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Witten's Volume Formula, Cohomological Pairings of Moduli Space of Flat Connections and Applications of Multiple Zeta Functions

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Abstract. We use Witten's volume formula to calculate the cohomological pairings of the moduli space of flat SU(3) connections. The cohomological pairings of moduli space of flat SU(2) connections is known from the work of Thaddeus-Witten-Donaldson, but for higher holonomy groups these pairings are largely unknown. We make some progress on these problems, and show that the pairings can be expressed in terms of multiple zeta functions.

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1. Introduction

This article emerges from the recently obtained connection between quantum field theory and algebraic geometry and it is devoted to the study of some cohomological properties of the moduli space of flat $SU(3)$ connections over a Riemann surface. Roughly speaking, one can study the cohomological pairings in three different ways, the first method was due to Thaddeus [32], the second one by Witten [38, 39] Donaldson [13] also proposed another method. The most updated one was proposed by Jeffrey and Kirwan [24, 25].

The moduli space $M(n, d)$ of semistable rank n degree d holomorphic vector bundles with fixed determinant on a compact Riemann surface Σ is a smooth Kähler manifold when n and d are coprime [2, 3, 12, 30]. Jeffrey and Kirwan [24, 25] gave full details of a mathematically rigorous proof of certain formulas for intersection pairings in the cohomology of moduli space $M(n, d)$ with complex coefficients. These formulas have been found by Witten by formally applying his version of nonabelian localization to the infinite-dimensional space $\mathcal{A}$ of all U(n)-connections on Σ and the group of gauge transformations. Jeffrey and Kirwan used nonabelian localization technique to a certain finite-dimensional extended moduli space from which moduli space $M(n, d)$ of semistable rank n degree d may be obtained by ordinary symplectic reduction. In this way they obtained the Witten's formulas. It has been known [23] that a moduli space of flat connections on principal G-bundles over Σ as a Marsden-Weinstein symplectic quotient of a finite-dimensional symplectic manifold [16] by a G-action.

We use the Verlinde's formula [32, 35] of conformal field theory and complex geometry. Verlinde's formula gives the dimension of the space of conformal blocks in the WZW model on a Riemann surface. E. Verlinde's [35] result on the diagonalization of the fusion algebra gives a compact formula for the dimension of the space of conformal blocks. This formula coincides with the dimension of $H^0(\mathcal{M}_G, L^{\otimes k})$ where $\mathcal{M}_G$ is the moduli space of flat G bundles over the Riemann surface Σ_g of genus g and L is the generator of $Pic(\mathcal{M}_G)$. The formula for the dimension of these spaces, which is independent of Riemann surface Σ, was proved by A. Tsuchiya, K. Ueno and Y. Yamada [33]. The Verline formula has given rise to a great deal of excitement and new mathematics of infinite-dimensional variety (it is an ind-scheme) [9, 14, 27].

We can write the celebrated Verlinde's formula as

$$dim H^0(\mathcal{M}_G, L^{\otimes k}) = \sum_{\alpha} \frac{1}{S_{0,\alpha}^{2g-2}}. \tag{1.1}$$

Here α runs over the representatives of G which are the highest weights of integrable representations of the corresponding affine group $\widehat{G}$ at level k and $S_{\alpha,\gamma}$ is a matrix arising from the modular transformation of the character of the affine group $\widehat{G}$ at level k. If $\chi_\alpha(\tau)$ is the character of $\widehat{G}$ at level k with highest weight α, then S is defined by the formula

$$\chi_\alpha(-1/\tau) = \sum_{\beta} S_{\alpha\beta}\chi_\beta(\tau).$$

As an example we see that when $G = SU(2)$ then

$$S_{ij} = (\frac{2}{k+2})^{1/2} sin \frac{\pi(i+1)(j+1)}{k+2}.$$

Hence we obtain

$$dim H^0(\mathcal{M}, L^{\otimes k}) = (\frac{k+2}{2})^{g-1} \sum_{j=0}^{k} (\frac{1}{sin \frac{\pi(j+1)}{k+2}})^{2g-2}. \tag{1.2}$$

The volume of the moduli space is obtained from the Verlinde's formula (1.1) and given by

$$Vol^F(\mathcal{M}) = lim_{k\to\infty} k^{-n} dim H^0(\mathcal{M}, L^{\otimes k}). \tag{1.3}$$

The Hirzebruch-Riemann-Roch formula is

$$dim\ H^0(\mathcal{M}, L^{\otimes k}) \ = < exp(kc_1(L))\dot{T}d(\mathcal{M}), \mathcal{M} >,$$

where $Td(\mathcal{M})$ denotes the Todd class. For large k, this yields (for $G = SU(2)$)

$$dim\ H^0(\mathcal{M}, L^{\otimes k}) \sim \frac{k^{3g-3}}{(3g-3)!} < c_1(L)^{3g-3}, \mathcal{M} > .$$

Since $c_1(L)$ is represented by the symplectic form ω in de Rham cohomology, hence $\frac{<c_1(L)^{3g-3}, \mathcal{M}>}{(3g-3)!}$ coincides with the volume of the moduli space $Vol(\mathcal{M})$.

Incidentally, Witten gave a volume formula for moduli space of flat connections for general G. It is given by

$$Vol(\mathcal{M}) = \frac{\sharp Z(G).(Vol(G))^{2g-2}}{(2\pi)^{dim\mathcal{M}}} \sum_{\alpha} \frac{1}{(dim\alpha)^{2g-2}} \tag{1.4}$$

where α runs over all the irreducible representations of G. Here $\sharp Z(G)$ is number of elements in the centre of G.

In principle, although Witten's volume formula is applicable to any G, but unfortunately there are some computational problems arise when $G = SU(n)$ for $n \geq 3$. In this case the main problem that one must face is to find out the matrix $S_{\alpha\beta}$ from the modular transformation of the Weyl-Kac character formula [18, 22].

In this article we obtain the volume formula for $G = SU(3)$ by computing the matrix $S_{\alpha\beta}$,

$$S_{0\lambda} = \frac{8}{\sqrt{6}(k+3)} sin\frac{\pi\lambda_1}{k+3} sin\frac{\pi\lambda_2}{k+3} sin\frac{\pi(\lambda_1+\lambda_2)}{k+3}. \tag{1.5}$$

Computation of volume formula is a *two step process.* At first, we obtain the Verlinde's formula for moduli space of $SU(3)$ flat connection by substituting the value of $S_{\alpha\beta}$ in (1). We obtain

Proposition 1.1.

$$\boxed{dimH^0(\mathcal{M}, L^{\otimes k}) = \frac{(k+3)^{2g-2}6^{g-1}}{2^{6g-6}} \sum_{\lambda_1,\lambda_2} \Big(\frac{1}{sin\frac{\pi\lambda_1}{k+3} sin\frac{\pi\lambda_2}{k+3} sin\frac{\pi(\lambda_1+\lambda_2)}{k+3}}\Big)^{2g-2}.}$$

In the next step using the above formula and Witten's prescription for large k limit, we obtain volume of the moduli space of flat $SU(3)$ connection this yields

Proposition 1.2.

$$\boxed{Vol(\mathcal{M})_{SU(3)}) = 3\frac{6^{g-1}}{(2\pi)^{6g-6}} \sum_{n_1,n_2}^{\infty} n_1^{-(2g-2)} n_2^{-(2g-2)} (n_1+n_2)^{-(2g-2)}.}$$

Here the zeta function appears in the volume of flat $SU(3)$ connection $Vol(\mathcal{M})_{SU(3)})$

$$\zeta_g(A, 2g-2) = \sum_{n_1,n_2}^{\infty} \frac{1}{n_1^{2g-2} n_2^{2g-2} (n_1+n_2)^{2g-2}} \tag{1.6}$$

is a member of a family of much larger class of zeta functions, known as multiple zeta function. The Euler-Zagier multiple zeta functions are nested generalizations of the Riemann zeta function [36, 37]. They are defined as

$$\zeta_k(s_1, \ldots, s_k) = \sum_{0<n_1<\cdots<n_k} n_1^{-s_1} \cdots n_k^{-s_k}.$$

Here, $s_1, \cdots, s_k \in \mathbb{Z}$, $s_1 \geq 2$, $s_j \geq 1$ for $2 \leq j \leq k$. For $k = 1$, this reduces to Riemann's zeta function. We call k the length or depth of $\underline{s}$, and $|\underline{s}| = \sum s_j$ the weight of $\underline{s}$.

Unlike Riemann zeta function one could determine several algebraic relations between the multiple zeta values (MZV). One type of such relations appears when one multiples two such series, in fact, one gets a linear combination of MZV. A simple example is stated below:

$$\zeta(s)\zeta(s') = \sum_{n\geq 1} \frac{1}{n^s} \sum_{m\geq 1} \frac{1}{m^{s'}}$$

$$= \sum_{n>m} + \sum_{n<m} + \sum_{n=m} = \zeta(s, s') + \zeta(s', s) + \zeta(s+s'),$$

this is a quadratic relation among zeta values.

In general the quadratic relation is given as

$$\zeta(\underline{s})\zeta(\underline{s'}) = \sum_{\underline{\sigma}} \underline{\sigma}.$$

This manipulation leads to define a product called stuffle product [6]. This is a formal sum defined recursively by

$$aP * bQ = a(P * bQ) + b(aP * Q) + (a+b)(P * Q).$$

A simple example is

$$\zeta(s)^2 = 2\zeta(s, s) + \zeta(2s).$$

Then for $s = 2$, $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, hence we obtain

$$\zeta(2, 2) = \sum_{m>n\geq 1} (mn)^{-2} = \frac{\pi^4}{120}.$$

Another example is $\zeta(2)\zeta(3) = \zeta(2, 3) + \zeta(3, 2) + \zeta(5)$.

Remark. In the same way as the stuffle product arises in the reorganization of multiple sums, multiple integrals lead to the definition of shuffle product of words over alphabet (with two letters) $X = \{x_0, x_1\}$. The words are given as $X^* = \{x_0^{a_1} x_1^{b_1} \cdots x_0^{a_k} x_1^{b_k}\}$. This product is defined by the same formula as the stuffle product except that last term in the sum is omitted. The algebraic relations between multiple polylogarithms

$$Li_{(s_1 \cdots s_k)}(z) = \sum_{n_1 > n_2 > \cdots n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}} \qquad |z| < 1 \; \forall s_j \geq 1$$

is generated by the shuffle relation. In fact these multiple polylogarithms can be expressed as iterated Chen integrals, and from this representation one obtains shuffle relations (for example, see [7]).

In our case, Don Zagier [40, 41] gave a formula calculating the values of this particular multiple zeta function. This is also derived using stuffle product. The *key formula* to compute our volume form is given by

$$\sum_{m,n}^{\infty} \frac{1}{m^s n^s (m+n)^s} = \frac{4}{3} \sum_{0 \leq r \leq s; r even} \begin{pmatrix} 2s - r - 1 \\ s - 1 \end{pmatrix} \zeta(r)\zeta(3s - r). \tag{1.7}$$

Witten's volume formula can be extended to the moduli space of vector bundles with marked points $z_1, z_2, ..., z_p \in \Sigma_g$. We associate to each marked point z_i an irreducible representation Γ of $G_{\mathbf{C}}$. If λ is the highest weight of Γ, then $(\lambda, \alpha_{max}) \leq k$ where α_{max} is the highest root and (,) is the basic inner product (see appendix, [22]): alternatively λ is in the fundamental domain of the action of the affine Weyl group at level $k + h$ on the Cartan subalgebra (Lie algebra of the maximal torus). We sum over representations Γ for which if λ is the highest weight of Γ_λ, the representation of dimension $(n + 1)$ and all the marked points are labelled by Γ_{n_r}. We associate a complex vector space to each labelled Riemann surface.

The generalized Verlinde's formula for a group G in the presence of marked points [38] is

$$dim H^0(\mathcal{M}_G, L^{\otimes k} \otimes \bigotimes_i \Gamma_{n_i}) = \sum_{j=0}^{k} \frac{1}{S_{0,j}^{2g-2+p}} \prod_{i=1}^{p} S_{n_i,j} \tag{1.8}$$

and the vector space $H^0(\mathcal{M}_G, L^{\otimes k} \otimes \bigotimes_i \Gamma_{n_i})$ is independent of the details of the positions of the marked points.

Similarly the volume of the generalized moduli space can be obtained from this generalized Verlinde's formula (1.8) by extracting the term at the large k limit (1.3).

The volume formula for $G = SU(2)$ with marked points is

$$Vol^F(\mathcal{M}_t) = 2.\frac{1}{2^{g-1}\pi^{2g-2+p}} \sum_{n=1}^{p} \frac{\prod_{i=1}^{p} sin(\pi n t_i)}{n^{2g-2+p}}. \tag{1.9}$$

Our strategy is to compute the volume from the Verlinde formula in the large k limit, rather than using Witten's volume directly and this will be our recipe to find the volume of the moduli space.

Unlike the $SU(2)$ case we obtain the volume formula for the moduli space of flat $SU(3)$ connection in terms of the multiple zeta function or double Bernoulli numbers [4, 5].

We obtain the volume formula of the moduli space of flat $SU(3)$ connection over one marked point Riemann surface.

Proposition 1.3.

$$\boxed{Vol^F(\mathcal{M}_t) = \frac{3.6^{g-1}}{2^{6g-6}\pi^{6g-3}} \sum \frac{\sin \pi n_1 t_1 . \sin \pi n_2 t_2 . \sin \pi (n_1+n_2)(t_1+t_2)}{n_1^{2g-1} n_2^{2g-1} (n_1+n_2)^{2g-1}}}$$

t_1 and t_2 is restricted to

$$0 < t_i < 1.$$

When we expand out the sine terms we obtain a comprehensive volume formula to find the intersection pairings of moduli space.

Witten's idea is based on the symplectic volume of the moduli space of flat connections. The moduli space $\mathcal{M}$ of flat connections of any semi-simple group G is a symplectic variety with a symplectic form ω. The volume of the moduli space of flat $SU(n)$ connections is

$$Vol^S(\mathcal{M}) = \frac{1}{r!} \int_{\mathcal{M}} \omega^r$$

where $r = (n^2-1)(g-1) \; = \; (g-1)$ dim G is the dimension of the moduli space.

Witten showed [38] that the Reidemeister torsion of a Riemann surface equipped with a flat connection determines a natural volume form on the moduli space of flat connections which agrees with the symplectic volume. Given a chain complex $C_\bullet$ that computes $H_*(\Sigma, ad(E))$, we define the torsion $\tau(C_\bullet)$ is a vector in

$$(det\ H_0(\Sigma, ad(E)))^{-1} \otimes det\ H_1(\Sigma, ad(E)) \otimes det\ H_2(\Sigma, ad(E)))^{-1}.$$

For an irredicible flat connection,

$$H_0(\Sigma, ad(E)) = H_2(\Sigma, ad(E)) = 0.$$

So $\tau(C_\bullet)$ defines a vector in det $H_1(\Sigma, ad(E))$. Witten [38] gave the actual road map to compute volume of $\mathcal{M}$.

Motivation. The result of this paper was first appeared in [19]. Aparantly one would ask why do we need another paper to study cohomological pairings when Jeffrey and Kirwan [24, 25] gave full details of a mathematically rigorous proof of Witten's formulas for intersection numbers in the moduli spaces of flat connections. Indeed the knowledge of the volume formula in principle allows us to calculate the full list of cohomology pairings for the moduli space of arbitrary rank.

Our article is an explicit example of the computation of intersection pairings in the cohomology of moduli space of flat $SU(3)$ connections. This involves the computation of multiple zeta functions and hence it is fairly difficult to compute intersection pairings for higher rank vector bundles. Our approach is based on the volume of moduli spaces of parabolic bundles prescribed by Witten. Jeffrey-Kirwan formulation yields formulas for all intersection numbers, whereas our approach yield formulas for the intersection numbers of restricted cases, for example we exclude some cases that could yield the intersection numbers of some algebraic cycles in the moduli spaces.

2. Background about moduli space

Let Σ_g be a compact Riemann surface of genus g. Let E be the G bundle over Σ_g — here G can be any compact Lie group. For simplicity we shall work with the special case $G = SU(n)$. Let us consider the space of flat G connections over a Riemann surface Σ_g. We consider the space $Hom(\pi, G)/G$ which parametrizes the conjugacy classes of homomorphisms

$$\pi_1(\Sigma_g) \longrightarrow G.$$

Now $\pi_1(\Sigma_g)$ has generators $A_1, A_2, ..., A_g, B_1, B_2, ..., B_g$ which satisfy

$$\prod_{i=1}^{g}[A_i, B_i] = 1.$$

It follows that $H^1(\Sigma_g, G)$ is the quotient by G of the subset of G^{2g} lying over 1 in the map $G^g \times G^g \to G$ given by $\prod[A_i, B_i]$. This shows clearly that $H^1(\Sigma_g, G)$ is a compact Hausdorff space.

Let us fix our structure group $G = SU(n)$. Let us consider a point $x \in \Sigma_g$. Suppose we cut out a small disc D around the point x. We fix the holonomy of the connection around the disc D to be $exp(2\pi ip/n)$, where p and n are coprime to each other. Actually this holonomy $exp(2\pi ip/n)$ around the point x ensures the irreducibility of the connection.

Consider a map

$$f_g : SU(n)^g \times SU(n)^g \longmapsto SU(n)$$

defined by

$$(A_1, B_1, \ldots, A_g, B_g) \longmapsto \prod_i^g A_i B_i A_i^{-1} B_i^{-1}$$

In particular we select the subspace $W_g = f_g^{-1}(exp(2\pi ip/n)$ of $SU(n)^{2g}$.

A point, say x, in the space $SU(n)^g \times SU(n)^g$ is considered to be reducible if there exists a matrix T in $SU(n)$ such that $(TA_iT^{-1}, TB_iT^{-1} \ldots)$ are all diagonal. When $n > 2$, we should include also those points where there exist matrices that can be simultaneously block diagonalised, for example in the case of $SU(3)$ this would go into $S(U(2) \times U(1))$. If x is a reducible point of $SU(n)^{2g}$ then $f_g(x) = I$ so the connections take values in the abelian subgroup of $SU(n)$.

Now it follows that the diagonal conjugation action of $SU(n)/Z(G) = PU(n)$ (where $Z(G)$ is the centre of $SU(n)$) on $SU(n)^{2g}$ clearly preserves W_g and also by Schur's Lemma the restriction of the action is free. Hence the quotient $W_g/PU(n)$ is a smooth compact Hausdorff space, it is a manifold of dimension $\mathbf{2(g-1)dimG}$.

We can give an *equivalent description* of this moduli space in the holomorphic way (see [2, 20, 21]). The space of connection $\mathcal{A}$ over E is an affine space modeled on $\Omega^1(\Sigma_g, adE)$, such that the tangent space of $\mathcal{A}$ at any point is canonically identified with $\Omega^1(\Sigma_g, adE)$. Let us consider a decomposition of

$$\Omega^1(\Sigma_g, adE) \otimes \mathbf{C} = \Omega^{1,0}(\Sigma_g, adE^C) \oplus \Omega^{0,1}(\Sigma_g, adE^C)$$

If we consider an isomorphism between $\Omega^1(\Sigma_g, adE)$ and $\Omega^{0,1}(\Sigma_g, adE^C)$, we obtain a complex structure on the modeled space of $\mathcal{A}$ and hence also on $\mathcal{A}$. We say $\mathcal{A}$ is the space of $\bar{\partial}$ operators on E^C. In the holomorphic picture we must restrict to a stable bundle [29] in order to obtain a smooth moduli space. A holomorphic vector bundle E is semi-stable over a Riemann surface, if for all sub-bundles F it satisfies

$$\frac{degF}{rankF} \leq \frac{degE}{rankE}$$

Here degree stands for the value of the first Chern class. The vector bundle E is a stable bundle if this inequality is strict. When the degree and the rank are coprime then all the semi stable bundles are stable. In this holomorphic picture the moduli space is interpreted as the space of gauge equivalence classes of stable vector bundles i.e. $\mathcal{M}(\Sigma, G) = \mathcal{A}^S/G^C$, where $Aut(E^C) = G^C$ acts on $\mathcal{A}^S$ with the constant scalars as the only isotropy group. The celebrated theorem of Narasimhan and Seshadri [30] connects both the pictures and it states that stable bundle arising from the representations of π_1 give irreducible representation.

Theorem 2.1 (Narasimhan-Seshadri). [30] *A holomorphic vector bundle of rank n is stable if and only if it arises from an irreducible projective unitary representation of the fundamental group. Moreover isomorphic bundles correspond to equivalent representation.*

The more general moduli space of flat $SU(n)$ connections over punctured Riemann surfaces have been studied by Mehta and Seshadri [28]. In the presence of a marked point on $\sum_g$ we associate a conjugacy class of $SU(n)$ to it.

$$\Gamma \sim \begin{pmatrix} e^{2\pi i\gamma_1/n} & 0 & \cdots & 0 \\ 0 & e^{2\pi i\gamma_2/n} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{2\pi i\gamma_n/n} \end{pmatrix}$$

for all $0 < \gamma_i < 1$, where $\sum_{i=1}^n \gamma_i = 0$. The holonomy around this marked point takes value in this conjugacy class. In presence of the marked points $z_1, z_2, ..., z_p$ we associate a set of conjugacy classes Γ_i of $SU(n)$. Consider a homomorphism

$$\pi_1(\Sigma_g - (z_1 \cup z_2 \cup .... \cup z_p)) \longrightarrow G$$

such that the loop around each z_i takes values in Γ_i, and the moduli space is the quotient by G of the fibre over 1 in the multiplication map

$$\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_p \longrightarrow SU(n).$$

In other words, when we factor out the conjugacy we obtain the moduli space of parabolic bundles with weight $(\gamma_1, \ldots, \gamma_n)$. The dimension of the generalized moduli space [3] is

$$2(g-1)dimG + \sum_{j=1}^{p} dim\Gamma_j.$$

This moduli space of parabolic bundles over the punctured Riemann surface can be given a holomorphic picture too. Mehta and Seshadri [28] have given the notion of stability in this case. This involves assigning weights given by the eigenvalues of Γ_i at each marked point.

3. Volume of the moduli space of $SU(2)$ flat connections

Let us quickly recapitulate the known case. We recall that $S_{\alpha\beta}$ is obtained from the modular transformation induced on the characters of level k. Let $\chi_\alpha(\tau)$ be the character of the affine group $\widehat{G}$ then by the modular transformation $\tau \to \frac{1}{\tau}$ ([DF],[Ka]) we obtain matrix $S_{\alpha\beta}$, where α is the highest weight.

$$\chi_\alpha(-\frac{1}{\tau}) = \sum_\beta S_{\alpha\beta}\chi_\beta(\tau)$$

The key way to construct this character is from Weyl-Kac formula (for example [11, 26]). We define the character of the representation $L(\lambda)$ to be the function

$$ch_\lambda(t) = tr_{L(\lambda)}exp(t)$$

where $t \in \hat{t}$ and $\hat{t}$ is the Cartan subalgebra of $\widehat{sl_n}$. The Weyl- Kac character formula is given by

$$tr_{L(\lambda)} exp(t) = \frac{\sum sign(w) exp(w(\lambda+\rho)|t)}{\sum sign(w) exp(w(\rho)|t)}$$

where summations run over w in the Weyl group. Weyl - Kac character formula is essentially same as Weyl character formula; It differs only two minor ways, i,e besides the usual root vectors, we also describe states by the number operator and the c-number term. Then k is the eigenvalue of the number operator.

The affine Weyl group W_{aff} is the semi-direct product of the ordinary Weyl group and the translation T_λ given by the co-root $\lambda^\vee$ of the highest root λ.

$$S_{ij} = (\frac{2}{k+2})^{\frac{1}{2}} \frac{sin\pi(i+1)(j+1)}{k+2}$$

Using Verlinde's formula we obtain

$$dim H^0(\mathcal{M}, L^k) = \sum_j (\frac{1}{S_{0,j}})^{2g-2}$$

$$R.H.S = \sum_j (\frac{1}{S_{0,j}})^{2g-2}$$

$$= (\frac{k+2}{2})^{g-1} \sum_{j=0}^{k} (\frac{1}{\frac{sin\pi(j+1)}{k+2}})^{2g-2}.$$

Since our goal is to obtain a formula for the volume of the moduli space $\mathcal{M}$ we need to extract a term proportional to $k^{dim_C \mathcal{M}} = k^{3g-3}$ for $k \longrightarrow \infty$. The two regions, namely, $j << k$ and $k - j << k$ make equal contributions. In order to see this we use asymptotic analysis.

3.1. Asymptotic analysis and computation of volume of moduli spaces

We want to show that this is asymptotic to

$$\frac{2k^{3g-3}}{2^{g-1}\pi^{2g-2}} . \sum_{r=1}^{\infty} \frac{1}{r^{2g-2}} \quad \text{as } k \longrightarrow \infty.$$

We will assume that $g \geq 2$ and write $n = 2g-2 \geq 2$. We will also replace k by $l = k+2$, so the sum is:

$$\Sigma_l = (\frac{l}{2})^{n/2} \sum_{j=1}^{l-1} (\frac{1}{sin(\pi j/l)})^n.$$

We divide this sum into the combination from $j \leq l/2$, $j \geq l/2$: these are essentially the same so it suffices to treat the first one. For $\epsilon > 0$, we write

$$\sum_{j=1}^{l/2} (\frac{1}{sin\ \pi j/l})^n = \sum_{j=1}^{[\epsilon l]} (\frac{1}{sin\ \pi j/l})^n + \sum_{[\epsilon l]+1}^{l/2} (\frac{1}{sin\ \pi j/l})^n$$

$$= S + T, \quad (\text{ say }) .$$

We want to compare the sum S with $S' = \sum_{j=1}^{\infty}(\frac{l}{\pi j})^n$. The difference $S-S'$ arises from two factors – approximating the sin function by its derivative and changing the range of summation. For the first case we have, for small ϵ and $j/l < \epsilon$,

$$\pi j/l \geq sin(\pi j/l) \geq \pi j/l - 1/6(\pi j/l)^3.$$

This implies that, for some constant C,

$$(\frac{l}{\pi j})^n \leq (\frac{1}{sin(\pi j/l)})^n \leq (\frac{k^n}{\pi j})^n(1 + C(\frac{j}{l})^2).$$

So

$$|\sum_{j=1}^{[\epsilon l]}(\frac{1}{sin(\frac{\pi j}{l})})^n - \sum_{j=1}^{[\epsilon l]}(\frac{l}{\pi j})^n|$$

$$\leq C\sum_{j=1}^{\epsilon l}\frac{l^{n-2}}{j^{n-2}} \leq C'l^{n-1},$$

for some C' (since these are $O(l)$ terms in the sum).

For the second factor:

$$\sum_{j=1}^{\infty}(\frac{l}{\pi j})^n - \sum_{j=1}^{[\epsilon l]}(\frac{l}{\pi j})^n$$

$$= \sum_{[\epsilon l]+1}^{\infty}(\frac{l}{\pi j})^n \;=\; O(l^{n-1})$$

by comparing with the integral $\int_{\epsilon l}^{\infty} x^{-n}dx$. So we see that $S-S'$ is $O(l^{n-1})$. Finally consider the other term T:

$$T \;=\; \sum_{[\epsilon l]+1}^{l/2}(\frac{1}{sin(\pi j/l)})^n.$$

In this sum

$$sin(\pi j/l) \geq \delta(\epsilon)$$

say so, for fixed ϵ, $T = O(l)$ (the number of terms in the sum). Putting all of this together we see that

$$\Sigma_l \;=\; (\frac{l}{2})^{n/2}.(2\sum_{j=1}^{\infty}(\frac{l}{\pi j})^n + O(l^{n-1}),$$

which gives the result required.

So for large k, we obtain

$$dim H^0(\mathcal{M}, L^k) \sim 2(\frac{k+2}{2})^{g-1}\sum_{j=o}^{k}(\frac{k+2}{\pi(j+1)})^{2g-2}$$

$$2\frac{k^{3g-3}}{2^{g-1}\pi^{2g-2}}\sum_{n=1}^{\infty}\frac{1}{n^{2g-2}}.$$

This finally yields

$$dimH^0(\mathcal{M},L^k) = 2\frac{k^{3g-3}}{2^{g-1}\pi^{2g-2}}\zeta(2g-2). \tag{3.1}$$

From the algebraic geometry point of view this dimension can be expressed via Hirzebruch-Riemann-Roch theorem [17]

$$dimH^0(\mathcal{M},L^k) =< exp(kc_1(L)).Td(\mathcal{M}),\mathcal{M} >$$

and for large k,

$$dimH^0(\mathcal{M},L^k) \sim \frac{k^{3g-3}}{(3g-3)!} < c_1(L)^{3g-3},\mathcal{M} > . \tag{3.2}$$

Now $c_1(L)$ is represented by the symplectic form ω in de-Rham cohomology. Hence

$$\frac{< c_1(L)^{3g-3},\mathcal{M} >}{(3g-3)!}$$

coincides with Vol($\mathcal{M}$) So equating the (10) and (11), we obtain

$$Vol(\mathcal{M}) = 2\frac{1}{(2\pi^2)^{g-1}}\sum_{n=1}^{\infty} n^{-(2g-2)}$$

$$= 2\frac{\zeta(2g-2)}{(2\pi^2)^{g-1}}$$

This is known as Witten's volume formula [38] of the moduli space of flat $SU(2)$ connection.

4. Volume of the moduli space of flat $SU(3)$ connections

We first recall some definitions of affine $\widehat{SU(3)}$ characters (for example, [11, 26]). The affine $\widehat{SU(3)}$ characters are labelled by a highest weight $\Lambda = \lambda_1\Lambda_1 + \lambda_2\Lambda_2$ where Λ_i are the fundamental weights and the set of components $\{\lambda_i\}$ are the non-negative integers. If the height of affine $\widehat{SU(3)}$ is $n = k + 3$ with level $k \geq 0$, the highest weights corresponding to unitary representations satisfy $\lambda_1 + \lambda_2 \leq k$. There are thus $\frac{(k+1)(k+2)}{2} = \frac{(n-1)(n-2)}{2}$ independent affine characters. To see these more explicitly, let us consider shifted weight $\lambda = \Lambda + \Lambda_1 + \Lambda_2 = p_1\Lambda_1 + p_2\Lambda_2$. Unitarity of the representations implies that λ belongs to the fundamental domain $\mathcal{W}$

$$\mathcal{W} = \{\lambda = p_1\Lambda_1 + p_2\Lambda_2 \ , \ p_i \geq 1 \text{ and } p_1 + p_2 \leq n-1\}$$

where Λ_i are the fundamental weights and the set of components $\{p_i\}$ are truncated by the level k. $\mathcal{W}$ is known as Weyl alcove (see for example, [8, 15, 26]).

Our starting point will be Weyl- Kac character for $\widehat{SU(3)}$. We obtain the matrix $S_{\alpha\beta}$(see details in [26]) from the modular transformations $\tau \longrightarrow -\frac{1}{\tau}$ of χ. The S matrix of $\widehat{SU(3)}$ is given below.

$$S_{0\lambda} = \frac{8}{\sqrt{6}(k+3)} sin\frac{\pi\lambda_1}{k+3} sin\frac{\pi\lambda_2}{k+3} sin\frac{\pi(\lambda_1+\lambda_2)}{k+3}, \tag{4.1}$$

which can be also derived from Weyl-Kac factorized form [11]:

$$\phi_\lambda = \frac{sin\frac{\pi\lambda_1}{k+3} sin\frac{\pi\lambda_2}{k+3} sin\frac{\pi(\lambda_1+\lambda_2)}{k+3}}{sin^2\frac{\pi}{k+3} sin\frac{2\pi}{k+3}}$$

after normalization .

Note that $k+3$ is the shifting of level k, and the shifting will be exactly equal to the Coxeter number of the group G. The Coxeter number of $SU(n)$ is n. Substituting the modular transformation $S_{0\lambda}$ in (1) we obtain the Verlinde's formula for the moduli space of flat $SU(3)$ connections.

$$dimH^0(\mathcal{M}, L^k) = \frac{(k+3)^{2g-2}6^{g-1}}{2^{7g-7}} \sum_{\lambda_1,\lambda_2} \left(\frac{1}{sin\frac{\pi\lambda_1}{k+3} sin\frac{\pi\lambda_2}{k+3} sin\frac{\pi(\lambda_1+\lambda_2)}{k+3}}\right)^{2g-2}. \tag{4.2}$$

Here the summation satisfies $\lambda_1 + \lambda_2 \leq k+2$.

To find the volume, again our goal is to extract the term proportional to

$$k^{dim_c\mathcal{M}} = k^{8g-8}$$

for $k \longrightarrow \infty$.

Like the $SU(2)$ case , here the contribution for large k comes from 3 different regions. Finally we obtain ,

$$dimH^0(\mathcal{M}, L^k) \sim 3\frac{k^{8g-8}}{(2\pi)^{6g-6}2^{g-1}} 6^{g-1} \sum_{\lambda_1=1,\lambda_2=1}^{\infty} \frac{1}{\lambda_1^{2g-2}} \frac{1}{\lambda_2^{2g-2}} \frac{1}{(\lambda_1+\lambda_2)^{2g-2}}$$

$$dimH^0(\mathcal{M}, L^k) \sim 3\frac{k^{8g-8}}{(2\pi)^{6g-6}2^{g-1}} 6^{g-1}\zeta_g(2g-2)$$

where this generalized zeta function

$$\zeta_g(2g-2) = \sum n_1^{-(2g-2)} n_2^{-(2g-2)} (n_1+n_2)^{-(2g-2)}$$

can be expressed in terms of double Bernoulli numbers [4, 5, 31] or multiple zeta functions [40, 41].

Hence using the Riemann-Roch formula , for large $k \longrightarrow \infty$ we obtain,

$$dimH^0(\mathcal{M}, L^k) \sim \frac{k^{8g-8}}{(8g-8)!} < c_1(L)^{8g-8}, \mathcal{M} >$$

Again $c_1(L)$ is represented by the symplectic form ω and

$$\frac{< c_1(L)^{8g-8}, \mathcal{M} >}{(8g-8)!}$$

coincides with $Vol(\mathcal{M})$. Hence we obtain

$$Vol(\mathcal{M})_{SU(3)}) = 3\frac{6^{g-1}}{(2\pi)^{6g-6}2^{g-1}}\zeta_g(2g-2)$$

Finally, using the formula [41] of the multiple zeta function

$$\sum_{m,n}^{\infty}\frac{1}{m^s n^s (m+n)^s} = \frac{4}{3}\sum_{0\leq r\leq s; reven}\begin{pmatrix} 2s-r-1 \\ s-1 \end{pmatrix}\zeta(r)\zeta(3s-r)$$

we obtain following examples.

Example For $g=2$ we know the value of the zeta function from Don Zagier [40, 41]. $\zeta_2(\lambda, 2) = (2\pi)^6/7!36$. So the volume is

$$Vol(\mathcal{M}) = 3.\frac{6}{(2\pi)^6.2}.\frac{(2\pi)^6}{7!36} = 1/4.7!$$

This is the first generalization of Witten's result [38] for moduli space of $SU(2)$ flat connection to moduli space of flat $SU(3)$ connections.

5. Cohomological pairings of the moduli space

This is the central theme of the whole talk. Our goal here is to find out the cohomological pairings of the moduli space of flat $SU(3)$ connections on the Riemann surface. Our recipe to find the volume will be to use a generalized Verlinde's formula (for the marked point case) (1.8) in the large k-limit. This volume formula contains all the information of certain cohomological pairings .

5.1. Review of Donaldson-Thaddeus-Witten's work on $SU(2)$ moduli space

Let $\mathcal{M}_1$ be the moduli space of flat $SU(2)$ connections. For a rational number $0 < t < 1$, we consider $\mathcal{M}_t$ to be the moduli space of flat connections on $\Sigma_g - x$, such that monodromy around x is in the conjugacy classes of $SU(2)$

$$\mathcal{T} = \begin{pmatrix} \exp(i\pi t) & 0 \\ 0 & exp(-i\pi t) \end{pmatrix}$$

One can show that for t close to 1, $\mathcal{M}_t$ is a CP^1 bundle over the moduli space $\mathcal{M}_1$.

$$\begin{array}{ccc} \mathbf{CP}^1 & \longrightarrow & \mathcal{M}_t \\ & & \downarrow \\ & & \mathcal{M}_1 \end{array}$$

For $\mathcal{M}_t$, we still have a natural symplectic structure ω but the periods of ω are no longer integers. Then ω is expressed by $a + th$ in $H^2(\mathcal{M}_t)$ generated by $a \in H^2(\mathcal{M}_1)$ and $h \in H^2(\mathcal{M}_t)$ takes value 1 on the fibre. Hence for small t, its symplectic volume will be

$$Vol^S(\mathcal{M}_t) =< \frac{1}{(3g-2)!}(a+th)^{3g-2}, [\mathcal{M}_t] >$$

Using the relation $h^2 = b \in H^4[\mathcal{M}_1]$ we can expand the above expression

$$\frac{1}{(3g-2)!} \sum_{j=0}^{(\frac{3g-2}{2})} \begin{pmatrix} 3g-2 \\ 2j+1 \end{pmatrix} t^{2j+1} a^{3g-3-2j} b^j [\mathcal{M}_1]. \tag{5.1}$$

On the other hand we use Witten's prescription [38] to obtain the volume of the moduli space of flat $SU(2)$ connections over the Riemann surface with p-marked points from the generalized Verlinde's formula (8) in the large k limit.

$$Vol^F(\mathcal{M}_t) = 2.\frac{1}{2^{g-1}\pi^{2g-2+p}} \sum_{n=1}^{p} \frac{\prod_{i=1}^{p} sin(\pi n t_i)}{n^{2g-2+p}}$$

This volume for the one marked point case is

$$Vol(\mathcal{M}_t) = \frac{2}{2^{g-1}\pi^{2g-1}} \sum_{n=1}^{\infty} \frac{sin(n\pi t)}{n^{2g-1}}. \tag{5.2}$$

Equating two expressions (5.1) and (5.2), one obtains the pairing in terms of Bernoulli numbers.

$$< a^m b^n, [\mathcal{M}] >= (-1)^g \frac{m!}{(g-1-m)!} 2^{1-g}(2^{g-1-m} - 2) B_{m-g+1}$$

where $m = 3g - 3 - 2j$ and we have used

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2.(2k)!} B_{2k}$$

This exactly coincides with Thaddeus formula [32] which is verified by Donaldson [13] using topological gluing techniques extracted from the Verlinde algebra.

After the demonstration of the known case we shall give our result in the remaining part of the article.

5.2. Cohomological pairings for $SU(3)$ connection

This is the final part of the article. Our goal is to obtain the cohomological pairings for the moduli space of flat $SU(3)$ connections.

To begin with, let $\mathcal{M}_t$ be the moduli space of flat $SU(3)$ connections over a Riemann surface $\Sigma_g - x$ having one marked point x such that the holonomy around x is characterized by two rational numbers t_1 , t_2 satisfying $0 < t_1 < 1$

and $0 < t_2 < 1$.The prescribed holonomy around x takes values in the conjugacy classes of $SU(3)$

$$\Theta \sim \begin{pmatrix} e^{2\pi i t_1/3} & 0 & 0 \\ 0 & e^{2\pi i t_2/3} & 0 \\ 0 & 0 & e^{-2\pi i(t_1+t_2)/3} \end{pmatrix}$$

Then for small values of t, $\mathcal{M}_t$ is the bundle over the ordinary smooth moduli space and the flag manifold is the fibre on it. It can be represented by

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{M}_t \\ & & \downarrow \\ & & \mathcal{M}_1 \end{array}$$

In other words, the fiber is a flag manifold

$$\mathcal{F} = \frac{SU(3)}{U(1) \times U(1)} = \frac{SL(3,C)}{B^+},$$

where B^+ is the Borel subgroup of $SL(3,C)$. We now give a brief description of the flag manifold from the classic Bott and Tu [10].

5.2.1. Flag manifolds and cohomology. We define a *flag* in a complex vector space V of dimension n as a sequence of subspaces

$$V_1 \subset V_2 \subset .... \subset V_n \quad , dim_{\mathbf{C}} A = i.$$

Let $Fl(V)$ be the collection of all flags in V. Any flag can be carried into any other flag in V by an element of the general linear group $GL(n,C)$, and the stabilizer of a flag is the Borel subgroup B^+ of the upper triangular matrices. Then a set $Fl(V)$ is isomorphic to the coset space $GL(n,C)/B^+$. The quotient of any smooth manifold by the free action of a compact Lie group is again a smooth manifold. Hence $Fl(V)$ is a manifold and it is called the flag manifold of V.

Similarly we can construct a flag structure on bundles. Let $\pi : E \longrightarrow M$ be a C^∞ complex vector bundle of rank n over a manifold M. The associated flag bundle $Fl(E)$ is obtained from E by replacing each fibre E_p by the flag manifold $Fl(E_p)$, the local trivialization

$$\phi_\alpha : E|_{U_\alpha} \simeq U_\alpha \times \mathbf{C}^n$$

induces a natural trivialization

$$Fl(E)|_{U_\alpha} \simeq U_\alpha \times Fl(\mathbf{C}^n).$$

Since $GL(n,C)$ acts on $Fl(\mathbf{C}^n)$ we may take the transition function of $Fl(E)$ to be those of E.

Let us discuss a few things about split manifold. Given a map $\sigma : Fl(E) \longrightarrow M$ we can define a split manifold as follows:

1. the pull back of E to $F(E)$ splits into a direct sum of line bundles

$$\sigma^{-1}E = L_1 \oplus .... \oplus L_n.$$

2. σ^* embeds $H^*(M)$ in $H^*(Fl(E))$.

The split manifold $Fl(E)$ is obtained by a sequence of $n-1$ projectivization. We shall now apply all these to obtain cohomology rings of flag manifolds.

Proposition 5.1. *The associated flag bundle $Fl(E)$ of a vector bundle is the split manifolds.*

Proof: Given in Bott [10] (chapter 4).
□

If E is a rank n complex vector bundle over M, then the cohomology ring of its projectivization is

$$H^*(P(E)) = H^*(M)[c_1, ..., c_n, d_1, ..., d_n]/\{C(S)C(Q) = \pi^* C(E)\}$$

where $c_1, ..., c_n$ are the Chern classes of universal subbundle S and $d_1, ..., d_n$ are those of the universal quotient bundle Q. Also $C(S)$ and $C(Q)$ denote the total Chern classes of S and Q respectively. The flag manifold is obtained from a sequence of $(n-1)$ projectivization

$$H^*(Fl(E))$$

$$= H^*(M)[C(S_1), ..., C(S_{n-1}), C(Q_1, ..., Q_{n-1})]/C(S_1)...C(S_{n-1})C(Q_{n-1}) = C(E)$$

If

$$h_i = C_1(S_i) i = 1...n-1$$

$$h_n = C(Q_{n-1})$$

then we have

$$H^*(Fl(E)) = H^*(M)[h_1, .., h_n]/(\prod_{i=1}^{n}(1+h_i)) = C(E))$$

In order to obtain the cohomology ring of the flag manifold F [10] we have to consider a trivial bundle over a point.

$$H^*(F) = R[h_1, ..., h_n]/(\prod_{i=1}^{n}(1+h_i) = 1)$$

For a special case , when $n = 3$, we obtain

$$H^*(F) = R[h_1, h_2, h_3]/(\prod_{i=1}^{3}(1+h_i) = 1).$$

5.2.2. Computation of the intersection pairings. We are going to apply our previous scheme. Since $\mathcal{M}_t$ is a bundle over $\mathcal{M}_1$, and so we can pull back the cohomology from the base manifold $\mathcal{M}_1$. In fact, it is not hard to see that the symplectic form ω represents the class

$$a + t_1 h_1 + t_2 h_2 \in H^2(\mathcal{M}_t),$$

where $a \in H^2(\mathcal{M}_1)$ and $h_i \in H^2(\mathcal{M}_t)$. So from this symplectic form the volume will be the following

$$Vol^S(\mathcal{M}_t) =< \frac{1}{(8g-5)!}(a + t_1 h_1 + t_2 h_2)^{8g-5}, [\mathcal{M}_t] >$$

When we expand out this expression we obtain the following results.

Proposition 5.2.

$$\boxed{Vol^S(\mathcal{M}_t) = \sum_{k,l} \frac{1}{(8g-5-k-l)!k!l!} t_1^k t_2^l < a^{8g-5-k-l} h_1^k h_2^l, [\mathcal{M}_t] >}$$

But to get the exact pairing we have to use the knowledge of Witten's volume function for small t_1, t_2 and also we use the following identities viz.
$h_1^2 = b \in H^4$; $h_2^2 = c \in H^4$; $-h_1 h_2 = d \in H^4$;

$h_1^2 h_2 = e \in H^6$; $-h_2^2 h_1 = f \in H^6$;

Note that $h_1^2 h_2 = -h_1 h_2^2 =$ fundamental class of Flag manifold, these are top cohomology modules. The key lemma for obtaining the cohomology ring over the moduli space $\mathcal{M}_1$ follows from the Leray-Hirsch theorem [10]

Theorem 5.3. *(Leray-Hirsch) Let $\mathcal{E}$ be a fibre bundle over a manifold M with fibre $\mathcal{F}$. Assume M has finite good cover and suppose there are global cohomology classes $e_1, e_2, ..., e_r$ on $\mathcal{E}$ which when restricted to each fibre freely generate the cohomology of the fibre. Then $H^*(\mathcal{E})$ is a free module over $H^*(M)$ with basis $\{e_1, e_2, ..., e_r\}$ i.e.*

$$\begin{aligned} H^*(\mathcal{E}) &\cong H^*(M) \otimes \mathbf{R}[e_1, e_2, ..., e_r] \\ &\cong H^*(M) \otimes H^*(\mathcal{F}). \end{aligned}$$

Now we state an important statement.

Lemma 5.4. *The fundamental classes of $\mathcal{M}_t$ is the product of the fundamental classes of the moduli space without marked point $\mathcal{M}_1$ and the fundamental classes of the flag manifold.*

We use the same recipe, i.e. extracting volume from the generalized Verlinde's formula (1.8) to find the volume of the moduli space.

If we feed the value of $S_{\alpha\beta}$ of $\widehat{SU(3)}$ obtained from the modular transformation of the Weyl-Kac character in (8) and repeat the derivation as in the previous section we obtain the torsion volume

$$Vol^F(\mathcal{M}_t) = \frac{3.6^{g-1}}{2^{7g-7}\pi^{6g-3}} \sum \frac{sin\pi n_1 t_1 sin\pi n_2 t_2 . sin\pi(n_1+n_2)(t_1+t_2)}{n_1^{2g-1} n_2^{2g-1} (n_1+n_2)^{2g-1}}.$$

This is the generalization of Witten's volume formula for the moduli space of flat $SU(3)$ connections. It is a volume of the moduli space of flat $SU(3)$ connections over a Riemann surface of genus g with one marked point.

After a tedious calculation which makes use of the Taylor expansion of $sin\pi n_1 t_1$, $sin\pi n_2 t_2$ and $sin\pi(n_1+n_2)(t_1+t_2)$ the above expression for small t_1, t_2 gives us a comprehensive formula.

Proposition 5.5.

$$\boxed{Vol(\mathcal{M}_t) = \frac{3.6^{g-1}}{2^{7g-7}\pi^{6g-3}} \sum_n \sum_j \frac{(-1)^{j_1+j_2+j_3} \pi^{2j_1+2j_2+j_3} t_1^{2(j_1+j_3-j_4)+1} t_2^{2(j_2+j_4)+2}}{(2j_1+1)!(2j_2+1)!(2j_3-2j_4)!(2j_4+1)! n_1^{2g-2j_1-2} n_2^{2g-2j_2-2} (n_1+n_2)^{2g-2j_3-2}}}$$

This is the key formula for getting the cohomology pairings of the moduli space of flat $SU(3)$ connections, this is Witten's volume formula for moduli space of flat $SU(3)$ connections. This formula is too big but can be handled for some lower genus cases.

5.3. Concrete examples

It is clear that the two volumes of the moduli space, namely the symplectic volume $Vol^S(\mathcal{M}_t)$ and the volume from Verline's formula $Vol(\mathcal{M}_t)$ in claim 9 are equal.

Using this simple prescription we obtain the *explicit examples* of the cohomological pairings of the moduli space of flat $SU(3)$ connections. These pairing is expressed in terms of multiple zeta function [40, 41] or double Bernoulli numbers [4, 5]. Equating the powers of $t_1^k t_2^l$ we obtain explicit pairings.

1. We consider $genus = 3$. Thus we obtain

$$< a^{10} e^1 f^1 [\mathcal{M}] >= \frac{3.7!5!3!.2^2.8.9.10}{2^8.(2\pi)^6} \zeta_{SU(3)}$$

From Zagier's formula we now come to know that the value this function is $(2\pi)^6./7!.36$. Hence

$$< a^{10} e^1 f^1 [\mathcal{M}] >= (10.9.3.6.2).(2^4.5!)/4.36.2^6 = 5.9.15 = 675$$

2. Once again consider $genus = 3$. We obtain

$$< a^{10} f^2 [\mathcal{M}] >= \frac{10.9.8.6.7!6!.2^2}{(2\pi)^6.2^8} {}_2\zeta(2, A)$$

i.e.

$$< a^{10} f^2 [\mathcal{M}] >= 10.9.8.5!.2^2/2^8 = 1350$$

Thus, we give two explicit examples of pairings. Indeed it is really hard to compute any arbitrary higher genus pairings. Hope our readers realise the degree of complications for further computations of intersection pairings.

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