Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Spectral geometry, homogeneous spaces, and differential forms with finite Fourier series

by

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Preprint no.: 72 2007

SPECTRAL GEOMETRY, HOMOGENEOUS SPACES, AND DIFFERENTIAL FORMS WITH FINITE FOURIER SERIES

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Abstract. Let G be a compact Lie group acting transitively on Riemannian manifolds M_i and let $\pi : M_1 \to M_2$ be a G-equivariant Riemannian submersion. We show that a smooth differential form ϕ on M_2 has finite Fourier series on M_2 if and only if the pull-back $\pi^*\phi$ has finite Fourier series on M_1.

1. Introduction

The spectral geometry of Riemannian submersions has been discussed by many authors; we refer, for example, to [4] for a more extensive discussion. In particular, it plays an important role in the study of non-bijective canonical transformations; see, for example, the discussion in [6].

Let M be a compact smooth closed Riemannian manifold of dimension m, and let Δ_M^p be the Laplace-Beltrami operator acting on the space $C^\infty(\Lambda^p M)$ of smooth p-forms. Let $\operatorname{Spec}(\Delta_M^p)$ be the spectrum of Δ_M^p; this is a discrete countable set of non-negative real numbers. The associated eigenspaces $E(\lambda, \Delta_M^p)$ are finite dimensional and there is a complete orthonormal decomposition

$$L^2(\Lambda^p M) = \oplus_{\lambda \in \operatorname{Spec}(\Delta_M^p)} E(\lambda, \Delta_M^p) \qquad (1.a)$$

which we may use to decompose a smooth p-form ϕ on M in the form $\phi = \sum_\lambda \phi_\lambda$ where $\phi_\lambda \in E(\lambda, \Delta_M^p)$. We say ϕ has *finite Fourier series* if this is a finite sum. If $p = 0$ and if $M = S^1$, then this yields, modulo a slight change of notation, the classical Fourier series decomposition $f(\theta) = \sum_n a_n e^{in\theta}$ and a function has a finite Fourier series in this setting if and only if it is a trigonometric polynomial. There is an extensive literature on the subject, a few representative items being [1, 3].

We say that M is a *homogeneous space* if there is a compact Lie group G which acts transitively on M by isometries; if H is the isotropy subgroup associated to some point $P \in M$, then we may identify $M = G/H$. We may choose a left-invariant metric $\tilde{g}$ on G so g is the induced metric or, equivalently, that $\pi : (G, \tilde{g}) \to (M, g)$ is a Riemannian submersion. The following is the main result of this paper:

Theorem 1.1. *Let $\pi : G \to G/H$ where H is a Lie subgroup of a compact Lie group G. Let $\tilde{g}$ be a left-invariant Riemannian metric on G and let g be the induced Riemannian metric on G/H. Then a p-form ϕ on G/H has finite Fourier series on G/H if and only if $\pi^*\phi$ has finite Fourier series on G.*

There is an associated Corollary which is useful in applications.

Corollary 1.2. *Let G be a compact Lie group acting transitively on Riemannian manifolds M_1 and M_2. Let $\pi : M_1 \to M_2$ be a G-equivariant Riemannian submersion. If ϕ is a smooth p-form on M_2, then ϕ has finite Fourier series on M_2 if and only if $\pi^*\phi$ has finite Fourier series on M_1.*

Key words and phrases. eigenform, finite Fourier series, homogeneous space, Laplace-Beltrami operator, Peter–Weyl theorem.

MSC 58J50; PACS numbers: 02.20.Qs, 02.30.Em, 02.30.Nw, 02.40.Vh.

Remark 1.3. The Hopf fibration $\pi : S^{2n+1} \to \mathbb{CP}^n$ is a $U(n+1)$ equivariant Riemannian submersion which is an important non-canonical transformation used to study the Coulumb problem, see, for example, the discussion in [2]. Corollary 1.2 shows ϕ has finite Fourier series on $\mathbb{CP}^n$ if and only if $\pi^*\phi$ has finite Fourier series on S^{2n+1}.

2. The proof of Theorem 1.1

The central ingredient is our discussion is the classical Peter–Weyl theorem [5]. Let $\operatorname{Irr}(G)$ be the collection of equivalence classes of irreducible finite dimensional representations of G; if $\rho \in \operatorname{Irr}(G)$, let V_ρ be the associated representation space. The Hilbert space structure on $L^2(G)$ depends on the particular Riemannian metric which is chosen; this space is invariantly defined as a Banach space, however. This is a minor distinction which will be useful, however, in Section 4. Left multiplication defines an action of G on $L^2(G)$. This action decomposes as a direct sum

$$L^2(\Lambda^p G) = \oplus_{\rho\in\operatorname{Irr}(G)} W_\rho \tag{2.a}$$

where each W_ρ is a finite dimensional irreducible subspace of $L^2(G)$ which is isomorphic to a finite number of copies of V_ρ. If Φ is a smooth p-form on G, we may use Equation (2.a) to decompose $\Phi = \sum_\rho \Phi_\rho$ for $\Phi_\rho \in W_\rho$. We say that Φ has *finite representation expansion* on G if this sum is finite; we emphasize that this notion is independent of the particular Riemannian metric chosen.

Since π is a submersion, π^* is an injective G-equivariant map from $L^2(\Lambda^p(G/H))$ to $L^2(G)$ with closed image. The decomposition

$$L^2(\Lambda^p G) = \pi^*(L^2(\Lambda^p(G/H))) \oplus \{\pi^*(L^2(\Lambda^p(G/H)))\}^\perp$$

is G-equivariant. We therefore have an orthogonal direct sum decomposition of $L^2(\Lambda^p(G/H))$ as a representation space for G in the form:

$$L^2(\Lambda^p(G/H)) = \oplus_{\rho\in\operatorname{Irr}(G)} X_\rho \quad \text{where} \tag{2.b}$$

$$\pi^* X_\rho = W_\rho \cap \pi^*(L^2(\Lambda^p(G/H)))\,. \tag{2.c}$$

We say that a p-form ϕ on G/H has *finite G-representation series* if the expansion $\phi = \sum_\rho \phi_\rho$ given by Equation (2.b) is finite. Theorem 1.1 will follow from the following:

Lemma 2.1. *Adopt the notation established above. Let ϕ be a smooth p-form on G/H. Fix a left-invariant $\tilde g$ metric on G and let g be the induced metric on G/H. The following assertions are equivalent:*

1. *ϕ has finite Fourier series on G/H.*
2. *ϕ has finite G-representation series on G/H.*
3. *$\pi^*\phi$ has finite Fourier series on G.*
4. *$\pi^*\phi$ has finite G-representation series on G.*

Proof. The equivalence of Assertions (ii) and (iv) is immediate from Equation (2.c). We argue as follows to prove that Assertion (i) implies Assertion (ii). Suppose that ϕ has finite Fourier series on G/H. Since G acts by isometries, G commutes with the Laplacian. Thus $E(\lambda, \Delta^p_{G/H})$ is a finite dimensional representation space for G. Only a finite number of representations occur in the representation decomposition of $E(\lambda, \Delta^p_{G/H})$ and thus any eigen p-form on G/H has finite G-representation series on G/H; more generally, of course, any finite sum of eigen p-forms on G/H has finite G-representation series on G/H. This shows that Assertion (i) implies Assertion (ii); a similar argument shows Assertion (iii) implies Assertion (iv).

Each representation appears with finite multiplicity in $L^2(\Lambda^p(G/H))$. Thus each representation appears in the decomposition of $E(\lambda, \Delta^p_{G/H})$ for only a finite number of λ. Thus any element of X_ρ has finite Fourier series and more generally any p-form on G/H with finite G-representation series has finite Fourier series. Thus Assertion (ii) implies Assertion (i); similarly, Assertion (iv) implies Assertion (iii). □

3. The proof of Corollary 1.2

Let $\pi : M_1 \to M_2$ be a G-equivariant Riemannian submersion; this means that we may express $M_i = G/H_i$ where $H_1 \subset H_2 \subset G$. Let $\pi_i : G \to G/H_i$ be the natural projections. We then have $\pi\pi_1 = \pi_2$ and thus $\pi_2^* = \pi_1^*\pi^*$. Let ϕ be a smooth p-form on G/H_2. We apply Theorem 1.1 to derive the following chain of equivalent statements from which Corollary 1.2 will follow:

(1) ϕ has finite Fourier series on G/H_2.
(2) $\pi_2^*\phi$ has finite Fourier series on G.
(3) $\pi_1^*(\pi^*\phi)$ has finite Fourier series on G.
(4) $\pi^*\phi$ has finite Fourier series on G/H_1.

4. Conclusions and open problems

Our methods in fact show a bit more. Let g_i be two left invariant metrics on G and let ϕ be a smooth p-form on G. Then ϕ has finite Fourier series with respect to g_1 if and only if ϕ has finite Fourier series with respect to g_2 since both conditions are equivalent to ϕ having finite representation series and this notion is independent of the particular metric chosen.

Cayley multiplication defines a Riemannian submersion $\pi : S^7 \times S^7 \to S^7$. The group of isometries commuting with this action does not, however, act transitively on $S^7 \times S^7$ and Theorem 1.1 is not applicable. Our research continues in this area as this example has important physical applications (see, for example, [6]).

Acknowledgments

Research of C. Dunn partially supported by a CSUSB faculty research grant. Research of P. Gilkey partially supported by the Max Planck Institute in the Mathematical Sciences (Leipzig, Germany). Research of both C. Dunn and P. Gilkey partially supported by the University of Santiago (Spain). Research of J.H. Park partially supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) KRF-2005-204-C00007.

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