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**On global attraction to quantum stationary states III. Klein-Gordon equation with mean field interaction**

by


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# On global attraction to quantum stationary states III. Klein-Gordon equation with mean field interaction

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**Abstract**

We consider a $\mathbf{U}(1)$-invariant nonlinear Klein-Gordon equation in dimension $n \geq 1$, self-interacting via the mean field mechanism. We analyze the long-time asymptotics of finite energy solutions and prove that, under certain generic assumptions, each solution converges as $t \to \pm\infty$ to the two-dimensional set of all "nonlinear eigenfunctions" of the form $\phi(x)e^{-i\omega t}$. This global attraction is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersive radiation.

## 1 Introduction and main results

In this paper, we establish the global attraction to the variety of all solitary waves for the complex Klein-Gordon field $\psi(x,t)$ with the mean field self-interaction:

$$\left\{ \begin{array}{l} \ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + \rho(x)F(\langle\rho,\psi(\cdot,t)\rangle), \qquad x \in \mathbb{R}^n, \quad n \geq 1, \quad t \in \mathbb{R}, \\ \psi|_{t=0} = \psi_0(x), \qquad \dot{\psi}|_{t=0} = \pi_0(x). \end{array} \right. \tag{1.1}$$

Above, $\rho$ is a smooth coupling function from the Schwartz class: $\rho \in \mathscr{S}(\mathbb{R}^n)$, $\rho \not\equiv 0$.

The long time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers by Segal [Seg63b, Seg63a], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and the local attraction to zero solution were proved. Local attraction to solitary waves, or *asymptotic stability*, in $\mathbf{U}(1)$-invariant dispersive systems was addressed in [SW90, BP93, SW92, BP95] and then developed in [PW97, SW99, Cuc01a, Cuc01b, BS03, Cuc03]. Global attraction to *static*, stationary solutions in dispersive systems *without* $\mathbf{U}(1)$ *symmetry* was first established in [Kom91, Kom95, KV96, KSK97, Kom99, KS00].

The present paper is our third result on the global attraction to solitary waves in $\mathbf{U}(1)$-invariant dispersive systems. In [KK07a], we proved such an attraction for the Klein-Gordon field coupled to one nonlinear oscillator. In [KK07b], we generalized this result for the Klein-Gordon field coupled to several oscillators. We are aware of only one other recent advance [Tao07] in the field of nonzero global attractors for Hamiltonian PDEs. In that paper, the global attraction for the nonlinear Schrödinger equation in dimensions $n \geq 5$ was considered. The dispersive wave was explicitly specified using the rapid decay of local energy in higher dimensions. The global attractor was proved to be compact, but it was neither identified with the set of solitary waves nor was proved to be of finite dimension [Tao07, Remark 1.18].

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In the present paper we are going to extend our theory to a higher dimensional setting, for the Klein-Gordon equation with the mean field interaction. This model could be viewed as a generalization of the $\delta$-function coupling [KK07a, KK07b] in higher dimensions. We follow the cairns of the approach we developed in [KK07a, KK07b]. The substantial modification is due to apparent impossibility to extract a dispersive component and get the convergence to the attractor in the local energy norm, as in [KK07a, KK07b]; the convergence we prove is $\varepsilon$-weaker. On the other hand, this allowed to avoid the technique of quasimeasures, considerably shortening the argument. The main ideas are the absolute continuity of the spectral density for large frequencies, compactness argument to extract the omega-limit trajectories, and then the usage of the Titchmarsh Convolution theorem to pinpoint the spectrum to just one frequency.

Let us give the plan of the paper. In the remainder of this section, we formulate the assumptions and the results. The absolute continuity of the spectrum is analyzed in Section 2. The proof of the Main Theorem takes up Section 2 (where we analyze the absolute continuity of the spectrum for large frequencies) and Section 3 (where we select omega-limit trajectories and analyze their spectrum with the aid of the Titchmarsh Convolution Theorem). The example of a multifrequency solitary waves in the situation when $\rho$ is orthogonal to some of the solitary waves is constructed in Section 4. In Appendix A we give a brief sketch of the proof of the global well-posedness for equation (1.1).

## 1.1 Hamiltonian structure

We set $\Psi(t) = (\psi(x,t), \pi(x,t))$ and rewrite the Cauchy problem (1.1) in the vector form:

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{bmatrix} \Psi(t) + \rho(x) \begin{bmatrix} 0 \\ F(\langle \rho, \psi(\cdot,t)\rangle) \end{bmatrix}, \qquad \Psi\big|_{t=0} = \Psi_0, \qquad x\in\mathbb{R}^n, \quad n\ge 1, \quad t\in\mathbb{R}, \tag{1.2}$$

where $\Psi_0 = (\psi_0, \pi_0)$. We assume that the nonlinearity $F$ admits a real-valued potential:

$$F(z) = -\nabla U(z), \quad z\in\mathbb{C}, \qquad U\in C^2(\mathbb{C}), \tag{1.3}$$

where the gradient is taken with respect to $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$. Then equation (1.2) formally can be written as a Hamiltonian system,

$$\dot{\Psi}(t) = J\, D\mathscr{H}(\Psi), \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $D\mathscr{H}$ is the variational derivative of the Hamilton functional

$$\mathscr{H}(\Psi) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\pi|^2 + |\nabla\psi|^2 + m^2|\psi|^2\right) d^n x + U(\langle\rho,\psi\rangle), \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}. \tag{1.4}$$

We assume that the potential $U(z)$ is $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group $e^{i\theta}$, $\theta\in\mathbb{R} \bmod 2\pi$: Namely, we assume that there exists $u\in C^2(\mathbb{R})$ such that

$$U(z) = u(|z|^2), \qquad z\in\mathbb{C}. \tag{1.5}$$

Conditions (1.3) and (1.5) imply that

$$F(z) = \alpha(|z|^2) z, \qquad z\in\mathbb{C}, \tag{1.6}$$

where $\alpha(\cdot) = -2u'(\cdot) \in C^1(\mathbb{R})$ is real-valued. Therefore,

$$F(e^{i\theta} z) = e^{i\theta} F(z), \qquad \theta\in\mathbb{R}, \quad z\in\mathbb{C}. \tag{1.7}$$

Due to the $\mathbf{U}(1)$-invariance, the Nöther theorem formally implies that the functional

$$\mathscr{Q}(\psi,\pi) = \frac{i}{2}\int_{\mathbb{R}^n} (\overline{\psi}\pi - \overline{\pi}\psi)\, d^n x, \qquad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}, \tag{1.8}$$

is conserved for solutions $\Psi(t)$ to (1.2).

We introduce the phase space of finite energy states for equation (1.2). Denote by $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^s}$ the norms in $L^2(\mathbb{R}^n)$ and the Sobolev space $H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, respectively. We also denote by $\|\cdot\|_{H^s_R}$, $R>0$, the norm in $H^s(\mathbb{B}^n(R))$, where $\mathbb{B}^n(R)$ is a ball of radius $R$ in $\mathbb{R}^n$. Let us fix an arbitrary $\varepsilon > 0$.

**Definition 1.1.** (*i*) $\mathscr{E} = H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ is the Hilbert space of the states $\Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}$, with the norm

$$\|\Psi\|_{\mathscr{E}}^2 := \|\pi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + m^2\|\psi\|_{L^2}^2. \tag{1.9}$$

(*ii*) $\mathscr{E}^{-\varepsilon} = H^{1-\varepsilon}(\mathbb{R}^n) \oplus H^{-\varepsilon}(\mathbb{R}^n)$ is the space with the norm

$$\|\Psi\|_{\mathscr{E}^{-\varepsilon}} = \|(1-\Delta)^{-\varepsilon/2}\Psi\|_{\mathscr{E}}. \tag{1.10}$$

(*iii*) $\mathscr{E}_F^{-\varepsilon}$ is the space with the Fréchet topology defined by the seminorms

$$\|\Psi\|_{\mathscr{E}^{-\varepsilon},R}^2 := \|\pi\|_{H_R^{-\varepsilon}}^2 + \|\nabla\psi\|_{H_R^{-\varepsilon}}^2 + m^2\|\psi\|_{H_R^{-\varepsilon}}^2, \qquad R > 0. \tag{1.11}$$

*Remark* 1.2. The space $\mathscr{E}_F^{-\varepsilon}$ is metrizable (but not complete). The metric can be introduced by

$$\|\Psi\|_{\mathscr{E}_F^{-\varepsilon}} = \sum_{R=1}^{\infty} 2^{-R}\|\Psi\|_{\mathscr{E}^{-\varepsilon},R}. \tag{1.12}$$

Equation (1.2) is formally a Hamiltonian system with the phase space $\mathscr{E}$ and the Hamilton functional $\mathscr{H}$. Both $\mathscr{H}$ and $\mathscr{Q}$ are continuous functionals on $\mathscr{E}$. We introduced into (1.9), (1.11) the factor $m^2 > 0$, so that $\mathscr{H}(\Psi) = \frac{1}{2}\|\Psi\|_{\mathscr{E}}^2 + U(\langle\rho,\psi\rangle)$.

## 1.2 Global well-posedness

To have a priori estimates available for the proof of the global well-posedness, we assume that

$$U(z) \ge A - B|z|^2 \quad \text{for } z \in \mathbb{C}, \quad \text{where } A \in \mathbb{R} \text{ and } 0 \le B < \frac{m^2}{2\|\rho\|_{L^2}^2}. \tag{1.13}$$

**Theorem 1.3.** *Let $F(z)$ satisfy conditions (1.3), (1.5), and (1.13). Then:*

(*i*) *For every $\Psi_0 \in \mathscr{E}$ the Cauchy problem (1.2) has a unique solution $\Psi \in C(\mathbb{R},\mathscr{E})$.*

(*ii*) *The map $W(t):\ \Psi_0 \mapsto \Psi(t)$ is continuous in $\mathscr{E}$ and $\mathscr{E}_F$ for each $t \in \mathbb{R}$.*

(*iii*) *The values of the energy and charge functionals are conserved:*

$$\mathscr{H}(\Psi(t)) = \mathscr{H}(\Psi_0), \qquad \mathscr{Q}(\Psi(t)) = \mathscr{Q}(\Psi_0), \qquad t \in \mathbb{R}. \tag{1.14}$$

(*iv*) *The following* a priori *bound holds:*

$$\|\Psi(t)\|_{\mathscr{E}} \le C(\Psi_0), \qquad t \in \mathbb{R}. \tag{1.15}$$

(*v*) *For any $\varepsilon \in [0,1]$,*

$$\Psi \in C^{(\varepsilon)}(\mathbb{R},\mathscr{E}^{-\varepsilon}), \tag{1.16}$$

*where $C^{(\varepsilon)}$ stands for the Hölder functional space.*

We prove this theorem in Appendix A.

## 1.3 Solitary waves

**Definition 1.4.** (*i*) The solitary waves of equation (1.1) are solutions of the form

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \text{where } \omega \in \mathbb{R},\ \ \phi_\omega(x) \in H^1(\mathbb{R}^n). \tag{1.17}$$

(*ii*) The solitary manifold is the set $\mathbf{S} = \{(\phi_\omega, -i\omega\phi_\omega)\colon \omega \in \mathbb{R}\}$, where $\phi_\omega$ are the amplitudes of solitary waves.

Identity (1.7) implies that the set $\mathbf{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$. Let us note that since $F(0) = 0$ by (1.6), for any $\omega \in \mathbb{R}$ there is a zero solitary wave with $\phi_\omega(x) \equiv 0$.

Define

$$V(x,\omega) = \mathscr{F}_{\xi\to x}\Big[\frac{\hat\rho(\xi)}{\xi^2+m^2-\omega^2}\Big], \qquad \omega \in \mathbb{C}^+ \cup (-m,m), \tag{1.18}$$

where $\mathbb{C}^+ = \{\omega \in \mathbb{C}\colon \operatorname{Im}\omega > 0\}$. Note that $V(\cdot,\omega)$ is an analytic function of $\omega \in \mathbb{C}^+$ with the values in $H^\infty(\mathbb{R}^n)$. Since $|V(x,\omega)| \le \mathrm{const}\,|\operatorname{Im}\omega|^{-1}$ for $\omega \in \mathbb{C}^+$, we can extend for any $x \in \mathbb{R}^n$ the function $V(x,\omega)$ to the entire real line $\omega \in \mathbb{R}$ as a boundary trace:

$$V(x,\omega) = \lim_{\varepsilon\to 0+} V(x,\omega+i0), \qquad \omega \in \mathbb{R}, \tag{1.19}$$

where the limit holds in the sense of tempered distributions.

**Proposition 1.5 (Existence of solitary waves).** *Assume that $F(z)$ satisfies (1.7), and that $\rho \in \mathscr{S}(\mathbb{R}^n)$, $\rho \not\equiv 0$. There may only be nonzero solitary wave solutions to (1.2) for $\omega \in [-m,m] \cup \Omega_\rho$, where*

$$\Omega_\rho = \{\omega \in \mathbb{R}\backslash[-m,m]\colon \hat\rho(\xi) = 0 \text{ for all } \xi \in \mathbb{R}^n \text{ such that } m^2+\xi^2 = \omega^2\}. \tag{1.20}$$

*The profiles of solitary waves are given by*

$$\hat\phi_\omega(\xi) = \frac{c\hat\rho(\xi)}{\xi^2+m^2-\omega^2},$$

*where $c \in \mathbb{C}$, $c \ne 0$ is a root of the equation*

$$\Sigma(\omega)\alpha(|c|^2|\Sigma(\omega)|^2) = 1, \tag{1.21}$$

*where*

$$\Sigma(\omega) = \langle\rho, V(\cdot,\omega)\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{|\hat\rho(\xi)|^2}{\xi^2+m^2-\omega^2}\,d^n\xi. \tag{1.22}$$

*The existence of such root is a necessary condition for the existence of nonzero solitary waves (1.17).*

*The condition (1.21) is also sufficient for $n \ge 5$ and for $|\omega| \ne m$, $n \ge 1$.*

*For $|\omega| = m$, $n \le 4$, the following additional condition is needed for sufficiency:*

$$\int_{\mathbb{R}^n}\frac{|\hat\rho(\xi)|^2}{\xi^4}\,d^n\xi < \infty. \tag{1.23}$$

*Remark* 1.6. As follows from (1.21) and (1.22), $\Sigma(\omega)$ is strictly positive for $|\omega| < m$ (since $\rho \not\equiv 0$) and takes finite nonzero values for all $\omega$ that correspond to solitary waves (for $n \le 4$, the finiteness of $\Sigma(\omega)$ at $\omega = \pm m$ follows if (1.23) is satisfied).

*Remark* 1.7. One can see that generically the solitary wave manifold is two-dimensional.

*Proof.* Substituting the ansatz $\phi_\omega(x)e^{-i\omega t}$ into (1.1) and using (1.6), we get the following equation on $\phi_\omega$:

$$-\omega^2\phi_\omega(x) = \Delta\phi_\omega(x) - m^2\phi_\omega(x) + \rho(x)F(\langle\rho,\phi_\omega\rangle), \qquad x \in \mathbb{R}^n. \tag{1.24}$$

Therefore, all solitary waves satisfy the relation

$$(\xi^2+m^2-\omega^2)\hat\phi_\omega(\xi) = \hat\rho(\xi)F(\langle\rho,\phi_\omega\rangle). \tag{1.25}$$

For $\omega \notin [-m,m]\cup\Omega_\rho$ the relation (1.25) leads to $\phi_\omega \notin L^2(\mathbb{R}^n)$ (unless $\phi_\omega \equiv 0$). We conclude that there are no nonzero solitary waves for $\omega \notin [-m,m]\cup\Omega_\rho$.

Let us consider the case $\omega \in [-m,m]\cup\Omega_\rho$. From (1.25), we see that

$$\hat\phi_\omega(\xi) = \frac{\hat\rho(\xi)}{\xi^2+m^2-\omega^2}F(\langle\rho,\phi_\omega\rangle). \tag{1.26}$$

Using the function $V(x,\omega)$ defined in (1.18), we may express $\phi_\omega(x) = cV(x,\omega)$, with $c \in \mathbb{C}$. Substituting this ansatz into (1.26) and using (1.6), we can write the condition on $c$ in the form (1.21).

For $n \le 4$, the finiteness of the energy of solitons corresponding to $\omega = \pm m$ is equivalent to the condition (1.23).

This finishes the proof of the proposition. $\square$

### 1.4 The main result

**Assumption A.** We assume that $\rho \in \mathscr{S}(\mathbb{R}^n)$, the set $\Omega_\rho$ is finite, and that

$$\Sigma(\omega) \neq 0, \qquad \omega \in \Omega_\rho. \tag{1.27}$$

Above, $\Omega_\rho$ and $\Sigma(\omega)$ are defined in (1.20) and (1.22).

*Remark* 1.8. Note that $\Sigma(\omega)$ is well-defined for $\omega \in \Omega_\rho$ since $\hat\rho\big|_{|\xi|=\sqrt{\omega^2-m^2}} \equiv 0$ then.

As we mentioned before, we need to assume that the nonlinearity is polynomial. This assumption is crucial in our argument: It will allow to apply the Titchmarsh convolution theorem. Now all our assumptions on $F$ can be summarised as follows.

**Assumption B.** $F(z)$ satisfies (1.3) with the polynomial potential $U(z)$, and also satisfies (1.5) and (1.13). This can be summarised as the following assumption on $U(z)$:

$$U(z) = \sum_{n=1}^{N} u_n |z|^{2n}, \qquad u_n \in \mathbb{R}, \quad N \ge 2, \quad u_N > 0. \tag{1.28}$$

Our main result is the following theorem.

**Theorem 1.9 (Main Theorem).** *Assume that the coupling function $\rho(x)$ satisfies Assumption A and that the nonlinearity $F(z)$ satisfies Assumption B. Then for any $\Psi_0 \in \mathscr{E}$ the solution $\Psi(t) \in C(\mathbb{R},\mathscr{E})$ to the Cauchy problem (1.2) converges to* $\mathbf{S}$ *in the space $\mathscr{E}_F^{-\varepsilon}$, for any $\varepsilon > 0$:*

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{E}_F^{-\varepsilon}}(\Psi(t),\mathbf{S}) = 0, \tag{1.29}$$

*where* $\operatorname{dist}_{\mathscr{E}_F^{-\varepsilon}}(\cdot,\cdot)$ *is the metric (1.12) and* $\operatorname{dist}_{\mathscr{E}_F^{-\varepsilon}}(\Psi,\mathbf{S}) := \inf\limits_{\Phi\in\mathbf{S}} \operatorname{dist}_{\mathscr{E}_F^{-\varepsilon}}(\Psi,\Phi)$.

*Remark* 1.10. The $\mathscr{E}_F^{-\varepsilon}$-convergence to the attractor stated in this theorem is weaker than the $\mathscr{E}_F$-convergence proved in [KK07a] and [KK07b], where we considered the Klein-Gordon field in dimension $n = 1$, coupled to nonlinear oscillators.

Obviously, it suffices to prove Theorem 1.9 for $t \to +\infty$.

## 2 Absolute continuity for large frequencies

### 2.1 Splitting of a dispersive component

First we split the solution $\psi(x,t)$ into $\psi(x,t) = \chi(x,t) + \varphi(x,t)$, where $\chi$ and $\varphi$ are defined as solutions to the following Cauchy problems:

$$\ddot\chi(x,t) = \Delta\chi(x,t) - m^2\chi(x,t), \qquad (\chi,\dot\chi)\big|_{t=0} = \Psi_0, \tag{2.1}$$

$$\ddot\varphi(x,t) = \Delta\varphi(x,t) - m^2\varphi(x,t) + \rho(x)f(t), \qquad (\varphi,\dot\varphi)\big|_{t=0} = (0,0), \tag{2.2}$$

where $\Psi_0$ is the initial data from (1.2), and

$$f(t) := F(\langle \rho, \psi(\cdot,t)\rangle). \tag{2.3}$$

Note that $\langle \rho, \psi(\cdot,t)\rangle$ belongs to $C_b(\mathbb{R})$ since $(\psi,\dot\psi) \in C_b(\mathbb{R},\mathscr{E})$ by Theorem 1.3 (*iv*). Hence,

$$f(\cdot) \in C_b(\mathbb{R}). \tag{2.4}$$

On the other hand, since $\chi(t)$ is a finite energy solution to the free Klein-Gordon equation, we also have

$$(\chi,\dot\chi) \in C_b(\mathbb{R},\mathscr{E}). \tag{2.5}$$

Hence, the function $\varphi(t) = \psi(t) - \chi(t)$ also satisfies

$$(\varphi,\dot\varphi) \in C_b(\mathbb{R},\mathscr{E}). \tag{2.6}$$

The following lemma reflects the well-known energy decay for the linear Klein-Gordon equation.

**Lemma 2.1.** *There is a local decay of $\chi$ in the $\mathscr{E}_F$ seminorms. That is,* $\forall R > 0$,

$$\|(\chi(t),\dot\chi(t))\|_{\mathscr{E},R} \to 0, \qquad t \to \infty. \tag{2.7}$$

### 2.2 Complex Fourier-Laplace transform

Let us analyze the complex Fourier-Laplace transform of $\varphi(x,t)$:

$$\tilde{\varphi}(x,\omega)=\mathscr{F}_{t\to\omega}[\Theta(t)\varphi(x,t)]:=\int_0^\infty e^{i\omega t}\varphi(x,t)\,dt,\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^n, \tag{2.8}$$

where $\mathbb{C}^+:=\{z\in\mathbb{C}:\ \mathrm{Im}\,z>0\}$. Due to (2.6), $\tilde{\varphi}(\cdot,\omega)$ is an $H^1$-valued analytic function of $\omega\in\mathbb{C}^+$. Equation (2.2) for $\varphi$ implies that

$$-\omega^2\tilde{\varphi}(x,\omega)=\Delta\tilde{\varphi}(x,\omega)-m^2\tilde{\varphi}(x,\omega)+\rho(x)\tilde{f}(\omega),\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^n,$$

where $\tilde{f}(\omega)$ is the Fourier-Laplace transform of $f(t)$:

$$\tilde{f}(\omega)=\mathscr{F}_{t\to\omega}[\Theta(t)f(t)]=\int_0^\infty e^{i\omega t}f(t)\,dt,\qquad \omega\in\mathbb{C}^+.$$

The solution $\tilde{\varphi}(x,\omega)$ is analytic for $\omega\in\mathbb{C}^+$ and can be represented by

$$\tilde{\varphi}(x,\omega)=V(x,\omega)\tilde{f}(\omega),\qquad \omega\in\mathbb{C}^+. \tag{2.9}$$

### 2.3 Traces of distributions for $\omega\in\mathbb{R}$

First we remark that

$$\Theta(t)\varphi(x,t)\in C_b(\mathbb{R},H^1(\mathbb{R}^n)) \tag{2.10}$$

by (2.6) since $\varphi(x,0+)=0$ by initial conditions in (2.2). The Fourier-Laplace transform of $\varphi$ in time, $\mathscr{F}_{t\to\omega}[\Theta(t)\varphi(\cdot,t)]$, is a tempered $H^1$-valued distribution of $\omega\in\mathbb{R}$ by (2.6). We will denote this Fourier-Laplace transform by $\tilde{\varphi}(\cdot,\omega)$, $\omega\in\mathbb{R}$, which is the boundary value of the analytic function $\tilde{\varphi}(\cdot,\omega)$, $\omega\in\mathbb{C}^+$, in the following sense:

$$\tilde{\varphi}(\cdot,\omega)=\lim_{\varepsilon\to0+}\tilde{\varphi}(\cdot,\omega+i\varepsilon),\qquad \omega\in\mathbb{R}, \tag{2.11}$$

where the convergence is in the space of $H^1$-valued tempered distributions of $\omega$, $\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}^n))$. Indeed,

$$\tilde{\varphi}(\cdot,\omega+i\varepsilon)=\mathscr{F}_{t\to\omega}[\Theta(t)\varphi(\cdot,t)e^{-\varepsilon t}],$$

while $\Theta(t)\varphi(\cdot,t)e^{-\varepsilon t}\underset{\varepsilon\to0+}{\longrightarrow}\Theta(t)\varphi(\cdot,t)$, with the convergence taking place in $\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}^n))$ which is the space of $H^1$-valued tempered distributions of $t\in\mathbb{R}$. Therefore, (2.11) holds by the continuity of the Fourier transform $\mathscr{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R})$. Similarly to (2.11), the distribution $\tilde{f}(\omega)$ for $\omega\in\mathbb{R}$ is the boundary value of the analytic in $\mathbb{C}^+$ function $\tilde{f}(\omega)$, $\omega\in\mathbb{C}^+$:

$$\tilde{f}(\omega)=\lim_{\varepsilon\to0+}\tilde{f}(\omega+i\varepsilon),\quad \omega\in\mathbb{R}, \tag{2.12}$$

since the function $\Theta(t)f(t)$ is bounded. The convergence holds in the space of tempered distributions $\mathscr{S}'(\mathbb{R})$.

Let us justify that the representation (2.9) for $\tilde{\varphi}(x,\omega)$ is also valid when $\omega\in\mathbb{R}$, $\omega\neq\pm m$, if the multiplication in (2.9) is understood in the sense of distributions.

**Proposition 2.2.** *For any fixed $x\in\mathbb{R}^n$, $V(x,\omega)$, $\omega\in\mathbb{R}\backslash\{-m,m\}$, is a smooth function, and the identity*

$$\tilde{\varphi}(x,\omega)=V(x,\omega)\tilde{f}(\omega),\qquad \omega\in\mathbb{R}\backslash\{-m,m\}, \tag{2.13}$$

*holds in the sense of distributions.*

*Proof.* Consider

$$V(x,\omega)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{i\xi x}\hat{\rho}(\xi)\,d\xi}{\xi^2+m^2-(\omega+i0)^2}=\int_0^\infty\frac{R(x,\eta)\,d\eta}{\eta^2+m^2-(\omega+i0)^2}, \tag{2.14}$$

where

$$R(x,\eta)=\frac{1}{(2\pi)^n}\int_{|\xi|=\eta}e^{i\xi x}\hat{\rho}(\xi)\,dS_\xi. \tag{2.15}$$

For each $x\in\mathbb{R}^n$, $R(x,\eta)$ is smooth for $\eta>0$ and satisfies $|R(x,\eta)|=O(\eta^{n-1})$. It follows that for each $x\in\mathbb{R}^n$, $V(x,\omega)$ is a smooth function of $\omega\in\mathbb{R}\backslash\{-m,m\}$, and hence is a multiplicator in the space of distributions. $\square$

## 2.4 Absolutely continuous spectrum

Let $\kappa(\omega)$ denote the branch of $\sqrt{\omega^2-m^2}$ such that $\operatorname{Im}\sqrt{\omega^2-m^2}\ge 0$ for $\omega\in\mathbb{C}^+$:

$$\kappa(\omega)=\sqrt{\omega^2-m^2},\qquad \operatorname{Im}\kappa(\omega)>0,\qquad \omega\in\mathbb{C}^+. \tag{2.16}$$

Then $\kappa(\omega)$ is the analytic function for $\omega\in\mathbb{C}^+$. We extend it to $\omega\in\overline{\mathbb{C}^+}$ by continuity.

**Proposition 2.3.** *The distribution $\tilde f(\omega+i0)$, $\omega\in\mathbb{R}$, is absolutely continuous for $|\omega|>m$ and satisfies*

$$\int_{|\omega|>m}|\tilde f(\omega)|^2\mathscr{M}(\omega)\,d\omega\le\text{const}<\infty, \tag{2.17}$$

*where* $\mathscr{M}(\omega)=\frac{\omega\mathscr{R}(\kappa(\omega))}{\kappa(\omega)}$, $\mathscr{R}(\eta)=\frac{1}{(2\pi)^n}\int_{|\xi|=\eta}|\hat\rho(\xi)|^2\,dS_\xi$, $\eta\in\mathbb{R}$.

*Remark* 2.4. Note that the function $\mathscr{M}(\omega)$ is non-negative for $|\omega|>m$. The set of zeros of $\mathscr{M}(\omega)$, $|\omega|>m$, coincides with $\Omega_\rho$ defined in (1.20).

*Remark* 2.5. Recall that $\tilde f(\omega)$, $\omega\in\mathbb{R}$, is defined by (2.12) as the trace distribution: $\tilde f(\omega)=\tilde f(\omega+i0)$.

*Proof.* We will prove that for any closed interval $I$ such that $I\cap([-m,m]\cup\Omega_\rho)=\emptyset$ the following inequality holds:

$$\int_I|\tilde f(\omega)|^2\mathscr{M}(\omega)\,d\omega\le C, \tag{2.18}$$

for some constant $C>0$ which does not depend on $I$. Since there is a finite number of connected components of $\mathbb{R}\backslash([-m,m]\cup\Omega_\rho)$, this will finish the proof of the proposition.

Let us prove (2.18). The Parseval identity applied to

$$\tilde\varphi(x,\omega+i\varepsilon)=\int_0^\infty\varphi(x,t)e^{i\omega t-\varepsilon t}\,dt,\qquad \varepsilon>0,$$

and a similar relation for $\partial_x\tilde\varphi(x,\omega+i\varepsilon)$ leads to

$$\int_{-\infty}^\infty\|\tilde\varphi(\cdot,\omega+i\varepsilon)\|_{H^1}^2\,d\omega=2\pi\int_0^\infty\|\varphi(\cdot,t)\|_{H^1}^2e^{-2\varepsilon t}\,dt.$$

Since $\sup_{t\ge0}\|\varphi(\cdot,t)\|_{H^1}<\infty$ by (2.6), we may bound the right-hand side by $C_1/\varepsilon$, with some $C_1>0$. Taking into account (2.9), we arrive at the key inequality

$$\int_{-\infty}^\infty|\tilde f(\omega+i\varepsilon)|^2\|V(\cdot,\omega+i\varepsilon)\|_{H^1}^2\,d\omega\le\frac{C_1}{\varepsilon}. \tag{2.19}$$

**Lemma 2.6.** *Assume that $I$ is a closed interval such that $I\cap([-m,m]\cup\Omega_\rho)=\emptyset$. Then there exists $\varepsilon_I>0$ such that*

$$\|V(\cdot,\omega+i\varepsilon)\|_{H^1}^2\ge\frac{C_2\mathscr{M}(\omega)}{\varepsilon},\qquad \omega\in I,\quad 0<\varepsilon\le\varepsilon_I, \tag{2.20}$$

*where $C_2$ does not depend on the interval $I$.*

*Proof.* Let us compute the $H^1$-norm using the Fourier space representation. Since $\hat V(\xi,\omega+i\varepsilon)=\frac{\hat\rho(\xi)}{\xi^2+m^2-(\omega+i\varepsilon)^2}$, we have:

$$\|V(\cdot,\omega+i\varepsilon)\|_{H^1}^2=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{(m^2+\xi^2)|\hat\rho(\xi)|^2|d^n\xi}{|\xi^2+m^2-(\omega+i\varepsilon)^2|^2}=\int_0^\infty\frac{(m^2+\eta^2)\mathscr{R}(\eta)\,d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2}\ge\int_{\mathrm{H}}\frac{(m^2+\eta^2)\mathscr{R}(\eta)\,d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2}, \tag{2.21}$$

where $\kappa(I)$ is given by

$$\kappa(I)=\{\eta>0\colon\sqrt{\eta^2+m^2}\in I\}. \tag{2.22}$$

For $\omega\in I$ given in the condition of the Lemma, we denote

$$\eta_0=\kappa(\omega)\in\kappa(I).$$

Since the denominator in the integral in the right-hand side of (2.21) vanishes when $\varepsilon = 0$ and $\eta = \eta_0$, the inequality in (2.20) is due to the contribution of a small neighbourhood of $\eta = \eta_0$ which we will specify shortly. Since the function $(m^2+\eta^2)\mathscr{R}(\eta)$ is smooth and strictly positive on $I$, there exists $\delta_I > 0$ (that does not depend on a particular $\omega \in I$) so that $(m^2+\eta^2)\mathscr{R}(\eta) \ge \frac{1}{2}(m^2+{\eta_0}^2)\mathscr{R}(\eta_0)$, for all $\eta \in \kappa(I)$ such that $|\eta-\eta_0| < \delta_I$. Hence, (2.21) takes the form

$$\|V(\cdot,\omega+i\varepsilon)\|^2_{H^1} \ge \frac{(\eta_0^2+m^2)\mathscr{R}(\eta_0)}{2} \int\limits_{\kappa(I)\cap[\eta_0-\delta_I,\eta_0+\delta_I]} \frac{d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2}. \tag{2.23}$$

We require that $\delta_I < |\kappa(I)|/2$; then either $[\eta_0-\delta_I,\eta_0] \subset \kappa(I)$ or $[\eta_0,\eta_0+\delta_I] \subset \kappa(I)$, or both. Therefore, the integral in the right-hand side of (2.23) restricted to $\kappa(I)\cap[\eta_0-\delta_I,\eta_0+\delta_I]$ becomes unboundedly large as $\varepsilon \to 0+$, and moreover there exists $\varepsilon_I > 0$ (which does not depend on a particular $\omega \in I$) such that

$$\int\limits_{\kappa(I)\cap[\eta_0-\delta_I,\eta_0+\delta_I]} \frac{d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2} > \frac{1}{2} \int\limits_{[\eta_0-\delta_I,\eta_0+\delta_I]} \frac{d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2} > \frac{1}{3}\int\limits_{\mathbb{R}} \frac{d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2}, \tag{2.24}$$

for all $\varepsilon > 0$, $\varepsilon \le \varepsilon_I$. The last integral, evaluated by the Cauchy theorem, is equal to $\pi/(2\varepsilon\omega\eta_0)+O(1)$. Therefore, we may assume that $\varepsilon_I > 0$ is so small (independently of a particular $\omega \in I$) that

$$\int\limits_{\kappa(I)\cap[\eta_0-\delta_I,\eta_0+\delta_I]} \frac{d\eta}{|\eta^2+m^2-(\omega+i\varepsilon)^2|^2} \ge \frac{1}{3}\frac{\pi}{3\varepsilon\omega\eta_0}, \qquad \omega \in I, \quad 0<\varepsilon\le\varepsilon_I. \tag{2.25}$$

Note that $\omega\eta_0 = \omega\kappa(\omega) > 0$ because, for $\omega \in \mathbb{R}\backslash[-m,m]$, $\kappa(\omega) \in \mathbb{R}$ and is of the same sign as $\omega$ is. Combining (2.23) and (2.25), we get:

$$\|V(\cdot,\omega+i\varepsilon)\|^2_{H^1} \ge \frac{\omega^2\mathscr{R}(\eta_0)}{2}\frac{1}{3}\frac{1}{\varepsilon\omega\eta_0} = \frac{1}{6\varepsilon}\frac{\omega\mathscr{R}(\eta_0)}{\eta_0}, \qquad \omega \in I, \quad 0<\varepsilon\le\varepsilon_I. \tag{2.26}$$

□

Substituting (2.20) into (2.19), we get

$$\int_I |\tilde f(\omega+i\varepsilon)|^2 \mathscr{M}(\omega)\,d\omega \le C_1/C_2, \qquad 0<\varepsilon\le\varepsilon_I. \tag{2.27}$$

We conclude that the set of functions $g_{I,\varepsilon}(\omega) = \tilde f(\omega+i\varepsilon)\sqrt{\mathscr{M}(\omega)}$, $0<\varepsilon\le\varepsilon_I$, defined for $\omega \in I$, is bounded in the Hilbert space $L^2(I)$, and, by the Banach Theorem, is weakly compact. The convergence of the distributions (2.12) implies the weak convergence $g_{I,\varepsilon} \xrightarrow[\varepsilon\to 0+]{} g_I$ in the Hilbert space $L^2(I)$. The limit function $g_I(\omega)$ coincides with the distribution $\tilde f(\omega)\sqrt{\mathscr{M}(\omega)}$ restricted onto $I$. This proves the bound (2.18) and finishes the proof of the proposition. □

# 3 Omega-limit trajectories

## 3.1 Compactness

We are going to prove compactness of the set of translations of the singular component, $\{\varphi(x,t+s)\colon s \ge 0\}$.

**Proposition 3.1.** *For any sequence $s_j \to \infty$ there exists an infinite subsequence (which we also denote by $s_j$) such that*

$$(\varphi(\cdot,t+s_j),\dot\varphi(\cdot,t+s_j)) \longrightarrow (\beta(\cdot,t),\dot\beta(\cdot,t)), \qquad j\to\infty, \tag{3.1}$$

*for some $\beta \in C_b(\mathbb{R},H^1(\mathbb{R}^n))$ with $\dot\beta \in C_b(\mathbb{R},L^2(\mathbb{R}^n))$.*

*In (3.1), the convergence holds in the topology of $C([-T,T],\mathscr{E}_F^{-\varepsilon})$, for any $T>0$ and any small $\varepsilon>0$. The following bound holds:*

$$\sup_{t\in\mathbb{R}} \|(\beta(\cdot,t),\dot\beta(\cdot,t))\|_{\mathscr{E}} < \infty. \tag{3.2}$$

This proposition is a consequence of Theorem 1.3 (v) which implies that $(\psi,\dot\psi)\in C^{(\varepsilon)}(\mathbb{R},\mathscr{E}^{-\varepsilon})$, $(\chi,\dot\chi)\in C^{(\varepsilon)}(\mathbb{R},\mathscr{E}^{-\varepsilon})$, and thus $(\varphi,\dot\varphi)\in C^{(\varepsilon)}(\mathbb{R},\mathscr{E}^{-\varepsilon})$.

We call *omega-limit trajectory* any function $\beta(x,t)$ that can appear as a limit in (3.1). Previous analysis demonstrates that the long-time asymptotics of the solution $\psi(x,t)$ in $\mathscr{E}_F$ depends only on the singular component $\varphi(x,t)$. By Proposition 3.1, to conclude the proof of Theorem 1.9, it suffices to check that every omega-limit trajectory belongs to the set of solitary waves; that is,

$$\beta(x,t)=\phi_{\omega_+}(x)e^{-i\omega_+ t},\qquad x\in\mathbb{R}^n,\quad t\in\mathbb{R}, \tag{3.3}$$

with some $\omega_+\in\mathbb{R}$.

## 3.2 Nonlinear Spectral analysis

The convergence (3.1) and equation (1.1), together with Lemma 2.1, imply that any omega-limit trajectory $\beta(x,t)$ is a solution to equation (1.1) (although $\varphi(x,t)$ is not!):

$$\ddot\beta(x,t)=\Delta\beta(x,t)-m^2\beta(x,t)+\rho(x)F(\langle\rho,\beta\rangle),\qquad x\in\mathbb{R}^n,\ t\in\mathbb{R}. \tag{3.4}$$

For a particular omega-limit trajectory $\beta(x,t)$, we denote $g(t)=F(\langle\rho,\beta(\cdot,t)\rangle)$.

**Proposition 3.2.** *There is the inclusion* $\operatorname{supp}\tilde g\subset[-m,m]\cup\Omega_\rho$, *where* $\Omega_\rho$ *is defined in (1.20).*

*Proof.* The convergence (3.1) implies that, for any $\alpha\in C_0^\infty(\mathbb{R}^n)$ and $\zeta\in C_0^\infty(\mathbb{R})$, $\langle\alpha,(\check\zeta*\varphi)(\cdot,t+s_j)\rangle\xrightarrow{\mathscr{S}'}\langle\alpha,(\check\zeta*\beta)(\cdot,t)\rangle$. Due to the continuity of the Fourier transform from $\mathscr{S}'(\mathbb{R})$ into itself, we also have

$$\zeta(\omega)\langle\alpha,\tilde\varphi(\cdot,\omega)\rangle e^{-i\omega s_j}\xrightarrow{\mathscr{S}'}\zeta(\omega)\langle\alpha,\tilde\beta(\cdot,\omega)\rangle,\qquad s_j\to\infty. \tag{3.5}$$

Assume that $\operatorname{supp}\zeta\cap([-m,m]\cup\Omega_\rho)=\emptyset$. Then, by Proposition 2.2, we may substitute $\zeta(\omega)\tilde\varphi(x,\omega)$ by $\zeta(\omega)V(x,\omega)\tilde f(\omega)$, getting

$$\zeta(\omega)\langle\alpha,V(\cdot,\omega)\rangle\tilde f(\omega)e^{-i\omega s_j}\xrightarrow{\mathscr{S}'}\zeta(\omega)\langle\alpha,\tilde\beta(\cdot,\omega)\rangle,\qquad s_j\to\infty. \tag{3.6}$$

Since $\tilde f$ is locally $L^2$ on $\mathbb{R}\backslash([-m,m]\cup\Omega_\rho)$ by Proposition 2.2, while (for each $x\in\mathbb{R}^n$) $V(x,\omega)$ is smooth for $\omega\in\mathbb{R}\backslash\{\pm m\}$, the product $\zeta(\omega)\langle\alpha,V(\cdot,\omega)\rangle\tilde f(\omega)$ is an absolutely continuous measure. Therefore the left-hand side of (3.6) converges to zero. It follows that $\tilde\beta(x,\omega)\equiv 0$ for $\omega\notin[-m,m]\cup\Omega_\rho$. □

**Proposition 3.3.** $\operatorname{supp}\tilde g\subset\operatorname{supp}\langle\rho,\tilde\beta(\cdot,\omega)\rangle$.

*Proof.* By Proposition 3.1, It follows that

$$f(t+s_j)=F(\langle\rho,\varphi(\cdot,t+s_j)\rangle)\xrightarrow{C([-T,T])}F(\langle\rho,\beta(\cdot,t)\rangle)=g(t),\qquad j\to\infty,$$

for any $T>0$. Using (2.13) and taking into account that $V(x,\omega)$ is smooth for $\omega\neq\pm m$, we obtain the following relation which holds in the sense of distributions:

$$\tilde\beta(x,\omega)=V(x,\omega)\tilde g(\omega),\qquad x\in\mathbb{R}^n,\qquad\omega\in\mathbb{R}\backslash\{\pm m\}. \tag{3.7}$$

Taking the pairing of (3.7) with $\rho$ and using definition of $\Sigma(\omega)$ (see (1.22)), we get:

$$\langle\rho,\tilde\beta(\cdot,\omega)\rangle=\Sigma(\omega)\tilde g(\omega),\qquad\omega\in\mathbb{R}\backslash\{\pm m\}. \tag{3.8}$$

First we prove Proposition 3.3 modulo the set $\omega=\{\pm m\}$.

**Lemma 3.4.** $\operatorname{supp}\tilde g\backslash\{\pm m\}\subset\operatorname{supp}\langle\rho,\tilde\beta(\cdot,\omega)\rangle$.

*Proof.* By Proposition 3.2, $\operatorname{supp}\tilde g\subset[-m,m]\cup\Omega_\rho$. Thus, the statement of the lemma follows from (3.8) and from noticing that $\Sigma(\omega)$ is smooth and positive for $\omega\in(-m,m)$ and moreover, by Assumption A, it is nonzero on $\Omega_\rho$. □

To finish the proof of Proposition 3.3, it remains to consider the contribution of $\omega=\pm m$.

**Lemma 3.5.** *If* $\omega_0=\pm m$ *belongs to* $\operatorname{supp}\tilde g$, *then* $\omega_0\in\operatorname{supp}\langle\rho,\tilde\beta\rangle$.

*Proof.* In the case when $\omega_0 = \pm m$ is not an isolated point in $[-m,m] \cap \operatorname{supp} \tilde{g}$, we use (3.8) to conclude that $\omega_0 \in \operatorname{supp}\langle \rho, \tilde{\beta} \rangle$ due to positivity of $\Sigma(\omega)$ for $|\omega| < m$ (which is apparent from (1.22)).

We are left to consider the case when $\omega_0 = m$ or $-m$ is an isolated point in $[-m,m] \cap \operatorname{supp} \tilde{g}$. We can pick an open neighbourhood $U$ of $\omega_0$ such that $U \cap \operatorname{supp} \tilde{g} = \{\omega_0\}$ since $\operatorname{supp} \tilde{g} \in [-m,m] \cup \Omega_\rho$ and $\Omega_\rho$ is a discrete finite set. Pick $\zeta \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \zeta \subset U$, such that $\zeta(\omega_0) = 1$. First we note that

$$\zeta(\omega)\tilde{g}(\omega) = M\delta(\omega - \omega_0), \qquad M \neq 0, \tag{3.9}$$

where the derivatives of the $\delta(\omega - \omega_0)$ are prohibited because $\check{\zeta} * g(t)$ is bounded. By (3.7), we have $U \cap \operatorname{supp}_\omega \tilde{\beta} \subset \{\omega_0\}$, hence

$$\zeta(\omega)\tilde{\beta}(x,\omega) = \delta(\omega - \omega_0)b(x), \qquad b \in H^1(\mathbb{R}^n). \tag{3.10}$$

Again, the terms with the derivatives of $\delta(\omega - \omega_0)$ are prohibited because $\langle \alpha, \check{\zeta} * \beta(\cdot,t) \rangle$ are bounded for any $\alpha \in C_0^\infty(\mathbb{R}^n)$, while the inclusion $b(x) \in H^1(\mathbb{R})$ is due to $\tilde{\beta} \in \mathscr{S}'(\mathbb{R}, H^1(\mathbb{R}))$.

Multiplying (3.4) by $\zeta(\omega)$ and taking into account (3.9), (3.10), and the relation $\omega_0^2 = m^2$, we see that the distribution $b(x)$ satisfies the equation

$$0 = \Delta b(x) + M\rho(x). \tag{3.11}$$

Therefore, $b(x) \not\equiv 0$ due to $M \neq 0$ and $\rho(x) \not\equiv 0$. Coupling (3.10) with $\rho$ and using (3.11), we get:

$$\zeta(\omega)\langle \rho, \tilde{\beta}(\cdot,\omega) \rangle = \delta(\omega - \omega_0)\langle \rho, b \rangle = -\delta(\omega - \omega_0)\frac{\langle \Delta b, b \rangle}{M} \neq 0, \tag{3.12}$$

since $b \in H^1(\mathbb{R}^n)$ is nonzero. This finishes the proof of Lemma 3.5. □

Lemmas 3.4 and 3.5 allow to conclude that $\operatorname{supp} \tilde{g}(\omega) \subset \operatorname{supp}\langle \rho, \tilde{\beta}(\cdot,\omega) \rangle$, finishing the proof of Proposition 3.3. □

Finally we reduce the spectrum of $\gamma(t)$ to one point.

**Lemma 3.6.** *$\langle \rho, \beta(\cdot,t) \rangle = 0$ or $\operatorname{supp}\langle \rho, \tilde{\beta}(\cdot,\omega) \rangle = \{\omega_+\}$, for some $\omega_+ \in [-m,m] \cup \Omega_\rho$.*

*Proof.* Denote

$$\gamma(t) = \langle \rho, \beta(\cdot,t) \rangle. \tag{3.13}$$

By (1.28), $g(t) := F(\gamma(t)) = -\sum_{n=1}^N 2nu_n |\gamma(t)|^{2n-2}\gamma(t)$. Then, by the Titchmarsh Convolution Theorem,

$$\sup\operatorname{supp} \tilde{g} = \max_{n \in \{n \le N,\, u_n \neq 0\}} \sup\operatorname{supp} \underbrace{(\tilde{\bar{\gamma}} * \tilde{\gamma}) * \ldots * (\tilde{\bar{\gamma}} * \tilde{\gamma})}_{n-1} * \tilde{\gamma} = N \sup\operatorname{supp} \tilde{\gamma} + (N-1)\sup\operatorname{supp} \tilde{\bar{\gamma}}. \tag{3.14}$$

*Remark* 3.7. The Titchmarsh theorem applies because $\operatorname{supp} \tilde{\gamma} \subset [-m,m] \cup \Omega_\rho$, and hence is compact.

Noting that $\sup\operatorname{supp} \tilde{\bar{\gamma}} = -\inf\operatorname{supp} \tilde{\gamma}$, we rewrite (3.14) as

$$\sup\operatorname{supp} \tilde{g} = \sup \tilde{\gamma} + (N-1)(\sup\operatorname{supp} \tilde{\gamma} - \inf\operatorname{supp} \tilde{\gamma}). \tag{3.15}$$

Taking into account Proposition 3.3 and (3.15), we get the following relation:

$$\sup\operatorname{supp} \tilde{\gamma} \ge \sup\operatorname{supp} \tilde{g} = \sup\operatorname{supp} \tilde{\gamma} + (N-1)(\sup\operatorname{supp} \tilde{\gamma} - \inf\operatorname{supp} \tilde{\gamma}). \tag{3.16}$$

This is only possible if $\operatorname{supp} \tilde{\gamma} \subset \{\omega_+\}$, for some $\omega_+ \in [-m,m] \cup \Omega_\rho$. □

## 3.3 Conclusion of the proof of Theorem 1.9

We need to prove (3.3). As follows from Lemma 3.6, $\tilde{\gamma}(\omega)$ is a finite linear combination of $\delta(\omega - \omega_+)$ and its derivatives. As the matter of fact, the derivatives could not be present because of the boundedness of $\gamma(t) := \langle \rho, \beta(\cdot,t) \rangle$ that follows from Proposition 3.1. Therefore, $\tilde{\gamma} = 2\pi C\delta(\omega - \omega_+)$, with some $C \in \mathbb{C}$. This implies the following identity:

$$\gamma(t) = Ce^{-i\omega_+ t}, \qquad C \in \mathbb{C}, \quad t \in \mathbb{R}. \tag{3.17}$$

The representation (3.7) implies that $\beta(x,t) = \beta(x,0)e^{-i\omega_+ t}$ since $\tilde{g} = 2\pi C\delta(\omega - \omega_+)$, $C \in \mathbb{C}$. Therefore, equation (3.4) and the bound (3.2) imply that $\beta(x,t)$ is a solitary wave. This completes the proof of Theorem 1.9.

# 4 Multifrequency solutions

Let us construct a multifrequency solution for the case when $\Sigma(\omega)$ vanishes at certain points of $\Omega_\rho$, and hence Assumption A is violated. Fix $\omega_1 \in (m, 3m)$. Pick $\rho \in \mathscr{S}(\mathbb{R}^n)$ so that the following two conditions are satisfied:

$$\hat{\rho}\big|_{|\xi|=\sqrt{\omega_1^2-m^2}} = 0, \tag{4.1}$$

$$\Sigma(\omega_1) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\hat{\rho}(\xi)|^2\, d^n\xi}{\xi^2+m^2-\omega_1^2} = 0. \tag{4.2}$$

**Lemma 4.1.** *There exist $a \in \mathbb{R}$, $b < 0$ so that equation (1.1) with the nonlinearity*

$$F(\gamma) = a\gamma + b|\gamma|^2\gamma, \qquad \gamma \in \mathbb{C},$$

*admits multifrequency solutions $\psi \in C(\mathbb{R}, H^1)$ of the form*

$$\psi(x,t) = \phi_0(x)\sin\omega_0 t + \phi_1(x)\sin\omega_1 t, \qquad \omega_0 = \frac{\omega_1}{3}, \quad \phi_0,\, \phi_1 \in H^1(\mathbb{R}^n),$$

*with both $\phi_0$ and $\phi_1$ nonzero.*

*Proof.* To make sure that the nonlinearity does not produce higher frequencies, we assume that

$$\langle \rho, \phi_1 \rangle = 0. \tag{4.3}$$

Due to this assumption,

$$F(\langle \rho, \psi \rangle) = F(\langle \rho, \phi_0 \rangle \sin\omega_0 t) = a\langle \rho, \phi_0 \rangle \sin\omega_0 t + b\langle \rho, \phi_0 \rangle^3 \frac{3\sin\omega_0 t - \sin 3\omega_0 t}{4}.$$

Collecting the terms with the factors of $\sin\omega_0 t$ and $\sin\omega_1 t = \sin 3\omega_0 t$, we rewrite the equation $\ddot{\psi} = \Delta\psi - m^2\psi + \rho F(\langle \rho, \psi \rangle)$ as two following equalities:

$$-\omega^2\phi_0 = \Delta\phi_0 - m^2\phi_0 + \rho(x)\Big(a\langle \rho, \phi_0 \rangle + \frac{3b\langle \rho, \phi_0 \rangle^3}{4}\Big), \tag{4.4}$$

$$-\omega_1^2\phi_1 = \Delta\phi_1 - m^2\phi_1 - \rho(x)\frac{b\langle \rho, \phi_0 \rangle^3}{4}. \tag{4.5}$$

We define $\phi_0$ by $\hat{\phi}_0(\xi) = \frac{\hat{\rho}(\xi)}{\xi^2+m^2-\omega^2}$, and pick $a$ and $b$ which satisfy (4.4). We take $b < 0$ so that Assumption B is satisfied. Then the function $\phi_1$ is defined by

$$\hat{\phi}_1(\xi) = -\frac{b\langle \rho, \phi_0 \rangle^3}{4}\frac{\hat{\rho}(\xi)}{\xi^2+m^2-\omega_1^2} = -\frac{b\Sigma(\omega_0)^3}{4}\frac{\hat{\rho}(\xi)}{\xi^2+m^2-\omega_1^2}.$$

Due to (4.1), $\phi_1 \in H^1(\mathbb{R}^n)$. Since $\langle \rho, \phi_1 \rangle = \mathrm{const}\,\Sigma(\omega_1) = 0$, the assumption (4.3) is indeed satisfied. □

# A Appendix: Global well-posedness

The global existence stated in Theorem 1.3 is obtained by standard arguments from the contraction mapping principle. To achieve this, we use the integral representation for the solutions to the Cauchy problem (1.2):

$$\Psi(x,t) = W_0(t)\Psi_0 + Z[\Psi](t), \qquad Z[\Psi](t) := \int_0^t W_0(t-s)\begin{bmatrix} 0 \\ \rho(\cdot)F(\langle \rho, \psi(\cdot,s) \rangle) \end{bmatrix} ds, \qquad \Psi = \begin{bmatrix} \psi \\ \pi \end{bmatrix}, \quad t \ge 0. \tag{A.1}$$

Here $W_0(t)$ is the dynamical group for the linear Klein-Gordon equation which is a unitary operator in the space $\mathscr{E}$. The bound

$$\|Z[\Psi_1](t) - Z[\Psi_2](t)\|_{\mathscr{E}} \le Ct \sup_{s\in[0,t]} \|\Psi_1(s) - \Psi_2(s)\|_{\mathscr{E}}, \qquad C > 0, \quad 0 \le t \le 1, \tag{A.2}$$

which holds for any two functions $\Psi_1, \Psi_2 \in C([0,1],\mathscr{E})$, shows that $Z[\psi]$ is a contraction operator in $C([0,t],\mathscr{E})$ if $t>0$ is sufficiently small.

To prove the a priori bound (1.15), we use (1.13) to bound $\|\Psi\|_{\mathscr{E}}$ in terms of the value of the Hamiltonian:

$$\|\Psi\|_{\mathscr{E}}^2 \le \frac{2m^2}{m^2-2B\|\rho\|_{L^2}^2}\left(\mathscr{H}(\Psi)-A\right), \qquad \Psi\in\mathscr{E}. \tag{A.3}$$

We now concentrate on the Hölder continuity of the solution to (1.2). First we consider the linear case.

**Lemma A.1.** *Let $u(x,t)$ be the solution to the Cauchy problem*

$$\ddot{u} = \Delta u - m^2 u, \qquad (u,\dot{u})|_{t=0} = (u_0, v_0)\in\mathscr{E}.$$

*Then $(u,\dot{u})\in C^{(\varepsilon)}(\mathbb{R},\mathscr{E}^{-\varepsilon}(\mathbb{R}^n))$ for $0<\varepsilon\le 1$.*

*Proof.* It suffices to prove the continuity stated in the lemma near the point $t=0$. We will only prove the estimate $\|u(\cdot,t)-u(\cdot,0)\|_{H^{1-\varepsilon}}\le \mathrm{const}\,|t|^{\varepsilon}$; the bound $\|\dot{u}(\cdot,t)-v(\cdot,0)\|_{H^{-\varepsilon}}\le\mathrm{const}\,|t|^{\varepsilon}$ is obtained similarly. The difference $\hat{u}(\xi,t)-\hat{u}(\xi,0)$ is given by

$$\hat{u}(\xi,t)-\hat{u}(\xi,0) = (\cos(t\sqrt{\xi^2+m^2})-1)\hat{u}_0(\xi) + \frac{\sin(t\sqrt{\xi^2+m^2})}{\sqrt{\xi^2+m^2}}\hat{v}_0(\xi). \tag{A.4}$$

Let us analyze the contribution into $\|u(\cdot,t)-u(\cdot,0)\|_{H^{1-\varepsilon}}^2$ of the second term from the right-hand side of (A.4) only (the first term is analyzed similarly). We have:

$$\int_{\mathbb{R}^n}(\xi^2+m^2)^{-\varepsilon}\sin^2(t\sqrt{\xi^2+m^2})|\hat{v}(\xi)|^2\,d^n\xi \le \sup_{\xi\in\mathbb{R}^n}\frac{\sin^2(t\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{\varepsilon}}\int_{\mathbb{R}^n}|\hat{v}(\xi)|^2\,d^n\xi \le \mathrm{const}\,|t|^{2\varepsilon}\|v\|_{L^2}^2,$$

where we used the inequality $|\sin z|\le z^{\varepsilon}$, valid for any $0<\varepsilon\le 1$ and $z\in\mathbb{R}$. This finishes the proof. □

Now we can prove the inclusion (1.16) stated in Theorem 1.3.

**Lemma A.2.** *The solution to (1.2) with $\Psi|_{t=0}\in\mathscr{E}$ satisfies $\Psi\in C^{(\varepsilon)}(\mathbb{R},\mathscr{E}^{-\varepsilon})$, $0<\varepsilon\le 1$.*

*Proof.* It suffices to prove the statement of the lemma near $t=0$. The representation (A.1) for $\Psi(t)$ yields

$$\Psi(t)-\Psi(0) = \big(W_0(t)\Psi(0)-\Psi(0)\big) + Z[\Psi](t). \tag{A.5}$$

Estimating the contribution into $\|\Psi(t)-\Psi(0)\|_{\mathscr{E}^{-\varepsilon}}$ of the first term in the right-hand side of (A.5) by Lemma A.1 and the contribution of the second term by the bound (A.2) (where we take $\Psi_1=\Psi$, $\Psi_2=0$), we get

$$\|\Psi(t)-\Psi(0)\|_{\mathscr{E}^{-\varepsilon}} \le C_1|t|^{\varepsilon}+C_2|t|, \qquad C_1,\, C_2>0.$$

□

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