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Abstract: We define the "volume" contained by pointed k-surfaces, first studied by the author in [9], and we show that this volume is always finite. Likewise, we show that the surface area of a pointed k-surface is always finite.

Key Words: immersed hypersurfaces, Plateau problem, Gaussian curvature, hyperbolic space, moduli spaces, Teichmüller theory.

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1 - Introduction.

Immersed hypersurfaces of constant Gaussian curvature are very classical objects of study which in recent years (in geometric terms) have found various fruitful applications to the study of negatively curved manifolds. In [6], Rosenberg and Spruck constructed a large class of examples by solving the boundary value problem. Labourie then showed in [4] that when the ambient manifold is three dimensional, constant Gaussian curvature surfaces may be studied in terms of pseudo-holomorphic curves in a contact manifold. The powerful techniques of this latter theory then allow the construction [5] of a much more general family of such surfaces, which are well adapted to a number of useful applications, such as the construction [3] of a canonical foliation of the non-compact ends of certain hyperbolic manifolds; the realisation of homomorphisms of compact Fuchsian groups into Kleinian groups as constant Gauss curvature immersions (ch.4 of [9]); and the canonical association [8] of a complete immersed surface in $\mathbb{H}^3$ to each ramified covering of the Riemann sphere, which is the case that we study in this paper.

Let $\mathbb{H}^3$ be three dimensional hyperbolic space. We identify the ideal boundary of $\mathbb{H}^3$ with the Riemann sphere $\hat{\mathbb{C}}$. Let Σ be a compact Riemann surface. Let $\mathcal{P}$ be a finite subset of Σ and denote $\Sigma' = \Sigma \setminus \mathcal{P}$. Let $\varphi : \Sigma \to \hat{\mathbb{C}}$ be a ramified covering with critical points contained in $\mathcal{P}$. The pair (Σ', φ) defines a Plateau problem in the sense of [5]. Since Σ' is of hyperbolic type, by [7], for all $k \in]0,1[$, there exists a unique immersion $i_k : \Sigma' \to \mathbb{H}^3$ of constant Gaussian curvature equal to k which is a solution to this Plateau problem (see section 5). In [8] we completely described the geometry of the immersed surface (Σ', i_k), showing that it is complete and asymptotically tubular of finite order (in the sense of [8]) near the critical points (see section 5). Heuristically, the immersed surface has only a finite number of point singularites, all of which wrap a finite number of times in a cusp shaped manner about a geodesic.

The aim of this paper is to study the area of and the "volume" contained by the immersed surface (Σ', i_k). The area is a relatively trivial matter, and we obtain:

Theorem 1.1

The area of (Σ', i_k) *is finite.*

We volume is more subtle, since (Σ', i_k) is not embedded, and therefore does not have a well defined interior. Nonetheless, we may define $\mathrm{Vol}(\Sigma', i_k)$ by integrating primitives of the volume form over (Σ', i_k). This still poses difficulties, since the primitive of the volume form is not necessarily L^1 over (Σ', i_k). However, we show that $\mathrm{Vol}(\Sigma', i_k)$ may be obtained as the limit of finite integrals. Indeed, using notation from section 5, for all $p \in \mathcal{P}$, let γ_p be a central geodesic for (Σ', i_k) at p. Let (α_p, Ω_p) be an asymptotically tubular chart for (Σ', i_k) about γ_p and let f_p be the graph function of (Σ', i_k) over this chart. For all $p \in \mathcal{P}$ and for all $t > 0$, we define $c_{t,p}(s) = f_p(t, s)$. For all $t > 0$, we define Σ'_t by:

$$\Sigma'_t = \Sigma' \setminus \bigcup_{p \in \mathcal{P}} \alpha_p(S^1 \times]t, +\infty[).$$

Let β be any primitive of the volume form of $\mathbb{H}^3$. Let $\Psi_{\gamma_p} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{H}^3$ be the parameterisation given by polar coordinates about the geodesic γ_p, is in section 2. We define $\mathrm{Vol}(\Sigma', i_k; t)$ by:

$$\mathrm{Vol}(\Sigma', i_k; t) = \int_{\Sigma'_t} i_k^* \beta + \sum_{p \in \mathcal{P}} \int_{\{t\} \times \mathbb{R}^2} \mathrm{Wind}(c_{t,p}, x) \Psi_{\gamma_p}^* \beta.$$

We obtain the following result:

Theorem 1.2

$\mathrm{Vol}(\Sigma', i_k; t)$ *converges to a finite limit as* t *tends to* $+\infty$*. Moreover, this limit is independant of the choices of* $(\gamma_p)_{p\in\mathcal{P}}$*,* $(\alpha_p, \Omega_p)_{p\in\mathcal{P}}$ *or* α*.*

This now allows us to define the volume of (Σ', i_k) as follows:

$$\mathrm{Vol}(\Sigma', i_k) = \mathop{\mathrm{Lim}}_{t\to+\infty} \mathrm{Vol}(\Sigma', i_k; t).$$

The key step in proving both these results lies in showing that the cusp ends of (Σ', i_k) taper off exponentially fast. Following the same philosophy as in [8], this is acheived by proving the result for the case $(D \setminus \{p\}, i)$ of a complete immersed disc with a unique singularity in its interior, and then showing that the general result may be deduced from this case.

These results allow us to define two interesting functions over the Teichmüller space of ramified coverings over the sphere, and provoke the following natural questions:

(i) How do the volume and the area vary as a function of φ?

(ii) What is the asymptotic behaviour of the volume and the area as k tends to 0 or 1?

(iii) How do the higher coefficents of the asymptotic series of the volume and the area vary as a function of φ?

(iv) How may these new functions be related to other known functions over the Teichmüller space?

This paper is structured as follows: in sections 2 and 3 we calculate polar coordinates about a geodesic in $\mathbb{H}^3$ in order to determine the Gaussian curvature of an arbitrary surface of revolution about that geodesic. In section 4, we use the resulting ODE to determine the asymptotic behaviour of a constant Gaussian curvature surface of revolution about a geodesic of some function f. Although we only need to know the decay rates of f and f' in order to control the volume and the area respectively, for no extra effort, we are also able to determine the decay rates of every derivative of f. In section 5, we recall the notion of immersed surfaces being tubular near a critical point, as defined in [8], and we adapt this notion to the current context. Finally, using elementary properties of convex curves in $\mathbb{R}^2$ obtained in section 6, we prove in section 7 the finiteness of the volume integral over each cusp, which allows us in section 8 to rapidly deduce Theorems 1.1 and 1.2.

I am grateful to François Labourie for introducing me to the study of Plateau problems and to Jean-Marc Schlenker for encouraging me to address this aspect of their geometry.

2 - Polar Coordinates About a Geodesic.

Let $\mathbb{H}^3$ be three dimensional hyperbolic space and let $g = g_{ij}$ be the hyperbolic metric. We begin by calculating g in terms of polar coordinates about a geodisic. We identify $\mathbb{H}^3$ with the three dimensional upper half space:

$$\begin{aligned} \mathbb{H}^3 &= \left\{(x, y, t) \in \mathbb{R}^3 \text{ s.t. } t > 0\right\}, \\ g_{ij} &= t^{-2}\delta_{ij}. \end{aligned}$$

Let $\gamma : \mathbb{R} \to \mathbb{H}^3$ be a geodesic. By applying an isometry of $\mathbb{H}^3$, we may assume that γ is the unique geodesic going from 0 to ∞ and that $\gamma(0) = (0, 0, 1)$. Let $N\gamma$ be the normal bundle over γ. We identify $N\gamma_0$, the fibre over 0, isometrically with $\mathbb{R}^2$. Using parallel transport, we obtain a bundle isometry $\tau_\gamma : \mathbb{R} \times \mathbb{R}^2 \to N\gamma$. Let $\mathrm{Exp} : T\mathbb{H}^3 \to \mathbb{H}^3$ be the exponential map. We now define Φ_γ by:

$$\Phi_\gamma = \mathrm{Exp} \circ \tau_\gamma.$$

This mapping is unique up to translation of the $\mathbb{R}$ coordinate and rotation of the $\mathbb{R}^2$ coordinate. Using polar coordinates of the $\mathbb{R}^2$ component, Φ_γ is given explicitely by:

$$\Phi_\gamma(t, r, \theta) = (e^t \tanh(r) \cos(\theta), e^t \tanh(r) \sin(\theta), e^t \cosh(r)^{-1}).$$

We have the following result:

Lemma 2.1

With respect to the basis $(\partial_t, \partial_r, \partial_\theta)$, the metric $\Phi_\gamma^* g$ is given by:

$$\Phi_\gamma^* g = \begin{pmatrix} \cosh^2(r) & & \\ & 1 & \\ & & \sinh^2(r) \end{pmatrix}.$$

Proof: The vectors ∂_t point along lines defined by r and θ being constant. These are straight lines in $\mathbb{R}^3$ leaving the origin. Likewise, the vectors ∂_r point along vertical circles in $\mathbb{R}^3$ having the origin as their centre. Finally, the vectors ∂_θ point along horizontal circles in $\mathbb{R}^3$ having their origin on the vertical line $t \mapsto (0, 0, t)$ which passes through the origin. These vectors are pairwise orthogonal in $\mathbb{R}^3$. Since the hyperbolic metric of $\mathbb{H}^3$ is conformally equivalent to the hyperbolic metric of $\mathbb{R}^3$, it follows that these vectors are also orthogonal in $\mathbb{H}^3$. It now remains to calculate the lengths of these vectors with respect to the hyperbolic metric.

We calculate these vectors over the point $(e^t \tanh(r) \cos(\theta), e^t \tanh(r) \sin(\theta), e^t \cosh(r)^{-1})$. Firstly:

$$\partial_t = (e^t \tanh(r) \cos(\theta), e^t \tanh(r) \sin(\theta), e^t \cosh(r)^{-1}).$$

Thus:

$$\begin{aligned} \|\partial_t\|^2 &= e^{-2t} \cosh^2(r)(e^{2t} \tanh^2(r) \cos^2(\theta) + e^{2t} \tanh^2(r) \sin^2(\theta) + e^{2t} \cosh^{-2}(r)) \\ &= \cosh^2(r). \end{aligned}$$

The result now follows by an analogous calculation for ∂_r and ∂_θ. $\square$

We define the mapping $\Psi_\gamma : \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi] \rightarrow \mathbb{H}^3$ by:

$$\Psi_\gamma(t, R, \theta) = \Phi(t, \operatorname{arcsinh}(R), \theta).$$

This mapping also yields a form of polar coordinates for $\mathbb{H}^3$ about a geodesic. However, the corresponding metric has a simpler formula, as the following lemma shows:

Lemma 2.2

With respect to the basis $(\partial_t, \partial_R, \partial_\theta)$, the metric $\Psi_\gamma^* g$ is given by:

$$\Phi_\gamma^* g = \begin{pmatrix} (1+R^2) & & \\ & (1+R^2)^{-1} & \\ & & R^2 \end{pmatrix}.$$

Proof: Since ∂_R is merely a rescaling of ∂_r, the pairwise orthogonality of the three coordinate vectors is preserved by this reparametrisation. Moreover, the lengths of the vectors ∂_t and ∂_θ also remain unchanged. Finally, since $r = \sinh(R)$, we have:

$$\begin{array}{rll} & \partial_r & = \cosh(R)\partial_R \\ \Rightarrow & \|\partial_R\|^2 & = \cosh(R)^{-2}\|\partial_r\|^2. \end{array}$$

Using the classical relation $\cosh^2(x) - \sinh^2(x) = 1$, we obtain the desired result. $\square$

We now calculate the action on this basis of the Levi-Civita covariant derivative of $\Psi_\gamma^* g$. We obtain the following result:

Lemma 2.3

The Levi-Civita covariant derivative of $\Psi_\gamma^* g$ is determined by the following relations:

$$\begin{array}{ll} \nabla_{\partial_t}\partial_t & = -R(1+R^2)\partial_R, \\ \nabla_{\partial_t}\partial_R & = R(1+R^2)^{-1}\partial_t, \\ \nabla_{\partial_t}\partial_\theta & = 0, \\ \nabla_{\partial_R}\partial_R & = -R(1+R^2)^{-1}\partial_R, \\ \nabla_{\partial_R}\partial_\theta & = R^{-1}\partial_\theta, \\ \nabla_{\partial_\theta}\partial_\theta & = -R(1+R^2)\partial_R. \end{array}$$

Proof: This follows directly from the preceeding lemma and the Kozhul formula. $\square$

3 - Surfaces of Revolution.

Let I be an interval in $\mathbb{R}$. Let $f : I \rightarrow]0,\infty[$ be a positive valued smooth function. We define $\Sigma_{f,\gamma} \subseteq \mathbb{H}^3$ by:

$$\Sigma_{f,\gamma} = \{\Psi_\gamma(t, f(t), \theta) \text{ s.t. } t \in I, \theta \in [0, 2\pi]\}\,.$$

$\Sigma_{f,\gamma}$ is a surface of revolution in $\mathbb{H}^3$ about the geodesic γ. We aim to obtain differential conditions on f for the surface Σ_f to have constant Gaussian curvature. Let $\kappa(t)$ be the Gaussian curvature of the surface Σ_f at the point $\Psi(t, f(t), 0)$. We have the following result:

Lemma 3.1

The Gaussian curvature, κ satisfies:

$$\kappa f((1+f^2)+(f')^2(1+f^2)^{-1})^{3/2} = (1+f^2)(-f''(1+f^2)+f(1+f^2)^2+3f(f')^2).$$

Proof: We work now in the coordinates of $\mathbb{R}\times]0,\infty[\times[0,2\pi]$. We define the function $\hat{f} : I\times[0,2\pi] \rightarrow \mathbb{R}\times]0,\infty[\times[0,2\pi]$ by:

$$\hat{f}(t,\theta) = (t, f(t), \theta).$$

We define the vector fields $\hat{\partial}_t = D\hat{f}\cdot\partial_t$ and $\hat{\partial}_\theta = D\hat{f}\cdot\partial_\theta$. These vector fields span the tangent space of Σ_f. We have:

$$\begin{aligned} \hat{\partial}_t(t,\theta) &= (1, f'(t), 0), \\ \hat{\partial}_\theta(t,\theta) &= (0,0,1). \end{aligned}$$

We now define the vector field $\hat{\mathsf{N}}$ by:

$$\hat{\mathsf{N}}(t,\theta) = (-f', (1+f^2)^2, 0).$$

This vector field spans the normal bundle to Σ_f. Moreover:

$$\|\hat{\mathsf{N}}\|^2 = (f')^2(1+f^2) + (1+f^2)^3.$$

By taking the covariant derivative of this vector field with respect to $\hat{\partial}_t$ and $\hat{\partial}_\theta$, we obtain the second fundamental form of Σ_f. Let D be the canonical flat connexion of $\mathbb{R}\times]0,\infty[\times[0,2\pi]$. We have:

$$\begin{aligned} D_{\hat{\partial}_t}\hat{\mathsf{N}} &= \partial_t\hat{\mathsf{N}} &&= (-f'', 4(1+f^2)ff', 0), \\ D_{\hat{\partial}_\theta}\hat{\mathsf{N}} &= \partial_\theta\hat{\mathsf{N}} &&= (0,0,0). \end{aligned}$$

By Lemma 2.2, we have:

$$\begin{aligned} \langle D_{\hat{\partial}_t}\hat{\mathsf{N}}, \hat{\partial}_t\rangle &= -f''(1+f^2)+4f(f')^2, \qquad & \langle D_{\hat{\partial}_t}\hat{\mathsf{N}}, \hat{\partial}_\theta\rangle &= 0, \\ \langle D_{\hat{\partial}_\theta}\hat{\mathsf{N}}, \hat{\partial}_t\rangle &= 0, & \langle D_{\hat{\partial}_\theta}\hat{\mathsf{N}}, \hat{\partial}_\theta\rangle &= 0. \end{aligned}$$

Let Ω be the connexion one form of ∇ with respect to D, so that, for any vector fields X and Y:

$$\nabla_X Y = D_X Y + \Omega(X, Y).$$

Using Lemmata 2.2 and 2.3. We obtain:

$$\begin{aligned} \langle\Omega(\hat{\partial}_t, \hat{\mathsf{N}}), \hat{\partial}_t\rangle &= f(1+f^2)^2 - f(f')^2, & \langle\Omega(\hat{\partial}_t, \hat{\mathsf{N}}), \hat{\partial}_\theta\rangle &= 0, \\ \langle\Omega(\hat{\partial}_\theta, \hat{\mathsf{N}}), \hat{\partial}_t\rangle &= 0, & \langle\Omega(\hat{\partial}_\theta, \hat{\mathsf{N}}), \hat{\partial}_\theta\rangle &= f(1+f^2)^2. \end{aligned}$$

Let N be the unit normal vector field to Σ_f:

$$\mathsf{N} = \|\hat{\mathsf{N}}\|^{-1}\hat{\mathsf{N}}.$$

Let II be the second fundamental form of Σ_f. That is, if X and Y are vector fields tangent to Σ_f:

$$II(X, Y) = \langle\nabla_X \mathsf{N}, Y\rangle.$$

If X and Y are both vector fields tangent to Σ_f, then $\langle\nabla_X \mathsf{N}, \mathsf{Y}\rangle = \|\hat{\mathsf{N}}\|^{-1}\langle\nabla_{\mathsf{X}}\hat{\mathsf{N}}, \mathsf{Y}\rangle$. We may thus calculate II:

$$\begin{aligned} II(\hat{\partial}_t, \hat{\partial}_t) &= ((f')^2(1+f^2) + (1+f^2)^3)^{-1/2} \times \\ &\qquad (-f''(1+f^2) + f(1+f^2)^2 + 3f(f')^2), \\ II(\hat{\partial}_t, \hat{\partial}_\theta) &= 0, \\ II(\hat{\partial}_\theta, \hat{\partial}_t) &= 0, \\ II(\hat{\partial}_\theta, \hat{\partial}_\theta) &= ((f')^2(1+f^2) + (1+f^2)^3)^{-1/2} f(1+(f')^2)^2. \end{aligned}$$

Observing that $\hat{\partial}_t$ and $\hat{\partial}_\theta$ are orthogonal to one another, we obtain:

$$\begin{aligned} \mathrm{Det}(\hat{\partial}_t, \hat{\partial}_\theta) &= \|\hat{\partial}_t\|^2\|\hat{\partial}_\theta\|^2 \\ &= f^2(1+f^2)((1+f^2) + f'(1+f^2)^{-2}). \end{aligned}$$

If we denote by A the matrix of II with respect to the basis $(\hat{\partial}_t, \hat{\partial}_\theta)$, then the Gaussian curvature, κ, satisfies:

$$\kappa = \mathrm{Det}(A)/\mathrm{Det}(\hat{\partial}_t, \hat{\partial}_\theta).$$

Thus:

$$\begin{aligned} &\kappa f^2(1+f^2)((1+f^2) + (f')^2(1+f^2)^{-1})^{3/2} \\ &\qquad = f(1+f^2)^2(-f''(1+f^2) + f(1+f^2)^2 + 3f(f')^2). \end{aligned}$$

The result now follows. $\square$

4 - Surfaces Of Constant Curvature.

We now study the asymptotique behaviour of solutions to the differential differential equation given by Lemma 3.1. We have the following result:

Lemma 4.1

Let k be a real number in $]0,1[$. Let $f : [0,\infty[\rightarrow]0,\infty[$ be such that the surface of revolution $\Sigma_{f,\gamma}$ is of constant Gaussian curvature equal to k. Suppose, moreover, that $f(t)$ and $f'(t)$ both tend to zero as t tends to $+\infty$. Then, for all $\delta > 0$, there exists $T > 0$ and constants $B > A > 0$ such that for all $t > T$:

$$Ae^{-(\lambda+\delta)t} \leqslant f(t), -f'(t), f''(t) \leqslant Be^{-(\lambda-\delta)t},$$

where $\lambda^2 = 1 - k$.

Proof: By Lemma 3.1, f, satisfies the following differential equation:

$$f''/f = (1-k) + \epsilon,$$

where $\epsilon : [0,\infty[\rightarrow]0,\infty[$ is a smooth function such that $\epsilon(t)$ tends to zero as t tends to infinity. We define the function $g(t) = \mathrm{Log}(f(t))$. Thus:

$$g' = f'/f, \qquad g'' = (ff'' - (f')^2)/f^2 = f''/f - (g')^2.$$

The function g therefore satisfies the following differential relation:

$$g'' + (g')^2 = (1-k) + \epsilon.$$

We define $h(t) = g'(t)$. We then obtain:

$$h' + h^2 = \lambda^2 + \epsilon.$$

Let $\delta > 0$ be such that $\delta < \lambda$. Let $T_0 > 0$ be such that for $t > T_0$:

$$\epsilon(t) < \delta^2.$$

Let $t > T_0$ be arbitrary and suppose that $h(t) \geqslant \lambda + \delta$. Then:

$$\left|\lambda^2 - h(t)^2\right| \geqslant \left|\delta^2 + 2\delta\lambda\right| \geqslant \delta\lambda.$$

Thus:

$$\left|\epsilon/(\lambda^2 - h(t)^2)\right| \leqslant \delta/\lambda.$$

Combining this with the differential equation for h, we obtain:

$$\left|h'(t)(\lambda^2 - h(t)^2)^{-1} - 1\right| \leqslant \delta/\lambda.$$

Consequently, if we define $\eta(t) = \lambda^{-1}\mathrm{arccotanh}(h(t)/\lambda)$, we obtain:

$$|\eta'(t) - 1| \leqslant \delta/\lambda.$$

It follows that, for any $t_1 > t_0 > T_0$, if $h(t) \geqslant \lambda + \delta$ for all t in the interval $[t_0, t_1]$, then:

$$\eta(t_1) \leqslant \eta(t_0) + (t_1 - t_0)(1 + \delta/\lambda).$$

Thus, under the same conditions:

$$h(t_1) \leqslant \lambda\mathrm{coth}(\lambda\eta(t_0) + (\lambda+\delta)(t_1 - t_0)).$$

This tells us that if $t > T_0$ and $h(t)$ is ever greater than $\lambda + \delta$, then, in finite time, it will fall below $\lambda + \delta$. Moreover, for all $t > T_0$, $h'(t) < 0$ whenever $h(t) = \lambda + \delta$. It thus follows that the function will never thereafter be greater than $\lambda + \delta$. Heuristically, we have shown that the function behaves like the positive branch of the hyperbolic cotangent. We thus conclude that, for any solution, h, there exists $T_1 > 0$ such that, for $t > T_1$:

$$h(t) \leqslant \lambda + \delta.$$

A similar analysis for the interval $]-\infty, -\lambda-\delta[$ reveals that there exists $T_2 > 0$ which depends only on the function ϵ such that if there exists $t > T_2$ with $h(t) < -\lambda - \delta$, then $h(t)$ tends to $-\infty$ in finite time (since, heuristically, it behaves like the negative branch of the hyperbolic cotangent). However, f exists and is positive for all time. Thus g and $g' = h$ both exist for all time, and this is not possible. It thus follows that, for $t > T_2$:

$$h(t) \geqslant -\lambda - \delta.$$

Finally, by considering the interval $]-\lambda+\delta, \lambda-\delta[$, there exists $T_3 > 0$ which depends only on ϵ and a constant $\Delta T_3 > 0$ which depends only on δ such that if there exists $t > T_3$ with $h(t) > -\lambda + \delta$, then, for all $t' > t + \Delta T_3$, $h(t') > \lambda - \delta$. This happens heuristically because the solution in this case behaves like the hyperbolic tangent.

We have thus shown that there exists $T_4 > 0$ such that, for all $t > T_4$, either $|h(t) - \lambda| < \delta$ or $|h(t) + \lambda| < \delta$.

We now exclude the case where $|h(t) - \lambda| < \delta$ when $t > T_4$. Indeed, suppose that $\delta < \lambda/2$. In this case, since $h(t) = g'(t)$, it follows that, for large values of t, the function $g(t)$ grows faster than $\lambda t/2$. Since f is the exponential of g it then follows that f tends to infinity as t tends to infinity, and this contradicts the hypotheses on f.

It thus follows that, for all $t > T_4$, $|h(t) + \lambda| < \delta$. Consequently, there exists a constant C_1 such that, for all $t > T_4$:

$$C_1 - (\lambda + \delta)t \leqslant g(t) \leqslant C_1 - (\lambda - \delta)t.$$

Taking the exponential of each of these functions, we see that there exists a constant C_2 such that, for $t > T_4$:

$$C_2 e^{-(\lambda+\delta)t} \leqslant f(t) \leqslant C_2 e^{-(\lambda+\delta)t}.$$

Since $f'' = f((1-k)+\epsilon)$, there exist constants $C_3, C_4 > 0$ and $T_5 \geqslant T_4$ such that for $t > T_5$:

$$C_3 e^{-(\lambda+\delta)t} \leqslant f''(t) \leqslant C_4 e^{-(\lambda+\delta)t}.$$

Finally, since $f'(t)$ tends to 0 as t tends to $+\infty$, we obtain the relation for $f'(t)$ by integrating $f''(t)$ back from $+\infty$. The result now follows. $\square$

We may also estimate the higher derivatives of f:

Corollary 4.2

With the hypothesis and notation of the previous lemma, for all $k \geqslant 2$, there exists $B_k > A_k > 0$ such that for $t \geqslant T$:

$$A_k e^{-(\lambda+\delta)t} \leqslant (-1)^k f^{(k)}(t) \leqslant B_k e^{-(\lambda-\delta)}t.$$

Proof: By induction, for all $k \geqslant 0$:

$$f^{(k+2)} = f^{(k)}(1-k) + \Sigma_{i=0}^{k}\epsilon_i f^{(i)},$$

where, for all i, $\epsilon_i(t)$ tends to zero as t tends to $+\infty$. The result now follows by induction. $\square$

This allows us to control the area of $\Sigma_{f,\gamma}$ and the volume that it contains:

Corollary 4.3

Let Area(t) *and* Vol(t) *be respectively that area of and the volume inside the restriction of* Σ_f *to* $[t,+\infty[$. *Then, for all* $\delta > 0$, *there exists* $T > 0$ *and* $B > A > 0$ *such that, for all* $t > T$*:*

$$\begin{array}{ll} Ae^{-(\lambda+\delta)t} & \leqslant \mathrm{Area}(t) \leqslant Be^{-(\lambda+\delta)t}, \\ Ae^{-2(\lambda+\delta)t} & \leqslant \mathrm{Vol}(t) \leqslant Be^{-2(\lambda+\delta)t}, \end{array}$$

where $\lambda^2 = 1-k$.

Proof: This follows directly by calculating the area and volume integrals, bearing in mind that $\Psi_\gamma^* g$ is uniformly equivalent to the Euclidian metric in an ϵ-neighbourhood of γ. $\square$

5 - Asymptotically Tubular Immersed Surfaces.

Let Σ be a compact Riemann surface and let $\mathcal{P}$ be a finite subset of Σ. Define $\Sigma' = \Sigma \setminus \mathcal{P}$. Let $\varphi : \Sigma \rightarrow \hat{\mathbb{C}}$ be a ramified covering such that the ramification points are contained in $\mathcal{P}$. The pair (Σ', φ) defines a Plateau problem in the sense of Labourie, [5].

Let $i : \Sigma \rightarrow \mathbb{H}^3$ be a convex immersion. Let $\mathsf{N}_i : \Sigma \rightarrow U\mathbb{H}^3$ be the exterior unit normal over i. We call this the Gauss lifting of i and in the sequel we denote it by $\hat{\imath}$. Let $\overrightarrow{n} : U\mathbb{H}^3 \rightarrow \partial_\infty \mathbb{H}^3 = \hat{\mathbb{C}}$ be the Gauss-Minkowski mapping. Thus, if $\gamma : \mathbb{R} \rightarrow \mathbb{H}^3$ is a unit speed geodesic in $\mathbb{H}^3$, then:

$$\overrightarrow{n}(\partial_t\gamma) = \gamma(+\infty).$$

Since i is convex, elementary hyperbolic geometry (see for example, [1]) allows us to show that $\overrightarrow{n} \circ \hat{\imath}$ is a local homeomorphism.

For $k \in]0,1[$, following [5], the pair (Σ', i) is said to be a solution of the Plateau problem (Σ', φ) with Gaussian curvature equal to k if and only if:

(i) the mapping i is a convex immersion with Gaussian curvature equal to k,

(ii) $(\Sigma', \hat{\imath})$ is complete in the sense of immersed surfaces, and

(iii) $\varphi = \overrightarrow{n} \circ \hat{\imath}$.

Since the surface Σ' is of hyperbolic conformal type, by [7], for all $k \in]0,1[$, there exists a unique solution (Σ', i_k) of the Plateau problem (Σ', φ) with Gaussian curvature equal to k.

Let p be an arbitrary point in $\mathcal{P}$. Let n be the order of ramification of φ at p. In [8], we defined the notion of a surface being asymptotically tubular of finite order near a point singularity, and we showed that the immersed surface $(\Sigma', \hat{i}_k)$ is asymptotically tubular of order n near p. This implies that there exists:

(i) a geodesic γ such that $\gamma(+\infty) = \varphi(p)$,

(ii) a smooth function $f : S^1 \times]0, +\infty[\to \mathbb{R}^2$,

(iii) a neighbourhood Ω of p in Σ containing no other point of $\mathcal{P}$, and

(iv) a diffeomorphism $\alpha : S^1 \times]0, +\infty[\to \Omega \setminus \{p\}$,

such that:

(i) $\Psi_\gamma(t, f(s,t)) = (i \circ \alpha)(s,t)$,

(ii) $\alpha(s,t)$ tends towards p as t tends to $+\infty$, and

(iii) $f(\cdot, t+\cdot)$ converges to 0 in the C^∞_{loc} topology as t tends to $+\infty$.

Moreover:

(iv) for all t, $f(\cdot,t)$ has index n in a sense that will be made clear shortly.

In the sequel, we refer to γ as a central geodesic for (Σ, i_k) at p, we refer to (α, Ω) as an asymtotically tubular chart for (Σ, i_k) about γ at p, and we refer to f as the graph function of (Σ, i_k) over this chart.

We have the following result:

Lemma 5.1

Let γ be a central geodesic for (Σ, i_k) at p and let (α, Ω) be an asymptotically tubular chart for (Σ, i_k) about γ at p. For all $t \geqslant 0$, $(\Omega \setminus \{p\}, i_k)$ is transverse to $\Psi_\gamma(\{t\} \times \mathbb{R}^2)$.

Proof: Let $p : \mathbb{H}^3 \to \gamma$ be the orthogonal projection. Let f be the graph function of (Σ, i_k) over (α, Ω). Then:

$$\begin{aligned} & (i_k \circ \alpha)(s,t) && = \Psi_\gamma(t, f(s,t)) \\ \Rightarrow\quad & (p \circ i_k \circ \alpha)(s,t) && = \gamma(t). \end{aligned}$$

The orthogonal projection onto the geodesic is thus surjective, and the result now follows. $\square$

For $t > 0$, we define the mapping $c_t = f(\cdot, t)$, and we obtain the following corollary:

Corollary 5.2

For $t > 0$, the mapping c_t is a smooth immersed curve. Moreover, for all t, c_t is convex with respect to the hyperbolic metric $\Psi_\gamma^* g$ over $\{t\} \times \mathbb{R}^2$.

Proof: By transversality, c_t is immersed. Since (Σ', i_k) is convex and $\{t\} \times \mathbb{R}^2$ is totally geodesic, c_t is also convex. The result now follows. $\square$

For all t, we define the exterior unit normal N_t of c_t. We then orient c_t such that N_t lies to its right hand side. By composing N_t with the Gauss-Minkowski mapping, we obtain a continuous mapping from S^1 into $\partial_\infty \Psi_\gamma(\{t\} \times \mathbb{R}^2)$, which itself is homeomorphic to S^1

(the orientation of $\partial_\infty\Psi_\gamma(\{t\}\times\mathbb{R}^2)$ may be explicitely specified although it is not very important). We thus define $\mathrm{Ind}(c_t)$, the index of c_t, by:

$$\mathrm{Ind}(c_t) = \mathrm{Ind}(\overrightarrow{n}\circ\mathsf{N}_t).$$

Condition (iv) may now be made explicit:

(iv) for all t, $\mathrm{Ind}(c_t) = n$.

6 - Convex Curves in Real and Hyperbolic Space.

We now require the following elementary results concerning the geometry of convex curves:

Lemma 6.1

Let M be either $\mathbb{R}^2$ or $\mathbb{H}^2$, so that $\partial_\infty M$ is homeomorphic to S^1. Let UM be the unitary bundle of M and let $\overrightarrow{n} : UM \to \partial_\infty M$ be the Gauss-Minkowski mapping.

Let $c : S^1 \to M$ be a smooth, closed, convex curve. Let $p \in M$ be any point in the complement of the image of c. If $\mathrm{Wind}(c,p)$ be the winding number of c about p, then:

$$0 \leqslant \mathrm{Wind}(c,p) \leqslant \mathrm{Ind}(c).$$

Proof: Let N be the exterior unit normal to c. We assume that c is oriented so that N lies to its right hand side. By deforming c by a small amount, we obtain a curve c' arbitrarily close to c in the C^∞ topology which is convex and intersects itself transversally. In particular, if c' is sufficiently close to c, then:

$$\mathrm{Ind}(c) = \mathrm{Ind}(c'), \qquad \mathrm{Wind}(c,p) = \mathrm{Wind}(c',p).$$

We thus assume that c intersects itself transversally. In particular, c only intersects itself at a finite number of points. We may therefore decompose c into a finite collection $c_1, ..., c_n$ of piecewise smooth, simple, closed curves which are convex except possibly at the apexes, where different curves join to each other. The number of apexes of c_i equals the number of distinct components of c comprising c_i. For each i let N_i be the restriction of N to c_i. The c_i may be labelled by the vertices of a tree, according to how they join to each other. The leaves are then precisely the curves with only one apex. By induction from the leaves downwards, we may show that each N_i only points into one of the connected components of the complement of c_i (i.e. it does not change sign at the apexes).

For each i, let Ω_i^0 and Ω_i^∞ be respectively the bounded and unbounded components of the complement of c_i in M. Let $\hat{\Omega}_i$ be the convex hull of c_i in M. Let Γ be a supporting geodesic of $\hat{\Omega}_i$. Γ intersects c_i non trivially. By the convexity of c_i, we may assume that that it intersects c_i away from the apexes. At this point of intersection, N_i points into the complement of $\hat{\Omega}_i$. Consequently, N_i always points into Ω_i^∞.

It follows that, for all i, N_i points outwards from Ω_i^0. Consequently, for each i, $\mathrm{Ind}(c_i)$ brings a contribution of $+1$ to $\mathrm{Ind}(c)$, and $\mathrm{Wind}(c_i,p)$ brings a contribution of 0 or $+1$ to $\mathrm{Wind}(c,p)$. The result now follows. $\square$

We now recall the following generalisation of Stokes theorem:

Lemma 6.2

Let $c : S^1 \to \mathbb{R}^2$ be a smooth, closed curve. If α is a 1-form over $\mathbb{R}^2$, then:

$$\int_c \alpha = \int_{\mathbb{R}^2} \mathrm{Wind}(c,x) d\alpha(x).$$

Proof: By deforming c a small amount, we obtain a curve c' arbitrarily close to c in the C^∞ topology such that c' is convex and intersects itself transversally. By choosing c' sufficiently close to c, we may assume that $\mathrm{Wind}(c,\cdot) = \mathrm{Wind}(c',\cdot)$ except on a set of arbitrarily small measure. We may thus assume that c intersects itself transversally. As in the proof of Lemma 6.1, we may decompose c into a finite collection $c_1, ...c_n$ of simple closed curves. By Stokes' theorem, the result holds for each c_i, and the general result holds by additivity. $\square$

This also allows us to obtain the derivative of the winding number of a smoothly varying family of curves as a distribution over $\mathbb{R}^2$:

Lemma 6.3

Let $c_t : S^1 \times]-\epsilon, \epsilon[$ be a smoothly varying family of smooth curves in $\mathbb{R}^2$. If β is a 2-form in $\mathbb{R}^2$, then:

$$\partial_t \int_{\mathbb{R}^2} \mathrm{Wind}(c_t,x)\beta(x) = \int_{c_t} i_{\partial_t c_t}\beta.$$

Proof: Let $\mathcal{L}$ denote the Lie derivative. Let $\beta' : \mathbb{R}^2 \to \mathbb{R}$ be a compactly supported 2-form such that:

$$\int_{\mathbb{R}^2} \beta' = 0.$$

Let γ be a primitive of β'. By Lemma 6.2, for all t, we have:

$$\int_{c_t} \gamma = \int_{\mathbb{R}^2} \mathrm{Wind}(c_t,x)\beta(x).$$

We have:

$$\begin{aligned} \partial_t \textstyle\int_{\mathbb{R}^2} \mathrm{Wind}(c_t,x)\beta'(x) &= \partial_t \textstyle\int_{c_t} \gamma \\ &= \textstyle\int_{c_t} \mathcal{L}_{\partial_t c_t}\gamma \\ &= \textstyle\int_{c_t} (d i_{\partial_t c_t} + i_{\partial_t c_t} d)\gamma \\ &= \textstyle\int_{c_t} i_{\partial_t c_t}\beta'. \end{aligned}$$

By reducing ϵ if necessary, we may construct a 2-form, β_0 such that $\mathrm{Supp}(\beta_0)$ is disjoint from c_t for all t, $\beta - \beta_0$ has compact support, and:

$$\int_{\mathbb{R}^2} \beta - \beta_0 = 0.$$

Since the integral of $\mathrm{Wind}(c_t,x)\beta_0(x)$ is constant, we obtain:

$$\partial_t \int_{\mathbb{R}^2} \mathrm{Wind}(c_t,x)\beta(x) = \int_{c_t} i_{\partial_t c_t}\beta.$$

The result now follows. $\square$

7 - The Volume Contained by a Cusp.

Let γ be a central geodesic for (Σ, i_k) at p. Let (α, Ω) be an asymtotically tubular chart for (Σ, i_k) about γ at p, and let f be the graph function of (Σ, i_k) over this chart. We begin by controlling the image of (Σ, i_k):

Lemma 7.1

For all $\delta > 0$, there exists $T > 0$ and $A > 0$ such that, for all $t \geqslant T$:

$$\|f(s,t)\| \leqslant Ae^{-(\lambda-\delta)t},$$

where $\lambda^2 = 1 - k$.

Proof: By applying an isometry of $\mathbb{H}^3$, we may suppose that γ is the unique geodesic in $\mathbb{H}^3$ joining 0 to ∞. Let D be a disc in $\hat{\mathbb{C}}$ centred about the origin such that no other point in $\varphi(\mathcal{P})$ lies in D. We may assume that D has unit radius. Let $j : D \setminus \{0\} \to \mathbb{H}^3$ be the unique solution of the Plateau problem $(D \setminus \{q\}, \mathrm{Id})$ with constant Gaussian curvature equal to k. We observe that this mapping is an embedding.

We will show that the immersed surface (Σ', i_k) lies entirely within the interior of $(D \setminus \{0\}, j)$. Indeed, for $t \in]0, 1]$ we define $D_t \subseteq \mathbb{C}$ and $k_t \in]0, 1[$ by:

$$\begin{aligned} D_t &= \{z \in \mathbb{C} \text{ s.t. } (1-t)/2 < |z| < (1+t)/2\}, \\ k_t &= (1-t) + tk. \end{aligned}$$

For all t, let $j_t : D_t \to \mathbb{H}^3$ be the unique solution to the Plateau problem (D_t, Id) with constant Gaussian curvature equal to k_t. We see that $(D_t, j_t)_{t \in]0,1[}$ defines a foliation of the exterior of (D, j). There exists $\epsilon > 0$ such that for $t < \epsilon$:

$$(\Sigma, i) \cap (D_t, j_t) = \emptyset.$$

Let us define $t_0 \in]0, 1]$ by:

$$t_0 = \mathrm{Inf}\{t \in]0,1] \text{ s.t. } (\Sigma, i) \cap (D_t, j_t) \neq \emptyset.\}.$$

Suppose that $t_0 < 1$. By compactness, there is some point in the closure of (Σ', i) in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ which lies in the image of (D_{t_0}, j_{t_0}). Since (D_{t_0}, j_{t_0}) does not intersect $\varphi(\mathcal{P})$, it follows that (Σ', i) intersects the image of (D_{t_0}, j_{t_0}) at some finite point of $\mathbb{H}^3$. However, since $(D_t, j_t)_{t \in]0,t_0[}$ forms a foliation of the exterior of (D_{t_0}, j_{t_0}), (Σ', i) lies in the interior of (D_{t_0}, j_{t_0}). However, this is impossible by the geometric maximum principal (see, for example, [5]), since the Gaussian curvature of (D_{t_0}, j_{t_0}) is greater than that of (Σ', i). Thus $t_0 = 1$, and (Σ', i_k) lies in the interior of $(D \setminus \{0\}, j)$.

The result now follows by Lemma 4.1, since, by uniqueness, $(D \setminus \{0\}, j)$ is a surface of revolution about γ. $\square$

We now obtain estimates concerning the "volume" bounded by (Ω, i). Let α be any primative of the volume of $\mathbb{H}^3$. We define the function $V : [T, +\infty[\to \mathbb{R}$ by:

$$\begin{aligned} V(t) &= \int_{[T,t]\times S^1} (i\circ f)^*\alpha \\ &\quad + \int_{\{T\}\times\mathbb{R}^2} \mathrm{Wind}(c_T, x)\alpha(T, x) \\ &\quad - \int_{\{t\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)\alpha(t, x). \end{aligned}$$

First, we have:

Lemma 7.2

Let dVol *be the hyperbolic volume element of* $\mathbb{R}\times\mathbb{R}^2$. *The function* $V(t)$ *satisfies:*

$$V(t) = \int_{[T,t]\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)\mathrm{dVol}(t, x).$$

Proof: Let $\mathcal{L}$ denote the Lie derivative. Let ∂_t denote the derivative in the direction of the first coordinate in $\mathbb{R}\times\mathbb{R}^2$, and let $\partial_t c_t$ denote the infinitesimal variation of c_t. We recall that:

$$\mathcal{L}_{\partial_t}\alpha = di_{\partial_t}\alpha + i_{\partial_t}d\alpha.$$

Thus:

$$\begin{aligned} \int_{[T,t]}\int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)(i_{\partial_t}d\alpha)(x)ds &= \int_{[T,t]}\int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)(\mathcal{L}_{\partial_t}\alpha)(x)ds \\ &\quad - \int_{[T,t]}\int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)(di_{\partial_t}\alpha)(x)ds. \end{aligned}$$

Since $d\alpha = \mathrm{dVol}$:

$$\int_{[T,t]}\int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)(i_{\partial_t}d\alpha)(x)ds = \int_{[T,t]\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)\mathrm{dVol}(t, x).$$

Next, using Lemmata 6.2 and 6.3, and taking care with orientations:

$$\begin{aligned} \int_{[T,t]}\int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)(di_{\partial_t}\alpha)(x)ds &= \int_{[T,t]}\int_{\{s\}\times S^1} (i\circ f)^*(i_{\partial_t}\alpha)(\theta)ds \\ &= -\int_{[T,t]\times S^1} (i\circ f)^*\alpha \\ &\quad - \int_{[T,t]}\int_{\{s\}\times S^1} (i\circ f)^*(i_{\partial_t c_t}\alpha)(\theta)ds \\ &= -\int_{[T,t]\times S^1} (i\circ f)^*\alpha \\ &\quad - \int_{[T,t]} \partial_t \int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)\alpha(x)ds \\ &\quad + \int_{[T,t]}\int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)(\mathcal{L}_{\partial_t}\alpha)(x)ds. \end{aligned}$$

Combining these relations, we obtain:

$$\begin{aligned} \int_{[T,t]\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)\mathrm{dVol}(t, x) &= \int_{[T,t]\times S^1} (i\circ f)^*\alpha \\ &\quad + \int_{[T,t]} \partial_t \int_{\{s\}\times\mathbb{R}^2} \mathrm{Wind}(c_t, x)\alpha(x)ds. \end{aligned}$$

The result now follows by integrating the last integral. □

This allows us to prove the convergence of $V(t)$:

Lemma 7.3

The function $V(t)$ *converges to a finite limit as* t *tends to* $+\infty$.

Proof: By the convexity of (Σ', i_k), corollary 5.2 and Lemmata 6.1 and 7.2, the function $V(t)$ is positive and increasing. We recall that the hyperbolic metric $\Psi_\gamma^* g$ is uniformly equivalent to the Euclidean metric in an ϵ-neighbourhood of γ. It thus follows by Lemmata 7.2 and 7.1 that V is bounded from above. The result now follows. □

8 - Finiteness of Area and Volume.

We are now in a position to prove Theorem 1.2:

Proof of Theorem 1.2: The existence and finiteness of this limit follows from Lemma 7.3. If α' is another primitive of the volume form, then $d(\alpha' - \alpha) = 0$. Thus, since the homology of $\mathbb{H}^3$ vanishes in dimension higher than zero, the integral of $\alpha' - \alpha$ over any closed surface vanishes and this limit does not depend on the choice of α. For any $p \in \mathcal{P}$, two different asymptotically tubular charts about γ_p differ only by a rotation of the S^1 coordinate and a translation of the $\mathbb{R}$ coordinate. It thus follows by Lemma 7.3 that this limit does not depend on the asymtotically tubular chart chosen. An analogous reasoning shows that the integral does not depend on the choice of the central geodesics. The result now follows. $\square$

The finiteness of the area of (Σ', i_k) is significantly simpler to prove:

Proof of Theorem 1.1: Since there are only a finite number of cusps, it suffices to prove that the area of each cusp is finite. Let p be a point in $\mathcal{P}$. Let n be the order of ramification of the function φ at p. We define $q = \varphi(p)$. Let D be a disc in $\hat{\mathbb{C}}$ about q which contains no other point of $\mathcal{P}$. By applying an isometry of $\mathbb{H}^3$, we may assume that $q = 0$ and that D is the unit disc about the origin. Let $j_{k,n} : D \setminus \{q\} \to \mathbb{H}^3$ be the solution of the Plateau problem $(D \setminus \{q\}, z \mapsto z^n)$ with Gaussian curvature equal to k. By Lemma 7.2.1 of [5], $(D \setminus \{q\}, j_{k,n})$ is a graph over (Σ', i_k). In otherwords, there exists a neighbourhood Ω of q in Σ', a diffeomorphism $\alpha : \Omega \setminus \{p\} \to D \setminus \{q\}$ and a smooth function $f : \Omega \to [0, +\infty[$ such that, if $\hat{\imath}_k$ is the Gauss lifting of i_k, then, for all $x \in \Omega \setminus \{p\}$:

$$j_k \circ \alpha(x) = \mathrm{Exp}_x(f(x)\hat{\imath}_k(x)).$$

Using elementary hyperbolic geometry (see, for example, [1]), the mapping α is dilating with respect to the metrics induced by the immersions. However, if $j_{k,1} : D \setminus \{q\} \to \mathbb{H}^3$ is the solution of the Plateau problem $(D \setminus \{q\}, z \mapsto z)$, then, by uniqueness of solutions, $j_{k,n}$ factors through as an n-fold covering of $j_{k,1}$. Finally, by Corollary 4.3, the area of the cusp end of $(D \setminus \{q\}, j_{k,1})$ is finite. Thus, the area of the cusp of (Σ', i_k) about p is finite. The result now follows. $\square$

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