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R^n_+-global stability of Cohen-Grossberg neural network system with nonnegative equilibria

by

Wenlian Lu, and Tianping Chen

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$\mathbf{R^n_+}$-global stability of Cohen-Grossberg neural network system with nonnegative equilibria ⋆

Wenlian Lu [a,b],Tianping Chen [∗,a],

[a]*Lab. of Nonlinear Mathematics Science, Institute of Mathematics, Fudan University, Shanghai, 200433, P. R. China.*

[b]*Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany.*

Abstract

In this paper, without assuming strict positivity of amplifier functions, boundedness of activation functions, or symmetry of connection matrix, the dynamical behaviors of delayed Cohen-Grossberg neural networks with nonnegative equilibrium are studied. Based on the theory of nonlinear complementary problem (NCP), a sufficient condition is derived guaranteeing existence and uniqueness of the nonnegative equilibrium in the NCP sense. Moreover, this condition also guarantees the R^n_+-global asymptotic stability of the nonnegative equilibrium in the first orthant. The result is compared with some previous results and a numerical example is presented to indicate the viability of our theoretical results.

Key words: Cohen-Grossberg neural networks, nonnegative equilibrium, $\mathbf{R^n_+}$-global stability

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∗ Corresponding author.
Email addresses: 021018041@fudan.edu.cn, wenlian@mis.mpg.de (Wenlian Lu), tchen@fudan.edu.cn (Tianping Chen).

1 Introduction and model description

It is well known that Cohen-Grossberg neural networks, proposed by Cohen and Grossberg (1983), have been extensively studied both in theory and applications. They have been successfully applied to signal processing, pattern recognition and associative memory, etc.

The success of these applications relies on understanding the underlying dynamical behavior of the models. A thorough analysis of the dynamics is a necessary step towards a practical design of neural networks. Such model can be formalized as follows:

$$\frac{dx_i(t)}{dt} = a_i(x_i(t))\Big[-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + I_i\Big],\ i = 1, \cdots, n, \tag{1}$$

where $x_i(t)$ denotes the state of the i-th neuron, $a_i(\cdot)$ is the amplifier function of the i-th neuron, $d_i(\cdot)$ is the behave function of the i-th neuron, and $g_j(\cdot)$ is the activation function of the i-th neuron, a_{ij} is the real weight coefficient for the connection between neuron i and j, and I_i is constant input to the i-th neuron.

As mentioned by Civalleri, Gilli, and Pabdolfi (1993), in hard implementation, time delays inevitably occur due to the finite switching speed of the amplifiers and propagation time. What's more, to process moving images, one must introduce time delay in the signals transmitted among the cells. Neural networks with time delay have much more complicated dynamics due to the incorporation of delay. Then, we should consider the delayed Cohen-Grossberg neural networks, which can be formalized as follows:

$$\frac{dx_i(t)}{dt} = a_i(x_i(t))\Big[-d_i(x_i) + \sum_{j=1}^{n} a_{ij} g_j(x(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau)) + I_i\Big], \quad i = 1, \cdots, n \tag{2}$$

in addition, where b_{ij} is the real weight coefficient for the delayed connection between neuron i and j and τ is the time delay.

This model is very generalized including a large class of existing neural field and evolution models. For instance, if assuming $a_i(\rho) > 0$ for all $\rho \in R$ and $i = 1, \cdots, n$, then this model can contain the famous Hopfield neural networks, which can be written as:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau)) + I_i, \ i = 1, \cdots, n,$$

with letting $a_i(\rho) = 1$, for all $\rho \in R$ and $i = 1, \cdots, n$ and $d_i(\rho) = d_i \rho$ for given $d_i > 0$, $i = 1, \cdots, n$. Recently, the global stability of this kind of Cohen-Grossberg neural networks (with assuming strictly positive amplifier functions) has been widely studied in the recent decades. See Lu & Chen (2003), Wang & Zou (2002), Cao & Liang (2004), Liao, Li, & Wong (2004), Chen & Rong (2003,2004), Hwang, Cheng, & Liao (2003), Wang (2005), Lu & Chen (2005) for examples.

However, all the results obtained in these papers were based on the assumption that amplifier function $a_i(\cdot)$ is always **positive** (see Lu a& Chen (2003, 2005), Chen & Rong (2003)), even greater than some positive number $a_i(\cdot) \geq \underline{a}_i > 0$ (see Wang & Zou (2002), Cao & Liang (2004), Liao, Li, & Wong (2004), Chen & Rong (2004), Wang (2005)). But in their original paper, Grossberg (1980), Cohen & Grossberg (1983), Grossberg (1988), they proposed this model as a kind of competitive-cooperation dynamical system for decision rules, pattern formation, and parallel memory storage. Hereby, each state of neuron x_i might be the population size, activity, or concentration, etc. of the ith species in the system, which is always nonnegative for all time. To guarantee the positivity of the states, one should assume $a_i(\rho) > 0$ for all $\rho > 0$ and $a_i(0) = 0$ for all $i = 1, \cdots, n$ (for more detailed, see Lemma 1 in this paper). It is clear that this subset of Cohen-Grossberg neural networks includes the famous Velterra-Lotka competitive-cooperation equations which can be formalized as follows:

$$\frac{dx_i}{dt} = A_i x_i \Big(I_i - \sum_{j=1}^{n} a_{ij} x_j \Big), \ i = 1, \cdots, n$$

with letting $a_i(\rho) = A_i \rho$, for all $\rho > 0$ and given $A_i > 0$, and $g_i(\rho) = \rho$, $i = 1, \cdots, n$. To our best knowledge, Cohen & Grossberg (1983) and Grossberg (1988) provided the pioneering study on the dynamics of such neural network model with assuming $a_i(\rho) > 0$ for all $\rho > 0$ and $a_i(0) = 0$ for all $i = 1, \cdots, n$ and without considering any time delay.

The aim of this paper is to continue studying the dynamics of Cohen-Grossberg neural networks without assuming the strict positivity of $a_i(\cdot)$, symmetry of connection matrix, or boundedness of activation functions, but with considering a time delay. Hereby, we focus our study of the dynamical behaviors on the first orthant: $R^n_+ = \{(x_1, \cdots, x_n)^\top \in R^n : \ x_i \geq 0, \ i = 1, \cdots, n\}$ and introduce the concept of R^n_+-global stability, which means that we consider all trajectories initiated in the first orthant R^n_+ instead of the whole space R^n. We point out that an asymptotically stable nonnegative equilibrium is closely related to the solution of a Nonlinear Complementary Problem (NCP). Based on the Linear Matrix Inequality (LMI) technique (more details about LMI, see Boyd, Ghaoui, Feron, and Balakrishnana (1994)) and the theory of Nonlinear Complementary Problem (NCP) (more details about NCP, we refer to Megiddo, Kojima (1977)), a sufficient condition for existence and uniqueness of nonnegative equilibrium is given. Moreover, the R^n_+-global asymptotic stability and exponential stability of the equilibrium are investigated, too.

This paper is organized as follows. In Section 2, we present some denotations, definitions and lemmas, which are used throughout the paper. In Section 3, we discuss existence and uniqueness of the nonnegative equilibrium by solution of a NCP. We discuss global stability in Section 4. A numerical example verifying our criteria and comparing them with previous works is provided in Section 5. We conclude this paper in Section 6.

2 Preliminaries

In this paper, we use the following denotations. $A^\top$ denotes the matrix transpose of A. A^s denotes the symmetric part of matrix A: $\frac{1}{2}(A + A^\top)$. $A > 0$ denotes that A is symmetric and positive definite. It is similar to denote $A \geq 0$, $A < 0$, and $A \leq 0$. $R^n_+ = \{x = (x_1, x_2, \cdots, x_n)^\top : x_i \geq 0, i = 1, \cdots, n\}$ denotes the first orthant. We denote the smallest and largest element of a set $K = \{t_1, t_2, \cdots, t_m\}$ by $\min K$ and $\max K$ respectively. $\lambda(A)$ denotes the spectrum set of matrix A. $\|\cdot\|$ denotes some norm of vector and matrix. In particular, $\|\cdot\|_2$ denotes the $2-$ norm by the way that $\|x\|_2 = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for any $x = (x_1, \cdots, x_n) \in R^n$ and $\|A\|_2 = \sqrt{\max \lambda(A^\top A)}$ for any $A \in R^{n,n}$.

We consider the Cohen-Grossberg neural network system with a time delay formalized as (2). Let $x(t) = (x_i(t), x_2(t), \cdots, x_n(t))^\top$, $d(x) = (d_i(x_i), d_2(x_2), \cdots, d_n(x_n))^\top$, $g(x) = (g_1(x_1), g_2(x_2), \cdots, g_n(x_n))^\top$, $a(x) = diag\{a_1(x_1), a_2(x_2), \cdots, a_n(x_n)\}$, $A = (a_{ij})$, $B = (b_{ij}) \in R^{n,n}$, and $I = (I_1, I_2, \cdots, I_n)^\top$. Then, the system (2) can be rewritten in matrix form:

$$\frac{dx(t)}{dt} = a(x)\Big[-d(x) + Ag(x(t)) + Bg(x(t-\tau)) + I \Big] \tag{3}$$

For the amplifier and activation functions, we have the following assumptions.

(1) $a(\cdot) \in \mathcal{A}_1$: If every $a_i(\rho)$ is continuous with $a_i(0) = 0$, and $a_i(\rho) > 0$, whenever $\rho > 0$

(2) $a(\cdot) \in \mathcal{A}_2$: $a(\cdot) \in \mathcal{A}1$, and for any $\epsilon > 0$, $\int_0^\epsilon \frac{d\rho}{a_i(\rho)} = +\infty$ holds for all $i = 1, \cdots, n$;

(3) $a(\cdot) \in \mathcal{A}_3 : a(\cdot) \in \mathcal{A}1$, and for any $\epsilon > 0$, $\int_\epsilon^\infty \frac{\rho d\rho}{a_i(\rho)} = +\infty$ holds for all $i = 1, \cdots, n$;

(4) $a(\cdot) \in \mathcal{A}_4 : a(\cdot) \in \mathcal{A}1$, and for any $\epsilon > 0$, $\int_0^\epsilon \frac{\rho d\rho}{a_i(\rho)} < +\infty$ holds for all $i = 1, \cdots, n$;

(5) $d(\cdot) \in \mathcal{D}$: If $d_i(\cdot)$ is continuous and satisfies

$$\frac{d_i(\xi) - d_i(\zeta)}{\xi - \zeta} \geq D_i \quad for\ all\ \xi \neq \zeta$$

where D_i are positive constants, $i = 1, \cdots, n$;

(6) $g(\cdot) \in \mathcal{G}$: If $g_i(\cdot)$ satisfies

$$0 \leq \frac{g_i(\xi) - g_i(\zeta)}{\xi - \zeta} \leq G_i \quad for\ \xi \neq \zeta$$

where G_i are positive constants, $i = 1, \cdots, n$.

Remark 1 *It can be seen that if $a_i(\cdot)$ has the form $a_i(\rho) = \rho^\alpha$ for some $\alpha > 0$, then $a(\rho) \in \mathcal{A}1$. If $\alpha \geq 1$, then $a(\rho) \in \mathcal{A}2$. If $\alpha \leq 2$, then $a(\rho) \in \mathcal{A}3$. If $\alpha < 2$, then $a(\rho) \in \mathcal{A}4$. However, it should be emphasized that in this paper, we do not assume that*

$a_i(\rho) > \underline{a}_i > 0$, which is required in Lu & Chen (2003, 2005), Wang & Zou (2002), Cao & Liang (2004), Liao, Li, & Wong (2004), Chen & Rong (2003, 2004), Hwang, Cheng, & Liao (2003), Wang (2005) and many others. Therefore, the results obtained in these papers fail to analyze dynamical behaviors of the Cohen-Grossberg neural networks without assuming $a_i(\rho) > \underline{a}_i > 0$.

First, we give following definition of positive solution by the component ways.

Definition 1 *A solution $x(t)$ of the system (3) is said being a positive solution, if for every positive initial condition $\phi(t) > 0$, $t \in [-\tau, 0]$, the trajectory $x(t) = (x_1(t), \cdots, x_n(t))^\top$ satisfies that $x_i(t) > 0$ holds for all $t \geq 0$ and $i = 1, \cdots, n$.*

Lemma 1 *(Positive Solution) If $a(\cdot) \in \mathcal{A}_2$, then the solution of the system (3) is a positive solution.*

proof: Suppose that the initial value $\phi(t) = (\phi_1(t), \cdots, \phi_n(t))^\top$ with all $\phi_i(t) > 0$ for $i = 1, \cdots, n$ and $t \in [-\tau, 0]$. If for some $t_0 > 0$ and some index i_0, $x_{i_0}(t_0) = 0$. Then, by the assumption $a(\cdot) \in \mathcal{A}_2$, we have

$$\int_0^{t_0} \Big[- d_i(x_i(t)) + \sum_{j=1}^n a_{ij} g_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t-\tau)) + I_i \Big] dt$$
$$= \int_0^{t_0} \frac{\dot{x}_i(t) dt}{a_i(x_i(t))} = - \int_0^{\phi_i(0)} \frac{d\rho}{a_i(\rho)} = -\infty$$

which is impossible due to the continuity of $x_i(\cdot)$ on $[0, t_0]$. Hence, $x_i(t) \neq 0$ holds for all $t \geq 0$ and $i = 1, \cdots, n$. This implies that $x_i(t) > 0$ for all $t \geq 0$ and $i = 1, \cdots, n$. $\sharp$

By this lemma, we can actually concentrate our study on the first orthant R^n_+. Then, we can go on investigating the equilibrium of the system (3) in R^n_+. If $a(\cdot) \in \mathcal{A}_1$, then any equilibrium in R^n_+ of the system (2) is a solution of the following equations:

$$x_i \Big[f_i(x) - I_i \Big] = 0, \ i = 1, \cdots, n \tag{4}$$

where $f_i(x) = d_i(x_i) - \sum_{j=1}^{n}(a_{ij} + b_{ij})g_j(x_j)$, $i = 1, \cdots, n$. Though the equations (4) might possess multiple solutions, we can show that an asymptotically stable nonnegative equilibrium is just a solution of a nonlinear complementary problem (NCP).

Proposition 1 *Suppose $a(\cdot) \in \mathcal{A}_1$. If $x^* = (x_i^*, \cdots, x_n^*)^\top \in R_+^n$ is an equilibrium of the system (3) and asymptotically stable, then it must be a solution of the following nonlinear complementary problem (NCP), i.e.,*

$$x_i^* \geq 0 \quad f_i(x^*) - I_i \geq 0 \quad x_i^*(f_i(x^*) - I_i) = 0, \quad i = 1, \cdots, n \tag{5}$$

where $f_i(x) = d_i(x_i) - \sum_{j=1}^{n}(a_{ij} + b_{ij})g_j(x_j)$, $i = 1, \cdots, n$.

Proof: Suppose that $x^* \in R_+^n$ is an asymptotically stable equilibrium of the system (3). Then $x_i^* > 0$ or $x_i^* = 0$. In case $x_i^* > 0$, we have $f_i(x_i^*) - I_i = 0$. If $x_i^* = 0$, we claim that $f_i(x_i^*) - I_i \geq 0$. Otherwise, if $f_{i_0}(x^*) - I_{i_0} < 0$ for some index i_0, then $\dot{x}_{i_0}(t) = a_i(x_{i_0}(t))\Big[-f_{i_0}(x_{i_0}(t)) + I_{i_0}\Big] > \frac{1}{2}a_i(x_{i_0}(t))\Big[-f_{i_0}(x^*) + I_{i_0}\Big] > 0$ when $x_{i_0}(t)$ is sufficiently close to x^*, which implies that $x_{i_0}(t)$ will never converge to 0. Therefore, x^* is unstable. $\sharp$

To study the solution of the equation (4), we should introduce the concept generalized of Nonlinear Complementarity Problem (NCP).

Definition 2 *A NCP is to find x^*, $i = 1, \cdots, n$ satisfying*

$$x_i^* \geq 0 \quad f_i(x^*) - I_i \geq 0 \quad x_i^*(f_i(x^*) - I_i) = 0 \quad for\ all\ i = 1, \cdots, n \tag{6}$$

where $f = (f_1(x), \cdots, f_n(x))^\top : R_+^n \to R^n$ is continuous and $I_i \in R$, $i = 1, \cdots, n$.

The following theorem gives a necessary and sufficient condition for the existence and uniqueness of the solution of a NCP.

Theorem 1 *(Theorem 2.3 in Megiddo and Kojima (1977)) The NCP (6) has a unique solution for every $I \in R^n$ if and only if $F(x)$ is norm-coercive, i.e.,*

$$\lim_{\|x\| \to \infty} \|F(x)\| = \infty$$

and locally univalent, where $F(x) : R^n \to R^n$ is defined as follows:

$$F(x) = f(x^+) + x^-$$

$x^+ = (x_1^+, x_2^+, \cdots, x_n^+)^\top$, $x^- = (x_1^-, x_2^-, \cdots, x_n^-)^\top$, *and*

$$x_i^+ = \begin{cases} x_i \ x_i \geq 0 \\ 0 \ \ x_i < 0 \end{cases} \quad x_i^- = \begin{cases} x_i \ x_i \leq 0 \\ 0 \ \ x_i > 0 \end{cases} \quad for\ i = 1, \cdots, n$$

Thus, we can propose the definition of a nonnegative equilibrium of the system (3) as follows.

Definition 3 *x^* is said to be a nonnegative equilibrium of the system (3) in the NCP sense, if x^* is the solution of the Nonlinear Complementarity Problem (NCP) (5); moreover, if $x_i^* > 0$, for all $i = 1, \cdots, n$, then x^* is said to be a positive equilibrium of system (3). In this case, x^* must satisfy*

$$d(x^*) - (A + B)g(x^*) + I = \mathbf{0},\ x_i^* > 0,\ i = 1, \cdots, n$$

where $\mathbf{0} = (0, \cdots, 0)^\top \in R^n$.

Definition 4 *A nonnegative equilibrium x^* of the system (3) in the NCP sense is said to be R_+^n-globally asymptotically stable if for any positive initial condition $\phi_i(t) > 0$ holds for all $t \in [-\tau, 0]$ and $i = 1, \cdots, n$, the trajectory $x(t)$ of the system (3) satisfies $\lim_{t\to\infty} x(t) = x^*$; moreover, if there exist constants $M > 0$ and $\epsilon > 0$ such that*

$$\|x(t) - x^*\| \leq Me^{-\epsilon t} \quad t \geq 0,$$

then x^* is said to be R_+^n-exponentially stable.

Finally, the following two matrix inequality lemmas are needed in the later sections.

Lemma 2 *(See Lemma 2 by Forti & Tesi (1995)) Let $D = diag\{D_1, \cdots, D_n\}$, $G =$*

$diag\{G_1, \cdots, G_n\}$. If there exists a positive definite diagonal matrix $P = diag\{P_1, P_2, \cdots, P_n\}$ such that

$$\{P[DG^{-1} - T]\}^s > 0$$

holds, then for any positive definite diagonal matrix $\bar{D} \geq D$ and nonnegative definite diagonal matrix K satisfying $0 \leq K \leq G$, we have $\det(\bar{D} - TK) \neq 0$, i.e., $\bar{D} - TK$ is nonsingular.

Lemma 3 *For given positive diagonal $R^{n,n}$ matrices D and G and given $R^{n,n}$ matrices A and B, if there exist a positive definite diagonal matrix P and a positive definite symmetric matrix Q such that*

$$Z_1 = \begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix} > 0 \tag{7}$$

holds, then we have

(1) (See Lemma 2 in Lu, Rong, & Chen (2003)).

$$\left\{P[DG^{-1} - (A + B)]\right\}^s > 0 \tag{8}$$

(2) There exists a constant $\beta > 0$ such that

$$Z_2 = \begin{bmatrix} 2\beta D & -\beta A & -\beta B \\ -\beta A^\top & 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -\beta B^\top & -B^\top P & Q \end{bmatrix} > 0 \tag{9}$$

Proof: Proof of Item 1. (For the convenience of readers, we place the proof here). By Shur Complement (see Boyd, Ghaoui, Feron, & Balakrishnana (1994)), the LMI (7) is equivalent to $2PDG^{-1} - PA - A^\top P - PBQ^{-1}B^\top P - Q > 0$. Then, we have $2PDG^{-1} - (PA +$

$A^\top P) > PBQ^{-1}B^\top P + Q$. By the inequality $[Q^{-1/2}(PB)^\top - Q^{1/2}]^\top[Q^{-1/2}(PB)^\top - Q^{1/2}] \geq 0$, we have $PBQ^{-1}B^\top P + Q \geq PB + B^\top P$. So, it becomes$2PDG^{-1} > PA + A^\top P + PB + B^\top P$, i.e.,

$$\Big\{P[DG^{-1} - (A+B)]\Big\}^s > 0.$$

Proof of Item 2. Let $\gamma_1 = \min \lambda(Z_1)$ and β be a positive constant satisfying $0 < \beta < \min\{\frac{\gamma_1}{4\|A\|_2^2\|D^{-1}\|_2}, \frac{\gamma_1}{4\|B\|_2^2\|D^{-1}\|_2}\}$. Then, we have

$$\begin{aligned}
&[x^\top, y^\top, z^\top]Z_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 2\beta x^\top Dx - 2\beta x^\top Ay - 2\beta x^\top Bz + \gamma_1 y^\top y + \gamma_1 z^\top z \\
&= \beta[\frac{1}{2}D^{1/2}x - 2D^{-1/2}Ay]^\top[\frac{1}{2}D^{1/2}x - 2D^{-1/2}Ay] \\
&\quad + \beta[\frac{1}{2}D^{1/2}x - 2D^{-1/2}Bz]^\top[\frac{1}{2}D^{1/2}x - 2D^{-1/2}Bz] \\
&\quad + [\gamma_1 y^\top y - 4\beta y^\top A^\top D^{-1}Ay] + [\gamma_1 z^\top z - 4\beta y^\top B^\top D^{-1}Bz] + \frac{3}{2}\beta x^\top Dx \\
&\geq [\gamma_1 y^\top y - 4\beta y^\top A^\top D^{-1}Ay] + [\gamma_1 z^\top z - 4\beta z^\top B^\top D^{-1}Bz] + \frac{3}{2}\beta x^\top Dx > 0
\end{aligned}$$

which means that the matrix Z_2 is positive definite. This completes the proof. $\sharp$

3 Existence and uniqueness of the nonnegative equilibrium

In this section, we discuss existence and uniqueness of the nonnegative equilibrium in the NCP sense.

Theorem 2 *(Existence and Uniqueness of Nonnegative Equilibrium) Suppose $a(\cdot) \in \mathcal{A}2$, $d(\cdot) \in \mathcal{D}$, and $g(\cdot) \in \mathcal{G}$. Let $D = diag\{D_1, \cdots, D_n\}$ and $G = diag\{G_1, \cdots, G_n\}$. If there exists a positive definite diagonal matrix $P = diag\{P_1, P_2, \cdots, P_n\}$ such that*

$$\left\{P[DG^{-1} - (A+B)]\right\}^s > 0 \tag{10}$$

holds, then for each $I \in R^n$, there exists a unique nonnegative equilibrium of the system (3) in the NCP sense.

Proof: Let

$$\begin{aligned} f_i(x) &= d_i(x_i) - \sum_{j=1}^{n}(a_{ij} + b_{ij})g_j(x_j), i = 1, \cdots, n, \; f(x) = (f_1(x), \cdots, f_n(x))^\top \\ F(x) &= f(x^+) + x^- \end{aligned}$$

where x^+ and x^- are defined as in Theorem 1.

According to Theorem 1, we only need to prove that $F(x)$ is norm-coercive and local univalent. First, we prove $F(x)$ is local univalent. For any $x = (x_1, \cdots, x_n) \in R^n$, without loss of generality, by some rearrangement of x_i, we can assume $x_i > 0$, if $i = 1, \cdots, p$; $x_i < 0$,if $i = p+1, \cdots, m$; $x_i = 0$, if $i = m+1, \cdots, n$, for some integers $p \leq m \leq n$. Moreover, if $y \in R^n$ is sufficiently close to $x \in R^n$, without loss of generality, we can also assume

$$\begin{array}{l} y_i > 0, \quad i = 1, \cdots, p \\ y_i < 0, \; i = p+1, \cdots, m \\ y_i > 0, i = m+1, \cdots, m_1 \\ y_i < 0, \; i = m_1, \cdots, m_2 \\ y_i = 0, i = m_2 + 1, \cdots, n \end{array}$$

for some integers $m \leq m_1 \leq m_2 \leq n$. It can be seen that

$$(x_i^+ - y_i^+)(x_i^- - y_i^-) = 0, \quad for\ i = 1, \cdots, n \tag{11}$$

and

$$\begin{aligned} F(x) - F(y) &= d(x^+) - d(y^+) - (A+B)[g(x^+) - g(y^+)] + (x^- - y^-) \\ &= [\bar{D} - (A+B)K](x^+ - y^+) + (x^- - y^-) \end{aligned}$$

where $\bar{D} = diag\{\bar{d}_i, \cdots, \bar{d}_n\}$ and $K = diag\{K_1, \cdots, K_n\}$ with

$$\bar{d}_i = \begin{cases} \frac{d_i(x_i^+)-d_i(y_i^+)}{x_i^+-y_i^+} & x_i^+ \neq y_i^+ \\ D_i & otherwise \end{cases} \qquad K_i = \begin{cases} \frac{g_i(x_i^+)-g_i(y_i^+)}{x_i^+-y_i^+} & x_i^+ \neq y_i^+ \\ G_i & otherwise \end{cases} \qquad i = 1, \cdots, n$$

Then, $\bar{d}_i \geq D_i$ and $K_i \leq G_i$, $i = 1, \cdots, n$, because $d(\cdot) \in \mathcal{D}$ and $g(\cdot) \in \mathcal{G}$.

If $F(x) - F(y) = 0$, then we have

$$x^- - y^- = -[\bar{D} - (A + B)K](x^+ - y^+) \tag{12}$$

By the equations (11), without loss of generality, we can assume

$$x^+ - y^+ = \begin{bmatrix} z_1 \\ 0 \end{bmatrix} \quad x^- - y^- = \begin{bmatrix} 0 \\ z_2 \end{bmatrix}$$

where $z_1 \in R^k$ and $z_2 \in R^{n-k}$, for some integer k.

Write

$$\bar{D} - (A + B)K = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

where $R_{11} \in R^{k,k}$, $R^{12} \in R^{k,n-k}$, $R_{21} \in R^{n-k,k}$, and $R_{22} \in R^{n-k,n-k}$. The equation (12) can be rewritten as

$$\begin{bmatrix} 0 \\ z_2 \end{bmatrix} = - \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \end{bmatrix}$$

which implies $R_{11}z_1 = 0$. By Lemma 2, R_{11} is nonsingular,which implies $z_1 = 0$ and $x^+ = y^+$. Similarly, we can prove $x^- = y^-$. Therefore, $x = y$, which means that $F(x)$ is locally univalent.

Second, we will prove that $F(x)$ is norm-coercive. Suppose that there exists a sequence $\{x_m = (x_{m,1}, \cdots, x_{m,n})^\top\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} \|x_m\|_2 = \infty$. Then, there exists some index i such that $\lim_{m\to\infty} |d_i(x_{m,i}^+) + x_{m,i}^-| = \infty$, which implies that $\lim_{m\to\infty} \|g(x_m^+)\|_2 = \infty$.

Some simple algebraic manipulations lead to

$$\begin{aligned} g(x^+)^\top PF(x) &= \sum_{i=1}^{n} g_i(x_i^+)P_i d_i(x_i^+) - g(x^+)^\top P(A+B)g(x^+) + \sum_{i=1}^{n} g_i(x_i^+)P_i x_i^- \\ &\geq g(x^+)^\top \{P[DG^{-1} - (A+B)]\}^s g(x^+) \\ &\geq \alpha g(x^+)^\top g(x^+) \end{aligned}$$

where $\alpha = \min \lambda(\Big\{P[DG^{-1} - (A+B)]\Big\}^s) > 0$. Therefore,

$$\|F(x_m)\|_2 \geq \alpha \|P\|_2^{-1} \|g(x_m^+)\|_2 \to \infty$$

which implies that $F(x)$ is norm-coercive. Combining with theorem 1, Theorem 2 is proved. $\sharp$

Following corollary is a direct consequence of Theorem 2 and Lemma 3.

Corollary 1 *Suppose $a(\cdot) \in \mathcal{A}2$, $d(\cdot) \in \mathcal{D}$, and $g(\cdot) \in \mathcal{G}$. Let $D = diag\{D_1, \cdots, D_n\}$ and $G = diag\{G_1, \cdots, G_n\}$. If there exist a positive definite diagonal matrix P and a positive definite symmetric matrix Q such that*

$$\begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix} > 0$$

holds, then there exists a unique nonnegative equilibrium for the system (3) in the NCP sense.

Remark 2 *It seems that conditions for the existence of nonnegative equilibrium in the NCP sense is similar to those for Hopfield neural networks (For example, see Lemma 2 in Lu, Rong, & Chen (2003)). However, the equilibrium for the Hopfield neural networks dis-*

cussed in (Lu, Rong, & Chen (2003)) is different than the nonnegative equilibrium discussed in this paper.

4 $\mathbf{R^n_+}$-global asymptotic stability of the nonnegative equilibrium

In this section, we discuss the global asymptotic stability of the nonnegative equilibrium defined in previous section. Let x^* be the nonnegative equilibrium of the system (3) in the NCP sense and $y(t) = x(t) - x^*$. Thus, the system (2) can be rewritten as

$$\frac{dy_i(t)}{dt} = a_i^*(y_i(t))\Big[-d_i^*(y_i(t)) + \sum_{j=1}^n a_{ij} g_j^*(y_j(t)) + \sum_{j=1}^n b_{ij} g_j^*(y_j(t-\tau)) + J_i\Big] \tag{13}$$

or in matrix form

$$\frac{dy(t)}{dt} = a^*(y(t))\Big[-d^*(y(t)) + Ag^*(y(t)) + Bg^*(y(t-\tau)) + J\Big] \tag{14}$$

where for $i = 1, \cdots, n$,

$$\begin{aligned}
a_i^*(s) &= a_i(s + x_i^*), \quad a^*(y) = diag\{a_1^*(y_1), \cdots, a_n^*(y_n)\} \\
d_i^*(s) &= d_i^*(s + x_i^*) - d_i^*(x_i^*), \quad d^*(y) = (d_1^*(y_1), \cdots, d_n^*(y_n))^\top \\
g_i^*(s) &= g_i^*(s + x_i^*) - g_i^*(x_i^*), \quad g^*(y) = (g_1^*(y_1), \cdots, g_n^*(y_n))^\top, \\
J_i &= \begin{cases} -d_i(x_i^*) + \sum\limits_{j=1}^n (a_{ij} + b_{ij}) g_j(x_j^*) + I_i & x_i^* = 0 \\ 0 & x_i^* > 0 \end{cases} \qquad J = (J_1, \cdots, J_n)^\top
\end{aligned}$$

Since x^* is the nonnegative equilibrium of (3) in the NCP sense, i.e., the solution of NCP (6), $J_i \le 0$ holds for all $i = 1, \cdots, n$ which implies that $g_i^*(y_i(t)) J_i \le 0$ holds for all $i = 1, \cdots, n$ and $t \ge 0$.

Theorem 3 *(R^n_+-Global Asymptotic Stability of the Nonnegative Equilibrium) Suppose $a(\cdot) \in \mathcal{A}2 \bigcap \mathcal{A}3 \bigcap \mathcal{A}4$, $d(\cdot) \in \mathcal{D}$, and $g(\cdot) \in \mathcal{G}$. Let $D = diag\{D_1, \cdots, D_n\}$ and $G =$*

$diag\{G_1, \cdots, G_n\}$. If there exist a positive definite diagonal matrix P and a positive definite symmetric matrix Q such that

$$\begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix} > 0 \tag{15}$$

holds , then the unique nonnegative equilibrium x^* for the system (3) in the NCP sense is R_+^n-globally asymptotically stable.

Proof : Without loss of generality, we assume

$$x_i^* = 0, \quad i = 1, 2, \cdots, p$$

$$x_i^* > 0, \; i = p+1, \cdots, n$$

for some integer p. By assumptions $\mathcal{A}_3$ and $\mathcal{A}_4$, it can be seen that

$$\int_0^{y_i(t)} \frac{\rho d\rho}{a_i^*(\rho)} < +\infty \qquad \int_0^{+\infty} \frac{\rho d\rho}{a_i^*(\rho)} = +\infty \qquad \int_0^{y_i(t)} \frac{g_i^*(\rho) d\rho}{a_i^*(\rho)} < +\infty$$

hold for $i = 1, \cdots, n$ and $t \geq 0$.

Let β, P, Q be the constant and matrices defined in Lemma 3 such that

$$Z = \begin{bmatrix} 2\beta D & -\beta A & -\beta B \\ -\beta A^\top & 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -\beta B^\top & -B^\top P & Q \end{bmatrix} > 0$$

and define

$$V(t)=2\beta\sum_{i=1}^{n}\int_{0}^{y_i(t)}\frac{\rho d\rho}{a_i^*(\rho)}+2\sum_{i=1}^{n}P_i\int_{0}^{y_i(t)}\frac{g_i^*(\rho)d\rho}{a_i^*(\rho)}+\int_{t-\tau}^{t}g^{*\top}(y(s))Qg^*(y(s))ds$$

It is easy to see that $V(t)$ is positive definite and radial unbounded.

Noting $g_i^*(y_i(t))J_i \le 0$, then

$$\begin{aligned}
&\frac{dV(t)}{dt}=2\beta\sum_{i=1}^{n}y_i(t)\Big[-d_i^*(y_i(t))+\sum_{j=1}^{n}a_{ij}g_j^*(y_j(t))+\sum_{j=1}^{n}b_{ij}g_j^*(y_j(t-\tau))+J_j\Big]\\
&+2\sum_{i=1}^{n}P_ig_i^*(y_i(t))\Big[-d_i^*(y_i(t))+\sum_{j=1}^{n}a_{ij}g_j^*(y_j(t))+\sum_{j=1}^{n}g_j^*(y_j(t-\tau))+J_j\Big]\\
&+g^{*\top}(y(t))Qg^*(y(t))-g^{*\top}(y(t-\tau))Qg^*(y(t-\tau))\\
&\le -2\beta\Big[y^\top(t)Dy(t)-y^\top(t)Ag^*(y(t))-y^\top(t)Bg^*(y(t-\tau)\Big]\\
&-2\Big[g^{*\top}(y(t))PDG^{-1}g^*(y(t))-g^{*\top}(y(t))PBg^*(y(t))-g^{*\top}(y(t))PBg^*(y(t-\tau))\Big]\\
&+g^{*\top}(y(t))Qg^*(y(t))-g^{*\top}(y(t-\tau))Qg^*(y(t-\tau))\\
=&-[y^\top(t),g^{*\top}(y(t)),g^{*\top}(y(t-\tau))]Z\begin{bmatrix}y(t)\\ g^*(y(t))\\ g^*(y(t-\tau))\end{bmatrix}\\
\le&-\delta y^\top(t)y(t)
\end{aligned}$$

where $\delta=\min\lambda(Z)>0$. Therefore, $\lim_{t\to\infty}\|y(t)\|_2=0$. $\sharp$

If the equilibrium is strictly positive, then the convergence is exponential.

Theorem 4 *(R_+^n-Global Exponential Stability) Suppose* $a(\cdot)\in\mathcal{A}2\bigcap\mathcal{A}3\bigcap\mathcal{A}4$, $d(\cdot)\in\mathcal{D}$, *and* $g(\cdot)\in\mathcal{G}$. *Let* $D=diag\{D_1,\cdots,D_n\}$ *and* $G=diag\{G_1,\cdots,G_n\}$. *If there exist a positive definite diagonal matrix* P *and a positive definite symmetric matrix* Q *such that*

$$\begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix} > 0$$

holds and x^* is the positive equilibrium of the system (3), then x^* is R^n_+-globally exponentially stable; moreover, the converge rate can be estimated by $O(e^{-\gamma t})$, where $\gamma > 0$ is determined by the following matrix inequality

$$\begin{bmatrix} 2\beta a(x^*)D - 4\gamma\beta I_n - 4\gamma a^{-1}(x^*)G & -\beta a(x^*)A & -\beta a(x^*)B \\ -\beta A^\top a(x^*) & 2PDG^{-1} - PA - A^\top P - Qe^{2\gamma\tau} & -PB \\ -\beta B^\top a(x^*) & -B^\top P & Q \end{bmatrix} > 0$$

for some $\beta > 0$.

Proof: By Theorem 3, we have $\lim_{t\to\infty} x(t) = x^*$. Because $x > 0$, then $a_i(x_i^*) > 0$, and $f_i(x^*) - I_i = 0$ hold for all $i = 1, \cdots, n$. Namely, for the system (13), $J_i = 0$ holds for $i = 1, \cdots, n$.

Let

$$L(t) = 2\beta y^\top(t)y(t)e^{2\gamma t} + 2\sum_{i=1}^n P_i e^{2\gamma t}\int_0^{y_i(t)} \frac{g_i^*(\rho)d\rho}{a_i^*(\rho)} + \int_{t-\tau}^t g^{*\top}(y(s))Qg^*(y(s))e^{2\gamma(s+\tau)}ds$$

Differentiating $L(t)$, we have

$$\frac{dL(t)}{dt} = 4\beta\gamma e^{2\gamma t}y^\top(t)y(t) + 2\beta e^{2\gamma t}y(t)a^*(y(t))\Big[-d^*(y(t)) + Ag^*(y(t)) + Bg^*(y(t-\tau))\Big]$$
$$+4\gamma e^{2\gamma t}\int_0^{y_i(t)} \frac{g_i^*(\rho)d\rho}{a_i^*(\rho)} + 2e^{2\gamma t}g^{*\top}(y(t))P\Big[-d^*(y(t)) + Ag^*(y(t)) + Bg^*g(y(t-\tau))\Big]$$
$$+g^{*\top}(y(t))Qg^*(y(t))e^{2\gamma(t+\tau)} - g^{*\top}(y(t-\tau))Qg^*(y(t-\tau))e^{2\gamma t}$$

Due to the convergence of $x(t)$, there exist a small $\varepsilon > 0$ and $T > 0$ such that for all

$i = 1, \cdots, n$ and $t \geq T$, we have

$$\int_0^{y_i(t)} \frac{g_i^*(\rho)d\rho}{a_i^*(\rho)} \leq G_i y_i^2(t)(a_i(x^*) - \varepsilon)^{-1}$$

and the following matrix, denoted by H,

$$\begin{bmatrix} 2\beta a^*(y(t))D - 4\gamma\beta I_n - 4\gamma(a(x^*) - \varepsilon)^{-1}G & -\beta a^*(y(t))A & -\beta a^*(y(t))B \\ -\beta A^\top a^*(y(t)) & 2PDG^{-1} - PA - A^\top P - Qe^{2\gamma\tau} & -PB \\ -\beta B^\top a^*(y(t)) & -B^\top P & Q \end{bmatrix}$$

is positive definite. Then

$$\frac{dL(t)}{dt} \leq [y^\top(t), {g^*}^\top(y(t)), {g^*}^\top(y(t-\tau))]H(t) \begin{bmatrix} y(t) \\ g^*(y(t)) \\ g^*(y(t-\tau)) \end{bmatrix} \leq 0$$

holds for all $t \geq T$. We have $L(t) \leq L(T)$. This implies $\|y(t)\|_2 = O(e^{-\gamma t})$. $\sharp$

Remark 3 *If all the components of the equilibrium x^* are positive, then x^* is R_+^n-exponentially stable. This is because there exists a sufficiently large T such that all $a_i(x_i(t)) > \eta > 0$ for $i = 1, \cdots, n$ and $t \geq T$, where η is a positive constant. Instead, if some components of x^* are zero, then $x(t)$ converges to x^*. However, the convergence is not exponential in general. For example, consider the following system:*

$$\begin{cases} \frac{du(t)}{dt} = u(t)[-u(t)] \\ u(0) = 1 \end{cases} \tag{16}$$

It can be seen that the system (16) satisfies conditions in Theorem 3, but does not satisfy the conditions in Theorem 4. Its solution is $u(t) = \frac{1}{t+1}$, which converges to zero but not exponentially.

5 Comparison and numerical example

In Chen & Rong (2003), the authors investigated global stability of delayed Cohen-Grossberg neural networks where amplifier functions are assumed strictly positive. By this way, there was not of much difference to deal with Cohen-Grossberg neural networks from cellular or Hopfield neural networks which do not contain any amplifier functions. The same linear matrix inequality as (15) were presented to guarantee the existence, uniqueness, and global stability of the equilibrium. In this paper, we consider that the amplifier functions are not always positive and can be zero. Thus, the Cohen-Grossberg neural networks have richer dynamical behaviors.

In the following, we compare the following two Cohen-Grossberg neural networks with a time delay:

$$\frac{dx_i(t)}{dt} = x_i(t)\Big[-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau)) + I_i\Big] \qquad i = 1, \cdots, n \tag{17}$$

$$\frac{dx_i(t)}{dt} = b_i(x_i(t))\Big[-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau)) + I_i\Big] \qquad i = 1, \cdots, n, \tag{18}$$

where $b_i(\rho) > 0$ for all $\rho \in R$, $i = 1, \cdots, n$. First, from Lemma 1, one can see that the first orthant R^n_+ is invariant through the evolution (17) under the assumption $\mathcal{A}_2$. Instead, the first orthant R^n_+ is not invariant for the system (18). Second, in the paper (Lu, Rong, & Chen (2003)), it was proved that under the assumption that $b_i(s) > 0$ and LMI (15), the system (18) has a unique equilibrium which is the solution of the equations

$$-d_i(x_i) + \sum_{j=1}^{n} a_{ij} g_j(x_j) + \sum_{j=1}^{n} b_{ij} g_j(x_j) + I_i = 0, \quad i = 1, \cdots, n \tag{19}$$

However, the system (17) has at most 2^n equilibria, which are the multiple solutions of the equations

$$x_i[-d_i(x_i) + \sum_{j=1}^{n} a_{ij} g_j(x_j) + \sum_{j=1}^{n} b_{ij} g_j(x_j) + I_i] = 0, \quad i = 1, \cdots, n \tag{20}$$

Among these equilibria, only one equilibrium is the solution of the corresponding NCP (5), which is R_+^n-globally asymptotically stable in R_+^n. In particular, if the LMI (15) is satisfied and each component of the unique equilibrium of the system (18) is nonnegative, then it must be the unique nonnegative equilibrium of the system (17) in the NCP sense and globally asymptotically stable. Otherwise, in case the unique equilibrium of the system (18) has some negative components, the nonnegative equilibrium of the system (17) in the NPC sense has at least a zero component. The following proposition summarize these comparisons.

Proposition 2 *Suppose that $b_i(s) > 0$ for all $s \in R$, $i = 1, \cdots, n$, $d_i(\cdot) \in \mathcal{D}$, and $g_i(\cdot) \in \mathcal{G}$, and the LMI (15) is satisfied. Denote the unique equilibrium of the system (18) by x^0, the equilibrium set of the system (17) by Ω, and the unique nonnegative equilibrium of the system (17) in the NCP sense by x^*. Thus, we conclude*

(1) For any index set $J \subset \{1, \cdots, n\}$, there exists an equilibrium $x^J \in \Omega$ of the system (17) such that $x_i^J = 0$, $i \in J$;

(2) If all components of x^0 are nonnegative, then $x^0 = x^*$;

(3) If some components of x^0 are negative, then x^* must have some zero components.

Proof: We only need to prove item 1. The remaining claims are direct consequences from the Theorems 1-4 in this paper and that in Chen, Rong (2003).

Without loss of generality, we assume $J = \{1, 2, \cdots, p\}$, where $p \leq n$. We consider the following equations

$$\begin{cases} x_i = 0 & i \in= 1, \cdots, p \\ -d_i x_i + \sum\limits_{j=1}^{p} (a_{ij} + b_{ij}) g_j(x_j) + \sum\limits_{j=p+1}^{n} (a_{ij} + b_{ij}) g_j(0) + I_i = 0, & i = p+1, \cdots, n \end{cases} \tag{21}$$

Applying Theorem 1 in Lu, Rong, & Chen (2003) to the complementary set J^c of J, we conclude that (21) has a unique equilibrium $x^J = [x_1^J, \cdots, x_p^J, x_{p+1}^J, \cdots, x_n^J]$, where $x_1 = \cdots = x_p = 0$. Let J range all the possible subset of $\{1, \cdots, n\}$, we obtain all 2^n equilibria of the system (17), which completes the proof. $\sharp$

In the following, we present a numerical example to verify the theoretical results obtained above and compare the dynamics of the systems (17) and (18).

Consider the dynamical behaviors of the following two systems:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\Big[-6x_1(t) + 2g(x_1(t)) - g(x_2(t)) \\ \qquad +3g(x_1(t-2)) + g(x_2(t-2)) + I_1 \Big] \\ \frac{dx_2(t)}{dt} = x_2(t)\Big[-6x_2(t) - 2g(x_1(t)) \\ \qquad +3g(x_2(t)) + \frac{1}{2}g(x_1(t-2)) + 2g(x_2(t-2)) + I_2 \Big] \end{cases} \tag{22}$$

$$\begin{cases} \frac{du_1(t)}{dt} = \frac{1}{|u_1(t)|+1}\Big[-6u_1(t) + 2g(u_1(t)) \\ \qquad -g(u_2(t)) + 3g(u_1(t-2)) + g(u_2(t-2)) + I_1 \Big] \\ \frac{du_2(t)}{dt} = \frac{1}{|u_2(t)|+1}\Big[-6u_2(t) - 2g(u_1(t)) \\ \qquad +3g(u_2(t)) + \frac{1}{2}g(u_1(t-2)) + 2g(u_2(t-2)) + I_2 \Big] \end{cases} \tag{23}$$

where $g(\rho) = \frac{1}{2}(\rho + \arctan(\rho))$ and $I = (I_1, I_2)^\top$ is the constant inputs which will be determined below. And,

$$D = 6 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}.$$

By the Matlab LMI and Control Toolbox, we obtain

$$P = \begin{bmatrix} 0.2995 & 0 \\ 0 & 0.3298 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.0507 & 0.3258 \\ 0.3258 & 0.9430 \end{bmatrix}$$

The eigenvalues of

$$Z = \begin{bmatrix} 2PDG^{-1} - PA - A^\top P - Q & -PB \\ -B^\top P & Q \end{bmatrix}$$

are 2.6490, 1.1343, 0.5302, and 0.0559, which implies Z is positive definite. By Theorem 3, for any $I \in R^2$, the system (22) has a unique nonnegative equilibrium x^* in the NCP sense which is R^2_+-globally asymptotically stable. By Theorem 1 in Lu, Rong, & Chen (2003), for any $I \in R^2$, system (23) has a unique equilibrium x^0 which is globally asymptotically stable in R^2.

In case $I = (1, 0.1)^\top$. The equilibria of the system (22) are $(0,0)^\top$, $(0.7414, 0)^\top$, $(0, 0.0992)^\top$, and $(0.7414, -0.7062)^\top$. Among them, $x^* = (0.7414, 0)^\top$ is the nonnegative equilibrium of the system (22) in the NCP sense and $x^0 = (0.7414, -0.7062)$ is the unique equilibrium of the system (23). Pick initial condition $\phi_1(t) = \frac{7}{2}(\cos(t)+1)$ and $\phi_2(t) = e^{-t}$, for $t \in [-2, 0]$. Figure 1 shows that the solution of the system (22) converges to $x^* = (0.7414, 0)^\top$, while the solution of the system (23) converges to $x^0 = (0.7414, -0.7062)$

If $I = (8, 10)^\top$, then $x^0 = (3.1908, 2.7770)^\top$, which implies $x^* = x^0$. Pick initial condition $\phi_1(t) = \sin(t) + 7$ and $\phi_2(t) = -2t + 1$, for $t \in [-2, 0]$. Figure 2 indicates that the solutions of both systems (22) and (23) converge to the same equilibrium $(3.1908, 2.7770)^\top$.

If $I = (-8, -10)^\top$, then the equilibria of the system (22) are $(0, 0)^\top$, $(0, -3.7951)^\top$, $(-3.1908, 0)^\top$, and $(-3.1908, -2.7770)^\top$. One can see that $x^* = (0, 0)^\top$ is the nonnegative equilibrium of the system (22) in the NCP sense and $x^0 = (-3.1908, -2.7770)^\top$ is the unique equilibirum of the system (23). Pick initial condition $\phi_1(t) = 7$ and $\phi_2 = 1$ for all $t \in [-2, 0]$. Figure 3 indicates that the solution of the system (22) converges to x^*, while the solution of the system (23) converges to x^0.

6 Conclusions

In this paper, we investigate the dynamics of the positive solutions of the Cohen-Grossberg neural networks with a time delay. Based on the theory of NCP and the LMI technique, a condition is obtained guaranteeing existence, uniqueness, and global stability of the nonnegative equilibrium in the NCP sense. If the equilibrium is positive, then the stability is globally exponential. Numerical example verifies the viability of the theoretical results.

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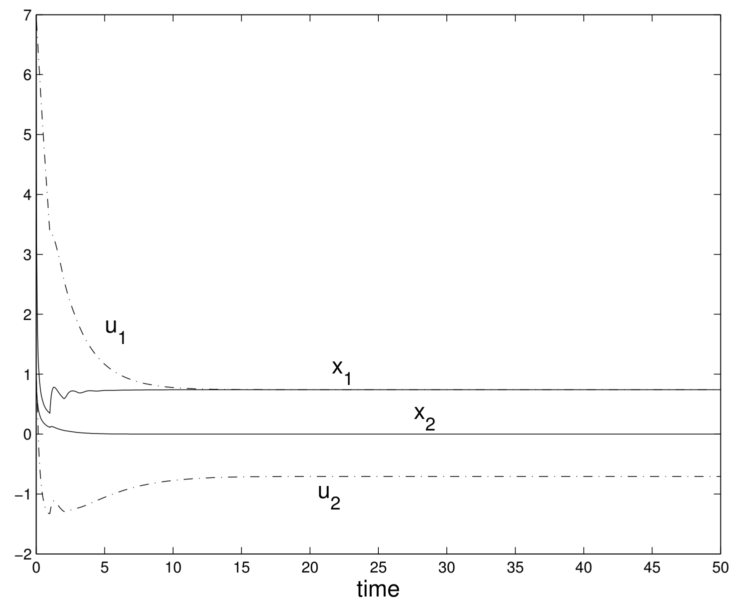

Figure 1: Dynamical behaviors of system (22), (23) with $I = (1, 0.1)^\top$.

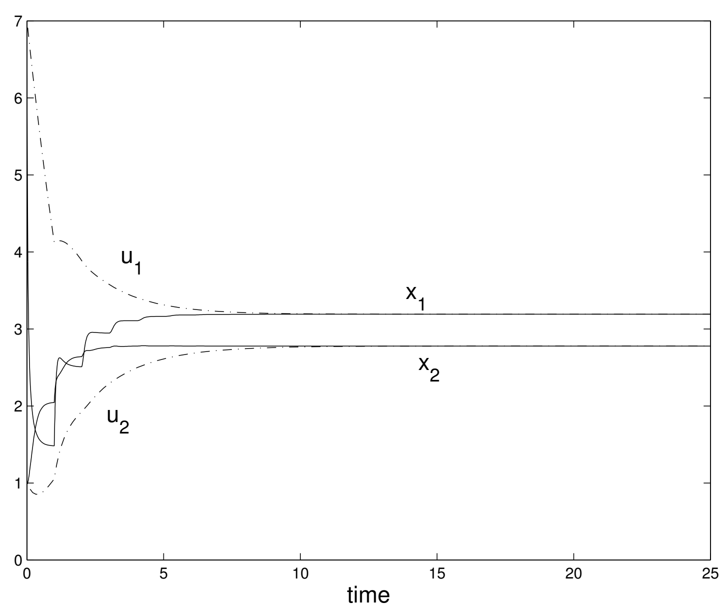

Figure 2: Dynamical behaviors of system (22), (23) with $I = (8, 10)^\top$.

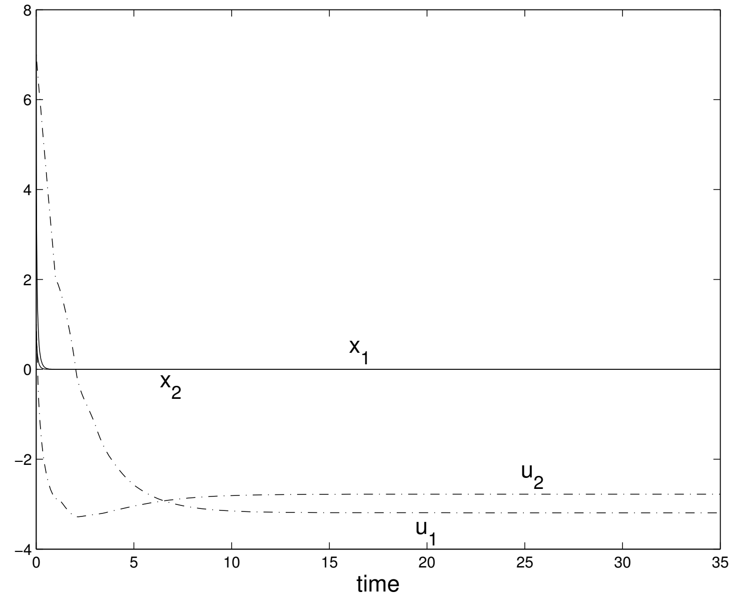

Figure 3: Dynamical behaviors of system (22), (23) with $I = (-8, -10)^\top$.