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Harnack's inequality for some nonlocal equations
and application

by

Jerome Coville

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HARNACK'S INEQUALITY FOR SOME NONLOCAL EQUATIONS AND APPLICATION

JÉRÔME COVILLE

ABSTRACT. In this paper, we establish a Harnack's inequality for positive solutions of the nonlocal inhomogeneous problem

$$\int_{\Omega} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy - a(x)u,$$

where $\Omega \subset \mathbb{R}^n$ is an open set, J is a probability density with compact support and g, b are positive bounded functions. For some particular $a(x)$, using the Harnack's Inequality, we also construct a positive solution of the above equation.

1. INTRODUCTION AND MAIN RESULTS

In the past few years much attention has been drawn to the study of nonlocal reaction diffusion equations, where the usual elliptic diffusion operator is replaced by a nonlocal one of the form

$$(1.1) \quad \mathcal{L}[u] := \int_{\Omega} k(x, y)u(y) dy - b(x)u,$$

where $\Omega \subset \mathbb{R}^n$, $k \geq 0$ satisfies $\int_{\mathbb{R}^n} k(y, x)dy = 1$ for all $x \in \mathbb{R}^n$ and $b(x) \in C(\Omega)$, see among other references [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 20, 21, 22]. Such types of diffusion processes have been widely used to describe the dispersal of a population through its environment in the following sense. As stated in [16, 17, 19] if $u(y, t)$ is thought of as a density at location y at time t and $k(x, y)$ as the probability distribution of jumping from location y to location x , then the rate at which individuals from all other places are arriving to location x is

$$\int_{\Omega} k(x, y)u(y, t) dy.$$

On the other hand, the rate at which individuals are leaving location x to travel to all other places is $-b(x)u(x, t)$.

Equation (1.1) can be seen as a nonlocal analog of the usual elliptic operator

$$\mathcal{M} := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x).$$

Indeed, let us rewrite equation (1.1) in the following way

$$(1.2) \quad \int_{\Omega} k(x, y)[u(y) - u(x)] dy - c(x)u = 0 \quad \text{in } \Omega,$$

with $c(x) := b(x) - \int_{\Omega} k(x, y)dy$. Setting $z = x - y$ and performing a formal Taylor expansion of u in the integral, we can rewrite the nonlocal operator as follows

$$\int_{x-\Omega} k(x, x-z)[u(x-z) - u(x)] dy = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + R[x, \partial_{i_j k}u]$$

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with $a_{ij}(x)$ and $b_i(x)$ defined by the following expressions

$$a_{ij}(x) = \frac{1}{2} \int_{x-\Omega} k(x, x-z) z_i z_j dz$$

$$b_i(x) = \int_{x-\Omega} k(x, x-z) z_i dz.$$

Therefore, on smooth functions, \mathcal{L} appears as a perturbation of \mathcal{M} involving higher derivatives.

For the uniform elliptic operator \mathcal{M} , it is well known that positive solutions of the equation

$$(1.3) \quad \mathcal{M}[u] = 0$$

satisfy a Harnack's inequality, see [15, 18]. That is,

Harnack's inequality

Let u be a positive solution of (1.3), then for any compact subset ω of Ω , such that $\omega \subset\subset \Omega$, there exist a constant $C(\omega)$ such that

$$\sup_{\omega} u \leq C \inf_{\omega} u.$$

In this work, we investigate the validity of such Harnack's inequality for positive solutions of (1.1) when the kernel $k(x, y)$ takes the form

$$(1.4) \quad k(x, y) = J \left(\frac{x-y}{g(y)} \right) \frac{1}{g^n(y)},$$

where J is a continuous probability density and the function g is bounded and positive. Such type of diffusion kernel was recently introduced by Cortazar *et al.* [8] in order to model a one dimensional non homogeneous dispersal process. More precisely, we are interested in finding simple conditions on J , g , b and Ω such that a Harnack's inequality holds for positive solutions of the following equation :

$$(1.5) \quad \int_{\Omega} J \left(\frac{x-y}{g(y)} \right) \frac{u(y)}{g^n(y)} dy - b(x)u = 0. \quad \text{in } \Omega.$$

That is to say:

Harnack's Inequality for nonlocal equation :

Let $u \in C(X, \mathbb{R})$ be a positive solution of (1.5). Then, for any compact subset ω of Ω such that $\omega \subset \Omega$, there exists a constant $C(\omega)$ such that

$$\sup_{\omega} u \leq C \inf_{\omega} u.$$

Harnack's inequality plays an important role in modern analysis of nonlocal equations because it provides various important *a priori* estimates. In particular, it is a key estimate in the construction of a positive solution of the principal eigenvalue problem

$$(1.6) \quad \int_{\mathbb{R}^n} J \left(\frac{x-y}{g(y)} \right) \frac{u(y)}{g^n(y)} dy - b(x)u = -\lambda_1 u \quad \text{in } \mathbb{R}^n,$$

see [8]. It is therefore of great interest to investigate the existence of such estimates.

Throughout this paper, we will always assume that J and g satisfy the following assumptions:

$$J \in C_c(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), J \geq 0, \int_{\mathbb{R}^n} J(z) dz = 1 \quad (H1)$$

$$g \in L^\infty(\Omega), 0 \leq g \leq \beta \quad (H2)$$

$$\frac{1}{g^n} \in L^1_{loc} \quad (H3)$$

$$b \in C(\bar{\Omega}), b(x) > 0 \quad (H4)$$

where $C_c(\mathbb{R}^n)$ denotes the set of continuous functions with compact support. Without loss of generality, we will also assume that the support of J is contained in the unit ball. Under the above assumptions on J , g , and b , we first prove the following:

Theorem 1.1. *Assume (H1–H4) hold and let J and g be such that $J(0) > 0$ and $g \geq \alpha > 0$. Then, for any compact set $\omega \subset\subset \Omega$, there exists $C(J, \omega, b, g)$ such that for all positive bounded solutions u of (1.5) we have*

$$u(x) \leq Cu(y) \quad \text{for all } x, y \in \omega.$$

Our next result concerns the case when g is allowed to vanish and/or the compact set ω can touch the boundary. In such cases, we usually don't have a pointwise Harnack's inequality. However, we still have some kind of uniform estimate. More precisely, with $\mathcal{S} \neq \emptyset$ the set of zeroes of g , we have the following:

Theorem 1.2. *Assume (H1–H4) hold and let J be such that $J(0) > 0$. Assume further that $\Omega \cap \mathcal{S} \subset\subset \Omega$ and let $\omega \subset \bar{\Omega}$ be a compact set. Then there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$ there exists a compact set $\omega' \subset\subset \Omega$ and a constant $C(J, \omega, \Omega, \omega', b, g, \eta)$ such that the following holds*

- (i) $\{x \in \Omega | d(x, \partial(\omega \cap W_\eta)) > \eta\} \subset \omega'$, where $W_\eta := \{x \in \Omega | g(x) > \eta\}$
- (ii) for all positive solution u of (1.5) the following inequality holds:

$$u(x) \leq Cu(y) \quad \text{for all } x \in \omega, y \in \omega' \cap \omega.$$

As a direct consequence of Theorem 1.2, when Ω is a bounded domain we get a uniform estimate on positive solutions u of (1.5). More precisely, we have

Corollary 1.3. *Let Ω , J and g be as in the Theorem 1.2 and assume that Ω is a bounded domain. Then there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$ there exists a compact set $\omega' \subset\subset \Omega$ and a constant $C(J, \Omega, \omega', b, g, \eta)$ such that the following hold*

- (i) $\{x \in \Omega | d(x, \partial W_\eta) > \eta\} \subset \omega'$
- (ii) for all positive bounded solution u of (1.5),

$$\sup_{\Omega} u \leq Cu(y) \quad \text{for all } y \in \omega'.$$

Under an extra assumption on the regularity of the compact set ω , we have a more precise description of the set ω' . Namely, we prove the following result

Theorem 1.4. *Let \mathcal{S} , J and g be as in Theorem 1.2 and let $\omega \subset \bar{\Omega}$ be a compact set satisfying a uniform inner cone condition. Then there exists a positive constant η^* , such that for any $0 < \eta \leq \eta^*$ there exists a constant $C(J, \omega, \Omega, b, g, \eta)$ such that for all positive solution u of (1.5) the following inequality holds:*

$$u(x) \leq Cu(y) \quad \text{for all } x \in \omega, y \in \omega \cap \{y \in \Omega | g(y) > 2\eta\}.$$

As a corollary of Theorem 1.4, we get a Harnack's inequality up to the boundary when $g > \alpha$ and Ω is a bounded regular domain. More precisely, we have:

Corollary 1.5. Let Ω , J , g and b be as in Theorem 1.1 and assume that Ω is a bounded domain satisfying a uniform inner cone condition. Then there exists a constant $C(\Omega, g, J, b)$ such that for all positive bounded solutions u of (1.5) the following holds

$$\sup_{\Omega} u \leq C \inf_{\Omega} u.$$

Our last result is an application of these Harnack's type estimates to the construction of a positive solution of (1.5) for a particular $b(x)$. More precisely, let us consider the equation

$$(1.7) \quad \int_{\mathbb{R}^n} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy - a(x)u = 0 \quad \text{in } \Omega,$$

where $a(x)$ is defined as follows:

$$a(x) := \begin{cases} \int_{\Omega} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)} & \text{if } x \notin \mathcal{S} \\ 1 & \text{otherwise} \end{cases}$$

For this equation, we have the following result

Theorem 1.6. Let Ω , \mathcal{S} , J and g be as in Theorem 1.2. Then there exists a positive solution p of (1.7).

1.1. Comments.

We first point out that such Harnack's inequalities extend easily to the case of dispersal kernels $k(x, y)$ of the form

$$k(x, y) = J\left(\frac{x_1 - y_1}{g_1(y)}; \frac{x_2 - y_2}{g_2(y)}; \dots; \frac{x_n - y_n}{g_n(y)}\right) \frac{1}{\prod_{i=1}^n g_i(y)}.$$

The above dispersal kernel is a natural generalization of (1.4) which takes into account that each coordinate is influenced differently by the environment.

From the proof, we note that the regularity condition imposed on the coefficient is not optimal and can be weakened. In particular, the results hold as well, when we assume that $b(x) \in L^\infty(\Omega)$ with $\inf_{\bar{\Omega}} b(x) \geq b_0 > 0$ and $J \in L^1(\mathbb{R}^n)$ is a non negative, compactly supported function satisfying the following condition:

$$\exists c_0 > 0, \epsilon_0 > 0 \text{ such that } \min_{y \in B(0, \epsilon_0)} J(y) > c_0.$$

Along the proof, we observe that the Harnack's inequalities extend also to non negative kernel $k(x, y)$ which are positive on the diagonal. More precisely, the Harnack's inequalities hold as well for $k(x, y)$ satisfying the following conditions

$$k(x, y) \in C_c(\Omega \times \Omega), k \geq 0, \int_{\Omega} k(x, y) dy < +\infty \quad \forall x \in \Omega \quad (\tilde{H}1)$$

$$\exists c_0 > 0, \epsilon_0 > 0 \text{ such that } \min_{x \in \Omega} \left(\min_{y \in B(x, \epsilon_0)} k(x, y) \right) > c_0. \quad (\tilde{H}2)$$

We also want to emphasize that since there is no condition on the open set Ω , the Harnack's inequalities (Theorems 1.1, 1.2 and 1.4) hold as well when $\Omega = \mathbb{R}^n$. In the later case, when

g is allowed to vanish, the assumption $\Omega \cap \mathcal{S} \subset \subset \Omega$ can be weakened. More precisely, we only require that for any subset $\tilde{\mathcal{S}}$ of \mathcal{S} there exists a ball $B(x_0, R)$ such that

$$B(x_0, R) \cap \tilde{\mathcal{S}} \subset \subset B(x_0, R).$$

Corollary 1.7. Let J, b, g be as in Theorem 1.2 and assume that \mathcal{S} satisfies the above condition. Let $\omega \subset \mathbb{R}^n$ be compact set. Then, there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$ there exists a constant $C(J, \omega, b, g, \eta)$ such that for all positive solutions u of (1.5) the following inequality holds:

$$u(x) \leq Cu(y) \quad \text{for all } x \in \omega, y \in \omega \cap \{y \in \Omega | g(y) > 2\eta\}.$$

We also want to stress that whereas the classical Harnack's inequality obtained in Theorem 1.1 is still true for positive solutions of the classical uniformly elliptic equation (1.3), its extension up to the boundary (Corollary 1.5) is not. The validity of such extension is a consequence of the nonlocal character of the equation we have considered.

We also want to point to recent work on Harnack's inequality for fractional operators by Bass and Kassmann [2] and by Cortazar *et al.* [8] in the particular case of a symmetric kernel in one dimension. In the later, the authors obtained a different type of Harnack's inequality for bounded positive solutions p of (1.4). Namely, they showed that

$$\forall (x, y) \in [-M, M] \times \mathbb{R}, \quad u(x) \leq A \int_{y-\beta}^{y+\beta} u(z) dz,$$

where the constant A depends only on M, J and g . An extension of this type of estimate to equation (1.5) is currently under investigation.

Let us now briefly explain our strategy for obtaining such Harnack's inequalities. It is well known (see [15, 18]) that harmonic functions (i.e $\Delta u = 0$) satisfy the mean value equality

$$u(x) = \int_{B(x,r)} u(y) \frac{dy}{|B(x,r)|},$$

which holds for any ball $B(x, r) \subset \subset \Omega$. Harnack's inequality is then easily derive from this property. Our main idea is to view a positive solution u of (1.5) as a positive function satisfying some mean value equality

$$u(x) = \frac{1}{b(x)} \int_{\Omega} u(y) d\mu(x, y),$$

for some given measure $d\mu$, and to use this formulation to obtain uniform estimates depending only on ω, J and b . However, the later mean value property is fundamentally different in at least two ways from the one satisfied by harmonic functions. First, the measure $d\mu(x, y)$ is not anymore homogeneous and may be singular in the variable y . Second, the solution of equation (1.5) satisfies the mean value equality for the fixed domain Ω , whereas for harmonic functions the mean value equality holds for any ball compactly included in Ω . All the difficulty in obtaining such estimates arises from these two differences.

Since Corollaries 1.3 and 1.7 come as a straightforward application of the main Theorems, we skip their proofs.

1.2. Organization of the paper.

The paper is organized as follows. In a first section, we establish some uniform estimates satisfied by positive solutions of (1.5). Then in section 3, we prove the different Harnack's inequalities in Corollary 1.5, Theorems 1.1, 1.2 and 1.4. Finally, in the last section we deal with the construction of a solution (Theorem 1.6).

2. LOCAL UNIFORM ESTIMATES

In this section we establish some local uniform estimates, which play an essential role in deriving the Harnack's Inequality.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a connected domain and u a continuous positive solution of (1.5). Let $\Omega' \subset \Omega$ be a compact set such that $g \geq \alpha > 0$ in Ω' . Then there exists $\epsilon^* > 0$ so that for all $\epsilon \leq \epsilon^*$, there exists Ω_ϵ and $C(\alpha, \beta, J, \epsilon, b)$ such that*

$$\int_{\Omega_\epsilon} u(y) dy \geq C \int_{\Omega'} u(y) dy.$$

Moreover, Ω_ϵ satisfies the following inclusion

$$\{x \in \Omega' \mid d(x, \partial\Omega') > \alpha\epsilon\} \subset \Omega_\epsilon \subset \left\{x \in \Omega' \mid d(x, \partial\Omega') > \frac{\alpha\epsilon}{2}\right\}.$$

Proof :

Since $\Omega' \subset \Omega$ and u is non negative, it follows that

$$(2.1) \quad \int_{\Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy - b(x)u(x) \leq 0 \quad \text{in } \Omega.$$

Since Ω' is compact and u continuous, we can integrate (2.1) over $\Omega_\epsilon \subset \subset \Omega'$ and it follows

$$\int_{\Omega_\epsilon} \int_{\Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy \leq \int_{\Omega_\epsilon} b(x)u(x) dx.$$

Using that $g \geq \alpha > 0$ in Ω' , Fubini's Theorem and setting $z = \frac{x-y}{g(y)}$, we end up with

$$(2.2) \quad \int_{\Omega_\epsilon} b(x)u(x) dx \geq \int_{\Omega'} \frac{u(y)}{g^n(y)} \left(\int_{\Omega_\epsilon} J\left(\frac{x-y}{g(y)}\right) dx \right) dy$$

$$(2.3) \quad \geq \int_{\Omega'} u(y) \left(\int_{\Omega_{\epsilon,y}} J(z) dz \right) dy$$

where $\Omega_{\epsilon,y} := \frac{\Omega_\epsilon - y}{g(y)}$. We claim that

Claim 2.1. There exists Ω_ϵ and $c_0 > 0$ so that for all $y \in \Omega'$,

$$\int_{\Omega_{\epsilon,y}} J(z) dz > c_0.$$

Observe that by proving the claim we end the proof of the lemma. Indeed, assuming the claim is proved, then from the above inequality we derive

$$\begin{aligned} \int_{\Omega_\epsilon} b(x)u(x) dx &\geq \int_{\Omega'} u(y) \left(\int_{\Omega_{\epsilon,y}} J(z) dz \right) dy \\ &\geq c_0 \int_{\Omega'} u(y) dy. \end{aligned}$$

Hence,

$$\int_{\Omega_\epsilon} u(x) dx \geq \frac{c_0}{\|b\|_\infty} \int_{\Omega'} u(y) dy.$$

□

Let us now prove the claim.

Proof of the claim

By assumption, since $J(0) > 0$, there exists $r_0 > 0$ and c_0 such that $\min_{B(0,r_0)} J > c_0$. Let $\epsilon \leq \min\{\frac{r_0}{2}; \frac{1}{4}\}$ be such that the two sets

$$(2.4) \quad \Omega'_\epsilon := \{x \in \Omega' \mid d(x, \partial\Omega') \geq \epsilon\alpha\}$$

$$(2.5) \quad \tilde{\Omega}'_\epsilon := \{x \in \tilde{\Omega}' \mid d(x, \partial\Omega') \leq \frac{\epsilon\alpha}{2}\}$$

are non empty disjoint sets. Choose Ω_ϵ smooth so that $\Omega'_\epsilon \subset \Omega_\epsilon$ and $\Omega_\epsilon \cap \tilde{\Omega}'_\epsilon = \emptyset$. By construction, we have for all $y \in \Omega'$, $d(y, \Omega_\epsilon) < \epsilon\alpha$ and $\bar{\Omega}_\epsilon$ is compact. Since Ω_ϵ is uniformly smooth there exists $\delta > 0$ small and $k \in \mathbb{N}^*$ so that for all $z \in \Omega_\epsilon$, $B(z, \delta) \subset \Omega'$ and $\exists z' \in \Omega_\epsilon \cap B(z, \delta)$ so that

$$(2.6) \quad B(z', \frac{\delta}{k}) \subset B(z, \delta) \cap \Omega_\epsilon.$$

Now, pick $y \in \Omega'$. Since $\bar{\Omega}_\epsilon$ is compact, there exists $z_0 \in \bar{\Omega}_\epsilon$ so that $\|y - z_0\| = d(y, \Omega_\epsilon)$. Using (2.6), it follows that

$$\frac{B(z'_0, \frac{\delta}{k}) - y}{g(y)} \subset \Omega_{\epsilon, y}.$$

Take now $s \in B(z'_0, \frac{\delta}{k})$ and compute $\frac{\|s-y\|}{g(y)}$:

$$\begin{aligned} \frac{\|s-y\|}{g(y)} &\leq \frac{\|s-z'_0\| + \|z_0-z'_0\| + \|z_0-y\|}{\alpha} \\ &\leq \frac{\delta}{k\alpha} + \frac{\delta}{\alpha} + \epsilon. \end{aligned}$$

By choosing $\delta \leq \frac{r_0\alpha}{4}$ small enough, since $k \geq 1$, we achieve

$$\frac{\|s-y\|}{g(y)} \leq r_0.$$

From the above construction, we have the following

$$\begin{aligned} \int_{\Omega_{\epsilon, y}} J(z) dz &\geq \int_{\frac{B(z'_0, \frac{\delta}{k}) - y}{g(y)}} J(z) dz \\ &\geq c_0 \int_{\frac{B(z'_0, \frac{\delta}{k}) - y}{g(y)}} dz \\ &\geq c_0 \mu(B(0, \frac{\delta}{k\beta})). \end{aligned}$$

Since the above computation is independent of $y \in \Omega'$, the claim is proved. □

Remark 2.2. Observe that from the computation, the parameter ϵ has a certain degree of freedom and can be chosen at our convenience.

Let us now show another important estimate.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a connected set and u a positive continuous function satisfying (1.5). Let $\Omega' \subset \Omega$ be such that $g \geq \alpha > 0$ in Ω' . Then for any $\Omega'' \subset \subset \Omega'$ there exists δ and a constant $C(\beta, J, \alpha, \delta, b)$ such that*

$$\forall x \in \Omega'', \quad u(x) \geq C \int_{B(x, \delta)} u(y) dy.$$

Proof:

Let $d := d(\Omega'', \partial\Omega')$ and by assumption $d > 0$. Since u satisfies (1.5) and is positive, at $x \in \Omega$ we have

$$\begin{aligned} b(x)u(x) &\geq \int_{\Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy \\ &\geq \int_{B(x, \beta) \cap \Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy. \end{aligned}$$

For $\delta < d$ small enough, since $J(0) > 0$, $\alpha \leq g \leq \beta$ in Ω' and $\|b\|_\infty < C$, we have

$$\begin{aligned} u(x) &\geq \int_{B(x, \delta) \cap \Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy \\ &\geq \frac{\min_{B(0, \delta)} J}{\beta^n \|b\|_\infty} \int_{B_\delta(x) \cap \Omega'} u(y) dy. \end{aligned}$$

Since $\delta < d$, it follows that for any $x \in \Omega''$, $B(x, \delta) \subset \Omega'$. Hence, the Lemma is proved. \square

Let us now prove our last estimates.

Lemma 2.4. *Let $\Sigma \subset \subset \Omega$ be regular and $\eta > 0$ such that*

$$\Omega_\eta := \bigcup_{x \in \Sigma} B(x, 2\eta) \subset \subset \Omega.$$

Let $u \in C(\Omega)$ be a positive function satisfying the following property:

$$(2.7) \quad \exists C_0 > 0 \quad \text{such that} \quad u(y) \geq C_0 \int_{B(y, \eta)} u(s) ds \quad \text{for all } y \in \Omega_\eta.$$

Then for all $x \in \Sigma$, there exists a constant $C(x)$ independent of u such that

$$\int_{B(x, \frac{\eta}{4})} u(s) ds \geq C(x) \int_{\Sigma} u(s) ds.$$

The proof of this Lemma relies essentially on the following technical Lemma that we prove later on.

Lemma 2.5. *Let $\Sigma \subset \subset \Omega$, $\eta > 0$ and $u \in C(\Omega)$ as in Lemma 2.4. Then for all $x \in \Sigma$ and $r \leq \eta$, there exists a constant $C(x, r)$ and $d \in \mathbb{R}$ independent of u such that*

$$\int_{B(x, r)} u(s) ds \geq C(x) \int_{B(x, r+d)} u(s) ds.$$

Assume for the moment that the above Lemma 2.5 holds and let us prove Lemma 2.4

Proof of Lemma 2.4:

Let $x \in \Sigma$ be fixed and let us denote $m = \int_{B(x, \frac{\eta}{4})} u(s) ds$. Now, define the following set:

$$\Gamma_x := \{y \in \Sigma \mid \exists C \text{ such that } m \geq C \int_{B(y, \eta)} u(s) ds\}.$$

Let us first show that Γ_x is non-empty. Since $x \in \Sigma$, then $B(x, \frac{\eta}{4}) \subset \Omega_\eta$ and for all $y \in B(x, \frac{\eta}{4})$,

$$u(y) \geq C_0 \int_{B(y, \eta)} u(s) ds.$$

Therefore,

$$(2.8) \quad m = \int_{B(x, \frac{\eta}{4})} u(y) dy$$

$$(2.9) \quad \geq C_0 \int_{B(x, \frac{\eta}{4})} \left(\int_{B(y, \eta)} u(s) ds \right) dy$$

$$(2.10) \quad \geq C_0 \int_{B(x, \frac{\eta}{4}) \cap \Sigma} \left(\int_{B(y, \eta)} u(s) ds \right) dy.$$

From the later inequality (2.10), it follows that there exists $y_0 \in B(x, \frac{\eta}{4}) \cap \Sigma$ such that

$$(2.11) \quad \frac{m}{C_0 \mu(B(x, \frac{\eta}{4}) \cap \Sigma)} \geq \int_{B(y_0, \eta)} u(s) ds.$$

Therefore, $y_0 \in \Gamma_x$ and $\Gamma_x \neq \emptyset$. Now let us consider

$$\omega_\infty := \bigcup_{y \in \Gamma_x} B(y, \frac{\eta}{2}).$$

We claim that

Claim 2.2. $\Sigma \subset \omega_\infty$.

Observe that by proving the claim, we also prove the Lemma. Indeed, assume the claim is proved. Since Σ is compact, there exists a finite number of balls $B(y_i, \frac{\eta}{2})$ covering Σ . That is to say, for some $N \in \mathbb{N}$,

$$\Sigma \subset \bigcup_{i=1}^N B(y_i, \frac{\eta}{2}).$$

Now since $y_i \in \Gamma_x$, it follows that

$$\begin{aligned}
m &\geq \sum_{i=1}^N \frac{C(y_i)}{N} \int_{B(y_i, \eta)} u(s) ds \\
&\geq \sum_{i=1}^N \frac{\inf_i C(y_i)}{N} \int_{B(y_i, \eta)} u(s) ds \\
&\geq \frac{\inf_i C(y_i)}{N} \int_{\bigcup_{i=1}^N B(y_i, \eta)} u(s) ds \\
&\geq \frac{\inf_i C(y_i)}{N} \int_{\Sigma} u(s) ds.
\end{aligned}$$

Hence the lemma is proved. □

Let us now turn our attention to the proof of the claim.

Proof of the claim

Since $\Gamma_x \neq \emptyset$, $\Sigma \cap \omega_\infty \neq \emptyset$. Assume now by contradiction that $\Sigma \not\subset \omega_\infty$ and choose $y \in \omega_\infty$ such that $B(y, \frac{\eta}{4}) \cap \mathbb{C}\omega_\infty \neq \emptyset$, where $\mathbb{C}\omega_\infty$ denotes the complementary set of ω_∞ . Then there exists $z \in \Sigma \cap \mathbb{C}\omega_\infty$ and $r > 0$ such that $B(z, r) \subset B(y, \frac{\eta}{4}) \cap \mathbb{C}\omega_\infty$. Now since $y \in \omega_\infty$, there exists $\tilde{y} \in \Gamma_x$ such that $y \in B(\tilde{y}, \frac{\eta}{2})$. Therefore $B(z, r) \subset B(\tilde{y}, \eta)$. Using that $\tilde{y} \in \Gamma_x$ it follows that

$$m \geq C(\tilde{y}) \int_{B(\tilde{y}, \eta)} u(s) ds \geq C(\tilde{y}) \int_{B(z, r)} u(s) ds.$$

Using now Lemma 2.5 with z and $B(z, r)$ yields

$$m \geq C(\tilde{y}) \int_{B(\tilde{y}, \eta)} u(s) ds \geq C'(\tilde{y}) \int_{B(z, r+d)} u(s) ds.$$

Now if $r + d < \eta$, using Lemma 2.5 with $B(z, r + d)$ instead of $B(z, r)$ yields

$$m \geq C \int_{B(z, r+2d)} u(s) ds.$$

By induction, since $z \in \Sigma$, there exists $p \in \mathbb{N}$ such that $r + pd \geq \eta$ and

$$m \geq C \int_{B(z, r+pd)} u(s) ds.$$

Thus, $z \in \Gamma_x$, which is a contradiction. Hence, $\Sigma \subset \omega_\infty$. □

Let us now turn our attention to the proof of the technical Lemma 2.5.

Proof of Lemma 2.5

Let us fix $y \in \Sigma$. Since $r \leq \eta$, (2.7) holds for any element $z \in B(y, r)$. Therefore, we have

$$(2.12) \quad \int_{B(y, r)} u(s) ds \geq C_0 \int_{B(y, r)} \left(\int_{B(z, \eta)} u(s) ds \right) dz.$$

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Let us now consider the annulus $A := A(y, r', r)$, for some $r' < r$ which will be chosen later on. Observe that A can be covered by a finite numbers of balls $B(z, r - r')$, where $z \in \partial B(y, r')$. Namely, we have for some $N(r - r') \in \mathbb{N}$,

$$A \subset \bigcup_1^{N(r-r')} B(z_i, (r - r')).$$

Now, choose r' close to r , such that $r - r' \leq \frac{\eta}{4}$ and for any $N(r - r')$ -tuplet $(\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{N(r-r)})$ such that $\tilde{z}_i \in B(z_i, r - r')$ we have

$$A(y, r', r' + \frac{\eta}{2}) \subset \bigcup_{i=1}^{N(r-r')} B(\tilde{z}_i, \eta).$$

Now consider, $A_i := B(z_i, (r - r')) \cap A$. For each A_i , it follows from (2.12) that

$$(2.13) \quad \int_{B(y,r)} u(s) ds \geq C_0 \int_{A_i} \left(\int_{B(z,\eta)} u(s) ds \right) dz.$$

Therefore, on each A_i there exists a point $\tilde{z}_i \in A_i$ such that

$$(2.14) \quad \int_{B(y,r)} u(s) ds \geq C_0 \mu(A_i) \int_{B(\tilde{z}_i, \eta)} u(s) ds.$$

Since $\mu(A_i) = \mu(A_j)$ for all i, j , we end up with

$$\begin{aligned} \int_{B(y,r)} u(s) ds &\geq \frac{C_0 \mu(A_i)}{N(r - r')} \sum_{i=1}^N \int_{B(\tilde{z}_i, \eta)} u(s) ds \\ &\geq \frac{C_0 \mu(A_i)}{N(r - r')} \int_{\bigcup_{i=1}^N B(\tilde{z}_i, \eta)} u(s) ds. \end{aligned}$$

Since, $\tilde{z}_i \in B(z_i, (r - r'))$ for all i , using the geometric condition it follows that

$$\int_{B(y,r)} u(s) ds \geq C_0 \mu(A_i) \int_{A(y, r', r' + \frac{\eta}{2})} u(s) ds.$$

Therefore,

$$\begin{aligned} \int_{B(y,r)} u(s) ds &\geq \frac{C_0 \mu(A_i)}{2} \int_{A(y, r', r' + \frac{\eta}{2})} u(s) ds + \frac{1}{2} \int_{B_r(y)} u(s) ds \\ &\geq C \int_{B(y,r) \cup A(y, r', r' + \frac{\eta}{2})} u(s) ds \\ &\geq C \int_{B(y, r' + \frac{\eta}{2})} u(s) ds, \end{aligned}$$

where $C := \min\{\frac{C_0 \mu(A_i)}{2}, \frac{1}{2}\}$.

Hence, with $d := \frac{\eta}{2} - (r - r') > 0$, we achieve

$$\int_{B(y,r)} u(s) ds \geq C \int_{B(y, r+d)} u(s) ds.$$

□

3. HARNACK'S INEQUALITIES

We are now in position to prove the different Harnack's inequalities, Theorems 1.1, 1.2 and 1.4. A simple proof of Theorem 1.1 can be obtained using Theorems 1.2 and 1.4, so let us first prove Theorem 1.2.

Proof of Theorem 1.2:

Before we begin, let us make some remarks and introduce some notation. First observe that if the estimates in Theorem 1.2 holds for a given compact set $\omega \subset \Omega$, then the estimates holds as well for any compact set $\tilde{\omega} \subset \omega$. Indeed, since estimates in Theorem 1.2 holds for ω , there exists a positive constant $\eta^*(\omega)$ such that for any $0 < \eta \leq \eta^*(\omega)$ there exists a compact set $\omega'(\eta) \subset \subset \Omega$ and a constant $C(J, \omega, \Omega, \omega', b, g, \eta)$ such that the following holds

- (i) $\{x \in \Omega | d(x, \partial(\omega \cap W_\eta)) > \eta\} \subset \omega'(\eta)$, where $W_\eta := \{x \in \Omega | g(x) > \eta\}$
- (ii) for all positive solution u of (1.5) the following inequality holds:

$$u(x) \leq Cu(y) \quad \text{for all } x \in \omega, y \in \omega' \cap \omega.$$

Since $\tilde{\omega} \subset \omega$, for $\eta < \eta^*(\omega)$ small enough we achieve

$$\{x \in \Omega | d(x, \partial(\tilde{\omega} \cap W_\eta)) > \eta\} \subset \subset \omega \cap W_\eta.$$

Let us now fix η . By the above inclusion, we can choose $\eta' < \eta^*(\omega)$ such that

$$\{x \in \Omega | d(x, \partial(\tilde{\omega} \cap W_\eta)) > \eta\} \subset \{x \in \Omega | d(x, \partial(\omega \cap W_{\eta'})) > \eta'\} \subset \omega'(\eta').$$

Using now (ii), it follows that any positive solutions of (1.5) satisfies

$$\sup_{\tilde{\omega}} u \leq \sup_{\omega} u \leq C(\eta') \inf_{\omega'(\eta') \cap \omega} u \leq C(\eta') \inf_{\omega' \cap \tilde{\omega}} u.$$

Therefore the estimates in Theorem 1.2 holds for $\tilde{\omega}$.

From the above observation, we can restrict our attention to compact set $\omega \subset \Omega$ such that $\mathcal{S} \subset \subset \omega$. Fix now ω and let us define the following sets

$$\begin{aligned} \omega_\eta &:= \bigcup_{x \in \omega} B(x, \eta) \cap \Omega \\ Z_\eta &:= \{y \in \Omega | g(y) < \eta\} \\ W_\eta &:= \{y \in \Omega | g(y) \geq \eta\}. \end{aligned}$$

Since $\frac{1}{g^n} \in L^1_{loc}$, let us choose η^* small enough such that

$$(3.1) \quad \int_{\omega \cap Z_{\eta^*}} \frac{dy}{g^n(y)} \leq \frac{\inf_{\omega} b}{2\|J\|_\infty}.$$

Since $\mathcal{S} \subset \subset \omega$, we can choose η^* smaller if necessary to achieve $\omega_{\eta^*} \cap Z_\eta \subset \omega$. Fix now, $0 < \eta \leq \eta^*$. We are now in a position to prove the Theorem.

Step 1: Now, define the following bounded set

$$\Omega(\omega) := \bigcup_{x \in \omega} B(x, \beta),$$

and set the measure $d\mu = \frac{dy}{g^n(y)}$, which is well defined since $\frac{1}{g^n} \in L^1_{loc}$. Since J is compactly supported, it follows that in ω , u satisfies

$$(3.2) \quad u(x) = \frac{1}{b(x)} \int_{\Omega(\omega) \cap \Omega} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y)$$

$$(3.3) \quad = \frac{1}{b(x)} \int_{\Omega(\omega) \cap Z_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y) + \frac{1}{b(x)} \int_{\Omega(\omega) \cap W_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y).$$

Observe that for $x \in \omega$, $y \in Z_\eta \cap (\Omega(\omega) \setminus \omega_\eta)$, we have

$$\left| \frac{x-y}{g(y)} \right| \geq 1.$$

Therefore since $\text{supp}(J) \subset B(0, 1)$, it follows that for $x \in \omega$

$$\frac{1}{b(x)} \int_{\Omega(\omega) \cap Z_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y) = \frac{1}{b(x)} \int_{\omega_\eta \cap Z_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y)$$

and from (3.3) we get

$$(3.4) \quad u(x) \leq \frac{1}{b(x)} \int_{\omega_\eta \cap Z_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y) + \frac{\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Since u is continuous and ω is compact, u achieves its maximum at some point, say $x_0 \in \omega$. At this point, from (3.4) we have:

$$(3.5) \quad u(x_0) \leq \frac{1}{b(x_0)} \int_{\omega_\eta \cap Z_\eta} J\left(\frac{x_0-y}{g(y)}\right) u(y) d\mu(y) + \frac{\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Using that $\omega_\eta \cap Z_\eta \subset \omega$ and (3.1), it follows that

$$(3.6) \quad u(x_0) \leq \frac{u(x_0)}{2} + \frac{\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Therefore,

$$(3.7) \quad u(x_0) \leq \frac{2\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

From (3.3), using that u , J and g are non-negative, it follows that for all $x \in \Omega$ we have

$$(3.8) \quad u(x) \geq \frac{1}{b(x)} \int_{\Omega(\omega) \cap W_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y).$$

Since $g \geq \eta$ in $\Omega(\omega) \cap W_\eta$, from Lemma 2.1 there exists $\epsilon^* > 0$ such that for all $0 < \epsilon \leq \epsilon^*$, there exists a non empty set $\Omega_\epsilon \subset \subset \Omega(\omega) \cap W_\eta$ and a constant $C(J, \eta, \Omega(\omega) \cap W_\eta, b)$ such that

$$\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \epsilon\eta\} \subset \Omega_\epsilon \subset \left\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \frac{\epsilon\eta}{2}\right\}$$

and

$$(3.9) \quad \int_{\Omega_\epsilon} u(y) d\mu(y) \geq C \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Recall that from the proof of Lemma 2.1, we also have $\epsilon \leq \epsilon^* \leq \frac{1}{4}$. Thus, we have

$$\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \eta\} \subset \{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \epsilon\eta\} \subset \Omega_\epsilon.$$

Step 2: Choose now $\delta < \min \left\{ \frac{\epsilon\eta}{8}, \frac{\epsilon}{8} \right\}$, where ϵ and η are defined by the previous Step, and consider the set

$$\Omega_\delta := \bigcup_{x \in \Omega_\epsilon} B(x, 2\delta).$$

By construction, $\Omega_\delta \subset \Omega(\omega) \cap W_\eta$ and $g \geq \eta$ in Ω_δ . Using Lemma 2.3, for any $x \in \Omega_\epsilon$ we have

$$(3.10) \quad u(x) \geq C \int_{B(x, \delta)} u(s) ds.$$

Step 3. Using now that Ω_ϵ is compact, we can cover it with a finite number of ball of radius $\frac{\delta}{4}$. That is, for some $N \in \mathbb{N}$, we have

$$\Omega_\epsilon \subset \bigcup_{i=1}^N B(x_i, \frac{\delta}{4}).$$

Now, for $x \in \Omega_\epsilon$ there exists x_i such that $x \in B(x_i, \frac{\delta}{4})$. Since $B(x_i, \frac{\delta}{4}) \subset B(x, \delta)$ by (3.10), it follows that

$$u(x) \geq C \int_{B(x, \delta)} u(s) ds \geq C \int_{B(x_i, \frac{\delta}{4})} u(s) ds.$$

Using now Lemma 2.4, we end up with

$$(3.11) \quad u(x) \geq C \int_{B(x, \delta)} u(s) ds \geq CC(x_i) \int_{\Omega_\epsilon} u(s) ds.$$

Setting $C_0 := \min C(x_i)$ and collecting (3.7), (3.8) and (3.11), it follows that for all $x \in \Omega_\epsilon$ we have

$$u(x_0) \leq CC_0 u(x).$$

Therefore, for all $y \in \omega$ and $x \in \Omega_\epsilon$ we have

$$u(y) \leq CC_0 u(x).$$

Hence, for all $y \in \omega$ and $x \in \Omega_\epsilon \cap \omega$ we have

$$u(y) \leq CC_0 u(x).$$

□

Let us now treat the case of smooth domains ω and prove Theorem 1.4.

Proof Theorem 1.4:

The proof follows essentially 4 steps.

Step 1: As above, we can restrict our attention to a compact set $\omega \subset \Omega$ such that $\mathcal{S} \subset \subset \omega$. Now let us define the sets $\Omega(\omega) := \bigcup_{x \in \omega} B(x, \beta) \cap \Omega$ and W_η as in the above proof. Following a similar argument, for a point $x_0 \in \omega$ where u achieves its maximum and for small enough η , say $\eta \leq \eta_1$, we have

$$(3.12) \quad u(x_0) \leq \frac{2\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Step 2: For any $\nu \in \mathbb{R}$, let us consider the set $\omega_\nu := \{x \in \omega \mid d(x, \partial\omega) \geq \nu\}$ and $C(x, \theta, a)$ be the cone issued from x with angle θ and height a . On one hand, since $\mathcal{S} \subset\subset \omega$, there exists $\nu_0 > 0$ such that $\omega \setminus \omega_{4\nu_0} \cap \mathcal{S} = \emptyset$. On the other hand, since ω satisfies an uniform inner cone condition, it follows that for ν small enough, say $\nu \leq \nu^*$, there exists $r(\nu) > 0$ such that for any $x \in \omega \setminus \omega_\nu$, there exists $\bar{x} \in \omega$ such that

$$\begin{aligned} B(\bar{x}, r) &\subset C_{x, \theta, a} \cap (\omega_\nu \setminus \omega_{4\nu}) \\ B(\bar{x}, r) &\subset B(x, \beta). \end{aligned}$$

Let us now fix $\nu \leq \min\{\nu_0, \nu^*\}$ and take $\eta^* := \min_{\Omega(\omega) \setminus \omega_{4\nu}} g$. By construction, $\eta^* > 0$.

Now take any $x \in \omega \setminus \omega_\nu$. From (1.5), using the uniform inner cone property, we have

$$(3.13) \quad u(x) = \frac{1}{b(x)} \int_{\Omega} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy$$

$$(3.14) \quad \geq \frac{1}{b(x)} \int_{C_{x, \theta, a} \cap B(x, \beta)} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy$$

$$(3.15) \quad \geq \frac{1}{b(x)} \int_{B(\bar{x}, r)} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} dy.$$

Recall that $g \geq \eta^*$ in $B(\bar{x}, r)$. Therefore, since $J(0) > 0$, $b > 0$, there exists δ_0 and C_0 independent of x such that $B(\bar{x}, \delta_0) \subset B(\bar{x}, r)$ and

$$(3.16) \quad u(x) \geq C_0 \int_{B(\bar{x}, \delta_0)} u(y) dy.$$

Fix now $\eta \leq \min\{\frac{\eta_1}{2}, \frac{\eta^*}{2}\}$ such that $W_\eta \setminus W_{2\eta} \subset\subset \omega_\nu$, and let

$$d := d\left(\omega_\nu \cap W_{2\eta}, \partial(\Omega(\omega) \cap W_\eta)\right).$$

By construction, we have $d > 0$. Indeed, since $\eta \leq \eta^*$, we have $\partial(\Omega(\omega) \cap W_\eta) = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \subset \overline{(\Omega(\omega) \setminus \omega_{\frac{\nu}{2}})}$ and $\Gamma_2 \subset \overline{(W_\eta \setminus W_{\frac{3\eta}{4}})}$. Therefore, for any $x \in \omega_\nu \cap W_{2\eta}$

$$d(x, \Gamma_1 \cup \Gamma_2) \geq d\left(x, \overline{(\Omega(\omega) \setminus \omega_{\frac{\nu}{2}})} \cup \overline{(W_\eta \setminus W_{\frac{3\eta}{4}})}\right) > 0.$$

Step 3: Let ν and η be defined by the above steps. Since $d > 0$, choosing ϵ small enough, say $\epsilon \leq \frac{d}{2\eta}$, it follows that

$$\omega_\nu \cap W_{2\eta} \subset \{x \mid d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \epsilon\eta\}.$$

Now, since $g \geq \eta$ in $\Omega(\omega) \cap W_\eta$, from Lemma 2.1 there exists ϵ^* so that for all $0 < \epsilon \leq \epsilon^*$ there exists a non empty set Ω_ϵ and a constant $C(J, \eta, \Omega(\omega) \cap W_\eta, b, \epsilon)$ such that

$$\{x \mid d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \epsilon\eta\} \subset \Omega_\epsilon \subset \left\{x \mid d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \frac{\epsilon\eta}{2}\right\}$$

and

$$(3.17) \quad \int_{\Omega_\epsilon} u(y) dy \geq C \int_{\Omega(\omega) \cap W_\eta} u(y) dy.$$

Observe that by choosing $\epsilon \leq \min\{\epsilon^*, \frac{d}{2\eta}\}$, we also have $\omega_\nu \cap W_{2\eta} \subset \Omega_\epsilon$.

We now fix $\epsilon \leq \min\{\epsilon^*, \frac{d}{2\eta}\}$ and choose $\delta < \frac{\epsilon\eta}{8}$, and consider the set

$$\Omega_\delta := \bigcup_{x \in \Omega_\epsilon} B(x, 2\delta).$$

By construction, we have $\Omega_\epsilon \subset \subset \Omega_\delta \subset \Omega(\omega) \cap W_\eta$ and $g \geq \eta$ in Ω_δ . Therefore, from Lemma 2.3 there exists δ_1 and C such that for any $x \in \Omega_\epsilon$, we have

$$(3.18) \quad u(x) \geq C \int_{B(x, \delta_1)} u(s) ds.$$

Step 4: Take now $\delta^* \leq \min\{\delta_0, \delta_1\}$, where δ_0 is defined in (3.16). Covering Ω_ϵ with a finite number of balls of radius $\frac{\delta^*}{4}$, (i.e. for some $N \in \mathbb{N}$, $\Omega_\epsilon \subset \bigcup_{i=1}^N B(x_i, \frac{\delta^*}{4})$), and using a similar argument as in the proof above yields

$$u(x_0) \leq CC_0 u(x) \quad \text{for all } x \in \Omega_\epsilon,$$

where $C := \min C(x_i)$. Since $\omega_\nu \cap W_{2\eta} \subset \Omega_\epsilon$, it follows that

$$(3.19) \quad u(x_0) \leq CC_0 u(x) \quad \text{for all } x \in \omega_\nu \cap W_{2\eta}.$$

From (3.16), we also get that for all $x \in \omega \setminus \omega_\nu$,

$$u(x) \geq C \int_{B(\bar{x}, \delta_0)} u(s) ds,$$

where $B(\bar{x}, \delta_0) \subset \omega_\nu \setminus \omega_{4\nu}$. Since $\bar{x} \in \omega_\nu \cap W_{2\eta} \subset \Omega_\epsilon$ and $\delta^* \leq \delta_0$, it follows that

$$u(x) \geq C \int_{B(\bar{x}, \delta_0)} u(s) ds \geq CC \int_{B(x_i, \frac{\delta^*}{2})} u(s) ds.$$

Using Lemma 2.4, it yields

$$(3.20) \quad u(x) \geq CC(x_i) \int_{\Omega_\epsilon} u(s) ds$$

$$(3.21) \quad \geq CC_0 \int_{\Omega_\epsilon} u(s) ds.$$

Combining now (3.21) and (3.12), we end up with

$$(3.22) \quad u(x_0) \leq CC_0 u(x) \quad \text{for all } x \in (\omega \setminus \omega_\nu).$$

Hence, from (3.22) and (3.19) we get

$$(3.23) \quad u(x) \leq CC_0 u(y) \quad \text{for all } x \in \omega, y \in \omega \cap W_{2\eta}.$$

□

Let us now prove Theorem 1.1.

Proof of Theorem 1.1:

The proof follows easily from Theorems 1.2 and 1.4. Indeed, let $\omega \subset \subset \Omega$ be a compact set. Then, since $d(\omega, \Omega) > \nu$ for some positive ν , there exists a regular compact set $\tilde{\omega}$ such that $\omega \subset \tilde{\omega}$. Applying now Theorem 1.4, there exists η^* such that for all positive $\eta \leq \eta^*$, there exists a constant $C(\eta)$ such that for any positive solution u of (1.5), we have

$$u(x) \leq C(\eta)u(y) \quad \text{for any } x \in \tilde{\omega}, y \in \tilde{\omega} \cap \{z \in \Omega | g(z) \geq \eta\}.$$

Recall that $g \geq \alpha$ in Ω . It follows that

$$u(x) \leq C(\alpha)u(y) \quad \text{for any } x, y \in \tilde{\omega}.$$

Now since, $\omega \subset \tilde{\omega}$, the above inequality holds as well on ω . □

Finally, let us prove Corollary 1.5.

Proof of Corollary 1.5

Since Ω is bounded and regular, using Theorem 1.4 and corollary 1.3, we have

$$\sup_{\Omega} u \leq C(\eta)u(y) \quad \text{for any } y \in \Omega \cap \{z \in \Omega | g(z) \geq \eta\}.$$

Recalling that $g \geq \alpha$ in Ω , it follows that

$$\sup_{\Omega} u \leq C(\alpha)u(y) \quad \text{for any } y \in \Omega.$$

Hence,

$$\sup_{\Omega} u \leq C(\alpha) \inf_{\Omega} u(y).$$
□

4. CONSTRUCTION OF NON TRIVIAL POSITIVE SOLUTION OF A PARTICULAR NONLOCAL EQUATION

In this section, we deal with the construction of a positive solution of (1.7) and prove Theorem 1.6.

Proof of Theorem 1.6:

We treat two cases

Case 1: Ω bounded.

First, let us assume that Ω is bounded. Let us define the operator $T \in \mathcal{L}(C(\Omega))$ by

$$Tu := \frac{1}{a(x)} \int_{\Omega} J \left(\frac{x-y}{g(y)} \right) \frac{u(y)}{g^n(y)} dy,$$

where

$$a(x) := \begin{cases} \int_{\Omega} J \left(\frac{y-x}{g(x)} \right) \frac{dy}{g^n(x)} & \text{for } x \notin \mathcal{S} \\ 1 & \text{otherwise.} \end{cases}$$

Since $\frac{1}{g^n(y)} \in L^1_{loc}$, T is a compact operator. Moreover T is positive since g, a and J are non-negative functions. Using now the Krein-Rutman Theorem, there exist an eigenvalue λ and an eigenfunction $\phi > 0$ such that

$$\frac{1}{a(x)} \int_{\Omega} J \left(\frac{x-y}{g(y)} \right) \frac{\phi}{g(y)} dy = \lambda \phi.$$

Integrating the equation over Ω , it follows that $\lambda = 1$, and ϕ is our desired solution.

Case 2: Ω unbounded.

Assume now that Ω is any open set and let Ω_n be an increasing sequence of bounded subsets such that $\lim_{n \rightarrow \infty} \Omega_n = \Omega$. Since $\mathcal{S} \subset \subset \Omega$, we can also assume that for all $n \in \mathbb{N}$, $\mathcal{S} \cap \Omega_n \subset \subset \Omega_n$. Let ϕ_n denote the associated solution to Ω_n with the normalization $\phi(x_0) = 1$ for some fixed $x_0 \in \Omega_n$ that we will choose later on. Since $n \in \mathbb{N}$, $\mathcal{S} \subset \subset \Omega_n$ and $(\Omega_n)_{n \in \mathbb{N}}$ is an increasing sequence of sets, for some η_1 small we have

$$\bigcap_{n \in \mathbb{N}} (\Omega_n \cap W_{\eta_1}) = \Omega_0 \cap W_{\eta_1} \neq \emptyset.$$

Let us choose $x_0 \in \Omega_0 \cap W_{\eta_1}$.

Let us now fix $n \in \mathbb{N}$ and consider the sequence of functions $(\phi_{n+k})_{k \in \mathbb{N}}$. By construction, ϕ_{n+k} satisfies the equation

$$\int_{\Omega_{n+k}} J\left(\frac{x-y}{g(y)}\right) \frac{\phi_{n+k}(y)}{g^n(y)} dy - a_{n+k}(x)\phi_{n+k} = 0.$$

Since Ω_n is an increasing sequence of bounded sets, for any $k \in \mathbb{N}$ we have $\Omega_n \subset \Omega_{n+k}$. Using Theorem 1.2 with Ω_n and ϕ_{n+k} , it follows that for any $k \in \mathbb{N}$ there exists a constant η_k^* , such that for all $\eta \leq \eta_k^*$ there exists ω'_{n+k} and a constant $C_{n+k}(J, g, \eta, \|a_{n+k}\|_\infty, \beta, \Omega_n)$ such that

$$(4.1) \quad \{x \in \Omega_{n+k} | d(x, \partial(\Omega_n \cap W_\eta)) > \eta\} \subset \omega'_{n+k}$$

$$(4.2) \quad \sup_{\Omega_n} \phi_{n+k} \leq C_{n+k}(\eta)\phi_{n+k}(x) \quad \text{for all } x \in \omega'_{n+k}.$$

For each $k \in \mathbb{N}$, let us choose η_k such that

$$\Omega_n \cap W_{\eta_1} \subset \{x \in \Omega_n | d(x, \partial W_{\eta_k}) > \eta_k\}.$$

Using the monotonicity of the sequence $(\Omega_n)_{n \in \mathbb{N}}$, it follows that

$$\Omega_0 \cap W_{\eta_1} \subset \Omega_n \cap W_{\eta_1} \subset \{x \in \Omega_n | d(x, \partial W_{\eta_k}) > \eta_k\} \subset \omega'_{n+k}.$$

Therefore, from the above set inclusion and (4.2), it follows that

$$(4.3) \quad \sup_{\Omega_n} \phi_n \leq C_{n+k}(\eta_k)\phi_n(x_0) \leq C_{n+k}(\eta_k).$$

Now, observe that the sequence of positive functions $(a_{n+k}(x))_{k \in \mathbb{N}}$ is increasing in Ω_n and uniformly bounded. The monotonicity property follows easily from the monotonicity of the Ω_n . Indeed, recall that for any $x \in \Omega_n \setminus \mathcal{S}$ we have

$$a_{n+k}(x) = \int_{\Omega_{n+k}} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)}.$$

Therefore, using that $\Omega_n \subset \Omega_{n+1}$ and that J, g are non negative functions, it follows that

$$a_{n+k}(x) = \int_{\Omega_{n+k}} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)} \leq \int_{\Omega_{n+k+1}} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)} = a_{n+k+1}(x).$$

On the other hand, for $x \in \mathcal{S}$, we have $a_n(x) = 1$ for all $n \in \mathbb{N}$. Thus, $a_{n+k} \leq a_{n+k+1}$ in Ω_n .

From the definition of a_n , we also get easily the uniform bound. For any $x \in \Omega_n \setminus \mathcal{S}$, using a change of variable we have

$$a_n(x) = \int_{\Omega_n} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)} \leq \int_{\frac{\Omega_n-x}{g(x)}} J(z) dz \leq 1.$$

Using that $(a_n(x))_n$ is uniformly bounded independent of n and increasing in Ω_n , we can make the constant C_{n+k} independent of k . Therefore, for all $k \in \mathbb{N}$, ϕ_{n+k} is uniformly bounded in Ω_n . Now, since ϕ_{n+k} is uniformly continuous on Ω_n , using Arzela-Ascoli's Theorem we can extract from $(\phi_{n+k})_{k \in \mathbb{N}}$ a subsequence which converges uniformly in Ω_n . By a standard diagonal argument, we can extract from $(\phi_n)_{n \in \mathbb{N}}$ a subsequence which converges to a function ϕ uniformly on every compact subset ω of Ω . Using that J has compact support and the Lebesgue dominated convergence theorem, passing to the limit in the equation yields

$$\int_{\Omega} J\left(\frac{x-y}{g(y)}\right) \frac{\phi}{g(y)} dy - a(x)\phi = 0.$$

□

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MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCE, INSELSTRASSE 22, D-04103 LEIPZIG, GERMANY

E-mail address: coville@mis.mpg.de