connected to the k-th component of vertex i by coefficient γ_{kl} if the vertices i and j are connected.

Two problems are studied in this paper. One is to synchronize the dynamical network (5). The other is to pin the whole dynamical network to certain object trajectory $s(t)$ by the following pinning controllers added to a small fraction of vertices [25, 26]:

$$\begin{cases} \dot{x}^i = g(x^i, t) + f(x^i, t)q^i + c_i \sum\limits_{j \in \mathcal{N}(i)} \Gamma(x^j - x^i) + u_i & i \in \mathcal{D}, \\ \dot{x}^k = g(x^k, t) + f(x^k, t)q^k + c_k \sum\limits_{l \in \mathcal{N}(k)} \Gamma(x^l - x^k) & k \notin \mathcal{D}, \\ u_i = \epsilon_i \Gamma(s - x^i) & i \in \mathcal{D}, \end{cases} \tag{6}$$

where u_i is the pinning controller to the pinned vertex i, ϵ_i is the pinning strength of the pinned vertex i, $i \in \mathcal{D}$, and $\mathcal{D} \subset \mathcal{N}$ is the pinned vertex set.

Based on the Lyapunov method, we design suitable adaptive updating laws to realize such aims. For each vertex, these adaptive updating laws only depend on the state information of its neighborhood and itself. This implies that these adaptive technique are realized to the dynamical network without breaking its intrinsic network topology. We can prove that connectivity of network is sufficient to guarantee synchronizing or stabilizing the coupled system (5) or (6) respectively since following these proposed adaptive updating laws, the potential difference between the collection of the states characterized by a candidate Lyapunov function decreases.

Furthermore, we will show that even though these adaptive technique can achieve desired synchronization or stability result in spite of the network topology, the network topology and the choice of the prior parameters actually affect the convergent rate and the average of the terminal coupling strengths which can be used to measure the synchronization or stability performance. We detailed analyze the dependence of the performance on the network topology and several prior parameters by estimating the upper-bounded of such quantities. We show that the smallest nonzero eigenvalue λ_2 of the corresponding Laplacian or the modified Laplacian plays the key role in the synchronization or pinning control problem respectively. A larger λ_2 might imply a faster convergence and a smaller coupling cost. In addition, the prior control parameters also affect these quantities.

This paper is organized as follows. In Section 2, we introduce the necessary definitions, lemmas, and notations which will be used throughout this paper. We give adaptive updating laws, prove that they can realize synchronization and pinning control if the network has connected property, and afterwards provide the upper-bounded estimations of the proposed quantities measuring performance, for synchronization and pinning control in Section 3 and 4 respectively. In Section 5, we present several numerical examples to verify our theoretical results and further discussions. We conclude this paper in Section 6.

II. PRELIMINARIES

In this section, we give necessary notations, definitions, and lemmas which will be used throughout the paper. If all eigenvalues of a matrix $A \in \mathbb{R}^{n,n}$ are real numbers, $\lambda_i(A)$ denotes its i-th smallest eigenvalue and $\lambda_{\max}(A)$ denotes the maximum one. $\top$ denotes the transpose of matrix. $\|v\|$ denotes some norm of vector v and $\|A\|$ denotes the norm of matrix A induced by this vector norm. For instance, $\|v\|_2 = \sqrt{\sum\limits_{i=1}^{n} |v_i|^2}$ for a vector $v = [v_1, \cdots, v_n]^\top \in \mathbb{R}^n$ and $\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)}$ for a matrix $A \in \mathbb{R}^{m,n}$. $\times$ denotes the Cartesian product of linear spaces. $\otimes$ denotes the Kroneck product, i.e., for two matrices $A = [a_{ij}]_{i,j=1}^{m} \in \mathbb{R}^{m,m}$ and $B = [b_{kl}]_{k,l=1}^{n} \in \mathbb{R}^{n,n}$, $A \otimes B = [a_{ij}B]_{i,j=1}^{m} \in \mathbb{R}^{mn,nm}$. $\mathbb{R}^+ = [0, +\infty)$. I_m denotes the identity matrix with order m. $\#\mathcal{D}$ denotes the element number of finite set $\mathcal{D}$. $\delta_{\mathcal{D}}(\cdot)$ denotes the characteristic function of set $\mathcal{D}$, i.e., $\delta_{\mathcal{D}}(i) = 1$ if $i \in \mathcal{D}$; otherwise, $\delta_{\mathcal{D}}(i) = 0$. $A \backslash B = A \cap B^c$ denotes the set difference from set A to set B. The approximation expression $a(t) \sim b(t)$ as $t \to \infty$ means $\lim_{t\to\infty} a(t)/b(t) = 1$ in the case $b(t) \neq 0$. We denote by $A \geq 0$ in the case that the symmetric matrix A is semi-positive definite. So it is with $\leq$, $>$, and $<$.

Define a graph by $G = [\mathcal{V}, \mathcal{E}]$, where $\mathcal{V} = \{1, \cdots, m\}$ denotes the vertex set and $\mathcal{E} = \{e(i,j)\}$ the edge set. $\mathcal{N}(i)$ denotes the neighborhood of vertex i in the sense $\mathcal{N}(i) = \{j \in \mathcal{N} : \ e(i,j) \in \mathcal{E}\}$. In this paper, graph G is supposed to be undirected ($e(i,j) \in \mathcal{E}$ implies $e(j,i) \in \mathcal{E}$) and simple (without loops and multiple edges). Let $L = [l_{ij}]_{i,j=1}^m$ be the Laplacian matrix of graph G defined as follows: for any pair $i \neq j$, $l_{ij} = l_{ji} = -1$ if $e(i,j) \in \mathcal{E}$; otherwise, $l_{ij} = l_{ji} = 0$. $l_{ii} = -\sum_{j=1, j\neq i}^m l_{ij}$ represents the degree of vertex i, for $i = 1, \cdots, m$. Thus, equivalently, the system (5) can be rewritten as:

$$\dot{x}^i = g(x^i, t) + f(x^i, t) q^i - c_i \sum_{j=1}^m l_{ij} \Gamma (x^j - x^i), \ i = 1, \cdots, m. \tag{7}$$

From the graph theory (see [50] for reference) and the Gerishgorin disc theorem [51], all eigenvalues of the Laplacian L corresponding graph G satisfy $0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_m(L)$. And, G is connected if and only if $\lambda_2(L) > 0$, namely, L is irreducible. Also, the following result can be derived.

Lemma 1 *For a graph $G = [\mathcal{V}, \mathcal{E}]$ and some vertex set $\mathcal{D} \subset \mathcal{V}$, the following statements are equivalent: (i) all vertices $\mathcal{V} \backslash \mathcal{D}$ can be accessible from the vertex set $\mathcal{D}$, i.e., for any vertex i in $\mathcal{V} \backslash \mathcal{D}$, there exists an vertex $j \in \mathcal{D}$ such that there exists an path between the vertices i and j; (ii) define a new graph $\tilde{G} = [\tilde{V}, \tilde{E}]$ where $\tilde{V} = \mathcal{V} \bigcup \{v_0\}$ and $\tilde{E} = \mathcal{E} \bigcup \{e(i,j) : \ i \in \mathcal{D} \text{ and } j = v_0\}$ such that $\tilde{G}$ is connected.*

A matrix-valued function $h(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{m,k}$ is said to be *locally uniformly bounded* on z with respect to t if for any compact $Z \subset \mathbb{R}^n$, there exists $M > 0$ such that $\|h(x,t)\| \leq M$ holds for all $x \in Z$ and $t \geq 0$. And $h(x,t)$ is said to be *locally uniformly Lipschitz continuous* on z with respect to t if for any compact $Z \subset \mathbb{R}^n$, there exists $K > 0$ such that $\|h(x,t) - h(y,t)\| \leq K\|x - y\|$ holds for all $x, y \in Z$ and $t \geq 0$. A vector-valued continuous function $p(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is said to be *uniformly decreasing for a matrix* $\Gamma \in \mathbb{R}^{n,n}$ if there exist $\alpha \in \mathbb{R}$ and $\delta > 0$ such that

$$(x - y)^\top \Big[p(x,t) - p(y,t) - \alpha \Gamma (x - y) \Big] \leq -\delta (x - y)^\top (x - y)$$

holds for all $x, y \in \mathbb{R}^n$ and $t \geq 0$.

Then, we give the definition of "persistently exciting" [44]. A function $\phi : \mathbb{R}^+ \to \mathbb{R}^{m\times n}$ is said to be *persistently exciting* if there exist $T > 0$, $\delta' > 0$ and $\delta > 0$ such that

$$\delta' I_m \geq \int_t^{t+T} \phi(\tau)\phi(\tau)^\top d\tau \geq \delta\, I_m$$

holds for all $t \geq 0$. As shown in [44, 52], the property of "persistent excitation" is firmly related to convergence of time-varying systems.

Lemma 2 *(Lemma A.1 in [52]) Given a system of the following form:*

$$\begin{cases} \dot{e}_1 = g(t)e_2 + f_1(t),\ e_1 \in \mathbb{R}^p \\ \dot{e}_2 = f_2(t),\ e_2 \in \mathbb{R}^q \end{cases}$$

such that (i) $\lim_{t\to\infty}\|e_1(t)\| = 0$, $\lim_{t\to\infty}\|f_1(t)\| = 0$, $\lim_{t\to\infty}\|f_2(t)\| = 0$*; (ii)* $g(t)$, $\dot{g}(t)$ *are bounded and* $g^\top(t)$ *is persistently exciting; then* $\lim_{t\to\infty}\|e_2(t)\| = 0$ *can hold.*

III. SYNCHRONIZATION ANALYSIS

In this section, we construct adaptive updating laws to achieve *global complete synchronization* of the coupled system (5). That is, for each initial data $x^i(0) \in \mathbb{R}^n$, $i = 1, \cdots, m$, $\lim_{t\to\infty}\|x^i(t) - x^j(t)\| = 0$ holds for all $i, j = 1, \cdots, m$. These adaptive updating laws have the form as (4), i.e., the adaption on each vertex only depends on its state and the states of its neighborhood and are constructed based on a candidate Lyapunov function indicating the potential difference between the states of vertices. And, we prove that these adaptive updating laws can synchronize any coupled system (5) with connected graph topology under several mild conditions since following the adaption, the candidate Lyapunov function actually decreases. Furthermore, we give the estimation of the synchronous performance also based on the Lyapunov function.

The following adaptive updating laws are proposed:

$$\begin{cases} \dot{q}^i(t) = Q^i f(x^i, t)^\top \sum\limits_{j\in\mathcal{N}(i)} (x^j - x^i), \\ \dot{c}^i = \eta_i \Big[\sum\limits_{j\in\mathcal{N}(i)} (x^j - x^i) \Big]^\top \Gamma \Big[\sum\limits_{j\in\mathcal{N}(i)} (x^j - x^i) \Big], \end{cases} \quad i = 1, \cdots, m, \tag{8}$$

where $Q^i \in \mathbb{R}^{d,d}$ is symmetric positive definite, and $\eta_i > 0$, $i = 1, \cdots, m$. One can see that for each vertex i, the adaption of q^i and c^i is actually only related to state information of its neighborhood and itself.

Theorem 1 *Suppose that (i). matrix Γ is symmetric and semi-positive definite; (ii). the common function $g(z,t)$ is continuous on $(z,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and locally uniformly bounded on $z \in \mathbb{R}^n$ with respect to $t \in \mathbb{R}^+$, $f(z,t)$ is continuous on $(z,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and locally uniformly bounded on z with respect to t, and there exists $\bar{q} \in \mathbb{R}^d$ such that $g(z,t) + f(z,t)\bar{q}$ is uniformly decreasing for the matrix Γ; (iii). for any initial data $x^i(0) \in \mathbb{R}^n$, $i = 1, \cdots, m$, there exists at least one index j_0 such that the trajectory $x^{j_0}(t)$ is bounded. If graph G is connected, then*

1. *the coupled system (5) associated with the feedback adaptive updating laws (8) can be globally completely synchronized, i.e., for any initial data $x^i(0) \in \mathbb{R}^n$, $i = 1, \cdots, m$,*

$$\lim_{t\to\infty} \|x^i(t) - x^j(t)\| = 0,\ i,j = 1,\cdots,m;$$

2. *the coupling strengths converge, i.e., $\lim_{t\to\infty} c_i = c_i^\infty$, for some $c_i^\infty \in \mathbb{R}$, $i = 1, \cdots, m$;*

3. *in addition, if supposing that (i). $g(z,t)$ and $f(z,t)$ are locally uniformly Lipschitz continuous on z with respect to t; (ii). $\partial f(z,t)/\partial t$ exists and is locally uniformly bounded on z with respect to t; (iii). $f^\top(x^{j_0}(t),t)$ is persistently exciting, then*

$$\lim_{t\to\infty} \|q^i(t) - q^j(t)\| = 0,\ ,i,j = 1,\cdots,m.$$

A proof is given in Appendix.

Remark 1 *As a preliminary, we suppose that the dynamics of the coupled system (5) associated with the adaptive laws (8) is bounded. This is very natural when studying chaotic dynamics and can be verified analytically for practical models or numerically for real-world applications. Also, we can conclude that the third hypothesis can guarantee that all $x^i(t)$ are bounded and the persistent excitation of $f^\top(x^{j_0}(t),t)$ can guarantee that each $f^\top(x^i(t),t)$ is persistently exciting.*

We introduce the following two quantities to measure performance of synchronization dynamics. One is the average of terminal coupling strengths:

$$c = \frac{1}{m}\sum_{i=1}^{m} c_i^\infty,$$

where $c_i^\infty = \lim_{t\to\infty} c_i(t)$, $i = 1, \cdots, m$, which denotes the average coupling cost of the network. The other comes from time average of the variance of the state collection $\{x^1(t), \cdots, x^m(t)\}$:

$$\sigma^2(t) = \frac{1}{t}\int_0^t \frac{1}{m-1}\sum_{i=1}^{m}[x^i(\tau) - \bar{x}(\tau)]^\top[x^i(\tau) - \bar{x}(\tau)]d\tau$$

with $\bar{x} = 1/m \sum_{j=1}^{m} x^j$. Obviously, for $\sigma^2(t)$, we have the approximation $\sigma^2(t) \sim \beta/t$ as $t \to \infty$, where

$$\beta = \int_0^\infty \frac{1}{m-1} \sum_{i=1}^{m} \left[x^i(\tau) - \bar{x}(\tau)\right]^\top \left[x^i(\tau) - \bar{x}(\tau)\right] d\tau,$$

which describes the convergence rate of the synchronization dynamics. Therefore, these two quantities: the terminal average coupling cost c and the convergent rate β can be used to measure the synchronous performance. A "good" performance means a high convergent rate (a small β) and low coupling cost (a small c).

Here, we pick $Q_i = \rho I_d$, $\eta_i = \eta$, where $\rho, \eta > 0$ and $c_i(0) = 0$ for all $i = 1, \cdots, m$. Then, we give estimations for the quantities β and c.

Proposition 1 *Under all the conditions in Theorem 1, picking $Q_i = \rho I_d$, $\eta_i = \eta$, and $c_i(0) = 0$ for all $i = 1, \cdots, m$, we have*

$$c \leq \hat{c},\ \hat{c} = \frac{\alpha}{\lambda_2(L)} + \sqrt{\frac{\alpha^2}{\lambda_2^2(L)} + \frac{2\eta m_0}{m}},$$

where $m_0 = 1/4 \sum_{i=1}^{m} \sum_{j \in \mathcal{N}(i)} \|e^{ij}(0)\|_2^2 + 1/(2\rho)\{\sum_{i=1}^{m} \|q^i(0) - \bar{q}\|_2^2 - \lim_{t\to\infty} \sum_{i=1}^{m} \|q^i(t) - \bar{q}\|_2^2\}$; in addition, if supposing $\Gamma = I_n$, then we have

$$\beta \leq \hat{\beta},\ \hat{\beta} = \frac{m}{(m-1)\lambda_2^2(L)} \left[\frac{\alpha}{\eta\lambda_2(L)} + \sqrt{\frac{\alpha^2}{\lambda_2^2(L)\eta^2} + \frac{2m_0}{m\eta}}\right].$$

A proof is given in Appendix. From the proof, one can see that if $c_i(0) \neq 0$, $\hat{c}$ estimates the increasing amount of the average coupling cost, i.e., $\hat{c} \geq 1/m \sum_{i=1}^{m} [c_i(\infty) - c_i(0)]$.

From the estimations $\hat{c}$ and $\hat{\beta}$, we can conclude that a larger $\lambda_2(L)$ implies smaller $\hat{c}$ and $\hat{\beta}$. This indicates that $\lambda_2(L)$ can describe the synchronizability of a network topology for the coupled model (5) associated with the adaptive updating laws (8). This conclusion is the similar to the linearly coupled network model (3) without feedback adaption as proposed in [3, 9]. However, there exists clear difference. In [3, 9], $\lambda_2(L)$ can denote the synchronized region of the coupling strength but in the adaptive model (5,8), $\lambda_2(L)$ affects the synchronization convergent rate since a connected graph is sufficient to synchronize an uncoupled chaotic system under several mild conditions. Moreover, it can also be shown that a larger ρ might imply smaller $\hat{c}$ and $\hat{\beta}$ because a larger ρ might roughly imply a smaller m_0 and a larger η might imply a smaller $\hat{\beta}$ but a larger $\hat{c}$ [56].

Remark 2 *In this paper, we use the similar ideas as in [27–30, 32–34, 48, 49] reviewed in Section 1 but clear differences exist. First, the models investigated are different. In the literature above, the authors study the coupled model (2) of which the output function is general and can be nonlinear and each vertex has identical dynamics. Our model concerns the case that the output function is linear but the dynamical property of each vertex is different, which is characterized by the mismatched parameter vector. Second, the adaptive method proposed in this section is universal, i.e., connectivity of network is sufficient for synchronization following such adaptive schemes and proved via rigorous mathematical art, i.e., the Lyapunov-based method. This implies that our adaptive scheme for the model (5) is universal under several mild conditions.*

IV. PINNING CONTROL

In this section, we consider the pinning control problem in dynamical networks. By the method proposed in [25, 26], we control the whole dynamical network to a given trajectory $s(t)$ of the uncoupled system

$$\begin{cases} \dot{s} = g(s,t) + f(s,t)\bar{q} \\ s(0) = s_0 \end{cases} \tag{9}$$

for a given $\bar{q} \in \mathbb{R}^d$ by pinning only a small fraction of vertices. Thus, the coupled system can be written as (6). For this purpose, we propose feedback adaptive updating laws to achieve global asymptotical stability of the object trajectory $s(t)$. A little difference from the synchronization problem is that the adaption on each pinned vertex should additionally depend on the external controllers. A candidate Lyapunov function indicating the potential difference between the network states and the objective trajectory is used to construct adaptive updating laws. We can prove that under several mild conditions, a pinned vertex set accessible from all other vertices can guarantee that all vertices are asymptotically pinned to the desired trajectory. This Lyapunov function method can also be used to analyze the pinning control performance dependently on the topology and prior parameters. Furthermore, this dependence might imply a method to select pinned vertices for better control performance.

The following adaptive updating laws are proposed:

$$\begin{cases} \dot{q}^i = Q^i f^\top(x^i,t)\Big[\sum_{j\in\mathcal{N}(i)}(x^j-x^i)+\delta_{\mathcal{D}}(i)(s-x^i)\Big] & i=1,\cdots,m, \\ \dot{c}_i = \eta_i\Big[\sum_{j\in\mathcal{N}(i)}(x^j-x^i)+\delta_{\mathcal{D}}(i)(s-x^i)\Big]^\top\Gamma\Big[\sum_{j\in\mathcal{N}(i)}(x^j-x^i)\Big] & i=1,\cdots,m, \\ \dot{\epsilon}_k = \kappa_k\Big[\sum_{l\in\mathcal{N}(k)}(x^l-x^k)+(s-x^k)\Big]^\top\Gamma(s-x^k) & k\in\mathcal{D}, \end{cases} \tag{10}$$

where Q^i, $i=1,\cdots,m$, are all symmetric and positive definite, η_i, $i=1,\cdots,m$, and κ_k, $k\in\mathcal{D}$ are positive numbers.

Theorem 2 *Suppose that (i). matrix Γ is symmetric and semi-positive definite; (ii). $g(z,t)$ is continuous on $(z,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and locally uniformly bounded, $f(z,t)$ is continuous on $(z,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and locally uniformly bounded on z with respect to t, and there exist $\alpha\in\mathbb{R}$ and $\bar{q}\in\mathbb{R}^d$ such that $g(z,t)+f(z,t)\bar{q}$ is uniform decreasing for Γ; (iii). the synchronized trajectory $s(t)$ is bounded. If all vertices in $\mathcal{V}\backslash\mathcal{D}$ can be accessible from the pinned vertex set $\mathcal{D}$, i.e., for any vertex i in $\mathcal{V}\backslash\mathcal{D}$, there exists an vertex j in $\mathcal{D}$ such that there exists an path between i and j, then*

1. *the pinning coupled system (6) with the adaptive updating laws (10) can be globally asymptotically stable at the trajectory $s(t)$, i.e., for any initial data $x^i(0)\in\mathbb{R}^n$, $i=1,\cdots,m$,*

$$\lim_{t\to\infty}\|x^i(t)-s(t)\|=0,\ i=1,\cdots,n;$$

2. *all coupling and pinning strengths converges as t goes to ∞., i.e., $\lim_{t\to\infty}c_i(t)=c_i^\infty$ holds for all $i=1,\cdots,m$ and $\lim_{t\to\infty}\epsilon_j(t)=\epsilon_j^\infty$ holds for $j\in\mathcal{D}$;*

3. *in addition, if supposing that (i). $g(z,t)$ and $f(z,t)$ are locally uniformly Lipschitz continuous on z with respect to t; (ii). $\partial f(z,t)/\partial t$ exists and is locally uniformly bounded on z with respect to t; (iii). $f(s(t),t)^\top$ is persistently exciting, then we have*

$$\lim_{t\to\infty}q^i(t)=\bar{q},\ i=1,\cdots,m.$$

A proof is given in Appendix.

Similar to Proposition 1, we can use two quantities to describe the stable performance and estimate their upper-bounds. One is the average of the coupling and pinning strengths:

$$\varepsilon=\frac{1}{m+\#\mathcal{D}}\Big[\sum_{i=1}^m c_i^\infty+\sum_{j\in\mathcal{D}}\epsilon_j^\infty\Big],$$

which describes the terminal coupling and pinning cost. The second comes from the variance from the state collection $\{x^1(t), \cdots, x^m(t)\}$ to $s(t)$:

$$\mathrm{var}^2(t) = \frac{1}{t}\int_0^t \frac{1}{m}\sum_{i=1}^{m}[x^i(\tau) - s(\tau)]^\top [x^i(\tau) - s(\tau)]d\tau$$

which can be estimated by $\mathrm{var}^2(t) \sim \chi/t$, as $t \to \infty$, where

$$\chi = \int_0^\infty \frac{1}{m}\sum_{i=1}^{m}[x^i(t) - s(t)]^\top [x^i(t) - s(t)]dt$$

describes the stability convergent rate and measure stable performance. Therefore, χ and ε can be used to measure the stability performance, i.e., a "good" performance implies a small ε and a small χ. Then, we will give the estimations for ε and χ.

Proposition 2 *Under all the conditions in Theorem 2, picking $Q^i = \rho I_n$, $\eta_i = \epsilon_j = \eta$, and $c_i(0) = \epsilon_j(0) = 0$, for all $i = 1, \cdots, m$ and $j \in \mathcal{D}$, we have*

$$\varepsilon \le \hat{\varepsilon},\ \hat{\varepsilon} = \frac{\alpha}{\lambda_2(\hat{L})} + \sqrt{\frac{\alpha^2}{\lambda_2^2(\hat{L})} + \frac{2\eta\hat{m}_0}{m + \#\mathcal{D}}};$$

in addition, if supposing $\Gamma = I_n$, we have

$$\chi \le \hat{\chi},\ \hat{\chi} = \frac{(m+1)(m+\#\mathcal{D})}{m\lambda_2^2(\hat{L})}\left[\frac{\alpha}{\eta\lambda_2(\hat{L})} + \sqrt{\frac{\alpha^2}{\eta^2\lambda_2^2(\hat{L})} + \frac{2\hat{m}_0}{\eta(m+\#\mathcal{D})}}\right],$$

where $\hat{m}_0 = V_1(\tilde{E}(0)) + V_2(Q(0)) - \lim_{t\to\infty} V_2(Q(t))$.

A proof is given in Appendix.

From Proposition 2, we can similarly conclude that a larger $\lambda_2(\hat{L})$ might imply smaller $\hat{\varepsilon}$ and $\hat{\chi}$. A larger ρ might imply smaller $\hat{\varepsilon}$ and $\hat{\chi}$ since a larger ρ might roughly imply a smaller m_0 and a larger η might imply a smaller $\hat{\chi}$ but a larger $\hat{\varepsilon}$.

Since $\lambda_2(\hat{L})$ can describe the stabilizability of a network topology with the pinned vertex set $\mathcal{D}$ through the coupled model (5) associated with the adaptive equations (10), if giving the network topology and the fraction of the pinned vertex set, the different choices of pinned vertex set $\mathcal{D}$ might lead different $\lambda_2(\hat{L})$, i.e., different stabilizabilities of the network with pinned vertex set $\mathcal{D}$. Hence, it is very significant to study how to select a vertex set (given the fraction) to achieve a "good" stabilizability, i.e., a large $\lambda_2(L)$. This will be further discussed in the following section.

V. NUMERICAL ILLUSTRATIONS

In this section, we present several numerical examples to verify the adaptive laws proposed in the sections above to synchronize or stabilize a complex network and discuss the dependence of synchronous and stable performance on the network topology and prior parameters. Here, on each vertex, we set the following Hopfield neural network as an intrinsic uncoupled system:

$$\frac{dv}{dt} = f(v, q) = -Dv + T(q)g(v). \tag{11}$$

Here, $v = [v_1, v_2, v_3]^\top$, $q = [q_1, q_2]^\top$, and

$$T(q) = \begin{bmatrix} 1.2500 & -3.200 & q_1 \\ -3.200 & q_2 & -4.4000 \\ -3.200 & 4.4000 & 1.000 \end{bmatrix},$$

where $q_{1,2}$ are unknown mismatched parameters. Also, $D = I_3$, and $g(v) = [g(v_1), g(v_2), g(v_3)]^\top$, where $g(s) = (|s+1| - |s-1|)/2$. In this section, ordinary differential equations (ODEs) are numerically solved by the fourth-order Runge-Kutta formula (RK4) with a fixed step length according to the network type and integrals are computed by the Simpson's rule.

A. Adaptive synchronization

First, we use the adaptive laws (8) to synchronize a complex network which can be formulated as follows:

$$\dot{x}^i = -Dx^i + T(q^i)g(x^i) + c_i \sum_{j\in\mathcal{N}(i)} (x^j - x^i),\ i = 1, \cdots, m. \tag{12}$$

And the adaptive laws (8) are realized as:

$$\begin{cases} \dot{q}_1^i = \rho\, g(x_3^i) \sum_{j\in\mathcal{N}(i)} (x_1^j - x_1^i) \\ \dot{q}_2^i = \rho\, g(x_2^i) \sum_{j\in\mathcal{N}(i)} (x_2^j - x_2^i) \\ \dot{c}^i = \eta \Big[\sum_{j\in\mathcal{N}(i)} (x^j - x^i) \Big]^\top \Big[\sum_{j\in\mathcal{N}(i)} (x^j - x^i) \Big],\ i = 1, \cdots, m, \end{cases} \tag{13}$$

where the $\rho, \eta > 0$ are prior control parameters.

The coupling graph used here is a small-world (SW) network, which is an intermediate between regular and random networks. It can be characterized as a regular network with a very small mean path length and also be characterized as a random graph with a high clustering coefficient. SW network was investigated in [12] and a little variance we follow here was proposed in [54]. It begins with m vertices and every vertex is connected with k nearest vertices on each side of its left and right hands. Then for each pair of vertices (i, j) for which $e(i, j)$ is not an edge at the initial time, we connect them with probability $p \ll 1$ as an edge with duplicate edges and self-connection avoided. Here, $m = 200$ and $k = 2$ are given, and the step length for solving the ODEs is picked as 0.005.

These two quantities β and c introduced in Proposition 1 are used to measure synchronous performance and numerically computed by $\beta = \int_0^T 1/(m-1)\sum_{i=1}^m [x^i(\tau) - \bar{x}(\tau)]^\top [x^i(\tau) - \bar{x}(\tau)] d\tau$ and $c = 1/m \sum_{i=1}^m c_i(T)$ for a large T picked as $T = 100$. As we have known, the smallest nonzero eigenvalue of the corresponding Laplacian: $\lambda_2(L)$, increases with respect to p. This implies that the estimated upper-bounds $\hat{\beta}$ and $\hat{c}$ decrease with respect to p. Figures 1 and 2 indicate that β and c decrease with respect to p from 0.002 to 0.2 with $\rho = 5$ and $\eta = 0.5$. Furthermore, the inner sub-figures of Figures 1 and 2 show that β and c actually decrease with respect to $\lambda_2(L)$. This coincides with from our theoretical analysis (Proposition 1). As an example, picking $p = 0.1$, Figure 3 shows that σ^2 converges to zero, which suggests achieving synchronization, q^i, $i = 1, \cdots, m$, converge to a uniform vector, and c^i, $i = 1, \cdots, m$, converge to certain different positive values.

We also investigate the variations of β and c with respect to η and ρ, respectively. Figures 4 and 5 show that β decreases with respect to η and ρ, and c increases with respect to η but decreases with respect to ρ. These phenomena coincide with our theoretical analysis well.

B. Adaptive pinning

Second, we use the adaptive laws (10) to pin a complex network of coupled Hopfiled neural networks to an object trajectory satisfying:

$$\frac{ds}{dt} = f(s, q) = -Ds + T(q^*)g(s),$$

where $s = [s_1, s_2, s_3]^\top$ and $q^* = [-3.2, 1.1]^\top$ is given. With initial data $s(0) = [0.1, 0.1, 0.1]^\top$, this system has a double-scrolling chaotic attractor [53]. We pin a selective vertex set $\mathcal{D}$ of the network and have the following pinned dynamical network:

$$\begin{cases} \dot{x}^i = -Dx^i + T(q^i)g(x^i) + c_i \sum\limits_{j\in\mathcal{N}(i)} (x^j - x^i) + \delta_{\mathcal{D}}(i)u_i & i = 1, \cdots, m, \\ u_k = \epsilon_k(s - x^k) & k \in \mathcal{D}. \end{cases} \tag{14}$$

The adaptive laws (10) proposed in Section 4 are realized as follows:

$$\begin{cases} \dot{q}^i_r = \rho\, g(x^i_{4-r}) \Big[\sum\limits_{j\in\mathcal{N}(i)} (x^j_r - x^i_r) + \delta_{\mathcal{D}}(i)(s_r - x^i_r) \Big] & i = 1, \cdots, m, r = 1, 2, \\ \dot{c}^i = \eta \Big[\sum\limits_{j\in\mathcal{N}(i)} (x^j - x^i) + \delta_{\mathcal{D}}(i)(s - x^i) \Big]^\top \Big[\sum\limits_{j\in\mathcal{N}(i)} (x^j - x^i) \Big] & i = 1, \cdots, m, \\ \dot{\epsilon}_k = \eta \Big[\sum\limits_{l\in\mathcal{N}(k)} (x^l - x^k) + (s - x^k) \Big]^\top (s - x^k) & k \in \mathcal{D}. \end{cases}$$

The wiring topology used here is a scale-free (SF) network, which is characterized by that the degree distribution obeys a power law that is observed in many real networks. Here, we follow the SF network model proposed in [13]. At the starting stage, it has k_0 vertices. Then at each time a new vertex is added and it is connected to the k already existing vertices with a probability proportional to the degree of that vertex. This process continues for a long time t_1, thus the degree distribution subordinates the power law $P(k) \sim k^{-\gamma}$ approximately, where $\gamma = 3$ is independent of k. Here, we pick $k_0 = k = 2$ and $t_1 = 100$, which implies the size of the final network is $m = 102$, and pick the step length for solving the ODEs as 0.002.

The quantities ε and χ are numerically computed as $\chi = \int_0^T 1/m \sum_{i=1}^m [x^i(\tau) - s(\tau)]^\top [x^i(\tau) - s(\tau)] d\tau$ and $\varepsilon = 1/(m + \#\mathcal{D}) \sum_{i=1}^m [c_i(T) + \sum_{j\in\mathcal{D}} \epsilon_j(T)]$ where T is picked as $T = 200$. Let $ff = \#\mathcal{D}/m$ be the fraction of the pinned vertex set of the whole network. We select $\lfloor ff \times m \rfloor$ vertices with equal probability composing the set $\mathcal{D}$, where $\lfloor z \rfloor$ is the ceiling function, i.e., the largest integer less than z. As an example, Figures 6 and 7 show the variations of $\mathrm{var}^2(t)$, $c_i(t)$, $\epsilon_j(t)$, $q^i(t)$, $i = 1, \cdots, m$, $j \in \mathcal{D}$, with respect to time with $ff = 0.2$, $\eta = 5$, and $\rho = 20$. One can see that $\mathrm{var}^2(t)$ converges to zero, which implies achieving stabilization, each $q^i(t)$ converges to the given vector q^*, and all coupling and pinning strengths converge to certain different positive values.

As shown in Figures 8 and 9, both $\lambda_2(\hat{L})$ and $\#\mathcal{D}$ increase with respect to ff, which implies that $\hat{\chi}$ and $\hat{\varepsilon}$ decreases with respect to ff. In particular, the inner sub-figure of Figures 8 and 9 gives the variance of χ and β with respect to $\lambda_2(\hat{L})$. Hence, if giving the pinned fraction ff, $\lambda_2(\hat{L})$ is a suitable quantity to measure the stabilizability of the network with the pinned vertex set $\mathcal{D}$. In Figures 10 and 11 , one can see that χ decreases with respect to ρ and η. But ε decreases with respect to ρ and increases with respect to η. These phenomena coincide with our analytical results well.

From the observation above, a question arises that for a given network and given pinned vertex fraction ff, which selection of pinned vertex set $\mathcal{D}$ with $\#\mathcal{D} = \lfloor ff \times m \rfloor$ can lead a high stabilizability: a large $\lambda_2(\hat{L})$, which suggests a fast convergence speed and low coupling and pinning cost. In [25, 26], the authors found out that for a scale-free network as introduced above, a degree-based selective pinning strategy is better than a random pinning strategy for pinning a linearly coupled differential systems without considering adaptive laws. Here, for the adaptive pinning problem, we also compare the stabilizability $\lambda_2(\hat{L})$ in two pinning strategies for a give scale-free network and the same pinning fraction ff: one is the random pinning strategy as mentioned above and the other is the selective pinning strategy which means that we pin the $\lfloor ff \times m \rfloor$ vertices which have the largest connection degrees. Figure 12 shows that the selective pinning strategy is actually better than the random pinning strategy for a large network size and a small fraction ff. This result coincides with that reported in [25, 26] without considering adaptive feedback laws.

VI. CONCLUSIONS

The adaptive technique proposed in this paper can be proved to achieve synchronization and pinning control for the graphs with connectivity. This technique has two advantages. First, enhance synchronizability or stability of a network. Synchronization or pinning control can be guaranteed under a very weak topology condition for the network: connectivity. Second, synchronization or pinning control can be obtained with any initial coupling and pinning strengths, and any initial mismatched parameter vectors. However, the enhancement is achieved by additional computation or realization expense for the adaption. Furthermore, the underlying network topology heavily affects the convergent performance. Therefore, the problem of a "good" network topology for synchronization or a "good" network topology attached with pinned vertex selection for pinning control is similar to the existing theoretical analysis in [9, 25, 26, 31] which concern dynamical networks without adaption. However, the obvious difference exists. The previous results concern whether the complex network can be synchronized or pinned but in this paper we consider the convergent rate and terminal coupling and pinning strengths with synchronization or pinning control guaranteed.

Also, we can see that these prior parameters used in real-world application should be properly chosen. In fact, there exists a trade-off when picking η and ρ. η should be picked as a relative small value since a large η might cause a large terminal coupling strengths which could lead divergence of the numerical computation of the ODEs. And, ρ might be picked as a relative large value since it could advance synchronization. However, both η and ρ can not be very large, otherwise it might cause divergence of the numerical computation. Therefore, in application, we always pick a relatively large ρ and a relatively small η to obtain both fast convergence and low terminal coupling cost.

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[56] It is obvious that the limit $\lim_{t\to\infty} q(t)$ also depend on η as well as ρ and $\lambda_2(L)$. But, we can analytically roughly conclude that m_0 might decrease with respect to ρ.

APPENDIX

Proof of Theorem 1: Since graph G is connected, the corresponding Laplacian L is irreducible, i.e., $\lambda_2(L) > 0$. And, picking $\bar{q}$ as defined in the second hypothesis, there must exist $\alpha \in \mathbb{R}$ and

$\delta > 0$ such that

$$(x-y)^\top\Big[(g(x,t)+f(x,t)\bar{q})-(g(y,t)+f(y,t)\bar{q})-\alpha\Gamma(x-y)\Big] \le -\delta(x-y)^\top(x-y) \quad (15)$$

holds for all $x, y \in \mathbb{R}^n$ and $t \ge 0$. Let a be a positive number such that $a\lambda_2(L) > \alpha$.

Let $x = [x^{1\top}, \cdots, x^{m\top}]^\top \in \mathbb{R}^{nm}$, $C = [c_1, \cdots, c_m] \in \mathbb{R}^m$, $Q = [q^1, \cdots, q^m] \in \mathbb{R}^{d,m}$, and $e^{ij} = x^i - x^j$, for all $i, j = 1, \cdots, m$. Since L is irreducible, we conclude that $(L \otimes I_n)x = 0$ if and only if $e^{ij} = 0$ holds for all $i, j = 1, \cdots, m$. Let $E = \{e^{ij}\}_{i,j=1}^m$ and define the following candidate Lyapunov function:

$$V(E,Q,C) = \frac{1}{4}\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}e^{ij} + \frac{1}{2}\sum_{i=1}^m (q^i-\bar{q})^\top Q^{i^{-1}}(q^i-\bar{q}) + \frac{1}{2}\sum_{i=1}^m \frac{1}{\eta_i}(c_i-a)^2.$$

Note that $1/2\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}e^{ij} = x^\top(L\otimes I_n)x$, $\sum_{i=1}^m(\sum_{j\in\mathcal{N}(i)} e^{ij})^\top\Gamma(\sum_{j\in\mathcal{N}(i)} e^{ij}) = x^\top(L^2\otimes\Gamma)x$, and $\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} c_j e^{ij\top}\Gamma\sum_{l\in\mathcal{N}(j)} e^{lj} = \sum_{i=1}^m c_i[\sum_{j\in\mathcal{N}(i)} e^{ji}]^\top\Gamma[\sum_{j\in\mathcal{N}(i)} e^{ji}]$.

Thus, differentiating $V(E,Q,C)$ along the coupled system (5) and adaptive laws (8) gives

$$\begin{aligned}
\frac{d}{dt}|_{(5,8)}V(E,Q,C) &= \frac{1}{2}\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Big[g(x^i,t)-g(x^j,t)+f(x^i,t)q^i-f(x^j,t)q^j \\
&+c_i\sum_{k\in\mathcal{N}(i)}\Gamma e^{ki} - c_j\sum_{l\in\mathcal{N}(j)}\Gamma e^{lj}\Big] + \sum_{i=1}^m\sum_{j\in\mathcal{N}(i)}(q^i-\bar{q})^\top f(x^i,t)^\top e^{ji} \\
&+\sum_{i=1}^m c_i\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big]^\top\Gamma\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big] - a\sum_{i=1}^m\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big]^\top\Gamma\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big] \\
&= \frac{1}{2}\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Big[(g(x^i,t)+f(x^i,t)\bar{q})-(g(x^j,t)+f(x^j,t)\bar{q})-\alpha\Gamma e^{ij}+f(x^i,t)(q^i-\bar{q}) \\
&-f(x^j,t)(q^j-\bar{q})\Big] + \sum_{i=1}^m\sum_{j\in\mathcal{N}(i)}(q^i-\bar{q})^\top f(x^i,t)^\top e^{ji} + \frac{\alpha}{2}\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Gamma e^{ij} \\
&-a\sum_{i=1}^m\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big]^\top\Gamma\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big] \\
&\le -\frac{\delta}{2}\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}e^{ij} + \frac{\alpha}{2}\sum_{i=1}^m\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Gamma e^{ij} - a\sum_{i=1}^m\Big(\sum_{j\in\mathcal{N}(i)} e^{ij}\Big)^\top\Gamma\Big(\sum_{j\in\mathcal{N}(i)} e^{ij}\Big) \\
&= -\delta\, x^\top\Big(L\otimes I_n\Big)x + x^\top\Big[(\alpha I_m - aL)L\otimes\Gamma\Big]x \le -\delta\, x^\top\Big(L\otimes I_n\Big)x, \qquad (16)
\end{aligned}$$

since $a\lambda_2(L) > \alpha$ implies that $(\alpha I_m - aL)L \le 0$. The derivation (16) gives

$$\int_0^t x^\top\Big(L\otimes I_n\Big)x \;\le\; \frac{1}{\delta}V(E(0),Q(0),c(0)) < +\infty,\ \forall\, t\ge 0 \qquad (17)$$

Since $\dot{V}(E,Q,C) \leq 0$, we can conclude that $e^{ij}(t)$, $q^i(t)$, and $c_i(t)$, $i,j = 1,\cdots,m$, are all bounded. Due to the boundedness of $x^{j_0}(t)$ and the irreducibility of L, we can conclude that $x^i(t)$, $i = 1,\cdots,m$, are all bounded. Note that for $i = 1,\cdots,m$,

$$\dot{e}^{ij_0} \;=\; g(x^i,t) - g(x^{j_0},t) + f(x^i,t)q^i - f(x^{j_0},t)q^{j_0} + c_i \sum_{k\in\mathcal{N}(i)} \Gamma e^{ki} - c_{j_0} \sum_{l\in\mathcal{N}(j_0)} \Gamma e^{lj_0}$$

implies that the derivatives $\dot{e}^{ij_0}$, $i = 1,\cdots,m$, are bounded. This implies that the derivatives $\dot{e}^{ij}$, $i,j = 1,\cdots,m$, are all bounded. Therefore, $e^{ij}(t)$, $i,j = 1,\cdots,m$, are all uniformly continuous. Namely, $x^\top(t)(L\otimes I_n)x(t)$ is uniformly continuous. Addition with the inequality (17) leads $\lim_{t\to\infty} x^\top(t)(L\otimes I_n)x(t) = 0$. Hence, we can conclude that $\lim_{t\to\infty} e^{ij}(t) = 0$ holds for all $i,j = 1,\cdots,m$, namely, the coupled system is globally completely synchronized. The first claim is proved.

The inequality (17) also implies that the trajectories $c_i(t) = \int_0^t [\sum_{j\in\mathcal{N}(i)} e^{ji}(\tau)]^\top \Gamma [\sum_{j\in\mathcal{N}(i)} e^{ji}(\tau)] d\tau$ is a Cauchy series as t goes to the infinity, $i = 1,\cdots,m$. Therefore, its convergence can be concluded. The second claim is proved.

Rewrite the equations for e^{ij_0} and $q^i - q^{j_0}$ as

$$\begin{cases} \dot{e}^{ij_0} = f_1^i(t) + f(x^{j_0}(t),t)(q^i - q^{j_0}), \\ \frac{d}{dt}[q^i - q^{j_0}] = f_2^i(t),\ i = 1,\cdots,m, \end{cases}$$

where $f_1^i(t) = [g(x^i,t) - g(x^{j_0},t)] + [f(x^i,t) - f(x^{j_0},t)]q^i + c_i \sum_{k\in\mathcal{N}(i)} \Gamma e^{ki} - c_{j_0} \sum_{l\in\mathcal{N}(j_0)} \Gamma e^{lj_0}$, and $f_2^i(t) = Q^i f(x^i,t)^\top \sum_{k\in\mathcal{N}(i)} e^{ki} - Q^{j_0} f(x^{j_0},t)^\top \sum_{l\in\mathcal{N}(j_0)} e^{lj_0}$, $i = 1,\cdots,m$. Since the $f^\top(x^{j_0}(t),t)$ is persistently exciting, $f(x^{j_0}(t),t)$ is bounded, and $df(x^{j_0}(t),t)/dt$ is essentially bounded, by Lemma 2, we can conclude that $\lim_{t\to\infty}(q^i - q^{j_0}) = 0$ holds for all $i = 1,\cdots,m$. This completes the proof of this theorem. $\square$

Proof of Proposition 1: Before the proof, we present the following matrix inequality.

Claim 1: Let L be the Laplacian matrix of the irreducible graph G. Then, the following matrix inequality holds: $(a - \alpha/\lambda_2(L))L^2 \leq a\,L^2 - \alpha\,L$.

In fact, let v_i be the eigenvector of L associated with the eigenvalue $\lambda_i(L)$ ordered by $0 = \lambda_1(L) < \lambda_2(L) \leq \lambda_3(L) \leq \cdots \leq \lambda_m(L)$. And, we properly pick the eigenvectors which correspond to the same eigenvalue with multiplicity such that v_1, v_2, $\cdots$, v_m compose an orthogonal standard basis of $\mathbb{R}^m$. For any $v \in \mathbb{R}^m$, we can write it as $v = \sum_{i=1}^m r_i v_i$ for some $r_i \in \mathbb{R}$,

$i = 1, \cdots, m$. Then,

$$v^\top \Big[(a - \frac{\alpha}{\lambda_2(L)})L^2 - (a\ L^2 - \alpha\ L)\Big] v = \sum_{i=1}^m v_i^\top v_i \Big[(a - \frac{\alpha}{\lambda_2(L)})\lambda_i^2(L) - a\lambda_i^2(L) + \alpha\lambda_i(L)\Big] r_i^2$$
$$+2\sum_{i=1}^m \sum_{j>i} v_i^\top \Big[(a - \frac{\alpha}{\lambda_2(L)})L^2 - (a\ L^2 - \alpha\ L)\Big] v_j r_i r_j = \sum_{i=2}^m v_i^\top v_i \Big[- \frac{\alpha}{\lambda_2(L)}\lambda_i^2(L) + \alpha\lambda_i(L)\Big] r_i^2 \leq 0$$

since $v_i^\top v_j = 0$ holds for all $i \neq j$. This proves the claim.

First, in order to estimate c, according to the adaptive updating laws (8), we have

$$\sum_{i=1}^m \int_0^\infty \dot{c}_i(t)dt = \int_0^\infty \sum_{i=1}^m \eta \Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big]^\top \Gamma \Big[\sum_{j\in\mathcal{N}(i)} e^{ji}\Big] dt.$$

This implies $c = \eta/m \int_0^\infty x^\top(t)(L^2 \otimes \Gamma)x(t)dt$. From the derivative (16), we have $\dot{V} \leq x^\top[(\alpha I_m - aL)L \otimes \Gamma]x$. Addition with Claim 1 implies

$$\begin{aligned}\int_0^\infty x^\top(\tau)\Big(L^2 \otimes \Gamma\Big)x(\tau)d\tau \ &\leq\ \frac{\lambda_2(L)}{a\lambda_2(L) - \alpha} \int_0^\infty x^\top(\tau)\Big[(aL - \alpha I_m)L \otimes \Gamma\Big]x(\tau)d\tau \\ &\leq\ -\frac{\lambda_2(L)}{a\lambda_2(L) - \alpha}\int_0^\infty \dot{V}dt = \frac{\lambda_2(L)}{a\lambda_2(L) - \alpha}[V_0 - V_\infty],\end{aligned}$$

where $V_0 = V(E(0), Q(0), C(0))$ and $V_\infty = \lim_{t\to\infty} V(E(t), Q(t), C(t))$. Note that

$$V_0 - V_\infty \ =\ m_0 + \frac{1}{2\eta}\sum_{i=1}^m (2ac_i^\infty - c_i^{\infty 2}) \leq m_0 + \frac{am}{\eta}c - \frac{m}{2\eta}c^2$$

where

$$m_0 = \frac{1}{2}\sum_{i=1}^m \sum_{j\in\mathcal{N}(i)} \|e^{ij}(0)\|_2^1 + \frac{1}{2\rho}\Big[\sum_{i=1}^m \|q^i(0) - \bar{q}\|_2^2 - \lim_{t\to\infty}\sum_{i=1}^m \|q^i(t) - \bar{q}\|_2^2\Big],$$

since $\sum_{i=1}^m c_i^{\infty 2} \geq 1/m(\sum_{i=1}^m c_i^\infty)^2$ where the equality holds if only if $c_i^\infty = c_j^\infty$ holds for all $i, j = 1, \cdots, m$. We have the following estimation:

$$c \leq \frac{\eta\lambda_2(L)}{m(a\lambda_2(L) - \alpha)}\Big(m_0 + \frac{am}{\eta}c - \frac{m}{2\eta}c^2\Big)$$

for all $a > \alpha/\lambda_2(L)$. By solving this quadratic inequality, we obtain $c \leq \hat{c}$, where $\hat{c}$ is an upper-bounded estimation of c:

$$\hat{c} = \frac{\alpha}{\lambda_2(L)} + \sqrt{\frac{\alpha^2}{\lambda_2^2(L)} + \frac{2\eta m_0}{m}}.$$

Second, we give an estimation for β. Let $M = (m_{ij})$ with $m_{ij} = -1/m$ if $i \neq j$ and $m_{ii} = 1 - 1/m$ for all $i = 1, \cdots, m$, and $W = 1/(m-1)M^\top M$. One can conclude that

$$\beta = \int_0^\infty x^\top(t)\Big(W \otimes I_n\Big)x(t)dt$$

holds and W has nonnegative off-diagonal elements and zero row sums. More careful calculations lead that the eigenvalues of W are 0 with multiplicity one and $1/(m-1)$ with multiplicity $m-1$. We demand the following matrix inequality:

Claim 2:

$$W \leq \frac{1}{(m-1)\lambda_2^2} L^2.$$

In fact, let $e = 1/\sqrt{m}[1, 1, \cdots, 1]^\top$. Then, for any vector $v \in \mathbb{R}^m$, we can write $v = r_e e + v_+$ for some $r_e \in \mathbb{R}$ and $v_+ \in \mathbb{R}^m$ satisfying $e^\top v_+ = 0$. Then, we have

$$\begin{aligned} & v^\top \left[W - \frac{1}{(m-1)\lambda_2^2(L)} L^2 \right] v = v_+^\top \left[W - \frac{1}{(m-1)\lambda_2^2(L)} L^2 \right] v_+ \\ & \leq \left[\frac{1}{m-1} - \frac{1}{(m-1)\lambda_2^2(L)} \lambda_2^2(L) \right] v_+^\top v_+ = 0 \end{aligned}$$

since $v_+^\top W v_+ \leq 1/(m-1) v_+^\top v_+$ and $v_+^\top L^2 v_+ \geq \lambda_2^2(L) v_+^\top v_+$. This proves the claim.

Therefore, we have

$$\beta \leq \frac{1}{(m-1)\lambda_2^2(L)} \int_0^\infty x^\top(t) \Big(L^2 \otimes I_n \Big) x(t) dt$$

If $\Gamma = I_n$, according to the estimation of c, we have $\beta \leq \hat{\beta}$, where $\hat{\beta}$ is an upper-bounded estimation of β which has the following form:

$$\hat{\beta} = \frac{m}{(m-1)\lambda_2^2(L)\eta} \hat{c} = \frac{m}{(m-1)\lambda_2^2(L)} \left(\frac{\alpha}{\eta \lambda_2(L)} + \sqrt{\frac{\alpha^2}{\lambda_2^2(L)\eta^2} + \frac{2m_0}{m\eta}} \right). \square$$

Proof of Theorem 2: Define $\hat{x} = [x^{1\top}, \cdots, x^{m\top}, s^\top]^\top \in \mathbb{R}^{n(m+1)}$ and matrix $\hat{L} = [\hat{l}_{ij}]_{i,j=1}^{m+1}$ as

$$\hat{l}_{ij} = \hat{l}_{ji} = \begin{cases} -1 & j \in \mathcal{N}(i), \ m \geq i, j \geq 1, \\ -1 & i = m+1, \ j \in \mathcal{D}, \ \text{or} \ j = m+1, \ i \in \mathcal{D}, \\ -\sum\limits_{j=1, j \neq i} l_{ij} & i = j, \\ 0 & \text{otherwise}. \end{cases}$$

This matrix can be regarded as the Laplacian of a graph $\hat{G} = [\hat{\mathcal{V}}, \hat{\mathcal{E}}]$, where $\hat{\mathcal{V}} = \mathcal{V} \cup \{m+1\}$ and $\hat{\mathcal{E}} = \mathcal{E} \cup \{(i,j), \ i \in \mathcal{D}, \ j = m+1\}$. Since all vertices in the graph G can be accessible from $\mathcal{D}$, according to Lemma 1, we can see that $\hat{G}$ is connected which implies that $\hat{L}$ is irreducible. Therefore, $\lambda_2(\hat{L}) > 0$. Let a be positive number satisfying $a\lambda_2(\hat{L}) > \alpha$, where α is as defined by the uniform decreasing condition in the second hypothesis.

For simple notations, let $e^{i,m+1} = x^i - s$, for all $i = 1, \cdots, m$, $\tilde{E} = E \cup \{e^{i,m+1}\}_{i\in\mathcal{D}}$, and $\epsilon = \{\epsilon_j : \ j \in \mathcal{D}\}$. Define the candidate Lyapunov function

$$\begin{aligned}
\hat{V}(\tilde{E}, Q, C, \epsilon) &= V_1(\tilde{E}) + V_2(Q) + V_3(C, \epsilon) \\
V_1(\tilde{E}) &= \frac{1}{4}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top} e^{ij} + \frac{1}{2}\sum_{i\in\mathcal{D}} e^{i,m+1\top} e^{i,m+1} \\
V_2(Q) &= \frac{1}{2}\sum_{i=1}^{m} (q^i - \bar{q})^\top Q^{i^{-1}} (q^i - \bar{q}) \\
V_3(C, \epsilon) &= \sum_{i=1}^{m} \frac{1}{2\eta_i}(c_i - a)^2 + \sum_{i\in\mathcal{D}} \frac{1}{2\kappa_i}(\epsilon_i - a)^2.
\end{aligned}$$

Note that $1/2\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}e^{ij} + \sum_{k\in\mathcal{D}}(x^k - s)^\top(x^k - s) = \hat{x}^\top\left(\hat{L}\otimes I_n\right)\hat{x}$ and

$$\sum_{i=1}^{m}\left[\sum_{j\in\mathcal{N}(i)} e^{ji} + \delta_{\mathcal{D}}(i)(s - x^i)\right]^\top \Gamma\left[\sum_{j\in\mathcal{N}(i)} e^{ji} + \delta_{\mathcal{D}}(i)(s - x^i)\right] = \hat{x}^\top\left(\hat{L}^2\otimes\Gamma\right)\hat{x}.$$

Differentiating $V_{1,2,3}$ gives

$$\begin{aligned}
&\frac{d}{dt}|_{(6)}V_1(\tilde{E}) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Bigg[g(x^i,t) - g(s,t) + f(x^i,t)q^i - f(x^j,t)q^j + c_i\sum_{k\in\mathcal{N}(i)}\Gamma e^{ki} \\
&-c_j\sum_{l\in\mathcal{N}(j)}\Gamma e^{lj} + \epsilon_i\Gamma(s - x^i) - \epsilon_j\Gamma(s - x^j)\Bigg] + \sum_{i\in\mathcal{D}}(x^i - s)^\top\Bigg[g(x^i,t) - g(s,t) + f(x^i,t)q^i \\
&-f(s,t)\bar{q} + c_i\sum_{j\in\mathcal{N}(i)}\Gamma e^{ji} + \epsilon_i\Gamma(s - x^i)\Bigg] \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Big[(g(x^i,t) + f(x^i,t)\bar{q}) - (g(s,t) + f(x^j,t)\bar{q}) - \alpha\Gamma e^{ij}\Big] + \sum_{i\in\mathcal{D}}(x^i - s)^\top \\
&\Big[(g(x^i,t) + f(x^i,t)\bar{q}) - (g(s,t) + f(s,t)\bar{q}) - \alpha\Gamma(x^i - s)\Big] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Big[f(x^i,t)(q^i - \bar{q}) \\
&-f(x^j,t)(q^j - \bar{q})\Big] + \sum_{i\in\mathcal{D}}(x^i - s)^\top f(x^i,t)(q^i - \bar{q}) - \sum_{i=1}^{m} c_i\Bigg[\sum_{j\in\mathcal{N}(i)} e^{ji} + \delta_{\mathcal{D}}(i)(s - x^i)\Bigg]^\top\Gamma\Bigg(\sum_{j\in\mathcal{N}(i)} e^{ji}\Bigg) \\
&-\sum_{i\in\mathcal{D}}\epsilon_i\Bigg[\sum_{j\in\mathcal{N}(i)} e^{ji} + (s - x^i)\Bigg]^\top\Gamma(s - x^i) + \frac{\alpha}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Gamma e^{ij} + \alpha\sum_{i\in\mathcal{D}}(x^i - s)^\top\Gamma(x^i - s) \\
&\le -\frac{\delta}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}e^{ij} - \delta\sum_{i\in\mathcal{D}}(x^i - s)^\top(x^i - s) + \frac{\alpha}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Gamma e^{ij} + \alpha\sum_{i\in\mathcal{D}}(x^i - s)^\top\Gamma(x^i - s) \\
&+\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top} f(x^i,t)(q^i - \bar{q}) + \sum_{i\in\mathcal{D}}(x^i - s)^\top f(x^i,t)(q^i - \bar{q})
\end{aligned}$$

$$-\sum_{i=1}^{m} c_i\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+\delta_{\mathcal{D}}(i)(s-x^i)\Big]^\top \Gamma\Big(\sum_{j\in\mathcal{N}(i)} e^{ji}\Big)-\sum_{i\in\mathcal{D}}\epsilon_i\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+(s-x^i)\Big]^\top\Gamma(s-x^i),$$

$$\frac{d}{dt}|_{(10)}V_2(Q)=\sum_{i=1}^{m}(q^i-\bar{q})^\top f^\top(x^i,t)\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+\delta_{\mathcal{D}}(i)(s-x^i)\Big],$$

$$\frac{d}{dt}|_{(10)}V_3(C,\epsilon)=\sum_{i=1}^{m}(c_i-a)\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+\delta_{\mathcal{D}}(i)(s-x^i)\Big]^\top\Gamma\Big(\sum_{j\in\mathcal{N}(i)} e^{ji}\Big)$$
$$+\sum_{i\in\mathcal{D}}(\epsilon_i-a)\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+(s-x^i)\Big]^\top\Gamma(s-x^i).$$

Therefore,

$$\frac{d}{dt}|_{(6,10)}\hat{V}(\tilde{E},Q,C,\epsilon)\le-\frac{\delta}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}e^{ij}-\delta\sum_{i\in\mathcal{D}}(x^i-s)^\top(x^i-s)+\frac{\alpha}{2}\sum_{i=1}^{m}\sum_{j\in\mathcal{N}(i)} e^{ij\top}\Gamma e^{ij}$$
$$+\alpha\sum_{i\in\mathcal{D}}(x^i-s)^\top\Gamma(x^i-s)-a\sum_{i=1}^{m}\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+\delta_{\mathcal{D}}(i)(s-x^i)\Big]^\top\Gamma\Big[\sum_{j\in\mathcal{N}(i)} e^{ji}+\delta_{\mathcal{D}}(i)(s-x^i)\Big]$$
$$=-\delta\hat{x}^\top\Big(\hat{L}\otimes I_n\Big)\hat{x}+\hat{x}^\top\Big[\hat{L}(\alpha I_m-a\hat{L})\otimes\Gamma\Big]\hat{x}\le-\delta\hat{x}^\top\Big(\hat{L}\otimes I_n\Big)\hat{x} \tag{18}$$

according to $a\lambda_2(\hat{L})>\alpha$ which implies that $\hat{L}(\alpha I_m-a\hat{L})\le 0$. Similar to the arguments in the proof of Theorem 1, we can conclude that the first claim. Moreover, the second and third claims can be proved as a repetition of the proof of Theorem 1. □

Proof of Proposition 2: This proof is quite similar to Proposition 1. First, from the adaptive equation (10), we have

$$(m+\#\mathcal{D})\varepsilon=\int_0^\infty\Big[\sum_{i=1}^{m}\dot{c}_i(t)+\sum_{j\in\mathcal{D}}\dot{\epsilon}_i(t)\Big]dt$$
$$=\eta\int_0^\infty\sum_{i=1}^{m}\big[\sum_{j\in\mathcal{N}(i)} e^{ji}(t)+\delta_{\mathcal{D}}(i)(s(t)-x^i(t))\big]^\top\Gamma\big[\sum_{j\in\mathcal{N}(i)} e^{ji}(t)+\delta_{\mathcal{D}}(i)(s(t)-x^i(t))\big]dt$$
$$=\eta\int_0^\infty\hat{x}^\top(t)\Big(\hat{L}^2\otimes\Gamma\Big)\hat{x}(t)dt$$

And, the derivative (18) of $\hat{V}(\tilde{E},Q,C,\epsilon)$ gives

$$\hat{x}^\top\Big[\hat{L}(a\hat{L}-\alpha I_m)\otimes\Gamma\Big]x\le-\dot{\hat{V}}(\tilde{E},Q,C,\epsilon).$$

Similarly by Claim 1, the inequality above implies

$$\varepsilon\le\frac{\eta}{m+\#\mathcal{D}}\int_0^\infty\hat{x}^\top(t)\Big(\hat{L}^2\otimes\Gamma\Big)\hat{x}(t)dt\le\frac{\eta\lambda_2(\hat{L})}{[a\lambda_2(\hat{L})-\alpha](m+\#\mathcal{D})}(\hat{V}_0-\hat{V}_\infty)$$

where $\hat{V}_0 = V(\tilde{E}(0), Q(0), C(0), \epsilon(0))$ and $\hat{V}_\infty = \lim_{t\to\infty} V(\tilde{E}(t), Q(t), C(t), \epsilon(t))$. Noting that

$$\begin{aligned}
\hat{V}_0 - \hat{V}_\infty \;&=\; \hat{m}_0 + \frac{1}{2\eta}\sum_{i=1}^{m}(2ac_i^\infty - c_i^{\infty 2}) + \frac{1}{2\eta}\sum_{k\in\mathcal{D}}(2a\epsilon_k^\infty - \epsilon_k^{\infty 2}) \\
&\le\; \hat{m}_0 + \frac{a(m+\#\mathcal{D})}{\eta}\varepsilon - \frac{m+\#\mathcal{D}}{2\eta}\varepsilon^2
\end{aligned}$$

for all $i = 1, \cdots, m$ and $k \in \mathcal{D}$ since

$$\sum_{i=1}^{m} c_i^{\infty 2} + \sum_{k\in\mathcal{D}} \epsilon_k^{\infty 2} \ge \frac{1}{m+\#\mathcal{D}}\Bigg(\sum_{i=1}^{m} c_i^\infty + \sum_{k\in\mathcal{D}} \epsilon_k^\infty\Bigg)^2.$$

Then, it gives

$$\varepsilon \le \frac{\eta\lambda_2(\hat{L})}{[a\lambda_2(\hat{L}) - \alpha](m+\#\mathcal{D})}\Bigg[\hat{m}_0 + \frac{a(m+\#\mathcal{D})}{\eta}\varepsilon - \frac{m+\#\mathcal{D}}{2\eta}\varepsilon^2\Bigg],$$

which implies that $\varepsilon \le \hat{\varepsilon}$ by solving the quadratic inequality above.

Second, define $T = [t_{ij}]_{i,j=1}^{m+1} \in \mathbb{R}^{m+1,m+1}$ as follows:

$$t_{ij} = \begin{cases} 1 & i = j \text{ and } i \le m \\ -1 & i \le m \text{ and } j = m+1 \\ 0 & \text{otherwise} \end{cases}$$

and let $\hat{W} = 1/m\; T^\top T$. One can see that $\hat{W}$ has eigenvalue 0 with multiplicity one, eigenvalue $1/m$ with multiplicity $m-1$, and eigenvalue $1 + 1/m$ with multiplicity one. Similarly by Claim 2, this gives

$$\begin{aligned}
\chi \;&=\; \int_0^\infty \hat{x}^\top(t)\Big(\hat{W}\otimes I_n\Big)\hat{x}(t)dt \le \frac{m+1}{m\lambda_2^2(L)}\int_0^\infty \hat{x}^\top(t)\Big(\hat{L}^2\otimes I_n\Big)\hat{x}(t)dt \\
&\le\; \frac{(m+1)(m+\#\mathcal{D})}{m\lambda_2^2(\hat{L})\eta}\varepsilon \le \frac{(m+1)(m+\#\mathcal{D})}{m\lambda_2^2(\hat{L})\eta}\hat{\varepsilon} = \hat{\chi}
\end{aligned}$$

since $\Gamma = I_n$. This completes the proof of this proposition. $\square$

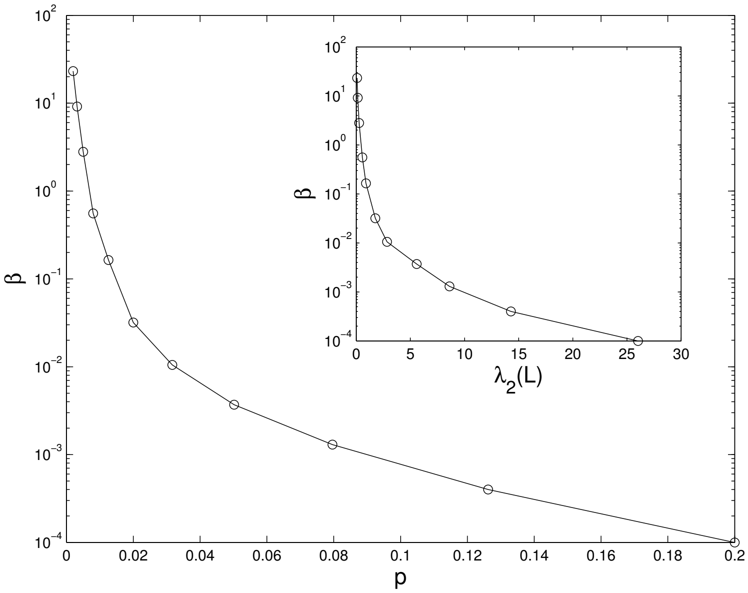

FIG. 1: β vs p and its corresponding $\lambda_2(L)$ (the inner sub figure) in a SW network with network size $m = 200$, where we pick $\eta = 0.5$, $\rho = 5$, and the same initial data.

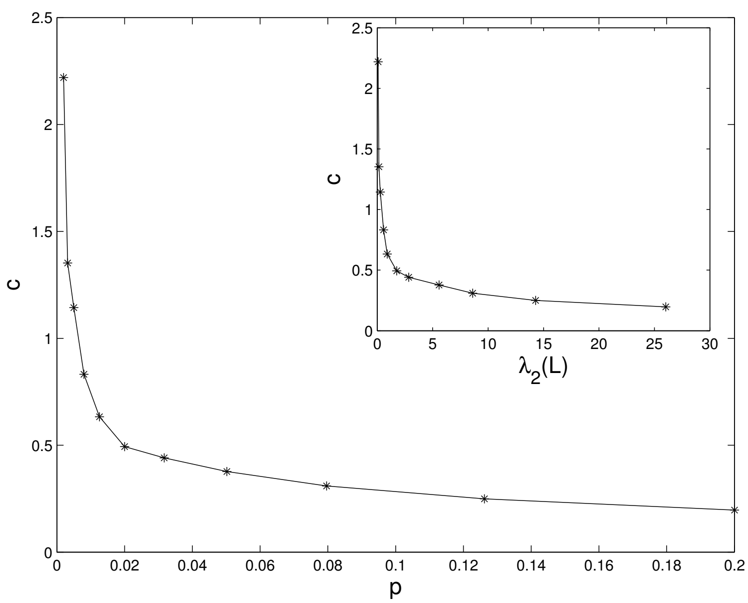

FIG. 2: c vs p and its corresponding $\lambda_2(L)$ (the inner sub figure) in a SW network with network size $m = 200$, where we pick $\eta = 5$, $\rho = 0.5$, and the same initial data.

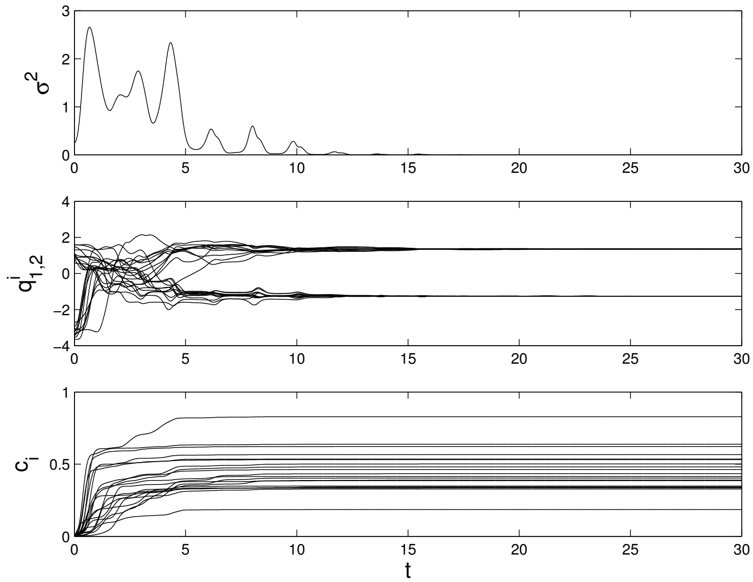

FIG. 3: The dynamics of the time-average variance $\sigma^2(t)$, the collective parameters $q^i_{1,2}$, and the collective coupling strengths c_i in the SW network with network size $m = 200$ and the connection probability $p = 0.1$, where we pick $\eta = 5$ and $\rho = 0.5$ and illustrate dynamics with random twenty indices i.

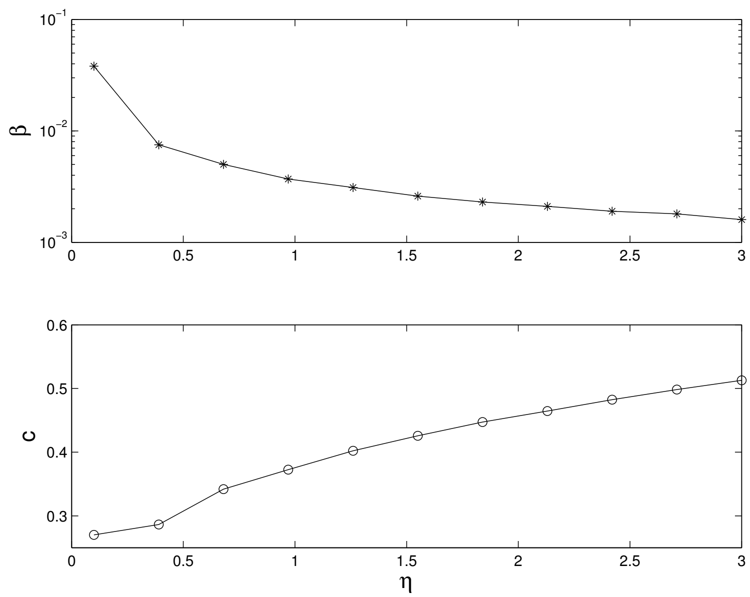

FIG. 4: β and c vs η in the SW network with network size $m = 200$ and the connection probability $p = 0.1$, where we pick $\rho = 0.5$ and the same initial data.

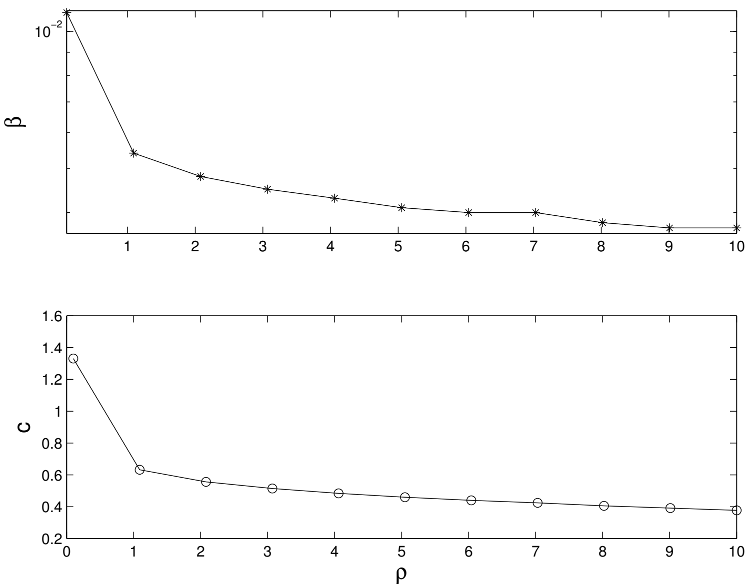

FIG. 5: β and c vs ρ in the SW network with network size $m = 200$ and the connection probability $p = 0.1$, where we pick $\eta = 5$ and the same initial data.

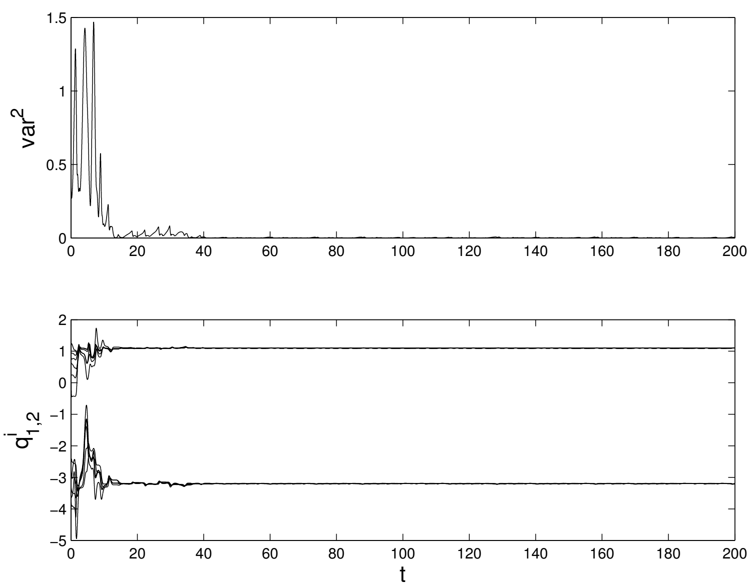

FIG. 6: The dynamics of the time-average variance $\mathrm{var}^2(t)$ and the parameters $q^i_{1,2}$ in the SF network with the network size $m = 102$ and pinning fraction $ff = 0.2$, where we pick $\eta = 10$ and $\rho = 20$ and illustrate dynamics with random ten indices i.

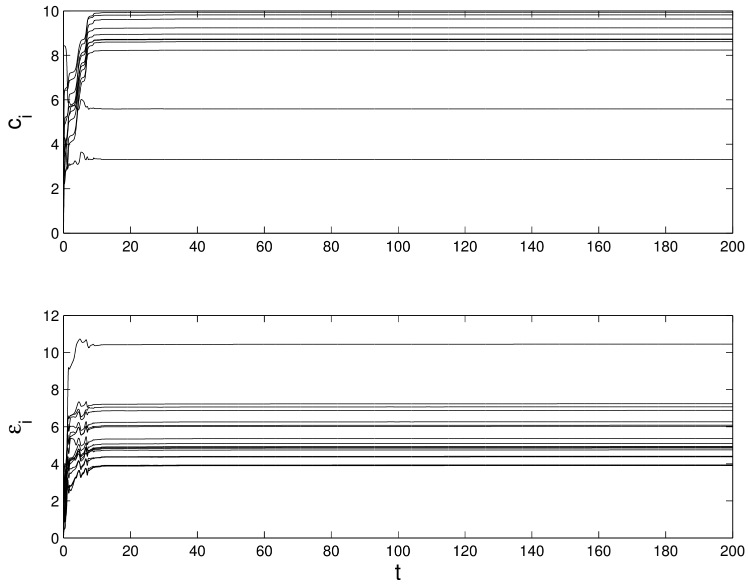

FIG. 7: The dynamics of the terminal coupling strengths c_i and the pinning strengths ϵ_j in the SF network with the network size $m = 102$ and the pinning fraction $ff = 0.2$, where we pick $\eta = 10$ and $\rho = 20$ and illustrate dynamics with random fifteen indices i and ten indices $j \in \mathcal{D}$.

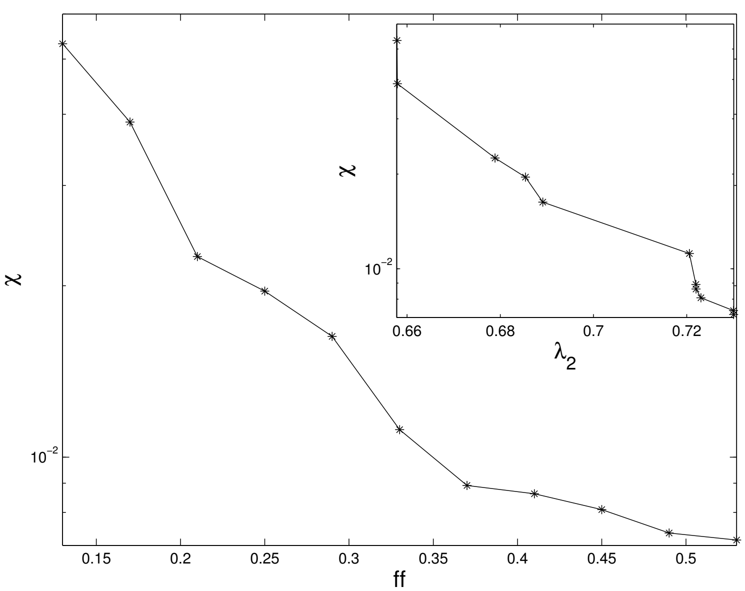

FIG. 8: χ vs the pinning fraction ff and its corresponding $\lambda_2(\hat{L})$ (the inner sub-figure) in SF network with the network size $m = 102$, where we pick $\eta = 10$, $\rho = 20$, and the same initial data.

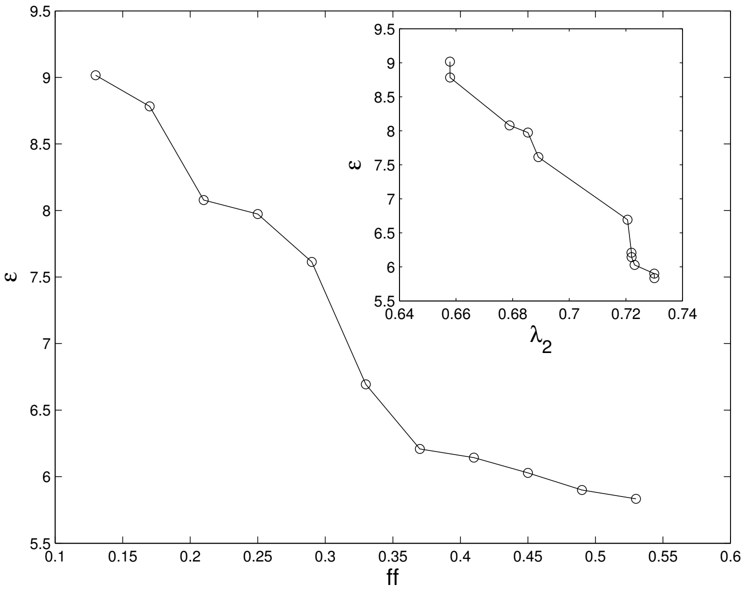

FIG. 9: ε vs ff and its corresponding $\lambda_2(\hat{L})$ (the inner sub-figure) in SF network with the network size $m = 102$, where we pick $\eta = 10$, $\rho = 20$, and the same initial data.

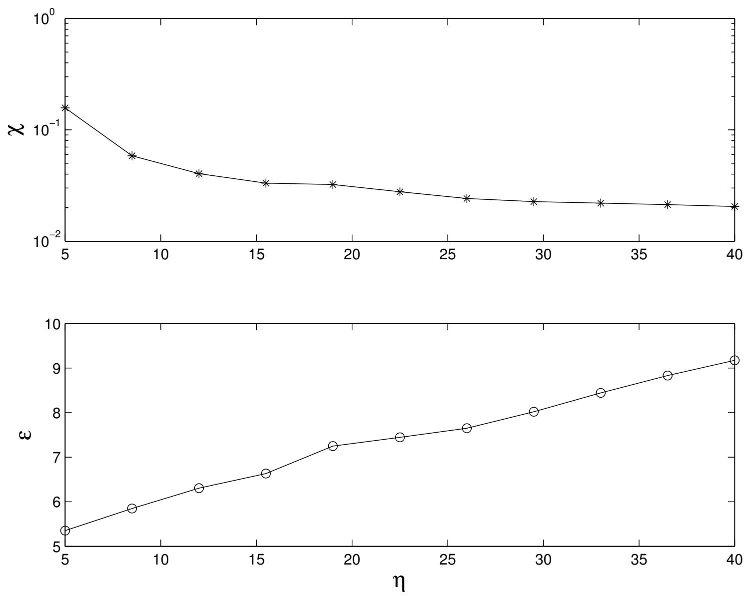

FIG. 10: χ and ε vs η in the SF network with the network size $m = 102$ and the pinning fraction $ff = 0.2$, where we pick $\rho = 20$ and the same initial data.

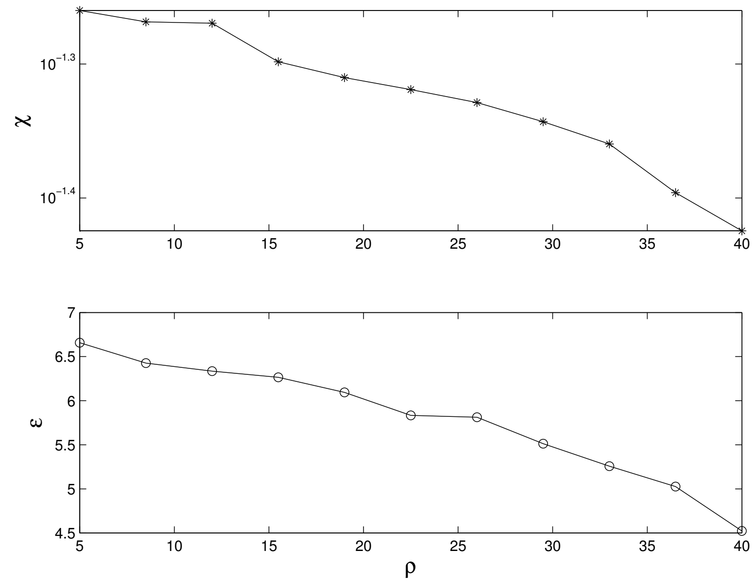

FIG. 11: χ and ε vs ρ in the SF network with the network size $m = 102$ and the pinning fraction $ff = 0.2$, where we pick $\eta = 10$ and the same initial data.

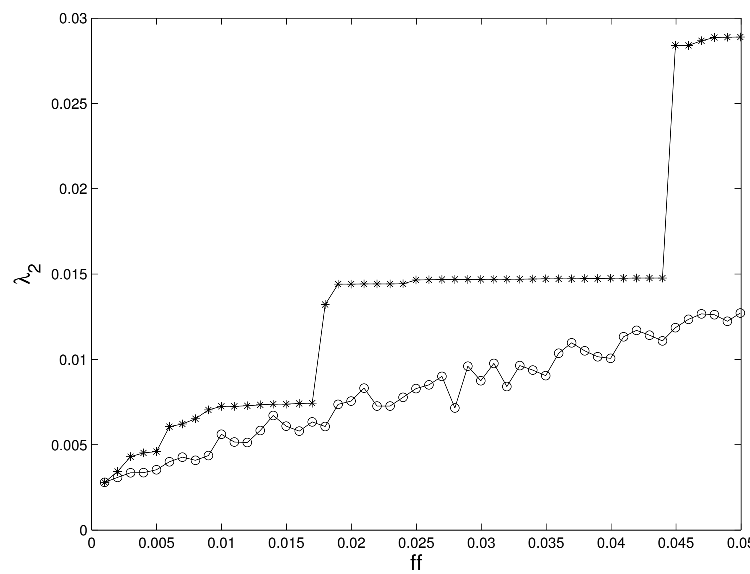

FIG. 12: $\lambda_2(\hat{L})$ vs the pinning fraction $ff \in [0, 0.05]$ under two pinning strategies: randomly pinning $(-o-)$ and selectively pinning based on degree distribution $(- * -)$ in SF network with network size $m = 1002$ and $k = 2$. The curves are plotted by averaging ten realizations.