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A Brief Note on Special Lagrangian Submanifolds in $\mathbb{R}^{2n}$

by

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Abstract: We study the differential geometry of the Grassmannian of Lagrangian subspaces in $\mathbb{R}^{2n}$, exhibiting in particular a convex funcion whose sublevel set is the space of positive (i.e. spacelike or null) Lagrangian subspaces. We then show how a special Lagrangian immersion in $\mathbb{R}^{2n}$ induces a harmonic immersion in this Grassmannian which allowing us to obtain a rigidity result for positive special Lagrangian submanifolds which are null on a single point.

Key Words: special Lagrangian, Grassmannian, harmonic maps.

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1 - Introduction.

In this note, we briefly study the differential geometries of the Grassmannian of Lagrangian planes in $\mathbb{R}^{2n}$ and of special Lagrangian submanifolds of $\mathbb{R}^{2n}$.

Let $\langle\cdot,\cdot\rangle$ be the canonical metric and let ω be the canonical symplectic form over $\mathbb{R}^{2n}$. The form ω may be expressed with respect to $\langle\cdot,\cdot\rangle$ in terms of a $2n\times 2n$ antisymmetric matrix. We assume that:

$$\omega = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

Let E be an n dimensional subspace of $\mathbb{R}^{2n}$. We say that E is Lagrangian if and only if the restriction of ω to E vanishes. We define $\mathrm{GLag}(2n)$ to be the Grassmannian of Lagrangian subspaces of $(\mathbb{R}^{2n},\omega)$.

We define the symmetric, bilinear form m over $\mathbb{R}^{2n}$ as follows:

$$m = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

The form m defines a Minkowski metric over $\mathbb{R}^{2n}$ of type (n,n). If E is an n dimensional subspace of $\mathbb{R}^{2n}$, then we say that E is positive (resp. non negative) if and only if the restriction of m to E is positive definite (resp. non negative). We define $\mathrm{GLag}^{>0}(2n)$ (resp. $\mathrm{GLag}^{\geqslant 0}(2n)$) to be the open subset of $\mathrm{GLag}(2n)$ consisting of positive (resp. non negative) Lagrangian subspaces.

We define the canonical isomorphism $c:\mathbb{R}^{2n}\to\mathbb{C}^n$ by:

$$c(x,y) = x + iy.$$

This mapping permits us to identify these two spaces, and we supress c in the sequel. For $\theta\in\mathbb{R}$, we define the n-form Ω_θ over $\mathbb{C}^n$ by:

$$\begin{aligned}\Omega_\theta &= e^{i\theta}dz^1\wedge\ldots\wedge dz^n \\ &= e^{i\theta}(dx^1+idy^1)\wedge\ldots\wedge(dx^n+idy^n).\end{aligned}$$

In [1], Harvey and Lawson proved that, for all θ, the pair $(\mathrm{Im}(\Omega_\theta),\omega)$ defines a calibration of n-dimensional real subspaces of $\mathbb{R}^{2n}$. We shall say that an n-dimensional real subspace of $\mathbb{R}^{2n}$ is special Lagrangian if and only if the restrictions of $\mathrm{Im}(\Omega_\theta)$ and ω to this subspace vanish. Likewise, if $\Sigma=(S,i)$ is an n dimensional immersed submanifold in $\mathbb{R}^{2n}$ (thus, if S is a manifold and $i:S\to M$ is an immersion), then we say that Σ is special Lagrangian and positive if and only if every tangent space of Σ is. Moreover, we say that Σ is nul at a given point if the restriction of m to the tangent space at that point is degenerate.

In [6], Smoczyk showed amongst other things that $\mathrm{GLag}^{\geqslant 0}(2n)$ is a convex subset of $\mathrm{GLag}(2n)$. This result is then used by Jost and Xin in [2] to obtain a Bernstein result for complete positive special Lagrangian submanifolds of $\mathbb{R}^{2n}$. The author in turn uses these results to obtain in [4] a compactness result for complete special Lagrangian or Legendrian submanifolds of a given manifold which are positive in a sense analogous to that given above.

In this note, we present an explicit function satisfying certian convexity conditions of which $\mathrm{GLag}^{\geqslant 0}(2n)$ is a sublevel set. Indeed, we define the function $\Phi : \mathrm{GLag}(2n) \to \mathbb{R}$ by:

$$\Phi(E) = \operatorname*{Inf}_{v \in E\setminus\{0\}} \frac{m(v,v)}{\|v\|^2}.$$

Trivially:

$$\begin{aligned} \mathrm{GLag}^{>0}(2n) &= \Phi^{-1}(]0,\infty[), \\ \mathrm{GLag}^{\geqslant 0}(2n) &= \Phi^{-1}([0,\infty[). \end{aligned}$$

We have the following result:

Theorem 1.1

Φ is concave over $\Phi^{-1}([0,\infty[)$ and strictly concave over $\Phi^{-1}(]0,\infty[)$.

The convexity of $\mathrm{GLag}^{\geqslant 0}(2n)$ in $\mathrm{GLag}(2n)$ is an immediate corollary of this result. We then obtain the following result concerning positive special Lagrangian submanifolds in $\mathbb{R}^{2n}$:

Theorem 1.2

Let $\Sigma = (S, i)$ be an immersed submanifold in $\mathbb{R}^{2n}$. Suppose, moreover, that Σ is special Lagrangian and non-negative. If Σ is nul at a single point of its interior, then Σ coincides with an open subset of a nul affine plane.

This result is the classical counterpart of the result [3] of Labourie concerning the apparition of so called curtain surfaces as the degenerate limits of the liftings of constant Gaussian curvature hypersurfaces into the unitary bundle of a given 3 dimensional manifold. In [4] (and [5]), the author placed this result into the more general setting of positive special Legendrian submanifolds in the unitary bundle of an arbitrary manifold. Since the problem in this latter case is more general, we are obliged to use different techniques. The relative weakness of these techniques becomes clear when their performance in the simpler case studied by Labourie is compareed with his own results. This offers the tantalising possibility of strengthening the results of [4] by applying the ideas presented in this paper, which are significantly closer in spirit to the original ideas of [3].

This paper is arranged as follows. In Sections 2, 3 and 4 we study the differential geometry of the Lagrangian Grassmannian, obtaining in particular formulae for the canonical metric and Levi-Civita connexion in a unitary chart. This study is further developed in Appendix A to obtain formulae for geodesics and the curvature which are interesting in themselves. Theorems 1.1 and 1.2 are then proven in sections 5 and 6 respectively.

2 - Coordinate Charts for the Grassmannian.

Let (E, F) be a pair of complementary Lagrangian subspaces of $\mathbb{R}^{2n}$. Thus:

$$\mathbb{R}^{2n} = E \oplus F.$$

Using the metric and the symplectic form, we define a canonical isomorphism $c : F \to E$ such that, for all $u \in E$ and $v \in F$:

$$\langle c(u), v\rangle = \omega(u, v).$$

We thus identify E and F in the sequel. Let U be an n dimensional subspace of $\mathbb{R}^{2n}$ which is a graph over E with respect to F. In otherwords, there exists a linear mapping $A : E \to F$ such that:

$$U = \{x + Ax \text{ s.t. } x \in E\}.$$

The subspace U is Lagrangian if and only if, for every $x, y \in E$:

$$\omega(x + Ax, y + Ay) = 0.$$

Since E and F are both Lagrangian, and bearing in mind the canonical identification of E with F, this is equivalent to:

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

In other words, A is a symmetric mapping of E. Thus, given a pair (E, F) of complementary Lagrangian subspaces of $\mathbb{R}^n$, there exists a canonical idendification sending the space of symmetric matrices over E, $\mathrm{Symm}(E)$ into an open subset of $\mathrm{GLag}(\mathbb{R}^n)$. These define a complete system of coordinate charts for the Grassmannian of Lagrangian subspaces.

In particular, we define the mapping $\Phi_0 : \mathrm{Symm}(\mathbb{R}^n) \to \mathrm{GLag}(2n)$ by:

$$\Phi_0(A) = \{(x, Ax) \text{ s.t. } x \in \mathbb{R}^n\}.$$

We define the open subset Ω_0 of $\mathrm{GLag}(2n)$ to be the image of $\mathrm{Symm}(\mathbb{R}^n)$ under this mapping. All of the coordinate charts discussed in this section are equivalent to (Ω_0, Φ^{-1}) up to a symplectomorphism of $\mathbb{R}^{2n}$.

3 - The Grassmannian as a Symmetric Space.

We define the complex structure J over $\mathbb{R}^{2n}$ by:

$$J = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

Let $U(2n)$ be the group of unitary transformations of $(\mathbb{R}^{2n}, J)$. That is, elements of $U(2n)$ are isometries preserving J. Equivalently, elements of $U(2n)$ are isometries preserving ω. The group $U(2n)$ acts transitively on $\mathrm{GLag}(2n)$.

Let M be an element of $U(2n)$ and suppose that M preserves $\mathbb{R}^n \times \{0\}$. Then, since M is a symplectomorphism and an isometry, there exists $A \in o(n)$ such that:

$$M = \begin{pmatrix} A & \\ & A \end{pmatrix}.$$

The stabiliser of $\mathbb{R}^n \times \{0\}$ in $U(2n)$ is thus an isomorphic copy of $O(n)$, and:

$$\mathrm{GLag}(2n) \cong U(2n)/O(n).$$

We view $U(2n)$ as a principal $O(n)$ bundle over $\mathrm{GLag}(2n)$. We say that a coordinate chart of $\mathrm{GLag}(2n)$ is unitary if and only if it is equivalent up to a unitary transformation to (Ω_0, Φ_0^{-1}). For any Lagrangian plane E in $\mathrm{GLag}(2n)$, there exists a unitary chart (Ω, Φ^{-1}) of $\mathrm{GLag}(2n)$ such that $\Phi(0) = E$, that is, such that this chart is centred about E.

Let $\pi : U(2n) \to \mathrm{GLag}(2n)$ be the canonical projection. We define the mapping $\Psi_0 : \mathrm{Symm}(\mathbb{R}^n) \to U(2n)$ by:

$$\Psi_0(A) = \begin{pmatrix} (I + A^2)^{-1/2} & -A(I + A^2)^{-1/2} \\ A(I + A^2)^{-1/2} & (I + A^2)^{-1/2} \end{pmatrix}.$$

If $A \in \mathrm{Symm}(\mathbb{R}^n)$, then:

$$\Phi_0(A) = \pi \circ \Psi_0(A).$$

The mapping Ψ_0 thus defines a section of $U(2n)$ over Φ_0. Conversely, let M be an element of $U(2n)$:

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

If $\pi(M) \in \Omega_0$, then A is invertible and:

$$\Phi_0^{-1} \circ \pi(M) = BA^{-1}.$$

4 - The Canonical Metric of the Grassmannian.

Let $\underline{u}(2n)$ be the Lie algebra of $U(2n)$:

$$\underline{u}(2n) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \text{ s.t. } A = -A^t,\ B = B^t \right\}.$$

Let us define $\underline{o}(n)$ by:

$$\underline{o}(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ s.t. } A = -A^t \right\}.$$

Thus $\underline{o}(n)$ is the copy of the Lie algebra of $O(n)$ sitting inside $\underline{u}(2n)$. We define a metric over $\underline{u}(2n)$ by:

$$\langle A, B \rangle = \mathrm{Tr}(A^t B).$$

This metric is preserved by the action of $U(2n)$ on $\underline{u}(2n)$. We thus extend it to a bi-invariant metric over $U(2n)$. The formula for this induced metric in the coordinate chart (Ω_0, Φ^{-1}) is given by the following result:

Lemma 4.1

Let A be an element of $\mathrm{Symm}(\mathbb{R}^n)$*. Let S be a symmetric matrix so that S_A is a tangent vector to* $\mathrm{Symm}(\mathbb{R}^n)$*. The Lagrangian Grassmannian metric over* $\mathrm{Symm}(\mathbb{R}^n)$ *is given by:*

$$\langle S_A, S_A\rangle = 2\mathrm{Tr}((1+A^2)^{-1}S(1+A^2)^{-1}S).$$

Proof: Let M be an element of $\underline{o}(n)^{\perp}$:

$$M = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}.$$

Let A be an element of $\mathrm{Symm}(\mathbb{R}^n)$. Then:

$$\Psi_0(A)\cdot\mathrm{Exp}(tM) = \Psi_0(A) + t\Psi_0(A)M + O(t^2).$$

Let us write $\hat{A} = (I+A^2)^{-1/2}$. Then:

$$\begin{aligned} & (\Phi_0^{-1}\circ\pi)(\Psi_0(A)\cdot\mathrm{Exp}(tM)) && = (A\hat{A} + t\hat{A}B)(\hat{A} - tA\hat{A}B)^{-1} + O(t^2) \\ & && = A + t\hat{A}^{-1}B\hat{A}^{-1} + O(t^2) \\ \Rightarrow\quad & \partial_t(\Phi_0^{-1}\circ\pi)(\Psi_0(A)\cdot\mathrm{Exp}(tM))|_{t=0} && = \hat{A}^{-1}B\hat{A}^{-1}. \end{aligned}$$

It thus follows that if S is a symmetric matrix so that S_A is a tangent vector to $\mathrm{Symm}(\mathbb{R}^n)$ at A, then, by definition of the metric on $\mathrm{GLag}(\mathbb{R}^{2n})$:

$$\langle S_A, S_A\rangle = 2\mathrm{Tr}(B^tB),$$

where B is a symmetric matrix such that $S = \hat{A}^{-1}B\hat{A}^{-1}$. Thus:

$$\langle S_A, S_A\rangle = 2\mathrm{Tr}(\hat{A}S\hat{A}^2S\hat{A}) = 2\mathrm{Tr}((1+A^2)^{-1}S(1+A^2)^{-1}S).$$

The result now follows. $\square$

We now calculate the Levi-Civita connection one form of this metric. We obtain the following result:

Lemma 4.2

Let S and T be symmetric matrices over $\mathbb{R}^n$. Let us denote also by S and T the vector fields over $\mathrm{Symm}(\mathbb{R}^n)$ *defined by these matrices. The Levi Civita covariant derivative of the Lagrangian Grassmannian metric over* $\mathrm{Symm}(\mathbb{R}^n)$ *satisfies:*

$$(\nabla_S T)(A) = -SA(I+A^2)^{-1}T - TA(I+A^2)^{-1}S.$$

Proof: This follows directly by applying the Kohzul formula to the metric obtained in Lemma 4.1. $\square$

In particular, when $A = 0$ this covariant derivative coincides with the canonical flat covariant derivative over $\mathrm{Symm}(\mathbb{R}^n)$. This is particularly conveniant since every Lagrangian plane E in $\mathrm{GLag}(\mathbb{R}^{2n})$ has a unitary coordinate chart centred about it.

5 - Concavity of Φ.

We now prove the concavity of Φ:

Proof of Theorem 1.1: First, we observe that $J^*m = J^t mJ = -m$. In otherwords:

$$J^*m = -m.$$

Let E be a Lagrangian plane in $\mathrm{GLag}(\mathbb{R}^{2n})$. By applying a transformation in $U(2n)$, we may assume that $E = \mathbb{R}^n \times \{0\}$. Such a transformation leaves the metric, the complex structure and the symplectic form invariant. However, it may change m. We thus use, instead of m, an arbitrary symmetric bilinear form μ such that $J^*\mu = \mu$. We now work in the coordinate chart (Ω_0, Φ^{-1}).

Let v be any non-zero vector in $\mathbb{R}^n$. We define the functions $f_v, g_v : \mathrm{Symm}(\mathbb{R}^n) \to \mathbb{R}$ by:

$$\begin{aligned} f_v(A) &= \mu((v, Av), (v, Av)), \\ g_v(A) &= \langle (v, Av), (v, Av) \rangle. \end{aligned}$$

Let S be a symmetric matrix over $\mathbb{R}^n$. Let us also denote by S the vector field over $\mathrm{Symm}(\mathbb{R}^n)$ defined by this matrix, and let S_0 be the value of this field in the fibre above 0. We have:

$$\begin{aligned} (d_S f_v)(A) &= 2\mu((v, Av), (0, Sv)), \\ (d_S g_v)(A) &= 2\langle (v, Av), (0, Sv) \rangle. \end{aligned}$$

In particular, these fields vanish when $A = 0$. Next:

$$\begin{aligned} (\nabla^2 f_v)(S_0, S_0) &= (\nabla_{S_0} df_v)(S_0) \\ &= (d_S d_S f_v)(0) - (d_{\nabla_S S} f_v)(0) \\ &= (d_S d_S f_v)(0) \\ &= 2\mu((0, Sv), (0, Sv)). \end{aligned}$$

Likewise:

$$(\nabla^2 f_v)(S_0, S_0) = \langle (0, Sv), (0, Sv) \rangle.$$

Now:

$$\begin{aligned} (\nabla^2 f_v/g_v) &= \nabla(1/g_v df_v - f_v/g_v^2 dg_v) \\ &= (-2/g_v^2) dg_v \otimes df_v + (1/g_v)\nabla df_v \\ &\quad +2f_v/g_v^3 dg_v \otimes dg_v - f_v/g_v^2 \nabla dg_v. \end{aligned}$$

Thus, bearing in mind that the differentials of f_v and g_v both vanish at $A = 0$, we have:

$$\begin{aligned} (\nabla^2 f_v/g_v)(S_0, S_0) &= (2/g_v)(0)\mu((0, Sv), (0, Sv)) - (2f_v/g_v^2)(0)\langle (0, Sv), (0, Sv) \rangle \\ &= 2\|v\|^{-4}(\|v\|^2 \mu((0, Sv), (0, Sv)) - f_v(0)\|S_v\|^2). \end{aligned}$$

Since the chart (Ω_0, Φ_0^{-1}) is unitary, $J(0, Sv) = (Sv, 0)$, and thus, since $J^*\mu = -\mu$:

$$\begin{aligned} (\nabla^2 f_v/g_v)(S_0, S_0) &= -2\|v\|^{-4}(\|v\|^2 \mu((Sv, 0), (Sv, 0)) + f_v(0)\|S_v\|^2) \\ &= -2\|v\|^{-4}(\|v\|^2 f_{Sv}(0) + \|S_v\|^2 f_v(0)). \end{aligned}$$

However:

$$\Phi(E) = \operatorname*{Inf}_{v \in E \setminus \{0\}} f_v(0)/g_v(0).$$

Thus, if $\Phi(E) \geqslant 0$, then, since it is the infimum of a family of concave functions, it is also concave. Likewise, if $\Phi(E) > 0$ then Φ is strictly concave. The result now follows. $\square$

6 - Special Lagrangian Submanifolds.

We now show that a special Lagrangian immersion induces a harmonic mapping into the Lagrangian Grassmannian:

Lemma 6.1

Let $\Sigma = (S, i)$ be an immersed Lagrangian submanifold in $\mathbb{R}^{2n}$ and let g be the metric over S induced by the immersion i. We define the mapping $A_i : S \to \mathrm{GLag}(2n)$ by:

$$A_i(p) = Di_p \cdot T_pS.$$

Then the mapping A_i is harmonic with respect to the metric g.

Proof: Let p be a point in S. By applying a translation, we may suppose that $i(p) = 0$, and by applying a unitary transformation, we may suppose that $Di_p \cdot T_pS = \mathbb{R}^n \times \{0\}$. Since Σ is Lagrangian, there exists a neighbourhood Ω of 0 in $\mathbb{R}^n$ and a function $f : \Omega \to \mathbb{R}$ such that Σ coincides near p with the graph of df over Ω. We define $\hat{f} : \Omega \to \mathbb{R}^{2n}$ by:

$$\hat{f}(x) = (x, f(x)).$$

We may assume that $(S, i) = (\Omega, \hat{f})$. We use the unitary chart (Ω_0, Φ_0^{-1}) of $\mathrm{GLag}(2n)$, and we identify A with the matrix $\Phi_0^{-1}(A)$. Thus:

$$A_{ij} = \partial_i\partial_j f.$$

In the sequel, we write $A_{ijk} = \partial_i A_{jk}$ and $A_{ijkl} = \partial_i\partial_j A_{kl}$. These are both symmetric. The metric g satisfies:

$$g_{ij} = (\mathrm{Id} + A^2)_{ij}.$$

In particular, $g_{ij}(0) = \delta_{ij}$. Let Γ^k_{ij} be the Levi Civita connexion of g. Using the Kohzul formula, we obtain:

$$\Gamma^k_{ij} = 2((I + A^2)^{-1})^{kp} A_{pq} A_{qij}.$$

In particular, $\Gamma^k_{ij}(0) = 0$. By Lemma 4.2, the Laplacian of the mapping A is given by:

$$\begin{aligned} \Delta A \quad &= g^{ij}\partial_i\partial_j A - g^{ij}\Gamma^k_{ij}\partial_k A \\ &\quad -2g^{ij}(\partial_i A)A(\mathrm{Id}A^2)^{-1}(\partial_j A). \end{aligned}$$

Thus:

$$(\Delta A)_{jk}(0) = \Sigma_{i=1}^n A_{iijk}(0).$$

Since Σ is special Lagrangian with respect to the form Ω_θ, we have:

$$\mathrm{Arg}(\mathrm{Det}(\mathrm{Id} + iA)) = -\theta.$$

Differentating this relation two times, we obtain, for all i and j:

$$\mathrm{Tr}(-2A(\mathrm{Id} + A^2)^{-1}\partial_i A(\mathrm{Id} + A^2)^{-1}\partial_j A + (\mathrm{Id} + A^2)^{-1}\partial_i\partial_j A) = 0.$$

Thus, for all i and j:

$$\begin{array}{rll} & \Sigma_{k=1}^{n} A_{ijkk}(0) & = 0, \\ \Rightarrow & \Delta A(p) & = 0. \end{array}$$

Since $p \in S$ is arbitrary, the result now follows. $\square$

Finally, this allows us to prove Theorem 1.2:

Proof of Theorem 1.2: We define the mapping $A : S \to \mathrm{GLag}(2n)$ by:

$$A(p) = Di_p \cdot T_p S.$$

By Lemma 6.1, this mapping is harmonic. The immersed submanifold Σ is non-negative at $p \in S$ if and only if:

$$A(p) \in \mathrm{GLag}^{\geqslant 0}(2n).$$

In otherwords, it is non-negative if and only if:

$$(\Phi \circ A)(p) \geqslant 0.$$

However, by Lemma 1.1, the mapping Φ is concave. Consequently, $\Phi \circ A$ is superharmonic. It follows by the maximum principal that if $\Phi \circ A$ vanishes at a single point, then it vanishes everywhere. The result now follows as in [4]. $\square$

A - Geodesics and Curvature of the Lagrangian Grassmannian.

The calculations made in this paper may be used to study in slightly more depth the differential geometry of the Lagrangian Grassmannian. First, we calculate the geodesic flow:

Lemma 1.1

Let A be a symmetric matrix over $\mathbb{R}^n$ and let $M :]-\epsilon, \epsilon[\to \mathrm{Symm}(\mathbb{R}^n)$ be the unique geodesic such that:

$$M(0) = 0, \qquad \nabla_{\partial_t M} \partial_t M|_{t=0} = A.$$

Then M is given by the formula:

$$M(t) = \tan(At).$$

Proof: By Lemma 4.2, M satisfies:

$$M''(t) = 2M'M(I + M^2)^{-1}M'.$$

Let $\lambda :]-\epsilon, \epsilon[\to \mathbb{R}$ be a real valued function such that:

$$\lambda'' = 2(\lambda')^2\lambda(1 + \lambda^{-2})^{-1}.$$

Suppose that $\lambda'(0) > 0$. Then, near zero:

$$\begin{array}{rll} & (\lambda')^{-1}\lambda'' & = 2\lambda'\lambda(1+\lambda^2)^{-1} \\ \Rightarrow & \partial_t \mathrm{Log}(\lambda') & = \partial_t \mathrm{Log}(1+\lambda^2) \\ \Rightarrow & \partial_t \mathrm{Log}(\lambda'(1+\lambda^2)^{-1}) & = 0. \end{array}$$

There thus exists $A > 0$ such that:

$$\begin{array}{rll} & \lambda'(1+\lambda^2)^{-1} & = A \\ \Rightarrow & \partial_t \mathrm{arctan}(\lambda) & = A. \end{array}$$

There thus exists $B > 0$ such that:

$$\lambda(t) = \arctan(At + B).$$

If $\lambda(0) = 0$ and $\lambda'(0) = A$, then:

$$\lambda(t) = \arctan(At).$$

An analogous reasoning permits us to obtain the same result when $\lambda'(0) < 0$ and when $\lambda'(0) = 0$. Let $e_1, ..., e_n$ be an orthonormal basis of eigenvectors for A and let $a_1, ..., a_n$ be their respective eigenvalues. For each i, we define $\lambda_i :]-\epsilon, \epsilon[\rightarrow \mathbb{R}$ by:

$$\lambda_i(t) = \arctan(a_i t).$$

For all t, let $M(t)$ be the symmetric linear map whose eigenvectors are the $(e_i)_{1 \leqslant i \leqslant n}$ and whose eigenvalues with respect to these eigenvectors are $(\lambda_i(t))_{1 \leqslant i \leqslant n}$ respectively. The function M then satisfies the geodesic equation for the Levi-Civita covariant derivative of the Lagrangian Grassmannian, and:

$$\nabla_{\partial_t M} \partial_t M|_{t=0} = A.$$

The result now follows. □

Similarly, we may calculate the curvature tensor of the Lagrangian Grassmannian:

Lemma 1.2

At the origin of a unitary chart, the Riemann curvature tensor is given by the formula:

$$\mathbb{R}_{XY} Z = [[XY]Z].$$

In over words:

$$\langle \mathbb{R}_{XY} Z, W \rangle = 2\mathrm{Tr}([XY][ZW]).$$

In particular, the sectional curvature of the plane spanned by the vectors X and Y is given by:

$$\langle \mathbb{R}_{XY} Y, X \rangle = 2\mathrm{Tr}([XY]^2).$$

Proof: At the origin, by Lemma 4.2, we have:

$$\nabla_X \nabla_Y Z(0) = \nabla_X(-YA(1+A^2)^{-1}Z - ZA(1+A^2)^{-1}Y)|_{A=0}.$$

The only terms that don't vanish are those where A is removed by differentiation. Thus:

$$\begin{array}{rll} & \nabla_X \nabla_Y Z(0) & = -XYZ - ZXY \\ \Rightarrow & R_{XY}Z(0) & = [[XY]Z]. \end{array}$$

The first result now follows, and the remaining two results follow by Lemma 4.1. □

In particular, we observe that the Lagrangian Grassmannian is of non-negative sectional curvature. This is no surprise, since it is a compact homogenous space. Moreover, the sectional curvature of a plane spanned by two matrices X and Y vanishes if and only if these matrices commute.

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