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Abstract

We study the equivalence of quantum states under local unitary transformations by using the singular value decomposition. A complete set of invariants under local unitary transformations is presented for several classes of tripartite mixed states in $C^K \otimes C^M \otimes C^N$ composite systems. Two density matrices in the same class are equivalent under local unitary transformations if and only if all these invariants have equal values for these density matrices.

1 Introduction

Quantum entanglement is playing very important roles in quantum information processing. Quantum entangled states are the key resource in quantum information processing [1] such as teleportation, super-dense coding, key distribution, error correction and quantum repeater. Therefore it is of great importance to classify and characterize the quantum states.

The nature of the entanglement among the parts of a composite system does not depend on the labeling of the basis states of the individual subsystems. It is therefore invariant under unitary transformations of the individual state spaces. Such transformations are referred to as local unitary transformations. The polynomial invariants of local unitary transformations have been discussed in [2, 3, 4]. General methods, which allow in principle to compute all such invariants, but are in fact not really operational, were introduced in [5, 6, 7, 8]. More explicit complete and partial solutions have been found for some special cases: two qubits [9] and three qubits [10, 11] systems, three qutrits [12], generic mixed states [13], special families of tripartite pure qudits [14, 15, 16].
The problem of classifying states under local unitary transformations can be solved completely for bipartite pure states. As the set of Schmidt coefficients forms a complete set of invariants under local unitary transformations, two bipartite pure states are equivalent under local unitary transformations if and only if they have the same Schmidt coefficients. For multiple composite system, there does not exist Schmidt decomposition in general. There are different generalizations for Schmidt decomposition in multipartite quantum pure states [17, 18, 19, 20, 21], but the results are not sufficient to solve the local equivalence problem. For multipartite mixed states, much less is known about the equivalence under local unitary transformations.

Another classification of quantum states is the one under stochastic local operations and classical communications (SLOCC). Invariants under SLOCC have been also extensively studied [22, 23, 24]. Recently, Lamata et al. [25] used the method of singular value decomposition and presented an inductive classification of multipartite qubit systems under SLOCC.

In this letter, we study the equivalence of multipartite mixed states under local unitary transformations by using the singular value decomposition. Let $H_1$ (resp. $H_2$) be $M$ (resp. $N$) dimensional complex Hilbert spaces ($M \leq N$). A mixed state $\rho$ in $H_1 \otimes H_2$ with rank $r(\rho) = n \leq M^2$ can be decomposed according to its eigenvalues $\lambda_i$ and eigenvectors $|\nu_i\rangle$, $i = 1, ..., n$:

$$\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle \langle \nu_i|.$$

In [26], a class of bipartite mixed states $\Gamma_0$ has been defined: $\Gamma_0$ contains all the states $\rho$ in $H_1 \otimes H_2$ satisfying

$$[\rho_i, \rho_j] = 0, \quad [\theta_i, \theta_j] = 0, \quad i, j = 1, 2, \ldots, n,$$

where $\rho_i$ are full rank matrices,

$$\rho_i = Tr_2 |\nu_i\rangle \langle \nu_i|, \quad \theta_i = (Tr_1 |\nu_i\rangle \langle \nu_i|)^*, \quad i = 1, \ldots, n,$$

$Tr_1$ (resp. $Tr_2$) denotes the partial trace over $H_1$ (resp. $H_2$). We denote by $^\dagger$, $^*$ and $^t$ the adjoint, complex conjugation and transposition, respectively.

It has been shown that two mixed states in $\Gamma_0$ are equivalent under local unitary transformations if and only if the following invariants ((a) or (b)) have the same values for both mixed states [26]:

(a) $Tr(\rho_i^\alpha), \quad Tr(\rho^\gamma), \quad \alpha = 1, 2, \ldots, M, \quad \gamma = 1, 2, \ldots, MN.$

(b) $Tr(\theta_i^\beta), \quad Tr(\rho^\gamma), \quad \beta = 1, 2, \ldots, N, \quad \gamma = 1, 2, \ldots, MN.$

The set of such states in $\Gamma_0$ is not trivial. In fact, $\Gamma_0$ is a subset of the Schmidt-correlated (SC) states [27]. The Schmidt-correlated (SC) states are defined as mixtures of pure states, sharing the same Schmidt bases. It was first appeared in [28],

2
named as maximally correlated state. For Schmidt-correlated states, for any classical measurement, two observers Alice and Bob will always obtain the same result. Two SC states can always be optimally discriminated locally. It is interesting that maximally entangled states (Bell state) can always be expressed in Schmidt correlated form. SC states naturally appear in a bipartite system dynamics with additive integrals of motion [29]. Hence, these states form an important class of mixed states from a quantum dynamical perspective. From the definition of SC state, we know the states in $\Gamma_0$ are all SC states. Therefore we can judge whether a state in $\Gamma_0$ is separable or not by calculating the negativity of this state [30].

Here we give a simple way to construct some families of states in $\Gamma_0$. For $M = N = 4$, one can set $|\psi_1\rangle = (|00\rangle + |12\rangle + |21\rangle + |33\rangle)/2$ and $|\psi_2\rangle = (|01\rangle + |10\rangle + |23\rangle + |32\rangle)/2$, where $|ij\rangle$, $i = 0, 1, ..., M - 1$, $j = 0, 1, ..., N - 1$, are the basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\rho = \alpha|\psi_1\rangle\langle\psi_1| + (1 - \alpha)|\psi_2\rangle\langle\psi_2|$ is a rank two state belonging to $\Gamma_0$ for $0 < \alpha < 1$. For general even $M = N = d + 1$, a state $\rho = \alpha|\psi_1\rangle\langle\psi_1| + (1 - \alpha)|\psi_2\rangle\langle\psi_2|$ is in $\Gamma_0$, where $|\psi_1\rangle = (|00\rangle + |12\rangle + |21\rangle + |34\rangle + |43\rangle + ... + |dd\rangle)/\sqrt{M}$ and $|\psi_2\rangle = (|01\rangle + |10\rangle + |23\rangle + |32\rangle + ... + |d - 1, d\rangle + |d, d - 1\rangle)/\sqrt{M}$.

For $M = N = 5$, one can set $|\phi_1\rangle = (|00\rangle + |12\rangle + |21\rangle + |34\rangle + |43\rangle)/\sqrt{5}$ and $|\phi_2\rangle = (|01\rangle + |10\rangle + |23\rangle + |32\rangle + |44\rangle)/\sqrt{5}$. Then $\rho = \alpha|\phi_1\rangle\langle\phi_1| + (1 - \alpha)|\phi_2\rangle\langle\phi_2|$ is a rank two state in $\Gamma_0$. For general odd $M = N$, $|\phi_1\rangle$ and $|\phi_2\rangle$ can be similarly constructed.

We can also construct higher rank states in $\Gamma_0$. For example, for $M = N = 4$, by adding $|\psi_3\rangle = (|11\rangle + |02\rangle + |20\rangle + |33\rangle)/2$, we have that $\rho = \alpha|\psi_1\rangle\langle\psi_1| + \beta|\psi_2\rangle\langle\psi_2| + (1 - \alpha - \beta)|\psi_3\rangle\langle\psi_3|$ is a state in $\Gamma_0$. For odd $M = N = 5$, we have $|\phi_3\rangle = (|04\rangle + |13\rangle + |22\rangle + |31\rangle + |40\rangle)/\sqrt{5}$ and $\rho = \alpha|\phi_1\rangle\langle\phi_1| + \beta|\phi_2\rangle\langle\phi_2| + (1 - \alpha - \beta)|\phi_3\rangle\langle\phi_3| \in \Gamma_0$.

The states constructed above are all distillable. The rank of reduced density matrices, which are in fact identity matrices, are greater than the rank of $\rho$ itself. They are all NPPT (non positive partial transpose) entangled states.

## 2 Tripartite Quantum Pure States

We first discuss the locally invariant properties of arbitrary dimensional tripartite pure quantum states. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be $K$, $M$ and $N$ dimensional complex Hilbert spaces with the orthonormal bases $\{|e_i\rangle\}_{i=1}^K$, $\{|f_i\rangle\}_{i=1}^M$ and $\{|h_i\rangle\}_{i=1}^N$, respectively.

$|\Psi\rangle$ can be regarded as a bipartite state by taking $\mathcal{H}_1$ (resp. $\mathcal{H}_2$, $\mathcal{H}_3$) and $\mathcal{H}_2 \otimes \mathcal{H}_3$ (resp. $\mathcal{H}_1 \otimes \mathcal{H}_3$, $\mathcal{H}_1 \otimes \mathcal{H}_2$) as the two subsystems. We denote these three bipartite decompositions as $1 - 23$ (resp. $2 - 13$, $3 - 12$). Let $a_{ijk}$ be the coefficients of $|\Psi\rangle$ in orthonormal bases $|e_i\rangle \otimes |f_j\rangle \otimes |h_k\rangle$. Let $A_1$ (resp. $A_2$, $A_3$) denote the matrix with respect to the bipartite state in $1 - 23$ (resp. $2 - 13$, $3 - 12$) decomposition,
i.e. taking the subindices \( i \) (resp. \( j, k \)) and \( jk \) (resp. \( ik, ij \)) of \( a_{ijk} \) as the row and column indices of \( A_1 \) (resp. \( A_2, A_3 \)).

Taking partial trace of \( |\Psi\rangle\langle\Psi| \) over the respective subsystems, we have \( \tau_1 = Tr_1|\Psi\rangle\langle\Psi| = A_1^+, A_1^+ \), \( \tau_2 = Tr_2|\Psi\rangle\langle\Psi| = A_2^+, A_2^+ \), \( \tau_3 = Tr_3|\Psi\rangle\langle\Psi| = A_3^+, A_3^+ \). The reduced matrices \( \tau_1, \tau_2 \) and \( \tau_3 \) can be decomposed according to their eigenvalues and eigenvectors, e.g.

\[
\tau_1 = \sum_{i=1}^{n_1} \lambda_i^1 \nu_i^1 \nu_i^1, \tag{1}
\]

where \( \lambda_i^1 \) resp. \( \nu_i^1 \), \( i = 1, ..., n_1 \), are the nonzero eigenvalues resp. eigenvectors of the density matrix \( \tau_1 \).

Let \( A_1^i \) denote the matrix with entries given by the coefficients of \( \nu_i^1 \) in the bases \( |f_k⟩ \otimes |h_l⟩ \). We have

\[
\rho_i^1 = Tr_3|\nu_i^1⟩⟨\nu_i^1| = A_i^+, A_i^+, \quad \theta_i^1 = (Tr_2|\nu_i^1⟩⟨\nu_i^1|)^* = A_i^+, A_i^+, \quad i = 1, ..., n_1.
\]

Set

\[
I_{\alpha}^1(|\Psi⟩) = Tr(\rho_{\alpha}^1), \quad \alpha = 1, 2, \cdots, M,
\]

\[
J_{\beta}^1(|\Psi⟩) = Tr(\theta_{\beta}^1), \quad \beta = 1, 2, \cdots, N,
\]

\[
K_{\gamma}^1(|\Psi⟩) = Tr(\tau_{\gamma}^1), \quad \gamma = 1, 2, \cdots, MN.
\]

It is easy to prove that \( I_{\alpha}^1(|\Psi⟩) \), \( J_{\beta}^1(|\Psi⟩) \) and \( K_{\gamma}^1(|\Psi⟩) \) are all invariants under local unitary transformations.

Let \( \Gamma_1 \) denote a class of tripartite pure states \( |\Psi⟩ \) satisfying

\[
[rho_i^1, rho_j^1] = 0, \quad [theta_i^1, theta_j^1] = 0 \tag{2}
\]

with \( rho_i^1 \) being full rank matrices, \( i, j = 1, 2, \cdots, n_1 \).

**[Theorem 1]** Two pure states in \( \Gamma_1 \) are equivalent under local unitary transformations if and only if the following invariants ((c) or (d)) have the same values for both states:

\[
(c) \quad I_{\alpha}^1(|\Psi⟩), \quad K_{\gamma}^1(|\Psi⟩), \quad \alpha = 1, 2, \cdots, M, \quad \gamma = 1, 2, \cdots, MN.
\]

\[
(d) \quad J_{\beta}^1(|\Psi⟩), \quad K_{\gamma}^1(|\Psi⟩), \quad \beta = 1, 2, \cdots, N, \quad \gamma = 1, 2, \cdots, MN.
\]

We only need to prove the sufficient part. Assume \( |\Psi⟩, |\Psi'⟩ \in \Gamma_1 \). \( K_{\gamma}^1(|\Psi⟩) = K_{\gamma}^1(|\Psi'⟩) \) imply that \( A_1 \) and \( A_1' \) have the same singular values, therefore there exists unitary matrices \( U_1 \) and \( U_{23} \) such that \( |\Psi'⟩ = U_1 \otimes U_{23} |\Psi⟩ \). If \( I_{\alpha}^1(|\Psi⟩) = I_{\alpha}^1(|\Psi'⟩) \) or \( J_{\beta}^1(|\Psi⟩) = J_{\beta}^1(|\Psi'⟩) \) holds, then \( \tau_1 \) and \( \tau_1' \) are equivalent under local unitary transformations by the sufficient condition of equivalence for bipartite states under local unitary transformations. While in [15] it has been proven that if \( |\Psi'⟩ = U_1 \otimes U_{23} |\Psi⟩ \), with \( U_1 \in U(H_1), U_{23} \in U(H_2 \otimes H_3) \) and \( Tr_1 (|\Psi'⟩⟨\Psi'|) = U_2 \otimes U_3 Tr_1 (|\Psi⟩⟨\Psi|) U_2^+ \otimes U_3^+ \).

4
We define \( \Gamma_2 \), where \( U_2 \in \mathcal{U}(\mathcal{H}_2) \) and \( U_3 \in \mathcal{U}(\mathcal{H}_3) \), then there exist matrices \( V_1 \in \mathcal{U}(\mathcal{H}_1) \), \( V_2 \in \mathcal{U}(\mathcal{H}_2) \), \( V_3 \in \mathcal{U}(\mathcal{H}_3) \) such that \( |\Psi') = V_1 \otimes V_2 \otimes V_3 |\Psi \rangle \), i.e., \( |\Psi \rangle \) and \( |\Psi' \rangle \) are equivalent under local unitary transformations.

Let us consider for example two states \( |\Psi \rangle = \sqrt{\frac{2}{3}} (|000\rangle + |012\rangle + |021\rangle) + \sqrt{\frac{1}{3}} (|101\rangle + |110\rangle + |122\rangle) \) and \( |\Psi' \rangle = \sqrt{\frac{2}{3}} (|000\rangle + |011\rangle + |022\rangle) + \sqrt{\frac{1}{3}} (|101\rangle + |112\rangle + |120\rangle) \) in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), for the case \( K = 2, M = N = 3 \). It is direct to verify that they are all states in \( \Gamma_1 \) with \( \rho^1_1 = \theta^1_1 = \frac{1}{3} I, i = 1, 2 \). As \( \tau_1 \) and \( \tau'_1 \) have the same eigenvalues, relation \( K^1_\gamma (|\Psi \rangle) = K^1_\gamma (|\Psi' \rangle) \) holds, from which and the following equations

\[
Tr(\rho^1_1) = Tr(\rho'^1_1) = 1, \quad Tr(\rho^1_1)^2 = Tr(\rho'^1_1)^2 = \frac{1}{3},
\]

by Theorem 1 we have that \( |\Psi \rangle \) and \( |\Psi' \rangle \) are equivalent under local unitary transformations. The same results can be also obtained from \( K^1_\gamma (|\Psi \rangle) = K^1_\gamma (|\Psi' \rangle) \) and the following facts:

\[
Tr(\theta^1_1) = Tr(\theta'^1_1) = 1, \quad Tr(\theta^1_1)^2 = Tr(\theta'^1_1)^2 = \frac{1}{3}.
\]

As an alternative example we consider two states \( |\Psi \rangle = \sqrt{\frac{2}{3}} (|000\rangle + |012\rangle + |021\rangle) + \sqrt{\frac{1}{3}} (|101\rangle + |110\rangle + |122\rangle) + \sqrt{\frac{1}{3}} (|202\rangle + |211\rangle + |220\rangle) \) and \( |\Psi' \rangle = \sqrt{\frac{2}{3}} (|000\rangle + |011\rangle + |022\rangle) + \sqrt{\frac{1}{3}} (|101\rangle + |112\rangle + |120\rangle) + \sqrt{\frac{1}{3}} (|202\rangle + |210\rangle + |221\rangle) \) in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), with \( K = M = N = 3, \alpha, \beta, \gamma \in \mathbb{R}, \alpha + \beta + \gamma = 1 \). One can prove that they are all states in \( \Gamma_1 \) with \( \rho^1_1 = \theta^1_i = \frac{1}{3} I, i = 1, 2, 3 \), and \( \tau_1, \tau'_1 \) have the same eigenvalues. As

\[
Tr(\rho^1_1) = Tr(\rho'^1_1) = 1, \quad Tr(\rho^1_1)^2 = Tr(\rho'^1_1)^2 = \frac{1}{3}, \quad Tr(\rho^1_1)^3 = Tr(\rho'^1_1)^3 = \frac{1}{9},
\]

from Theorem 1 we have \( |\Psi \rangle \) and \( |\Psi' \rangle \) are equivalent under local unitary transformations. Moreover by using the generalized concurrence \([31]\), we have \( C^3_3 \neq 0 \), hence \( |\Psi \rangle \) and \( |\Psi' \rangle \) are entangled.

**Remark** We can also similarly define the set of states \( \Gamma_2 \). Let \( \tau_2 \) be a reduced density matrix by tracing \( |\Psi \rangle \langle \Psi| \) over the second system. \( \tau_2 \) can be decomposed according to its eigenvalues and eigenvectors:

\[
\tau_2 = \sum_{i=1}^{n_2} \lambda^2_i |\nu^2_i \rangle \langle \nu^2_i|,
\]

where \( \lambda^2_i \) resp. \( |\nu^2_i \rangle, i = 1, ..., n_2 \), are the nonzero eigenvalues resp. eigenvectors of the density matrix \( \tau_2 \). Define \( \{\rho^2_i\}, \{\theta^2_i\} \),

\[
\rho^2_i = Tr_3 |\nu^2_i \rangle \langle \nu^2_i|, \quad \theta^2_i = (Tr_1 |\nu^2_i \rangle \langle \nu^2_i|)^*, \quad i = 1, ..., n_2.
\]

We define \( \Gamma_2 \) to be a set of tripartite pure states \( |\Psi \rangle \) satisfying

\[
[\rho^2_i, \rho^2_j] = 0, \quad [\theta^2_i, \theta^2_j] = 0 \quad (3)
\]

\[
5
\]
[Theorem 2] Two pure states in $\Gamma_2$ are equivalent under local unitary transformations if and only if the following invariants ((e) or (f)) have the same values for both states:

$$(e) \quad I_\alpha^2(|\Psi\rangle), \quad K_\gamma^2(|\Psi\rangle), \quad \alpha = 1, 2, \cdots, K, \quad \gamma = 1, 2, \cdots, KN,$$

$$(f) \quad J_\beta^2(|\Psi\rangle), \quad K_\gamma^2(|\Psi\rangle), \quad \beta = 1, 2, \cdots, N, \quad \gamma = 1, 2, \cdots, KN,$$

where $I_\alpha^2(|\Psi\rangle) = Tr(\rho_\alpha^2), J_\beta^2(|\Psi\rangle) = Tr(\theta_\beta^2), K_\gamma^2(|\Psi\rangle) = Tr(\tau_\gamma^2)$.

The set of states $\Gamma_3$ can be defined in a similar way and the corresponding theorem (like theorem 1 and 2) can be obtained similarly.

The results above can be generalized to general many partite systems. As each $n$ partite pure states can be treated as a bipartite one: the $j$th system and rest $n-1$ partite system, by using the results of Lemma 2 in [15], one can similarly obtain a complete set of invariants for some classes of multipartite pure states.

3 Tripartite Quantum Mixed States

We consider now mixed states in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. We assume $K \leq M, N$. Let $\rho$ be a density matrix defined on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ with $r(\rho) = n \leq K^3$. $\rho$ can be decomposed according to its eigenvalues and eigenvectors:

$$\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle \langle \nu_i|,$$

where $\lambda_i$, resp. $|\nu_i\rangle$, $i = 1, \ldots, n$, are the nonzero eigenvalues resp. eigenvectors of the density matrix $\rho$. We introduce

$$\rho_i = Tr_1|\nu_i\rangle \langle \nu_i|, \quad \theta_i = Tr_2|\nu_i\rangle \langle \nu_i|, \quad \gamma_i = Tr_3|\nu_i\rangle \langle \nu_i|.$$

If we treat $|\nu_i\rangle$ as a bipartite state $|\omega_i\rangle$ in $1-23$ system, let $A_{1i}$ denote the matrix with entries given by the coefficients of $|\omega_i\rangle$ in the bases $|e_k\rangle \otimes |g_l\rangle$, where $|g_l\rangle = |f_l\rangle \otimes |h_s\rangle$, $l = ts; \ t = 1, \cdots, M, \ s = 1, \cdots, N$. According to the result of bipartite system, we have

$$Tr_2|\omega_i\rangle \langle \omega_i| = A_{1i}A_{1i}^\dagger, \quad (Tr_1|\omega_i\rangle \langle \omega_i|)^* = A_{1i}^\dagger A_{1i}, \quad i = 1, \ldots, n.$$

As $Tr_2|\omega_i\rangle \langle \omega_i| = Tr_3(Tr_2|\nu_i\rangle \langle \nu_i|)$ and $Tr_1|\omega_i\rangle \langle \omega_i| = Tr_1|\nu_i\rangle \langle \nu_i|$, we have

$$\theta_i^{23} = A_{1i}A_{1i}^\dagger, \quad \rho_i = (A_{1i}^\dagger A_{1i})^*,$$

where $\theta_i^{23} = Tr_3(Tr_2|\nu_i\rangle \langle \nu_i|)$.
$\rho_i$ can be again decomposed according to its eigenvalues and eigenvectors:

$$\rho_i = \sum_{j=1}^{m_i} \alpha_j^i |\mu_j^i\rangle \langle \mu_j^i|,$$

where $\alpha_j^i$ resp. $|\mu_j^i\rangle$, $j = 1, \ldots, m_i$, are the nonzero eigenvalues resp. eigenvectors of the reduced density matrix $\rho_i$. Let $B_j^i$ denote the matrix with entries given by coefficients of $|\mu_j^i\rangle$ in the bases $|f_k\rangle \otimes |h_l\rangle$. We further introduce $\{\xi_j^i\}, \{\eta_j^i\}$,

$$\xi_j^i = Tr_B |\mu_j^i\rangle \langle \mu_j^i| = B_j^i B_j^{i\dagger}, \quad \eta_j^i = (Tr_B |\mu_j^i\rangle \langle \mu_j^i|)^* = B_j^{i\dagger} B_j^i, \quad j = 1, \ldots, m_i.$$

Let $\Gamma$ denote a class of tripartite mixed states satisfying

$$[\rho_i, \rho_k] = 0, \quad [\theta^2_{23}^i, \theta^2_{23}^k] = 0$$

with $\theta^2_{23}^i$ being full rank matrices, $i, k = 1, 2, \ldots, n$, and

$$[\xi_j^i, \xi_j^k] = 0, \quad [\eta_j^i, \eta_j^k] = 0$$

with $\xi_j^i$ being full rank matrices, $\forall i, k = 1, 2, \ldots, n, \ t = 1, 2, \ldots, m_i, \ l = 1, 2, \ldots, m_k$.

[Theorem 3] Two mixed states in $\Gamma$ are equivalent under local unitary transformations if and only if the following invariants ((g) or (h)) have the same values for both mixed states:

$$\begin{align*}
(g) & \quad Tr(\rho_i)^\alpha, \quad Tr(\xi_j^k)^\alpha, \quad Tr(\rho^\gamma), \quad \alpha = 1, 2, \ldots, M, \quad \gamma = 1, 2, \ldots, MN. \\
(h) & \quad Tr(\theta^2_{23}^i)^\beta, \quad Tr(\eta_j^k)^\beta, \quad Tr(\rho^\gamma), \quad \beta = 1, 2, \ldots, N, \quad \gamma = 1, 2, \ldots, MN.
\end{align*}$$

[Proof]: If $\rho$ and $\rho' \in \Gamma$ are equivalent under the local unitary transformation $u \otimes v \otimes w$, $\rho' = u \otimes v \otimes w \ \rho u^\dagger \otimes v^\dagger \otimes w^\dagger$, then $|\nu_i'\rangle = u \otimes V |\nu_i\rangle$, where $V = v \otimes w$, namely $A_{1i}$ is mapped to $A_{1i}' = u A_{1i} V^\dagger$. Therefore

$$\theta^2_{23}^i = A_{1i}' A_{1i}^\dagger = u A_{1i} A_{1i}^\dagger u^\dagger = u \theta^2_{23} u^\dagger,$$

$$\rho_i' = (A_{1i}' A_{1i}^\dagger)^* V (A_{1i}' A_{1i}) V^\dagger = V \rho_i V^\dagger = v \otimes w \rho_i v^\dagger \otimes w^\dagger.$$

Thus $\rho_i$ and $\rho_i'$ are equivalent under the local unitary transformation $v \otimes w$, from the results of bipartite system [26] we have $Tr(\xi_j^k)^\alpha = Tr(\xi_j^k)^\alpha$ and $Tr(\eta_j^k)^\beta = Tr(\eta_j^k)^\beta$. Therefore (g) and (h) hold.

Conversely, $Tr(\rho^\gamma) = Tr(\rho'^\gamma)$ imply that $\rho$ and $\rho'$ have the same eigenvalues. We now prove that there exist common unitary matrices $V_1, V_2, V_3$ such that $|\nu_i'\rangle = V_1 \otimes V_2 \otimes V_3 |\nu_i\rangle$ by using Lemma 2 in [15].

From the relation $Tr(\rho_i)^\alpha = Tr(\rho_i')^\alpha$ in (g) and the condition (4), we have common unitary matrices $U_1$ and $U_{23}$ for all $i$ such that $|\nu_i'\rangle = U_1 \otimes U_{23} |\nu_i\rangle$. 

7
The relation $Tr(\xi^k) = Tr(\xi'^k)$ in (g) and the condition (5) imply that $\rho_i$ and $\rho'_i$ are equivalent under local unitary transformations, $\rho'_i = U_i \otimes V_i \rho_i U_i^\dagger \otimes V_i^\dagger$ according to the results of bipartite system [26]. For the case $i \neq k$ in condition (5), (5) implies that there exist common unitary matrices $U$ and $V$ such that $\rho'_i = U \otimes V \rho_i U^\dagger \otimes V^\dagger$. To elucidate this we just show the case $n = 2$. For a rank-two state $\rho$ we have

$$\rho_1 = \sum_{j=1}^{m_1} \alpha^1_j \beta^1_j, \quad \rho_2 = \sum_{j=1}^{m_2} \alpha^2_j \beta^2_j.$$  

$$Tr(\xi^1) = Tr(\xi'^1)$$ implies that $\xi^1$ and $\xi'^1$ are equivalent under unitary transformations. Therefore $B^1_j$ and $B'^1_j$ have the same singular values.

$$[\xi^1, \xi'^1] = 0$$  \hspace{1cm} (6)

and

$$[\eta^1, \eta'^1] = 0$$  \hspace{1cm} (7)

imply that (from singular value decomposition) there exist common unitary matrices $U_1, U'_1$ and $V_1, V'_1$ such that

$$U_1 B^1_j V_1 = U'_1 B'^1_j V'_1.$$  \hspace{1cm} (8)

While

$$[\xi^2, \xi'^2] = 0$$  \hspace{1cm} (9)

and

$$[\eta^2, \eta'^2] = 0$$  \hspace{1cm} (10)

imply that there exist common unitary matrices $U_2, U'_2$ and $V_2, V'_2$ such that

$$U_2 B^2_j V_2 = U'_2 B'^2_j V'_2.$$  \hspace{1cm} (11)

From (6) and (9), we have $U_1 = U_2$. From (7) and (10), we have $V_1 = V_2$. Hence $B^i_j = U B^i_j V^i$, and $|\mu^i_j\rangle = U \otimes V |\mu^i_j\rangle$, $j = 1, ..., m_i$. Therefore, $\rho'_i = U \otimes V \rho_i U^\dagger \otimes V^\dagger$. Hence $\rho'_i$ and $\rho_i$ are equivalent under local unitary transformations.

Therefore, from Lemma 2 in [15] we have that tripartite states $|\nu_i\rangle$ and $|\nu'_i\rangle$ are equivalent under local unitary transformations. In fact there exist common unitary matrices $V_i, i = 1, 2, 3$, such that $|\nu'_i\rangle = V_1 \otimes V_2 \otimes V_3 |\nu_i\rangle$, where $V_1 = WU_1, V_2 = U, V_3 = V$ ($[\theta^2, \theta^3] = 0$ imply that there exists $W$ for different $\nu_i$). Therefore, we have $\rho' = V_1 \otimes V_2 \otimes V_3 \rho V_1^\dagger \otimes V_2^\dagger \otimes V_3^\dagger$.

Thus from (g) we get that $\rho$ and $\rho'$ are equivalent under local unitary transformations. One can similarly prove $\rho$ and $\rho'$ are equivalent under local unitary transformations from (h).

We have discussed the local invariants for arbitrary dimensional tripartite quantum mixed states in $\mathbb{C}^K \otimes \mathbb{C}^M \otimes \mathbb{C}^N$ composite systems and have presented sets of
invariants under local unitary transformations for some classes of tripartite mixed states. The invariants in a set is not necessarily independent, but they are sufficient to judge if two states in $\Gamma$ or $\Gamma_i$, $i = 1, 2, 3$, are equivalent under local unitary transformations. For three qubits case, $K = M = N = 2$, a set of invariants has been presented in [10, 11] for a special class of states. By using the method in [14, 15], the results can be generalized to detect local equivalence for some special classes of general multipartite states.

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