Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Quasi-Hamiltonian structure and Hojman construction II: Nambu Mechanics and Nambu-Poisson structures

by

José Cariñena, Partha Guha, and Manuel Rañada

Preprint no.: 89 2007

Quasi-Hamiltonian structure and Hojman construction II: Nambu Mechanics and Nambu-Poisson structures

José F. Cariñena† a), Partha Guha‡§ b, and Manuel F. Rañada† c)

† *Departamento de Física Teórica, Facultad de Ciencias*
Universidad de Zaragoza, 50009 Zaragoza, Spain

‡ *Max Planck Institute for Mathematics in the Sciences*
Inselstrasse 22, D-04103 Leipzig, Germany

§ *S.N. Bose National Centre for Basic Sciences, JD Block*
Sector-3, Salt Lake, Calcutta-700098, India

22 de Mayo, 2007

Abstract

In this paper we extend Hojman's approach of the non standard Hamiltonian structure to the Nambu-Poisson case. In particular the relationship between the generalized non standard Hamiltonian structure and the degenerate quasi Nambu-Hamiltonian structures is unveiled in this paper. We also provide ample applications of our construction.

Keywords: Nambu Mechanics. Hojman. Hamiltonian systems. quasi-Hamiltonian systems. superintegrability.

PACS numbers: 45.20.Jj 02.30Ik 05.45.Ac

MSC Classification: 37J35 70H06 70H20.

a) *E-mail address:* jfc@unizar.es
b) *E-mail address:* partha@bose.res.in
c) *E-mail address:* mfran@unizar.es

1 Introduction

The geometric approach to dynamical systems by means of vector fields suggests us to consider additional compatible structures. In a recent paper [1] Hojman proposed a general technique for finding an admissible Hamiltonian structure for a given equation of motion using one infinitesimal symmetry transformation and one constant of motion, valid for systems of both ordinary and partial differential equations, and it was extended in subsequent papers by him and his coworkers [2, 3] for dealing with dynamical systems in field theory without using any Lagrangian. For a recent updating of Hojman's approach see e.g. [4].

The geometric approaches to mechanics used first symplectic and later presymplectic and Poisson structures. Later, Nambu proposed in 1973 [5] a generalisation of the classical Hamiltonian formalism for the study of a system defined on a three-dimensional phase space with coordinates (x_1, x_2, x_3) and a new class of bracket for three functions (f_1, f_2, f_3) given by

$$\{ f_1, f_2, f_3\} = \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)},$$

where the right hand side denotes the Jacobian determinant, allowing us to express the time evolution of a function f by

$$\frac{df}{dt} = \{f, h_1, h_2\}\,.$$

Here h_1 and h_2 are two 'Hamiltonian' or 'Nambu' functions for such dynamics.

Like Poisson geometry, the existence of a Nambu-Poisson bracket is equivalent to the existence of a skewsymmetric contravariant tensor N of order m satisfying a condition equivalent to the fundamental identity.

Shortly after that, Takhtajan [6] introduced the concept of Nambu-Poisson (or simply Nambu) structure using an axiomatic formulation for the n-bracket operation and this paper motivated a series of papers on the same subject (see e.g. [7, 8, 9, 10]. Another generalisation was the so-called generalised Poisson brackets [11, 12, 13].

Like Poisson geometry, the existence of a Nambu-Poisson bracket is equivalent to the existence of a skewsymmetric contravariant tensor N of order m satisfying a condition equivalent to the fundamental identity. It has been proved [14, 15] that a Nambu-Poisson tensor N of order $m \geq 3$ is decomposable, as a consequence a Nambu-Poisson manifold is locally foliated.

Our aim in this paper is to analyse the existence of a Nambu structure for a given dynamical system, Γ, and how we can find such a structure when two commuting infinitesimal symmetries of the dynamical vector field and two constants of motion are known.

The organization of the paper is as follows: Section 2 is devoted to introduce notation and basic definitions and we summarise the relevant properties of Nambu–Poisson manifolds. The possibility of finding a Nambu structure making the vector field Γ Hamiltonian when two commuting infinitesimal symmetries of Γ and two constants of the motion are known is proved in Sections 3 and finally Section 4 contains several illustrative examples.

2 Notation and basic definitions

Let M be a smooth n-dimensional manifold and $C^\infty(M)$ denote the algebra of differentiable real-valued functions on M. A Nambu structure is given by an m-dimensional multivector

$$N\,:\,\bigwedge{}^1(M)\times\overset{m}{\cdots}\times\bigwedge{}^1(M)\ \to\ \mathcal{F}(M)$$

which in local coordinates $(x_1,x_2,\dots,x_m)$ is given by

$$N=n_{i_1\dots i_m}(x)\,\frac{\partial}{\partial x_{i_1}}\wedge\frac{\partial}{\partial x_{i_2}}\dots\wedge\frac{\partial}{\partial x_{i_m}},$$

where summation over repeated indices is understood, which allows us to define the bracket of m functions by

$$\{\,f_1,f_2,\dots,f_m\}=N(df_1,df_2,\dots,df_m),$$

in local coordinates $(x_1,\dots,x_n)$ this is

$$\{f,f_1,\dots,f_{n-1}\}\ =n_{i_0i_1,\dots i_{m-1}}\phi_{i_0}f\phi_{i_1}f_1\dots\phi_{i_{n-1}}f_{m-1},$$

in such a way that the following conditions be satisfied:

1. Skew-symmetry
$$\{\,f_1,f_2,\dots,f_m\}=(-1)^{\epsilon(\sigma)}\{\,f_{\sigma(1)},f_{\sigma(2)},\dots,f_{\sigma(m)}\},$$
where $\sigma\in S_m$ (symmetric group of m elements) and $\epsilon(\sigma)$ denotes its parity.

2. Multilinearity: if k_a and k_b are real numbers,
$$\{\,k_ag_a+k_bg_b,f_2,\dots,f_m\}=k_a\{\,g_a,f_2,\dots,f_m\}+k_b\{\,g_b,f_2,\dots,f_m\}$$
for any $m+1$ functions $g_a,g_b,f_1,\dots,f_{m-1}$.

3. Leibniz rule
$$\{\,g_a\,g_b,f_2,\dots,f_m\}=g_a\,\{\,g_b,f_2,\dots,f_m\}+\{\,g_a,f_2,\dots,f_m\}\,g_b.$$

4. Generalised Jacobi identity, usually called Fundamental identity (shortened as F.I.):
$$\begin{aligned}\{\,f_1,\dots,f_{m-1},\{\,g_m,\dots,g_{2m-1}\}\}\ &=\ \{\{\,f_1,\dots,f_{m-1},g_m\},g_{m+1},\dots,g_{2m-1}\}+\dots\\&\quad\ \dots+\{\,g_m,\dots,g_{2m-2},\{\,f_1,\dots,f_2,g_{2m-1}\}\}\end{aligned}$$

The last property, (iv), that is also known as the "Takhtajan identity", is the appropriate generalisation of the Jacobi identity characterising the standard Poisson bracket. As an example for $m=3$ and $m=4$ it reduces to

$$\{\,f_1,f_2,\{\,g_3,g_4,g_5\}\}=\{\{\,f_1,f_2,g_3\},g_4,g_5\}+\{\,g_3,\{\,f_1,f_2,g_4\},g_5\}+\{\,g_3,g_4,\{\,f_1,f_2,g_5\}\}$$

and

$$\begin{aligned}\{\,f_1,f_2,f_3,\{\,g_4,g_5,g_6,g_7\}\}\ &=\ \{\{\,f_1,f_2,f_3,g_4\},g_5,g_6,g_7\}+\{\,g_4,\{\,f_1,f_2,f_3,g_5\},g_6,g_7\}\\&+\ \{\,g_4,g_5,\{\,f_1,f_2,f_3,g_6\},g_7\}+\{\,g_4,g_5,g_6,\{\,f_1,f_2,f_3,g_7\}\}.\end{aligned}$$

Lemma 1 *Let $\{\cdot,\cdot,\cdots,\cdot\} : C^\infty(M)\times C^\infty(M)\cdots\times C^\infty(M)\longrightarrow C^\infty(M)$ be a multi- derivation that satisfies FI iff*

1. *$\{\cdot,\cdot,\cdots,\cdot\}$ satisfies FI for generators.*
2. *It satisfies quadratic identities.*

$$\sum_{k=1}^{n}\{\phi,f_1,\cdots,f_{n-2},f_{n+k-1}\}\{\phi',f_n,\cdots,\widehat{f_{n+k-1}},\cdots,f_{2n-1}\} \tag{1}$$
$$+\{\phi',f_1,\cdots,f_{n-2},f_{n+k-1}\}\{\phi,f_{n+1},\cdots,\widehat{f_{n+k-1}},\cdots,f_{2n-1}\}=0.$$

Note that a set of $m-1$ functions, $f_1,\ldots,f_{m-1}$, defines a vector field, to be denoted $X_{f_1,\ldots,f_{m-1}}$, by contracting N with $df_1\wedge\cdots\wedge df_{m-1}$, i.e. $X_{f_1,\ldots,f_{m-1}}g=\{f_1,\ldots,f_{m-1},g\}$. Such a vector field satisfies $\mathcal{L}_{X_{f_1,\ldots,f_{m-1}}}N=0$, which is equivalent to the F.I.. Actually $\mathcal{L}_{X_{f_1,\ldots,f_{m-1}}}N=0$ means that

$$X_{f_1,\ldots,f_{m-1}}\{f_m,\ldots,f_{2m-1}\}=\{X_{f_1,\ldots,f_{m-1}}f_m,\ldots,f_{2m-1}\}+\cdots+\{f_m,\ldots,X_{f_1,\ldots,f_{m-1}}f_{2m-1}\}.$$

Definition 1 *$f\in C^\infty(M)$ is a first integral of $X_{f_1,\cdots f_{m-1}}$ if and only if*

$$\{f,f_1,f_2,\cdots,f_{m-1}\} \;=\; 0.$$

Note also that as a consequenece of the F.I. the Nambu Bracket of m constants of the motion for a Hamiltonian vector field is a constant of motion too.

The vector field $X_{f_1,\ldots,f_{m-1}}$ is said to be Hamiltonian. A vector field Y in M for which there exist m functions $g,f_1,\ldots,f_{m-1}$ such that $gY=X_{f_1,\ldots,f_{m-1}}$ is said to be quasi-Hamiltonian. The functions f_i defining such a vector field are constants of the corresponding motion by the skew-symmetry of N.

More generally for any $k\le n$ we can define a map

$$N^{\#}:\Gamma(\bigwedge{}^{k}(T^*M))\to\Gamma((\bigwedge{}^{n-k}(TM)))$$

by contraction of N with each k-form in M.

An interesting particular case is when m is equal to the dimension of the manifold M. For instance, if Q is a n-dimensional manifold and $M=T^*Q$ is endowed with its natural symplectic form ω_0, then the multivector in M which is dual of the $2n$-form $\omega\wedge\overset{n}{\cdots}\wedge\omega$ defines a Nambu structure (in this case it is the dual of the Liouville structure).

Finally the F.I. also implies that [10]

$$[X_{f_1,\cdots,f_{m-1}},X_{f_m,\cdots,f_{2m-2}}]=\sum_{i=1}^{m-1}X_{f_m,\cdots,f_{m+k-2},X_{f_1,\cdots,f_{m-1}}f_{m+k-1},\cdots,f_{2m-2}}.$$

A remarkable property is that, when n is an even number, the F.I. holds if and only if the Schouten Bracket $[N,N]$ vanishes (see e.g. [11, 13, 16])). Moreover we recall that the Nambu Bracket of two decomposable m-vectors is given by

$$[X_1\wedge\cdots\wedge X_m,Y_1\wedge\cdots\wedge Y_m]=\sum_{i,j=1}^{m}(-1)^{i+j}[X_i,Y_j]\wedge X_1\cdots\wedge\widehat{X}^i\wedge\cdots X_m\wedge Y_1\wedge\cdots\wedge\widehat{Y}_j\wedge\cdots\wedge\widehat{Y}_m\,, \tag{2}$$

where $\widehat{X}^i$ means that the vector field X_i is omitted and the same for $\widehat{Y}_j$.

3 The main theorem

Theorem 1 *Let Γ be a dynamical system on a manifold M. Suppose that*

1. *Γ possesses two commuting infinitesimal symmetries represented by the vector fields X_1 and X_2.*

2. *There exist two functions, h_1 and h_2, which are constants of the motion for Γ.*

Then the 3-vector field $N_{012} = \Gamma \wedge X_1 \wedge X_2$ is a Nambu structure on M and the dynamical system Γ is 'quasi-Hamiltonian' with respect to N_{012}. Moreover a new Nambu structure J, proportional to N_{012}, can be defined so that Γ is the Hamiltonian vector field of the functions h_1 and h_2 with respect to J.

Proof.: The expression

$$N_{012} = \Gamma \wedge X_1 \wedge X_2 \tag{3}$$

defines a 3-vector field on M and the fact that X_1 and X_2 are infinitesimal symmetries of Γ,

$$[X_1, \Gamma] = 0\,, \qquad [X_2, \Gamma] = 0\,, \tag{4}$$

implies that it is invariant under Γ

$$\mathcal{L}_\Gamma\,(\Gamma \wedge X_1 \wedge X_2) = \Gamma \wedge [\Gamma, X_1] \wedge X_2 + \Gamma \wedge X_1 \wedge [\Gamma, X_2)] = 0\,.$$

Using the expression (2) we see that the vanishing of the Lie Bracket of the two symmetries, that is $[X_1, X_2] = 0$, together with (4) leads to the vanishing of the Schouten Bracket

$$[\,N_{012}, N_{012}\,] = 0\,,$$

and thus N_{012} is a Nambu structure invariant under the dynamics.

Denote by (x_1, x_2, x_3) a local set of coordinates in a 3-dimensional manifold M and suppose the following coordinate expressions for the three vector fields

$$\Gamma = f^a(x)\,\frac{\partial}{\partial x_a}\,, \qquad X_1 = z_1^b(x)\,\frac{\partial}{\partial x_b}\,, \qquad X_2 = z_2^c(x)\,\frac{\partial}{\partial x_c}\,.$$

Then N_{012} is given by

$$N_{012} = n_{abc}\,\frac{\partial}{\partial x_a} \wedge \frac{\partial}{\partial x_b} \wedge \frac{\partial}{\partial x_c}\,, \quad n_{abc} = \det \begin{vmatrix} f^a & f^b & f^c \\ z_1^a & z_2^b & z_3^c \\ z_2^a & z_2^b & z_3^c \end{vmatrix}\,.$$

The action of $N_{012}^{\#}$ on the two differentials, dh_1 and dh_2, of the two assumed constants of motion for Γ is

$$N_{012}^{\#}(dh_1, dh_2) = h_{12}\,\Gamma\,,$$

where the function h_{12} is given by

$$h_{12} = X_1(h_1)X_2(h_2) - X_1(h_2)X_2(h_1)\,.$$

Hence the dynamical vector field Γ is 'quasi-Hamiltonian' with respect the Nambu structure N_{012}. On the other side the vanishing of the Lie brackets $[X_i,\Gamma]$ means that the corresponding Lie derivatives, $\mathcal{L}_X$ and $\mathcal{L}_\Gamma$, also commute. Therefore the function h_{12} is a constant of the motion for Γ because from $\mathcal{L}_\Gamma\mathcal{L}_{X_i} - \mathcal{L}_{X_i}\mathcal{L}_\Gamma = 0$ we obtain that

$$\mathcal{L}_\Gamma h_{12} = \mathcal{L}_\Gamma(\mathcal{L}_{X_1}(h_1)\ \mathcal{L}_{X_2}(h_2) + \mathcal{L}_{X_1}(h_2)\mathcal{L}_{X_2}(h_1)) = 0\,.$$

When $h_{12} \neq 0$, we can define a new structure J by

$$J = \frac{1}{h_{12}}\,N_{012}.$$

The property $\Gamma(h_{12}) = 0$ implies that J is also Nambu and that

$$\Gamma = J^{\#}(dh_1, dh_2).$$

Thus Γ is the Hamiltonian vector field of the functions h_1 and h_2 with respect to J.

4 Three Examples

4.1 Isotropic harmonic oscillator

Consider a six-dimensional phase space M with local coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the dynamical vector field

$$\Gamma = X_1 + X_2 + X_3\,, \qquad X_i = y_i\,\frac{\partial}{\partial x_i} + \omega^2 x_i\,\frac{\partial}{\partial y_i}\,, \quad i = 1, 2, 3.$$

Note that $[\Gamma, X_2] = [\Gamma, X_3] = 0$, and $[X_i, X_j] = 0$ for $i, j = 1, 2, 3$. Then we can define the 3-vector field N by

$$N_{023} = \Gamma \wedge X_2 \wedge X_3$$

or, in an equivalent way, $N_{023} = N_{123}$ with N_{123} being given by

$$N_{123} = X_1 \wedge X_2 \wedge X_3\,.$$

On the other hand the functions

$$h_2 = x_3 y_1 - x_1 y_3 \quad \text{ and } \quad h_3 = x_1 y_2 - x_2 y_1\,,$$

are Γ-invariant, i.e. $\Gamma(h_2) = \Gamma(h_3) = 0$.

The vector field defined by the functions h_2 and h_3 is

$$N^{\#}_{012}(dh_2, dh_3) = h_{23}\,\Gamma\,,$$

where the function h_{23} is given by

$$h_{23} = -\,(y_1 y_2 + \omega^2 x_1 x_2)(y_1 y_3 + \omega^2 x_1 x_3)\,.$$

Such a function is also Γ-invariant, $\Gamma(h_{23}) = 0$ and consequently

$$J = \frac{1}{h_{23}}\, N_{023} \quad \text{is such that} \quad [J, J] = 0\,.$$

Therefore J is also Nambu structure and the dynamical vector field corresponding to J is

$$\Gamma = J^{\#}(dh_2, dh_3)\,.$$

4.2 Kepler problem

In a similar way, if we remove the points $(0, 0, 0, y_1, y_2, y_3)$ in the preceding phase space and consider the dynamical vector field

$$\Gamma = y_1\,\frac{\partial}{\partial x_1} + y_2\,\frac{\partial}{\partial x_2} + y_3\,\frac{\partial}{\partial x_3} + \frac{k}{r^3}\left(x_1\,\frac{\partial}{\partial y_1} + x_2\,\frac{\partial}{\partial y_2} + x_3\,\frac{\partial}{\partial y_3}\right), \qquad r^2 = x_1^2 + x_2^2 + x_3^2\,,$$

then the vector fields X_2 and X_{123} given by

$$\begin{aligned} X_2 &= x_3\,\frac{\partial}{\partial x_1} - x_1\,\frac{\partial}{\partial x_3} + y_3\,\frac{\partial}{\partial y_1} - y_1\,\frac{\partial}{\partial y_3} \\ X_{123} &= J_1\,X_1 + J_2\,X_2 + J_3\,X_3 \end{aligned} \tag{5}$$

are infinitesimal symmetries of Γ such that $[X_2, X_{123}] = 0$ and we can define a 3-vector field N_{023} by

$$N_{023} = \Gamma \wedge X_2 \wedge X_{123}$$

which is invariant under Γ.

Moreover, one can check that the functions

$$h_2 = R_2\,, \quad h_3 = R_3\,, \quad \text{and} \quad R_i = \epsilon_{ijl}\,J_j\,y_l - k\,\frac{x_i}{r}$$

are Γ-invariant, i.e. $\Gamma(h_2) = \Gamma(h_3) = 0$. The Hamiltonian vector field defined by the functions h_2 and h_3 and the Nambu tensor N_{023} is given by

$$N_{023}^{\#}(dh_2, dh_3) = h_{23}\,\Gamma\,,$$

where the function h_{23} is given by

$$h_{23} = R_1(J_1 R_3 - R_2 J_3)$$

and is Γ-invariant, i.e. $\Gamma(h_{23}) = 0$.

The 3-vector field

$$J = \frac{1}{h_{23}}\, N_{023}\,,$$

is then a Nambu structure as well and Γ is Hamiltonian with respect to J

$$\Gamma = J^{\#}(dh_2, dh_3)\,.$$

4.3 Calogero-Moser system

Consider now a six-dimensional phase space M with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the dynamical system in M given by

$$\Gamma = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3} + 2c_0 \left[\Big(\frac{1}{x_{21}^3} - \frac{1}{x_{13}^3} \Big) \frac{\partial}{\partial y_1} + \Big(\frac{1}{x_{32}^3} - \frac{1}{x_{21}^3} \Big) \frac{\partial}{\partial y_2} + \Big(\frac{1}{x_{13}^3} - \frac{1}{x_{32}^3} \Big) \frac{\partial}{\partial y_3} \right],$$

where use has been made of the notation $x_{ij} = x_i - x_j$.

Let N be the multivector

$$N_{023} = \Gamma \wedge X_2 \wedge X_3 \,,$$

where X_2 and X_3 are given by

$$\begin{aligned} X_2 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \\ X_3 &= (y_1^2 + V_{21} + V_{13}) \frac{\partial}{\partial x_1} + (y_2^2 + V_{32} + V_{21}) \frac{\partial}{\partial x_2} + (y_3^2 + V_{13} + V_{32}) \frac{\partial}{\partial x_3} \\ &+ \Gamma(y_1^2 + V_{21} + V_{13}) \frac{\partial}{\partial y_1} + \Gamma(y_2^2 + V_{32} + V_{21}) \frac{\partial}{\partial y_2} + \Gamma(y_3^2 + V_{13} + V_{32}) \frac{\partial}{\partial y_3} \end{aligned}$$

and V_{ij} denotes the function $V_{ij} = c_0/x_{ij}^2$.

Note that the vector fields X_1 and X_2 commute: $[X_2, X_3] = 0$.

Moser proved [32] that the n-dimensional Calogero system can be presented as a Lax equation and that a fundamental set of constants of the motion is given by

$$I_k = \frac{1}{k} \operatorname{tr} A^k \,, \qquad A = A_1 + \mathrm{i}\, c_0 A_2 \,, \quad k = 1, 2, \ldots, n \,,$$

where A_1 and A_2 denote the diagonal and nondiagonal matrices

$$A_1 = \text{diagonal}\,[y_1, y_2, \ldots, y_n] \,, \quad (A_2)_{ij} = \left[(1 - \delta_{ij}) \frac{1}{x_{ij}} \right].$$

Wojciechowski proved the superintegrability of this system [33] by showing the existence of an additional family of integrals (see also [34, 35, 36, 37]). If we make use of the matrix Q defined by

$$Q = \text{diagonal}\,[q_1, q_2, \ldots, q_n],$$

then the additional constants of the motion can be given as the traces of products of the matrices Q and A [35]. In the particular case we are considering, if we denote by L_{ij} the functions $L_{ij} = x_i y_j - x_j y_i$, the following two functions

$$\begin{aligned} h_2 &= [\operatorname{tr}(QA)]\, I_1 - [\operatorname{tr}(Q)]\, (2I_2) \\ &= L_{21}(y_2 - y_1) + L_{32}(y_3 - y_2) + L_{13}(y_1 - y_3) + \text{ terms of lower order} \\ h_3 &= [\operatorname{tr}(QA^2)]\, I_1 - [\operatorname{tr}(Q)]\, (3I_3) \\ &= L_{21}(y_2^2 - y_1^2) + L_{32}(y_3^2 - y_2^2) + L_{13}(y_1^2 - y_3^2) + \text{ terms of lower order} \end{aligned} \tag{6}$$

are Γ-invariant, i.e. $\Gamma(h_2) = \Gamma(h_3) = 0$. The action of N_{023} on the 1-forms dh_2 and dh_3 is

$$N_{023}^{\#}(dh_2, dh_3) = h_{23}\, \Gamma \,,$$

where the function h_{23} is given by

$$h_{23} = (y_2 - y_1)^2 (y_3 - y_2)^2 (y_1 - y_3)^2 (y_1 + y_2 + y_3) + \text{ terms of lower order.}$$

Taking into account that $\Gamma(h_{23}) = 0$ we obtain

$$J = \frac{1}{h_{23}} N_{023}\,, \quad [J, J] = 0\,.$$

In other words J is a Nambu structure and the dynamical vector field Γ is Hamiltonian with respect to J.

$$\Gamma = J^{\#}(dh_2, dh_3)\,.$$

5 Conclusion and outlook

This is a second paper of the project to understand Hojman's programme of nonstandard construction of Hamiltonian structures. Recall that this construction is a general technique to find a Hamiltonian structure for a given equation of motion using one infinitesimal transformation and one constant of motion. In this paper we have studied the generalization of degenerate quasi-Hamiltonian towards the Nambu-Poisson case. We have given several interesting examples in support of our construction. In future we will study the noncommutative generalization of degenerate quasi-Hamiltonian structure.

Acknowledgments.

We are grateful to Peter Leach for careful reading of the manuscript. JFC and MFR acknowledge support from research projects MTM-2006-10531, FPA-2003-02948, E24/1 (DGA), and MTM-2005-09183. PG thanks the Departamento de Fisica Teõrica de la Universidad de Zaragoza for its hospitality and acknowledges support from Max Planck Institute for Mathematics in the Sciences, Leipzig.

References

[1] S. Hojman, *The construction of a Poisson structure out of a symmetry and a conservation law of a dynamical system*, J. Phys. A: Math. Gen. **29** (1996) 667–674.

[2] A. Gomberoff and S. Hojman, *Non-standard construction of Hamiltonian structures*, J. Phys. A: Math. Gen. **30**, 5077–5084 (1997).

[3] M. Herrera and S. Hojman, *Construction of alternative Hamiltonian structures for field equations*, J. Phys. A: Math. Gen. **34**, 6135–6141 (2001).

[4] J.F. Cariñena, P. Guha and M.F. Rañada, *Quasi-Hamiltonian Structure and Hojman Construction*, J. Math. Anal. Appl. **332**, 975–88 (2007).

[5] Y. Nambu, *Generalized Hamiltonian Dynamics*, Phys. Rev. **D 7**, 2405-2412 (1973).

[6] L. Takhtajan, *On foundation of the generalized Nambu mechanics*, Comm. Math. Phys. **160**, no. 2, 295–315 (1994).

[7] F. Bayen and M. Flato, *Remarks concerning Nambu's generalized mechanics*, Phys. Rev. **D 11**, 3049-3053 (1975).

[8] N. Mukunda and E.C.G. Sudarshan, *Relation between Nambu and Hamiltonian mechanics*, Phys. Rev. **D 13**, 2846-2850 (1976).

[9] A.J. Kálnay and R. Tascón, *Lagrange, Hamilton-Dirac, and Nambu mechanics*, Phys. Rev. **D 17**, 1552-1562 (1978).

[10] P. Guha, Quadratic Poisson structures and Nambu mechanics. Nonlinear Anal. **65** (2006), no. 11, 2025–2034.

[11] J.A. de Azcárraga, A.M. Perelomov and J.C. Pérez-Bueno, *New generalized Poisson structures*, J. Phys. A: Math. Gen. **29**, 627–49 (1996)

[12] J.A. de Azcárraga, A.M. Perelomov and J.C. Pérez-Bueno, *Schouten-Nijenhuis brackets, cohomology, and generalized Poisson structures*, J. Phys. A: Math. Gen. **29**, 7993–8009 (1996)

[13] J.A. de Azcárraga, J.M. Izquierdo and J.C. Pérez-Bueno, *On the higher order generalizations of Poisson structures*, J. Phys. A: Math. Gen. **30**, L 607–16 (1997)

[14] D. Alekseevsky and P. Guha, *On decomposability of Nambu-Poisson tensor*, Acta Math. Univ. Commenianae **65**, 1–10 (1996).

[15] P. Gautheron, *Some remarks concerning Nambu mechanics*, Lett. Math. Phys. **37**, 103–116 (1996).

[16] P. Michor and I. Vaisman, *A note on n-ary Poisson brackets*, in Proceedings of the 19th Winter School "Geometry and Physics," 1999; Rend. Circ. Mat. Palermo (2) Suppl. No. 63, 165 (2000).

[17] S. Codriansky, *Developments in Nambu mechanics*, J. Phys. A: Math. Gen. **27**, no. 7, 2565–2578 (1994).

[18] S. Codriansky, R. Navarro and M. Pedroza, *The Liouville condition and Nambu mechanics*, J. Phys. A: Math. Gen. **29**, no. 5, 1037–1044 (1996).

[19] P. Morando, *Liouville condition, Nambu mechanics, and differential forms*, J. Phys. A: Math. Gen. **29**, L 329–L331 (1996).

[20] P. Guha, *Volume-preserving integrable systems and Nambu-Poisson manifolds*, Class. Quantum Grav. **14**, L 37–L43 (1997).

[21] J. Hietarinta, *Nambu tensors and commuting vector fields*, J. Phys. A: Math. Gen. **30**, no. 3, L27–L33 (1997).

[22] S.A. Pandit and A.D. Gangal, *On generalized Nambu mechanics*, J. Phys. A: Math. Gen. **31**, no. 12, 2899–2912 (1998).

[23] J. Grabowski and G. Marmo, *Generalized n-Poisson brackets on a symplectic manifold*, Modern Phys. Lett. **A 13**, no. 39, 3185-3192 (1998).

[24] J. Grabowski and G. Marmo, *Remarks on Nambu-Poisson and Nambu-Jacobi brackets*, J. Phys. A: Math. Gen. **32**, no. 23, 4239–4247 (1999).

[25] R. Ibáñez, M. de León, J.C. Marrero and D. Martn de Diego, *Dynamics of generalized Poisson and Nambu-Poisson brackets*, J. Math. Phys. **38**, no. 5, 2332–2344 (1997).

[26] R. Ibáñez, M. de León, J.C. Marrero and E. Padrón, *Nambu-Jacobi and generalized Jacobi manifolds*, J. Phys. A: Math. Gen. **31**, no. 4, 1267–1286 (1998).

[27] R. Ibáñez, M. de León, J.C. Marrero and E. Padrón, *Leibniz algebroid associated with a Nambu-Poisson structure*, J. Phys. A: Math. Gen. **32**, no. 46, 8129–8144 (1999).

[28] R. Ibáñez, M. de León, B. Lopez, J.C. Marrero and E. Padrón, *Duality and modular class of a Nambu-Poisson structure*, J. Phys. A: Math. Gen. **34**, no. 17, 3623–3650 (2001).

[29] T. Ogawa and T. Sagae, *Nambu mechanics in the Lagrangian formalism*, International J. Theor. Phys. **39** (12), 2875-2890 (2000).

[30] D. Minic and C.H. Tze, *Nambu quantum mechanics: a nonlinear generalization of geometric quantum mechanics*, Phys. Lett. **B 536** (3-4), 305-314 (2002).

[31] G. Gaeta and P. Morando, *A variational principle for volume-preserving dynamics*, J. Nonlinear Math. Phys. **10** (4), 539-554 (2003).

[32] J. Moser, *Three integrable Hamiltonian systems connected with isospectral transformations*, Adv. Math. **16**, 197–220 (1975).

[33] S. Wojciechowski, *Superintegrability of the Caloger-Moser system*, Phys. Lett. **A 95**, 279–281 (1983).

[34] C. Gonera, *On the superintegrability of Calogero–Moser–Sutherland model*, J. Phys. A: Math. Gen. **31**, 4465–4472 (1998).

[35] M.F. Rañada, *Superintegrability of the Calogero-Moser system: constants of motion, master symmetries, and time-dependent symmetries*, J. Math. Phys. **40**, 236–247 (1999).

[36] M.F. Rañada, *On a second Lax structure for the Calogero–Moser system: time-dependent constants and superintegrability*, Phys. Lett **A 277**, 219–22 (2000).

[37] R. Caseiro, J.P. Franoise and R. Sasaki, *Quadratic algebra associated with rational Calogero -Moser models*, J. Math. Phys. 42, 5329–5340 (2001).