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LIOUVILLE THEOREMS FOR DIRAC-HARMONIC MAPS

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Abstract. We prove Liouville theorems for Dirac-harmonic maps from the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ and a Riemannian manifold $\mathfrak{S}^n$ ($n \geq 3$) with the Schwarzschild metric to any Riemannian manifold N.

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1. Introduction

Let (M^n, g) be a Riemannian manifold with fixed spin structure, ΣM its spinor bundle, on which we chose a Hermitian metric $\langle \cdot, \cdot \rangle$. Let ∇ be the Levi-Civita connection on ΣM compatible with $\langle \cdot, \cdot \rangle$ and g. Let ϕ be a smooth map from M to a Riemannian manifold (N, h) of dimension $n' \geq 2$ and $\phi^{-1}TN$ the pull-back bundle of TN by ϕ. On the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ there is a metric (still denoted by $\langle \cdot, \cdot \rangle$) induced from the metrics on ΣM and $\phi^{-1}TN$. There is also a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$. In local coordinates $\{x_\alpha\}$ and $\{y^i\}$ on M and N respectively, we write the section ψ of $\Sigma M \otimes \phi^{-1}TN$ as

$$\psi(x) = \psi^j(x) \otimes \partial_{y^j}(\phi(x)),$$

where ψ^i is a spinor on M and $\{\partial_{y^j}\}$ is the natural local basis on N, and $\widetilde{\nabla}$ can be written as

$$\widetilde{\nabla}\psi(x) = \nabla\psi^i(x) \otimes \partial_{y^i}(\phi(x)) + \Gamma^i_{jk}\nabla\phi^j(x)\psi^k(x) \otimes \partial_{y^i}(\phi(x)).$$

Here and in the sequel, we use the summation convention.

The *Dirac operator along the map* ϕ is defined as

$$\begin{aligned} \not{D}\psi &:= e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi \\ &= \not{\partial}\psi^i(x) \otimes \partial_{y^i}(\phi(x)) + \Gamma^i_{jk}\nabla_{e_\alpha}\phi^j(x) e_\alpha \cdot \psi^k(x) \otimes \partial_{y^i}(\phi(x)), \end{aligned}$$

where $\{e_\alpha\}$ is the local orthonormal basis of M and $\not{\partial} := e_\alpha \cdot \nabla_{e_\alpha}$ is the usual Dirac operator on M. The Dirac operator $\not{D}$ is formally self-adjoint, i.e.,

$$\int_M \langle \psi, \not{D}\xi \rangle = \int_M \langle \not{D}\psi, \xi \rangle, \tag{1.1}$$

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for all $\psi, \xi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$. For properties of the spin bundle ΣM and the Dirac operator $\not\partial$, we refer the readers to [7] or [6].

Let us consider the functional

$$L(\phi, \psi) := \frac{1}{2}\int_M [|d\phi|^2 + \langle \psi, \not D\psi\rangle], \tag{1.2}$$

where $\langle \psi, \xi\rangle := h_{ij}(\phi)\langle \psi^i, \xi^j\rangle$, for $\psi, \xi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$.

The Euler-Lagrange equations of L are (see [2]):

$$\tau^i(\phi) = \frac{1}{2}R^i{}_{jkl}(\phi)\langle \psi^k, \nabla\phi^j \cdot \psi^l\rangle, \tag{1.3}$$

$$\not D\psi^i := \not\partial\psi^i + \Gamma^i_{jk}(\phi)\partial_\alpha\phi^j e_\alpha \cdot \psi^k = 0, \tag{1.4}$$

$i = 1, 2, \cdots, n' := \dim N$, where $\tau(\phi)$ is the tension field of the map ϕ.

Denoting

$$\mathcal{R}(\phi, \psi) := \frac{1}{2}R^i{}_{jkl}(\phi)\langle \psi^k, \nabla\phi^j \cdot \psi^l\rangle \otimes \partial_{y^i},$$

then (1.3) and (1.4) can be written as:

$$\tau(\phi) = \mathcal{R}(\phi, \psi), \tag{1.5}$$

$$\not D\psi = 0. \tag{1.6}$$

We call solutions (ϕ, ψ) of the coupled system (1.3) and (1.4) Dirac-harmonic maps from M into N. The system (1.3, 1.4) arises from the supersymmetric nonlinear sigma model of quantum field theory by making all variables commuting (see [2] and [3]). Thus, Dirac-harmonic constitute a natural extension of the harmonic maps thoroughly studied in geometric analysis. An obvious question then is to what extent the structural theory of harmonic maps generalizes to Dirac-harmonic maps.

In the present paper, our starting point in this direction is [5], where, motivated again by considerations from quantum field theory, it was proved that any harmonic map of finite energy from the Euclidean space $\mathbb{R}^n$ ($n \geq 3$) into a Riemannian manifold N must be constant. In [11], this vanishing property was shown for the case of the domain manifold $\mathbb{H}^n$, the hyperbolic space. These Liouville theorems are a consequence of the non-invariance of the energy functional under conformal transformations, and the fact that there exist conformal vector fields on the domains. In [10], these results were extended to the case where the domain is a Riemannian manifold $\mathfrak{S}^n$ with the Schwarzschild metric (see definitions and notations in section 3).

In contrast to harmonic maps, the integrands in the functional L for Dirac-harmonic maps are not nonnegative in general, and the energy functional should be chosen as follows (c.f. [2] and [3]):

$$E(\phi, \psi) := \int_M [|d\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}}].$$

Our aim is to extend the previous Liouville theorems to the case of Dirac-harmonic maps. We will prove the following

Theorem 1.1. *Let M^n be one of $\mathbb{R}^n$, $\mathbb{H}^n$, $\mathfrak{S}^n$, $n \geq 3$, N be any Riemannian manifold. Let $\phi : M \to N$ be a map and $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$. If (ϕ, ψ) is a Dirac-harmonic map with finite energy:*

$$E(\phi, \psi) := \int [|d\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}}] < \infty, \tag{1.7}$$

then ϕ must be constant and $\psi \equiv 0$.

In fact, the supersymmetric σ-model in superstring theory includes an additional curvature term in addition to (1.2). Turning again the components of ψ, which in quantum field theory take values in some Grassmann algebra and anticommute with each other, into ordinary spinor fields on M, we have the following functional:

$$L_c(\phi, \psi) := \frac{1}{2}\int_M [|d\phi|^2 + \langle\psi, \not{D}\psi\rangle - \frac{1}{6}R_{ikjl}\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle]. \tag{1.8}$$

We call the critical points (ϕ, ψ) of L_c *Dirac-harmonic maps with curvature term.* We should point out that the factor $-\frac{1}{6}$ in front of the curvature term in (1.8) is dictated by supersymmetry. Since in our treatment of the functional, we shall not utilize this symmetry, the value of this coupling constant will not be essential for us, except that changing it from negative to positive values would also change the sign in the curvature condition in Theorem 1.2 below. In other words, with a positive instead of a negative coupling constant, we would obtain a vanishing for negatively curved targets.

The Euler-Lagrange equations of the functional L_c are (see section 2 below):

$$\tau^m(\phi) - \frac{1}{2}R^m{}_{lij}\langle\psi^i, \nabla\phi^l \cdot \psi^j\rangle + \frac{1}{12}h^{mp}R_{ikjl;p}\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle = 0, \tag{1.9}$$

$$\not{D}\psi^m = -\frac{1}{3}R^m{}_{jkl}\langle\psi^j, \psi^k\rangle\psi^l, \qquad m = 1, 2, \cdots, n'. \tag{1.10}$$

For solutions of this system, we also have a Liouville theorem. However, due to the presence of the curvature term in the functional L_c, we will need a condition on the curvature of the target N, namely that N has positive sectional curvature.

Theorem 1.2. *Let M, N, ϕ and ψ be as in Theorem 1.1, suppose N has positive sectional curvature. If (ϕ, ψ) is a Dirac-harmonic map with curvature term with finite energy, then ϕ must be constant and $\psi \equiv 0$.*

2. The Euler-Lagrange equations for L_c

Let us first derive the Euler-Lagrange equations for L_c. We put

$$A := h_{ij}(\phi)g^{\alpha\beta}\frac{\partial\phi^i}{\partial x_\alpha}\frac{\partial\phi^j}{\partial x_\beta}, \quad B := h_{ij}(\phi)\langle\psi^i, \not{D}\psi^j\rangle, \quad R := -\frac{1}{6}R_{ikjl}(\phi)\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle,$$

and have

$$L_c = \frac{1}{2}\int_M (A+B+R).$$

First, noting that

$$\begin{aligned}\delta_\psi B &= 2\langle\delta\psi, \not{D}\psi\rangle \\ &= 2h_{ij}\langle\delta\psi^i, \not{D}\psi^j\rangle\end{aligned}$$

and

$$\begin{aligned}\delta_\psi R &= -\frac{1}{6}R_{ijkl}[\langle\delta\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle + \langle\psi^i,\delta\psi^k\rangle\langle\psi^j,\psi^l\rangle \\ &\quad +\langle\psi^i,\psi^k\rangle\langle\delta\psi^j,\psi^l\rangle + \langle\psi^i,\psi^k\rangle\langle\psi^j,\delta\psi^l\rangle] \\ &= -\frac{2}{3}R_{ijkl}\langle\delta\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle,\end{aligned}$$

we have

$$\begin{aligned}\delta_\psi L_c &= \frac{1}{2}\int_M [2h_{ij}\langle\delta\psi^i, \not{D}\psi^j\rangle - \frac{2}{3}R_{ijkl}\langle\delta\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle] \\ &= \int_M [\langle\delta\psi^i, h_{ij}\not{D}\psi^j\rangle - \frac{1}{3}R_{ijkl}\langle\delta\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle],\end{aligned}$$

which implies that

$$h_{ij}\not{D}\psi^j - \frac{1}{3}R_{ijkl}\psi^k\langle\psi^j,\psi^l\rangle = 0.$$

Thus, we obtain *the ψ-equation for L*:

$$\not{D}\psi^m = -\frac{1}{3}R^m{}_{jkl}\langle\psi^j,\psi^k\rangle\psi^l. \tag{2.1}$$

Second, consider the ϕ-variation $\{\phi_t\}$ with $\phi_0 = \phi$ and $\frac{d\phi_t}{dt}|_{t=0} = \xi$, we have

$$\begin{aligned}\frac{dL_c(\phi_t)}{dt}|_{t=0} &= \frac{1}{2}\int_M \frac{\partial}{\partial t}|d\phi_t|^2|_{t=0} + \frac{1}{2}\int_M \frac{\partial}{\partial t}\langle\psi,\not{D}\psi\rangle|_{t=0} \\ &\quad -\frac{1}{12}\int_M \frac{\partial}{\partial t}(R_{ijkl}\langle\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle)|_{t=0} \\ &:= I_1 + I_2 + I_3. \end{aligned}\tag{2.2}$$

For the term I_1 it is well-known that (see e.g. [12] or [6])

$$I_1 = -\int_M h_{im}\tau^i(\phi)\xi^m. \tag{2.3}$$

For I_2 we choose an orthonormal basis $\{e_\alpha|\alpha = 1,2,\cdots,n\}$ on M with $[e_\alpha, \partial_t] = 0$. Note that

$$\begin{aligned}\frac{\partial}{\partial t}\langle\psi,\not{D}\psi\rangle &= \langle\widetilde{\nabla}_{\partial_t}\psi, \not{D}\psi\rangle + \langle\psi, \widetilde{\nabla}_{\partial_t}\not{D}\psi\rangle \\ &:= \langle\psi_t, \not{D}\psi\rangle + \langle\psi, \widetilde{\nabla}_{\partial_t}\not{D}\psi\rangle.\end{aligned}\tag{2.4}$$

One can compute

$$\begin{aligned}\widetilde{\nabla}_{\partial_t} \not{D}\psi &= \widetilde{\nabla}_{\partial_t}(e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi)\\ &= e_\alpha \cdot \nabla_{e_\alpha}\psi^i \otimes \nabla_{\partial_t}\partial_{y^i} + e_\alpha \cdot \psi^i \otimes \nabla_{\partial_t}\nabla_{e_\alpha}\partial_{y^i}\\ &= e_\alpha \cdot \nabla_{e_\alpha}\psi^i \otimes \nabla_{\partial_t}\partial_{y^i} + e_\alpha \cdot \psi^i \otimes [\nabla_{e_\alpha}\nabla_{\partial_t}\partial_{y^i} + R(\partial_t, e_\alpha)\partial_{y^i}]\\ &= e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}(\psi^i \otimes \nabla_{\partial_t}\partial_{y^i}) + e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha))\partial_{y^i}\\ &= \not{D}\psi_t + e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha))\partial_{y^i}.\end{aligned}$$

It follows that

$$\langle \psi, \widetilde{\nabla}_{\partial_t}\not{D}\psi\rangle = \langle \psi, \not{D}\psi_t\rangle + \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha))\partial_{y^i}\rangle. \tag{2.5}$$

Since

$$\begin{aligned}R^N(d\phi(\partial_t), d\phi(e_\alpha))\partial_{y^i}|_{t=0} &= R^N(\xi^m\partial_{y^m}, \phi^l_\alpha\partial_{y^l})\partial_{y^i}\\ &= \xi^m\phi^l_\alpha R^j_{iml}\partial_{y^j},\end{aligned}$$

we have

$$\begin{aligned}\langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha))\partial_{y^i}\rangle|_{t=0} &= \langle \psi, \xi^m\phi^l_\alpha R^j_{iml}\partial_{y^j} \otimes e_\alpha \cdot \psi^i\rangle\\ &= \langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}\xi^m.\end{aligned}$$

From this formula and (2.5) we have

$$\langle \psi, \widetilde{\nabla}_{\partial_t}\not{D}\psi\rangle|_{t=0} = \langle \psi, \not{D}\psi_t\rangle|_{t=0} + \langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}\xi^m.$$

Combining this with (2.4) we obtain

$$\frac{\partial}{\partial t}\langle \psi, \not{D}\psi\rangle|_{t=0} = \langle \psi_t, \not{D}\psi\rangle|_{t=0} + \langle \psi, \not{D}\psi_t\rangle|_{t=0} + \langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}\xi^m.$$

Thus, we have

$$I_2 = \frac{1}{2}\int_M [\langle \psi_t, \not{D}\psi\rangle + \langle \psi, \not{D}\psi_t\rangle]|_{t=0} + \frac{1}{2}\int_M \langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}\xi^m. \tag{2.6}$$

From

$$\psi_t = \widetilde{\nabla}_{\partial_t}(\psi^i \otimes \partial_{y^i})|_{t=0} = \psi^i \otimes \nabla_{d\phi(\partial_t)}\partial_{y^i}|_{t=0} = \xi^m\psi^i \otimes \Gamma^k_{im}\partial_{y^k},$$

we have

$$\begin{aligned}\langle \psi_t, \not{D}\psi\rangle|_{t=0} &= \langle \xi^m\psi^i\Gamma^k_{im} \otimes \partial_{y^k}, \not{D}\psi^l \otimes \partial_{y^l}\rangle\\ &= \langle \xi^m\psi^i\Gamma^k_{im}, \not{D}\psi^l h_{kl}\rangle\\ &= \langle \psi^i, \not{D}\psi^j\rangle\xi^m\Gamma_{im,j},\end{aligned}$$

where $\Gamma_{im,j} := \Gamma^k_{im}h_{kj}$. Therefore,

$$I_2 = \int_M \langle \psi^i, \not{D}\psi^j\rangle\xi^m\Gamma_{im,j} + \frac{1}{2}\int_M \langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}\xi^m. \tag{2.7}$$

From (2.3) and (2.7) we obtain

$$I_1 + I_2 = \frac{1}{2}\int_M [-2h_{im}\tau^i(\phi) + 2\langle \psi^i, \not{D}\psi^j\rangle\xi^m\Gamma_{im,j} + \langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}]\xi^m. \tag{2.8}$$

Using the ψ-equation,

$$\begin{aligned} 2\langle\psi^i, \not{D}\psi^j\rangle\xi^m\Gamma_{im,j} &= 2\langle\psi^i, \not{D}\psi^p\rangle\xi^m\Gamma_{im,p} \\ &= \frac{2}{3}\Gamma_{mi,p}R^p_{jkl}\langle\psi^i, \psi^k\rangle\langle\psi^j, \psi^l\rangle, \end{aligned}$$

we have

$$I_1 + I_2 = \frac{1}{2}\int_M [-2h_{im}\tau^i(\phi) + \frac{2}{3}\Gamma_{mi,p}R^p{}_{jkl}\langle\psi^i, \psi^k\rangle\langle\psi^j, \psi^l\rangle + \langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle R_{mlij}]\xi^m. \tag{2.9}$$

The term I_3 is easy to compute.

$$I_3 = -\frac{1}{2}\int_M \frac{1}{6}R_{ijkl,m}\langle\psi^i, \psi^k\rangle\langle\psi^j, \psi^l\rangle\xi^m.$$

Substituting this and (2.9) into (2.2) yields

$$\begin{aligned} \frac{dL_c(\phi_t)}{dt}|_{t=0} &= \frac{1}{2}\int_M [-2h_{im}\tau^i(\phi) + \frac{2}{3}\Gamma_{mi,p}R^p{}_{jkl}\langle\psi^i, \psi^k\rangle\langle\psi^j, \psi^l\rangle \\ &\quad +\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle R_{mlij} - \frac{1}{6}R_{ijkl,m}\langle\psi^i, \psi^k\rangle\langle\psi^j, \psi^l\rangle]\xi^m \\ &= \frac{1}{2}\int_M [-2h_{im}\tau^i(\phi) + \langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle R_{mlij} \\ &\quad -\frac{1}{6}R_{ijkl;m}\langle\psi^i, \psi^k\rangle\langle\psi^j, \psi^l\rangle]\xi^m. \end{aligned} \tag{2.10}$$

Here, $R_{ijkl;m}$ denotes the covariant derivative of the curvature tensor R_{ijkl} with respect to $\frac{\partial}{\partial y^m}$. Therefore, we obtain *the ϕ-equation for L_c*:

$$\tau^m(\phi) - \frac{1}{2}R^m{}_{lij}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle + \frac{1}{12}h^{mp}R_{ikjl;p}\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle = 0. \tag{2.11}$$

3. Proofs of theorems

Now we start to prove our main Theorems. Suppose $X \in \Gamma(TM)$ is a conformal vector field on (M, g), namely,

$$L_X g = 2fg, \tag{3.1}$$

where $f \in C^\infty(M)$. Here L_X denotes the Lie derivative with respect to X. The vector field X generates a family of conformal diffeomorphisms

$$F_t := exp(tX) : M \to M.$$

We will consider the variation of the functionals L and L_c under this family of diffeomorphisms.

In the Euclidean space $\mathbb{R}^n$, the vector field $X(x) := x$ is conformal with $f = 2$. Consider $\mathbb{R}^n$ equipped with a metric

$$g = b^2(dr^2 + a^2 d\Theta^2),$$

where a, b are radial functions, (r, Θ) are polar coordinates centered at the origin, and $d\Theta^2$ stands for the standard metric on the unit sphere $\mathbb{S}^{n-1}$. Then the vector field $X := a(r)\partial_r$ satisfies: $L_X g = 2fg$ with $f = (ab)'/b$, that is, X is a conformal vector field (c.f. [10]). Besides the standard Euclidean space $\mathbb{R}^n$, we also consider the following cases:

(i) The hyperbolic space $\mathbb{H}^n = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} | 1 + x^2 = t^2\}$: $b^2 = 1/(r^2+1)$ and $a = r/b$. In this case, $f \geq 1$ and $|X(r)| \leq r$.

(ii) $\mathfrak{S}^n$ with the Schwarzschild metric: a constant slice of the outer region ($r > r_0$ $:= 2m$) of $n+1$-dimensional Schwarzschild space, $b = 1/\sqrt{1 - \frac{r_0}{r}}$ and $a = r/b$, where m is the mass of a black hole. In this case, $0 < f \leq 1$ and $|X(r)| \leq r$.

Recall the definition of L:

$$L(\phi, \psi, g) = \frac{1}{2} \int_M [|d\phi|^2 + \langle \psi, \not{D}\psi \rangle] v_g,$$

where $v_g := \sqrt{det g_{\alpha\beta)}} dx$ is the volume form of M.

$$\Omega := (|d\phi|^2 + \langle \psi, \not{D}\psi \rangle) v_g$$

is an n-form on M. We note that for any $\eta \in C_0^\infty(M)$,

$$0 = \int_M d[(\imath_X \Omega)\eta] = \int_M \eta d(\imath_X \Omega) + \int_M d\eta \wedge \imath_X \Omega = \int_M \eta L_X \Omega + \int_M d\eta \wedge \imath_X \Omega,$$

that is,

$$\int_M \eta L_X \Omega = -\int_M d\eta \wedge \imath_X \Omega, \tag{3.2}$$

where $\imath_X$ stands for the inner product with the vector X.

Now let us compute $L_X \Omega$. We first recall the following

Lemma 2.1(c.f. [4]**).** *Let $\phi : M \to N$ be a map, and X any smooth vector field on M. Then*

$$L_X(\frac{1}{2}|d\phi|^2 v_g) = \langle d\phi, \nabla(d\phi(X)) \rangle v_g + \frac{1}{2} \langle L_X g, S_\phi \rangle v_g, \tag{3.3}$$

$$L_X v_g = \frac{1}{2} \langle L_X g, g \rangle v_g, \tag{3.4}$$

where $S_\phi := \frac{1}{2}|d\phi|^2 g - \phi^* h$ is the stress-energy tensor of ϕ.

Second, we note that

$$\begin{aligned} L_X(\langle \not{D}\psi, \psi \rangle v_g) &= (L_X \langle \not{D}\psi, \psi \rangle) v_g + \langle \not{D}\psi, \psi \rangle L_X v_g \\ &= \langle L_X(\not{D}\psi), \psi \rangle v_g + \langle \not{D}\psi, L_X \psi \rangle v_g + \frac{1}{2} \langle \not{D}\psi, \psi \rangle \langle L_X g, g \rangle v_g. \end{aligned} \tag{3.5}$$

To continue, we recall that

$$\begin{aligned} \not{D}\psi &= e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi \\ &= \not{\partial}\psi^i \otimes \partial_{y^i}(\phi) + (e_\alpha \cdot \psi^i)\phi^j_\alpha \nabla_{\partial_{y^j}}\partial_{y^i}(\phi), \end{aligned}$$

The variation of $\not{D}\psi$ consists of two parts: one with respect to the metric g, the other with respect to the parameterization p of M caused by X, namely,

$$\frac{d}{dt}(\not{D}\psi)|_{t=0} = \delta_g(\not{D}\psi) + \delta_p(\not{D}\psi). \tag{3.6}$$

Lemma 2.2. *The first variation is:*

$$\delta_g(\not{D}\psi) = -\frac{1}{2}e_\alpha \cdot \widetilde{\nabla}_{K(e_\alpha)}\psi + A \cdot \psi, \tag{3.7}$$

where $A := \frac{1}{4}[\mathrm{div}_g k + d(\mathrm{Tr}_g k)]$, $k := L_X g$ *and* K *is a (1,1)-tensor on* M *defined by*

$$g(K(e_\alpha), e_\beta) := k(e_\alpha, e_\beta) = L_X g(e_\alpha, e_\beta).$$

Proof. The proof follows closely [1]. See also [8]. In order to obtain (3.7), we first note that given any real $n-$dimensional vector space V equipped with a metric g, then for any other metric g' on V, there exists a unique positive endomorphism H on V such that $g'(\cdot,\cdot) = g(H(\cdot),\cdot)$. It is clear that $b_{g',g} := H^{-1/2}$ transforms $g-$orthonormal frames to $g'-$orthonormal frames. And consequently, we have an SO_n-equivariant map from the manifold $P(g)$ of $g-$orthonormal frames to the manifold $P(g')$ of $g'-$orthonormal frames.

Since M is spin, the map $b_{g',g}$ can be lifted to a $Spin_n$-equivariant map $\beta_{g',g}$: $\tilde{P}(g) \to \tilde{P}(g')$. Extend $b_{g',g}$ and $\beta_{g',g}$ to an SO_n-equivariant map $b_{g',g} : P_{SO}(M,g) \to P_{SO}(M,g')$ and a $Spin_n$-equivariant map $\beta_{g',g} : P_{Spin}(M,g) \to P_{Spin}(M,g')$ respectively. Denote the spin bundles with respect to g and g' by $\Sigma_g M$ and $\Sigma_{g'} M$ respectively, then the map $\beta_{g',g}$ extends to an isometry $\beta_{g',g} : \Sigma_g M \to \Sigma_{g'} M$ of Hermitian bundles. Clearly, $\beta^{-1}_{g',g} = \beta_{g,g'}$.

For the Dirac operator $\not{D}$, we consider the transformation operator acting on the spin bundle $\Sigma_g M$:

$$\not{D}_{g',g} := \beta^{-1}_{g',g}\not{D}_{g'}\beta_{g',g},$$

where $\not{D}_{g'}$ denotes the Dirac operator $\not{D}$ with respect to the metric g' on M, namely,

$$\begin{aligned} \not{D}_{g'}\psi &= \not{\partial}_{g'}\psi^i \otimes \partial_{y^i}(\phi) + (e_{\alpha,g'} \cdot \psi^i) \otimes \nabla_{e_{\alpha,g'}}\phi^j \nabla_{\partial_{y^j}}\partial_{y^i}(\phi) \\ &= \not{\partial}_{g'}\psi^i \otimes \partial_{y^i}(\phi) + (\mathrm{grad}_{g'}\phi^j \cdot \psi^i) \otimes \nabla_{\partial_{y^j}}\partial_{y^i}(\phi); \end{aligned}$$

here, $\{e_{\alpha,g'}\}$ denotes the $g'-$orthornormal frame, which is transformed via $b_{g,g'}$ to the $g-$orthornormal frame $\{e_\alpha\}$:

$$b_{g,g'}(e_{\alpha,g'}) = e_\alpha.$$

Hence,

$$\begin{aligned}
\not{D}_{g',g}\psi &= \beta_{g',g}^{-1}\not{\partial}'_{g}\beta_{g',g}\psi^i\otimes\partial_{y^i}+\beta_{g',g}^{-1}(\mathrm{grad}_{g'}\phi^j)\beta_{g',g}\cdot\psi^i\otimes\nabla_{\partial_{y^j}}\partial_{y^i}\\
&= \not{\partial}_{g',g}\psi^i\otimes\partial_{y^i}+b_{g,g'}(\mathrm{grad}_{g'}\phi^j)\cdot\psi^i\otimes\nabla_{\partial_{y^j}}\partial_{y^i}\\
&= \not{\partial}_{g',g}\psi^i\otimes\partial_{y^i}+b_{g,g'}(\nabla_{e_{\alpha,g'}}\phi^j e_{\alpha,g'})\cdot\psi^i\otimes\nabla_{\partial_{y^j}}\partial_{y^i}\\
&= \not{\partial}_{g',g}\psi^i\otimes\partial_{y^i}+\nabla_{e_{\alpha,g'}}\phi^j(e_\alpha\cdot\psi^i)\otimes\nabla_{\partial_{y^j}}\partial_{y^i}. \qquad (3.8)
\end{aligned}$$

For the variation $\{g_t\}$ of g with $\frac{dg_t}{dt}|_{t=0}=L_Xg$, we have

$$\not{D}_{g_t,g}\psi=\not{\partial}_{g_t,g}\psi^i\otimes\partial_{y^i}+\nabla_{e_{\alpha,g_t}}\phi^j(e_\alpha\cdot\psi^i)\otimes\nabla_{\partial_{y^j}}\partial_{y^i},$$

from which we have

$$\frac{d}{dt}(\not{D}_{g_t,g}\psi)|_{t=0}=\frac{d}{dt}(\not{\partial}_{g_t,g})|_{t=0}\psi^i\otimes\partial_{y^i}+\nabla_{\frac{d}{dt}(b_{g_t,g})|_{t=0}(e_\alpha)}\phi^j(e_\alpha\cdot\psi^i)\otimes\nabla_{\partial_{y^j}}\partial_{y^i}. \qquad (3.9)$$

Since $b_{g_t,g}=(Id+tK)^{-1/2}$, it follows that

$$\frac{d}{dt}(b_{g_t,g})|_{t=0}=-\frac{1}{2}K. \qquad (3.10)$$

On the other hand, Theorem 21 in [1] gives us

$$\frac{d}{dt}(\not{\partial}_{g_t,g})|_{t=0}\psi^i=-\frac{1}{2}e_\alpha\cdot\nabla_{K(e_\alpha)}\psi^i+A\cdot\psi^i,\qquad i=1,2,\cdots,n'. \qquad (3.11)$$

Inserting (3.10) and (3.11) into (3.9) then yields

$$\begin{aligned}
\frac{d}{dt}(\not{D}_{g_t,g}\psi)|_{t=0} &= [-\frac{1}{2}e_\alpha\cdot\nabla_{K(e_\alpha)}\psi^i+A\cdot\psi^i]\otimes\partial_{y^i}-\frac{1}{2}\nabla_{K(e_\alpha)}\phi^j(e_\alpha\cdot\psi^i)\otimes\nabla_{\partial_{y^j}}\partial_{y^i}\\
&= -\frac{1}{2}e_\alpha\cdot[\nabla_{K(e_\alpha)}\psi^i\otimes\partial_{y^i}+\psi^i\otimes\nabla_{K(e_\alpha)}\phi^j\nabla_{\partial_{y^j}}\partial_{y^i}]+A\cdot\psi\\
&= -\frac{1}{2}e_\alpha\cdot\widetilde{\nabla}_{K(e_\alpha)}\psi+A\cdot\psi.
\end{aligned}$$

This proves Lemma 2.2. Q.E.D.

Thus, from (3.7) we have

$$\delta_g(\not{D}\psi)=-\frac{1}{2}(L_Xg)(e_\alpha,e_\beta)(e_\alpha\cdot\widetilde{\nabla}_{e_\beta}\psi)+A\cdot\psi. \qquad (3.12)$$

Now we compute the second variation

$$\begin{aligned}
\delta_p(\not{D}\psi) &= \delta_p[\not{\partial}\psi^i\otimes\partial_{y^i}(\phi)+(e_\alpha\cdot\psi^i)\otimes\nabla_{e_\alpha}\partial_{y^i}]\\
&= \not{\partial}\psi^i\otimes\delta_\phi(\partial_{y^i}(\phi))+(e_\alpha\cdot\psi^i)\otimes\delta_\phi(\nabla_{e_\alpha}\partial_{y^i})+\not{D}(L_X\psi^i\otimes\partial_{y^i})\\
&= [\not{\partial}\psi^i\otimes\nabla_{\partial_t}\partial_{y^i}+(e_\alpha\cdot\psi^i)\otimes\nabla_{\partial_t}\nabla_{e_\alpha}\partial_{y^i}]|_{t=0}+\not{D}(L_X\psi^i\otimes\partial_{y^i})\\
&= [\not{\partial}\psi^i\otimes\nabla_{\partial_t}\partial_{y^i}+(e_\alpha\cdot\psi^i)\otimes\nabla_{e_\alpha}(\nabla_{\partial_t}\partial_{y^i})\\
&\quad +(e_\alpha\cdot\psi^i)\otimes R^N(\frac{d\phi}{dt},\phi_\alpha)\partial_{y^i}]|_{t=0}+\not{D}(L_X\psi^i\otimes\partial_{y^i})\\
&= \not{D}(L_X\psi)+[(e_\alpha\cdot\psi^i)\otimes R^N(\frac{d\phi}{dt},\phi_\alpha)\partial_{y^i}]|_{t=0}. \qquad (3.13)
\end{aligned}$$

Therefore,

$$\begin{aligned} L_X(\not{D}\psi) &= \delta_g(\not{D}\psi) + \delta_p(\not{D}\psi) \\ &= -\frac{1}{2}(L_X g)(e_\alpha, e_\beta)(e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi) + A \cdot \psi \\ &\quad + \not{D}(L_X\psi) + [(e_\alpha \cdot \psi^i) \otimes R^N(\frac{d\phi}{dt}, \phi_\alpha)\partial_{y^i}]|_{t=0} \\ &= -\frac{1}{2}(L_X g)(e_\alpha, e_\beta)(e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi) + A \cdot \psi \\ &\quad + \not{D}(L_X\psi) + (e_\alpha \cdot \psi^i) \otimes R^N(d\phi(X), d\phi(e_\alpha))\partial_{y^i}, \end{aligned} \tag{3.14}$$

from which we have

$$\begin{aligned} \langle L_X(\not{D}\psi), \psi\rangle &= -\frac{1}{2}(L_X g)(e_\alpha, e_\beta)\langle e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi, \psi\rangle + \langle \not{D}(L_X\psi), \psi\rangle \\ &\quad + \langle (e_\alpha \cdot \psi^i) \otimes R^N(d\phi(X), d\phi(e_\alpha))\partial_{y^i}, \psi\rangle. \end{aligned} \tag{3.15}$$

The last term in the above equality can be calculated as follows

$$\begin{aligned} \langle (e_\alpha \cdot \psi^i) \otimes R^N(d\phi(X), d\phi(e_\alpha))\partial_{y^i}, \psi\rangle &= \langle e_\alpha \cdot \psi^i, \psi^j\rangle \langle R^N(\partial_{y^m}, \partial_{y^l})\partial_{y^i}, \partial_{y^j}\rangle X(\phi^m)\phi^l_\alpha \\ &= \langle \nabla\phi^l \cdot \psi^j, \psi^i\rangle R_{mlij} X(\phi^m) \\ &= 2\langle \frac{1}{2} R^k{}_{lij}\langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle \partial_{y^k}, d\phi(X)\rangle \\ &= 2\langle \mathcal{R}(\phi, \psi), d\phi(X)\rangle. \end{aligned} \tag{3.16}$$

Thus, we have

$$\begin{aligned} \langle L_X(\not{D}\psi), \psi\rangle &= -\frac{1}{2}(L_X g)(e_\alpha, e_\beta)\langle e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi, \psi\rangle + \langle \not{D}(L_X\psi), \psi\rangle \\ &\quad + 2\langle \mathcal{R}(\phi, \psi), d\phi(X)\rangle. \end{aligned} \tag{3.17}$$

Finally, we have

$$\begin{aligned} L_X(\langle \not{D}\psi, \psi\rangle v_g) &= -\frac{1}{2}(L_X g)(e_\alpha, e_\beta)\langle e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi, \psi\rangle v_g + \langle \not{D}(L_X\psi), \psi\rangle v_g \\ &\quad + 2\langle \mathcal{R}(\phi, \psi), d\phi(X)\rangle v_g + \langle \not{D}\psi, L_X\psi\rangle v_g + \frac{1}{2}\langle \not{D}\psi, \psi\rangle \langle L_X g, g\rangle v_g. \end{aligned} \tag{3.18}$$

Proof of Theorem 1.1. Assume that X is a conformal vector field: $L_X g = 2fg$, and (ϕ, ψ) is a Dirac-harmonic map: $\tau(\phi) = \mathcal{R}(\phi, \psi)$, $\not{D}\psi = 0$. Then from (3.18) we have

$$\begin{aligned} L_X(\langle \not{D}\psi, \psi\rangle v_g) &= -f\langle \not{D}\psi, \psi\rangle v_g + \langle \not{D}(L_X\psi), \psi\rangle v_g \\ &\quad + 2\langle \mathcal{R}(\phi, \psi), d\phi(X)\rangle v_g \\ &= \langle \not{D}(L_X\psi), \psi\rangle v_g + 2\langle \mathcal{R}(\phi, \psi), d\phi(X)\rangle v_g. \end{aligned} \tag{3.19}$$

From Lemma 2.1, we have

$$\begin{aligned} L_X(|d\phi|^2 v_g) &= 2\langle d\phi, \nabla(d\phi(X))\rangle v_g + f\langle g, S_\phi\rangle v_g \\ &= 2\langle d\phi, \nabla(d\phi(X))\rangle v_g + \frac{n-2}{2} f |d\phi|^2 v_g. \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) yields

$$\begin{aligned}\int_M \eta L_X\Omega &= \int_M \eta\langle \not{D}(L_X\psi),\psi\rangle v_g + \frac{n-2}{2}\int_M \eta f|d\phi|^2 v_g\\ &\quad +2\int_M \eta\langle d\phi,\nabla(d\phi(X))\rangle v_g + 2\int_M \eta\langle \mathcal{R}(\phi,\psi), d\phi(X)\rangle v_g. \end{aligned}\tag{3.21}$$

Note that

$$\begin{aligned}\int_M \eta\langle d\phi,\nabla(d\phi(X))\rangle v_g &= \int_M \eta\langle d\phi(e_\alpha),\nabla e_\alpha(d\phi(X))\rangle v_g\\ &= \int_M \nabla_{e_\alpha}(\eta\langle d\phi(e_\alpha), d\phi(X)\rangle)v_g - \int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g\\ &\quad -\int \eta\langle \nabla_{e_\alpha} d\phi(e_\alpha), d\phi(X)\rangle v_g\\ &= -\int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g - \int_M \eta\langle \tau(\phi), d\phi(X)\rangle v_g. \end{aligned}\tag{3.22}$$

Putting this into (3.21), we obtain

$$\begin{aligned}\int_M \eta L_X\Omega &= \int_M \eta\langle \not{D}(L_X\psi),\psi\rangle v_g + \frac{n-2}{2}\int_M \eta f|d\phi|^2 v_g\\ &\quad -2\int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g - 2\int \eta\langle \tau(\phi)-\mathcal{R}(\phi,\psi), d\phi(X)\rangle v_g\\ &= \int_M \eta\langle \not{D}(L_X\psi),\psi\rangle v_g + \frac{n-2}{2}\int_M \eta f|d\phi|^2 v_g\\ &\quad -2\int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g. \end{aligned}\tag{3.23}$$

But

$$\begin{aligned}\int_M \eta\langle \not{D}(L_X\psi),\psi\rangle v_g &= \int_M \langle L_X\psi, \not{D}(\eta\psi)\rangle v_g\\ &= \int_M \langle L_X\psi, \nabla\eta\cdot\psi + \eta\not{D}\psi\rangle v_g\\ &= \int_M \langle L_X\psi, \nabla\eta\cdot\psi\rangle v_g, \end{aligned}\tag{3.24}$$

therefore,

$$\begin{aligned}\int_M \eta L_X\Omega &= \int_M \langle L_X\psi, \nabla\eta\cdot\psi\rangle v_g + \frac{n-2}{2}\int_M \eta f|d\phi|^2 v_g\\ &\quad -2\int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g. \end{aligned}\tag{3.25}$$

Using the equation (1.4), i.e., $\not{D}\psi = 0$, we have

$$-\int_M d\eta\wedge \imath_X\Omega = -\int_M d\eta\wedge \imath_X(|d\phi|^2 v_g).\tag{3.26}$$

Putting (3.25) and (3.26) into (3.2) yields

$$\begin{aligned} \frac{n-2}{2}\int_M \eta f|d\phi|^2 &= 2\int_M \langle d\phi(\nabla\eta, d\phi(X)\rangle v_g - \int_M \langle L_X\psi, \nabla\eta\cdot\psi\rangle v_g \\ &\quad - \int_M (d\eta\wedge \imath_X v_g)|d\phi|^2. \end{aligned} \tag{3.27}$$

(1) $M=\mathbb{R}^n, \mathbb{H}^n (n\geq 3)$: For any $R>0$, choose a cut-off function η_R such that $0\leq \eta_R\leq 1$,

$$\eta_R = \begin{cases} 1 & B_R, \\ 0 & M\setminus B_{2R}, \end{cases}$$

and $|\eta_R'|\leq 2/R$. Inserting this into (3.27) yields

$$\begin{aligned} \frac{n-2}{2}\int_M \eta_R f|d\phi|^2 &\leq C[\int_{B_{2R}\setminus B_R} (|d\phi|^2 + |d\phi||\psi|^2 + |\psi||\nabla\psi|)] \\ &\leq C\int_{B_{2R}\setminus B_R} (|d\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}}). \end{aligned} \tag{3.28}$$

Now in the previous formula letting $R\to+\infty$ and using the finiteness of the energy, we have

$$\int_M f|d\phi|^2 = 0$$

which implies $\phi\equiv const.$ for $f>0$.

Now we fix coordinates (y^i) at $\phi(M)$. Then from the ψ-equation: $\not{D}\psi=0$ we have

$$\not{\partial}\psi^i = 0, \qquad \int_M |\psi^i|^4 < \infty, \quad i=1,2,\cdots,n'.$$

Denote $\xi := \psi^i$, then $\xi\in\Gamma(\Sigma M)$ and

$$\not{\partial}\xi = 0, \qquad \int_M |\xi|^4 < \infty. \tag{3.29}$$

By the Weitzenböck formula:

$$\frac{1}{2}\Delta|\xi|^2 = |\nabla\xi|^2 + \frac{1}{4}R_M|\xi|^2,$$

we have

$$\Delta|\xi|^2 \geq -C|\xi|^2,$$

where R_M is the scalar curvature of M.

By a Morrey-type estimate (see e.g. [9], Theorem 5.3.1), we conclude that for any $x_0\in M$ and $\rho>0$,

$$\sup_{B_{x_0}(\rho)} |\xi|^4 \leq \frac{C}{R^n}\int_{B_{x_0}(\rho+R)} |\xi|^4 \to 0 \quad (R\to+\infty),$$

hence $\xi\equiv 0$ on M. We have proved the theorem for the euclidean and hyperbolic case.

(2) $M = \mathfrak{S}^n(n \geq 3, r > r_0)$: For any $R >> 1$, choose a cut-off function η_R as in (1), and another cut-off function ζ_ε such that

$$\zeta_\varepsilon = \begin{cases} 0 & B_{r_0+\varepsilon}, \\ 1 & M \setminus B_{r_0+2\varepsilon}, \end{cases}$$

and $|d\zeta_\varepsilon| \leq 2/\varepsilon$. The functions $d\eta_R$ and $d\zeta_\varepsilon$ are supported in $B_{2R} \setminus B_R$ and $B_{r_0+2\varepsilon} \setminus B_{r_0+\varepsilon}$ respectively. Using $\eta = \eta_R \zeta_\varepsilon$ in (3.27), similar to (3.28), we have

$$\begin{aligned} \frac{n-2}{2} \int_{B_R \setminus B_{r_0+2\varepsilon}} f|d\phi|^2 &\leq C[\int_{B_{2R}\setminus B_R} (|d\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}}) \\ &\quad + \int_{B_{r_0+2\varepsilon}\setminus B_{r_0+\varepsilon}} (|d\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}})]. \end{aligned} \tag{3.30}$$

Letting $R \to +\infty$ and $\varepsilon \to 0$, we obtain

$$\int_M f|d\phi|^2 = 0$$

which implies $\phi \equiv const$. Similar to (1), we then conclude that $\psi \equiv 0$ on M. This completes the proof of Theorem 1.1. Q.E.D.

Proof of Theorem 1.2. Denote

$$\Omega_c := (|d\phi|^2 + \langle\psi, \not{D}\psi\rangle - \frac{1}{6}R_{ikjl}\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle)v_g.$$

Similar to (3.2) we have

$$\int_M \eta L_X \Omega_c = -\int_M d\eta \wedge \imath_X \Omega_c. \tag{3.31}$$

As for the left hand side of this equality, from (3.18) and the conformality of the vector field X, we have

$$\begin{aligned} L_X(\langle\not{D}\psi, \psi\rangle v_g) &= \langle\not{D}(L_X\psi), \psi\rangle v_g + 2\langle\mathcal{R}(\phi, \psi), d\phi(X)\rangle v_g \\ &\quad + \langle\not{D}\psi, L_X\psi\rangle v_g + (n-1)f\langle\not{D}\psi, \psi\rangle v_g. \end{aligned} \tag{3.32}$$

Using (3.4) and the conformality of X again, we have

$$\begin{aligned} L_X(-\frac{1}{6}R_{ikjl}\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle v_g) &= -\frac{1}{6}R_{ikjl;p}X(\phi^p)\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle v_g \\ &\quad -\frac{2}{3}R_{ikjl}\langle L_X\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle v_g \\ &\quad -\frac{2}{3}\langle\psi^i \otimes L_X\partial_{y^i}, R^m{}_{klj}\langle\psi^k, \psi^l\rangle\psi^j\partial_{y^m}\rangle \\ &\quad -\frac{1}{6}nfR_{ikjl}\langle\psi^i, \psi^j\rangle\langle\psi^k, \psi^l\rangle v_g. \end{aligned} \tag{3.33}$$

Recall (3.20):

$$\begin{aligned} L_X(|d\phi|^2 v_g) &= 2\langle d\phi, \nabla(d\phi(X))\rangle v_g + f\langle g, S_\phi\rangle v_g \\ &= 2\langle d\phi, \nabla(d\phi(X))\rangle v_g + \frac{n-2}{2}f|d\phi|^2 v_g. \end{aligned} \tag{3.34}$$

Noting that

$$\begin{aligned}\int_M \langle \not{D}(L_X\psi), \eta\psi\rangle v_g &= \int_M \langle L_X\psi, \nabla\eta\cdot\psi + \eta\not{D}\psi\rangle v_g \\ &= \int_M \langle L_X\psi, \nabla\eta\cdot\psi\rangle v_g + \int_M \langle L_X\psi, \eta\not{D}\psi\rangle v_g,\end{aligned} \tag{3.35}$$

and recall (3.22):

$$\int_M \eta\langle d\phi, \nabla(d\phi(X))\rangle v_g = -\int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g - \int_M \eta\langle\tau(\phi), d\phi(X)\rangle v_g, \tag{3.36}$$

Combining (3.32)-(3.36), and using the Euler-Lagrange equations (2.1) and (2.11), we obtain

$$\begin{aligned}\int_M \eta L_X\Omega_c &= \frac{n-2}{2}\int_M \eta f|d\phi|^2 v_g + (n-1)\int_M \eta f\langle\not{D}\psi,\psi\rangle v_g \\ &\quad -\frac{n}{6}\int_M \eta f R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle v_g + \int_M \langle L_X\psi, \nabla\eta\cdot\psi\rangle v_g \\ &\quad -2\int_M \langle d\phi(\nabla\eta), d\phi(X)\rangle v_g.\end{aligned} \tag{3.37}$$

On the other hand, the right hand side of (3.31)

$$\begin{aligned}-\int_M d\eta\wedge\imath_X\Omega_c &= -\int_M (d\eta\wedge\imath_X v_g)(|d\phi|^2 + \langle\not{D}\psi,\psi\rangle - \frac{1}{6}R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle) \\ &= -\int_M (d\eta\wedge\imath_X v_g)(|d\phi|^2 + \frac{1}{6}R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle).\end{aligned} \tag{3.38}$$

Putting (3.37) and (3.38) into (3.31), and using the same argument as in the proof of Theorem 1.1, we have

$$\begin{aligned}\frac{n-2}{2}\int_M f|d\phi|^2 v_g &+ (n-1)\int_M f\langle\not{D}\psi,\psi\rangle v_g \\ &-\frac{n}{6}\int_M f R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle v_g = 0.\end{aligned} \tag{3.39}$$

Substituting the ψ-equation (2.1) into it, we obtain the following equality:

$$\int_M f|d\phi|^2 v_g + \frac{1}{3}\int_M f R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle v_g = 0. \tag{3.40}$$

Denote $a_{ij} := \langle\psi^i,\psi^j\rangle$. The symmetric matrix (a_{ij}) is semi-positive, therefore we can write

$$a_{ij} = b_{ip}b_{jp},$$

where (b_{ij}) is a real $n'\times n'$ matrix. Set $b^p := (b_{1p}, b_{2p}, \cdots, b_{n'p})$, then

$$\begin{aligned}R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle &= R_{ikjl}a_{ij}a_{kl} \\ &= R_{ikjl}b_{ip}b_{jp}b_{kq}b_{lq} \\ &= R(b^p, b^q, b^p, b^q).\end{aligned}$$

Using the assumption that N has positive sectional curvature and noting that $f > 0$, we immediately conclude that ϕ is constant and ψ vanishes. This completes the proof of Theorem 1.2. Q.E.D.

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