Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Reconstruction of the intertwining operator and new striking examples added to "Isospectral pairs of metrics on balls and spheres with different local geometries"

by

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Preprint no.: 19 2008

Reconstruction of the intertwining operator and new striking examples added to "Isospectral pairs of metrics on balls and spheres with different local geometries"

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Hatvanadik születésnapomra

Abstract

The intertwining operator constructed in [Sz1, Sz2] [1] does not appear in the right form. It is established there by using only the anticommutators J_1 and J'_1. The correct operator involves all endomorphisms, J_α, which are unified by the Z-Fourier transform. Although some of the correct elements of the previous constructions are kept, this idea is established by a new technique which yields the various isospectrality theorems stated in the papers on a much larger scale. The new results include new isospectrality examples living on sphere×ball- and sphere×sphere-type manifolds. Among them, there are such discrete isospectrality families where one of the members is homogeneous while the others are locally inhomogeneous (striking examples). Furthermore, a large class of new isospectrality families are constructed by σ deformations.

1 Introduction.

In papers [Sz1, Sz2], the intertwining operator is constructed by the complex linear correspondence

$$\kappa^* : \varphi(|X|,Z)\Theta_{Q_1}(X,J_1)...\Theta_{Q_p}(X,J_1)\overline{\Theta}_{Q_{p+1}}(X,J_1)\dots\overline{\Theta}_{Q_{p+q}}(X,J_1) \quad (1)$$
$$\to \varphi(|X|,Z)\Theta_{Q_1}(X,J'_1)\dots\Theta_{Q_p}(X,J'_1)\dots\overline{\Theta}_{Q_{p+1}}(X,J'_1)\dots\overline{\Theta}_{Q_{p+q}}(X,J'_1),$$

*Partially supported by NSF grant DMS-0604861

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[1] Annals of Mathematics, 154(2001), 437-475; and 161(2005), 343-395

where $Q_1, \dots, Q_{p+q}$ are arbitrary X-vectors, furthermore, $\Theta_Q(X, J_1) = \langle Q + \mathbf{i}J_1(Q), X\rangle$ and the corresponding $\Theta'_Q(X, J'_1)$ are defined by the anticommutators J_1 and J'_1 respectively. These anticommutators σ-relate to each other, meaning, that the X-space is a direct sum, $\mathbf{v} = \mathbf{v}^{(a)} \oplus \mathbf{v}^{(b)}$, of $k^{(a)}$ resp. $k^{(b)}$ real-dimensional subspaces such that the components are invariant under the action of the anticommutators and they agree on the first component while $J_1 = -J'_1$ holds on the second one. The complex linear property guaranties that it intertwines the operators $\partial_1 D_1 \bullet$ and $\partial_1 D'_1 \bullet$, which appear in the Laplacians involved to the angular momentum operators $\mathbf{M} = \sum_{\alpha=1}^{l} \partial_\alpha D_\alpha \bullet$ resp. $\mathbf{M}'$. One of the ultimate goals is to intertwine the complete Laplacians.

It was overlooked by the author that one can not allow arbitrary vectors, Q_i, in the definition of κ^* because the operator becomes ill-defined. In fact, well-defined complex linear map can be introduced by choosing a system, $\{E_i^{(a)}, E_j^{(b)}\}$, where $1 \leq i \leq k^{(a)}/2$; $1 \leq j \leq k^{(b)}/2$, of independent vectors in the corresponding component spaces $\mathbf{v}^{(a)}$ and $\mathbf{v}^{(b)}$ which form a complex linear basis with respect to both complex structures J_1 and J'_1. Then, the admissible Q's are the vectors laying in the real subspace spanned by all these $E_r^{(c)}$'s. This natural complex linear map can be described in terms of the complex coordinates $\{z_r\}$ resp. $\{z'_r\}$ determined by the same basis $\{E_r | 1 \leq r \leq k^{(a)}/2 + k^{(b)}/2 = k/2\} = \{E_i^{(a)}\} \cup \{E_j^{(b)}\}$, for the two complex structures respectively such that the image of a polynomial written up in terms of the coordinates $\{z_r\}$ is the polynomial of the same form but written up in terms of the other coordinates $\{z'_r\}$. One can easily see that these well-defined maps depend on the real subspaces spanned by $\{E_r\}$, meaning that maps defined for different real subspaces correspond different elements to the very same element in general. Thus, by allowing all possible Q's, the above κ^* is ill-defined indeed.

Unfortunately, this problem can not be eliminated by replacing the ill-defined operator by one of these well-defined ones. In this case a much more serious difficulty appears, namely, such a well-defined operator does not intertwine $\partial_\alpha D_\alpha \bullet$ and $\partial_\alpha D'_\alpha \bullet$ satisfying $\alpha > 1$, which are also parts of the corresponding Laplacians. Let it also be mentioned that the rest parts of the Laplacians as well as the boundary conditions are intertwined by it. In the papers, the proof of intertwining of the above parts of $\mathbf{M}$ resp. $\mathbf{M}'$ explores the false assumption claiming that the κ^* operates on functions defined for imaginary Q's in the same way as for the real ones.

One of the reasons causing this blunder was that, instead of $\mathbf{M}$ and $\mathbf{M}'$, one was focusing just on $D_\alpha \bullet$ and $D'_\alpha \bullet$, i. e., tried to define intertwining operator using only the X-space. An other reason was that the intertwining operator was defined just with endomorphisms J_1 and J_1'. The corrected operator, involving all J_α and focusing on $\mathbf{M}$ and $\mathbf{M}'$, is defined by choosing a basis $\{E_1, \dots, E_{k/2}\}$ and using Q_i's laying in the real span of these basis vectors. More precisely, this correspondence is:

$$\kappa : \int_{\mathbf{z}} e^{\mathbf{i}\langle Z,V\rangle}\varphi(|X|,V)\Theta_{Q_{1\dots p}}(X,V_u)\overline{\Theta}_{Q_{p+1\dots p+q}}(X,V_u)dV \to \tag{2}$$
$$\int_{\mathbf{z}} e^{\mathbf{i}\langle Z,V\rangle}\varphi(|X|,V)\Theta'_{Q_{1\dots p}}(X,V_u)\overline{\Theta}'_{Q_{p+1\dots p+q}}(X,V_u)dV,$$

where $\Theta_{Q_{1\dots p}}(X,V_u) := \Theta_{Q_1}(X,V_u)\dots\Theta_{Q_p}(X,V_u)$ and the corresponding $\Theta'_{Q_{1\dots p}}(X,V_u)$ are defined by the endomorphisms J_{V_u} and J'_{V_u} belonging to the unit Z-vectors V_u respectively. This operator associates functions defined by the Z-Fourier transform to each other. Appropriate intertwining, κ^-, can be established by using $e^{-\mathbf{i}\langle Z,V\rangle}$ in the above formulas. Then also $\kappa^{\mathbf{R}} = (\kappa + \overline{\kappa}^-)/2$ is going to be an intertwining operator. However, this paper proceeds only with the first version.

The well-definedness of this operator follows from reasons such as only Q's laying in the real span of vectors E_i are used in its definition, furthermore, the Fourier transform is an isometry on the corresponding L^2 Hilbert spaces. The intertwining of the Laplacians is due to the Z-Fourier transform implemented into the formulas. The addition of this Z-Fourier transform to the original idea makes the mathematical situation much more complex, requiring a complete rethinking of the original construction. For instance, beyond proving the intertwining of the Laplacians, it is much more difficult to prove the intertwining of the boundary conditions for this operator. Actually, the proof of the latter statement combines the Z-Fourier transform with an independent idea incorporated into the *Independence Theorems*. The ill-definedness of (1) was recognized by H. Fürstenau whose observation triggered the author's thorough rethinking of his complete construction. The much deeper problems hidden under the cover of ill-definedness came to the light during this revision process. The reborn operator presented here saves all the previous results and provides also new interesting isospectrality examples. The main goal is to establish

Theorem 1.1. *The ball- resp. sphere-type manifolds, which have the same radius function $\varphi(|X|,|Z|)$ and are defined on H-type groups $H_l^{(a,b)}$ having the same parameters $a+b$ and l, are isospectral. This statement extends to a large class of general 2-step nilpotent Lie groups where an isospectrality family is defined by σ-deformations.*

On ball-type manifolds this statement includes the well-definedness of (2) and the intertwining of the Laplacians as well as the boundary conditions. Since the Dirichlet condition is intertwined, by restrictions, the operator induces bijections between the function spaces defined on the boundaries. By observing that it is enough to use functions satisfying the Z-Neumann condition on the ambient manifolds, one can prove the intertwining property also on the boundary manifolds.

All these statements extend onto the solvable extensions of 2-step nilpotent Lie groups. Furthermore, new examples, not discussed in the original papers, are also constructed. They live on sphere$\times$ball- and sphere$\times$sphere-type manifolds. Among them two are particularly interesting. Namely, the isospectrality family of sphere$\times$sphere-type manifolds, constructed both on $H_3^{(a,b)}$ and $SH_3^{(a,b)}$, the metric is homogeneous for the manifold belonging to the pair $(a+b,0)$ or $(0,a+b)$, while the metrics satisfying $ab \neq 0$ are locally inhomogeneous. Also the dimensions of groups of isometries acting on the members are different. These are new contributions to the old list of striking examples constructed on the sphere$\times$torus-type manifolds of $H_3^{(a,b)}$. resp. geodesic spheres of $SH_3^{(a,b)}$.

These theorems are established, first, on H-type groups and their solvable extensions. Then, they are extended to those 2-step nilpotent groups and their solvable extensions which are defined by endomorphism spaces obtained by perturbing the endomorphism space of a given H-type group $H_l^{(a,b)}$.

The perturbation process mentioned above is as follows. The endomorphisms, J_Z, on H-type groups are defined by endomorphisms, j_Z, acting on the irreducible components, by the formula $J_Z = (j_Z, \dots, j_Z, -j_Z, \dots, -j_Z)$. The perturbation primarily concerns the endomorphisms j_Z, i. e., close to the Cliffordian one, a new linear space of endomorphisms is chosen with elements denoted by $\tilde{j}_Z$. Then, the groups, $\tilde{H}_l^{(a,b)}$ and $\tilde{SH}_l^{(a,b)}$, resulted by such a perturbation arise from endomorphisms $\tilde{J}_Z$ defined by the same formula in terms of $\tilde{j}_Z$.

Such a perturbation results a new family, defined by the same $(a+b)$ and l, whose members are obviously σ-equivalent. By a theorem proved in the last section, the κ always intertwines the Laplacians, however, the same statement for the boundary conditions is not guarantied. This problem is solved by the independence theorems which state the independence of certain subspaces formed by functions. This independence is established, originally, for H-type groups defined by Cliffordian endomorphism spaces and remains true for groups defined by endomorphism spaces which are close enough to the Cliffordian ones. The latter spaces constitute an open set whose members are called the *small perturbations* of a given Cliffordian endomorphism space. The complete isospectrality theorems are established only on those σ-equivalent groups whose endomorphism spaces are produced by small perturbations of the Cliffordian ones. It is also important to understand that the independence theorems alone do not validate the intertwining property for the boundary conditions. They can guarantee it just for intertwining operators defined by σ-deformations.

The new features in this mathematical process include, first of all, the integral formula by which the intertwining operator is defined. This formula deeply roots in quantum theory [Sz4, Sz5], furthermore, it can be used also for explicit computations of the eigenfunctions and spectra. This rooting in physics is exhibited by the surprising fact that the Laplacian on the investigated manifolds can be identified with the Landau-Zeeman operator attached to electron-positron systems where these particles are orbiting in constant magnetic fields. Above, the endomorphisms $-j_Z$ resp. j_Z correspond to electrons resp. positrons, and σ-deformations is interpreted such that some of the electrons are exchanged for positrons. Yet, the spectra on all submanifolds investigated in this paper are not changing during this exchange-process. The local geometry, however, is dramatically changing. The manifolds in the striking examples, for instance, are homogeneous for systems having particles of the same type, while, they are locally inhomogeneous for mixed particles.

This is a physical interpretation of the above isospectrality theorems. The perturbation can be interpreted such that, instead of a system of identically charged particles, one considers ones which are charged distinctly. Then, the isospectrality theorems are established also for systems produced by small perturbation of the charge. The fact of non-changing spectra during electron-positron-exchanges is well known in physics. However, the statement of this form concerns the spectra considered on a non-compact manifold. Our statement claims much more than just this. Namely, the spectra remain the same also on a large class of compact submanifolds. An other distinguishing feature is that no attached local geometries are considered in quantum theory.

The intertwining of the Laplacians can be established by using only the integral formula. For proving the intertwining property for the boundary conditions, a new idea, appearing in the Independence Theorems, is involved. These theorems are also important new features in this field.

The methods developed in this paper apply only for σ-deformations. A characteristic feature of these discrete deformations is that they do not change the Ricci curvature. This fact is strongly used in proving the intertwining regarding the boundary conditions. This experience strongly indicates that the submanifolds considered in this paper are not isospectral on the Gordon-Wilson [GW] examples where continuous isospectral deformations with changing Ricci tensor are established on groups defined by 2-dimensional Z-spaces. Other arguments supporting this statement are explained in the end of the last section.

The much more general Theorem 1.1 replaces the Isospectrality Theorems of the articles. Only the construction of the intertwining operator (cf. pages 461-465 in [Sz1] and 371-375 in [Sz2]) is effected by this problem. The major non-effected part includes all the Non-Isometry resp. Rigidity Theorems and the preparatory part of Sections 4. resp. 3.. This problem with solution was announced at the CUNY Geometric Analysis Conference, in 2006 [Sz3].

2 Technicalities.

The constructions are performed on 2-step nilpotent metric Lie groups and their solvable extensions. The nilpotent groups are defined on corresponding orthogonal direct sums, $\mathbf{v} \oplus \mathbf{z}$, of Euclidean spaces where the components, $\mathbf{v} = \mathbb{R}^k$ and $\mathbf{z} = \mathbb{R}^l$, are called X- and Z-space respectively. The Lie algebra is completely determined by the linear space, $J_{\mathbf{z}}$, of skew endomorphisms whose actions on the X-space are defined by the relation $\langle [X,Y],Z\rangle = \langle J_Z(X),Y\rangle$, where $X,Y \in \mathbf{v}$ and J_Z is the endomorphism associated with $Z \in \mathbf{z}$. The Riemannian metric, g, is the left invariant extension of the natural Euclidean metric on the Lie algebra. The exponential map identifies the Lie algebra with the group itself, thus also the group can be considered to be defined on the same (X,Z)-space. Each group, (N,g), extends into a solvable group (SN,g_s), where a point is represented by (t,X,Z).

Particular 2-step nilpotent Lie groups are the so called Heisenberg-type groups, defined by endomorphisms satisfying the Clifford condition $J_Z^2 = -|Z|^2 id$. These metric groups are attached to Clifford modules, thus the classification of these modules provides classification also for the H-type groups. In this case the X-space decomposes into the product $\mathbf{v} = (\mathbf{R}^{r(l)})^{a+b} = \mathbf{R}^{r(l)a} \times \mathbf{R}^{r(l)b}$ and endomorphisms J_Z are defined by endomorphisms j_Z acting on the smaller space $\mathbf{R}^{r(l)}$ such that they act on $\mathbf{R}^{r(l)a}$ resp. $\mathbf{R}^{r(l)b}$ according to the Cartesian product $j_Z \times \cdots \times j_Z$ resp. $-j_Z \times \cdots \times -j_Z$. The H-type groups are denoted by the symbol $H_l^{(a,b)}$, which indicates the above decomposition. The solvable extensions of H-type groups are denoted by $SH_l^{(a,b)}$.

The Laplacian on a H-type group is of the form

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \sum_{\alpha=1}^{r} \partial_\alpha D_\alpha \bullet, \tag{3}$$

where $D_\alpha \bullet$ denotes directional derivative along the vector field $X \to J_\alpha(X) = J_{Z_\alpha}(X)$ defined for each element, Z_α, of an orthonormal basis of the Z-space. The integral curves of this field are the Hopf circles defined for the complex structure J_α. In the isospectrality constructions performed in this paper one should deal with this compound operator. Earlier, the constructions were performed on center periodic H-type groups, $\Gamma\backslash H$, defined by factorizing the center of the group with a Z-lattice $\Gamma = \{Z_\gamma\}$. In this case the L^2 function space is the direct sum of function spaces W_γ spanned by functions of the form $\Psi_\gamma(X,Z) = \psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$. Each W_γ is invariant under the action of the Laplacian, i. e., $\Delta\Psi_\gamma(X,Z) = \Box_\gamma\psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$, where operator $\Box_\gamma$, acting on $L^2(\mathbf{v})$, is of the form

$$\Box_\gamma = \Delta_X + 2\pi\mathbf{i}D_\gamma \bullet -4\pi^2|Z_\gamma|^2(1 + \frac{1}{4}|X|^2). \tag{4}$$

Note that the first operator involves all endomorphisms J_α while the second one involves, regarding each invariant subspace W_γ, only J_γ.

Remark. There is pointed out in [Sz3, Sz4, Sz5] that operator (4) is nothing but the Zeeman-Hamilton operator of a free charged particle (the 2D version is called Landau Hamiltonian), which was used for explaining the Zeeman effect. Term involving $D_\gamma \bullet$ is the so called angular momentum operator, which represents a preliminary version of the spin concept. The non-periodic metric group (N, g) strongly relates to Dirac's relativistic multi-time model, which, in order to furnish relativistic features on the quantum level, endowed the particles with individual self-times. In the H-type model the multi-time is represented by the multi-dimensional center of the group. Regarding this relativistic interpretation, which is not the same as the classical relativism, Laplacian (3) on the total space (space-time) corresponds to the Klein-Gordon operator. Note, however, that this multi-time operator is an elliptic one. This fact points to the distinctive features of the multi-time and classical relativism. Operator, $\mathbf{M}$, involving all angular momentum operators is called compound angular momentum operator.

3 Isospectrality constructions.

The isospectrality constructions are performed on H-type groups, $H_l^{(a,b)}$, and on their solvable extensions, $SH_l^{(a,b)}$, first. It is only the last section where they are extended to σ-equivalent groups whose endomorphism spaces are constructed by perturbing the Clifford endomorphism spaces. The main goal is to describe these constructions on non-periodic groups, however, in order to see both the similarities and differences, they are briefly reviewed here also in the center periodic cases. For fixed $a+b$ and l, these groups are defined on the same (X,Z)- resp. (t,X,Z)-space. There is established in many different ways that the metrics, $g_l^{(a,b)}$, in a family have completely different local geometries (cf., for instance, the striking examples), yet they are isospectral on a wide range of submanifolds.

3.1 Constructing the ball$\times$torus- and sphere$\times$torus-examples.

These examples are constructed for a family, $\Gamma\backslash H_l^{(a,b)}$, of Z-periodic manifolds. The submanifolds considered are torus bundles over a ball (resp. sphere) around the origin of the X-space. An intertwining operator can be constructed such that, for each invariant space W_γ constructed above, just an orthogonal transformation conjugating $J_\gamma^{(a,b)}$ to $J_\gamma^{(a',b')}$ on the X-space should be considered. The intertwining operator on W_γ is defined by the map induced on functions $\psi(X)$, defined in the Fourier-Weierstrass decomposition, by this point transformation. This transformation clearly intertwines $\Box_\gamma$ with $\Box'_\gamma$ such that it keeps also the boundary conditions. (The boundary conditions can be described in terms of radial functions. The intertwining of boundary conditions is then due to the invariance of these functions under the action of the operator.) It induces an appropriate intertwining operator also on the boundary manifolds. The striking examples appear on the quaternionic families $H_3^{(a,b)}$, in which case the sphere$\times$torus-type boundary manifolds in $\Gamma\backslash H_3^{(a+b,0)}$ are homogeneous while the others in the family are locally inhomogeneous. Note that the simplicity of this case is due to the fact that the intertwining operator is constructed, on each invariant space W_γ separately, by a single endomorphism, J_γ.

3.2 The ball- and sphere-type domains.

These examples were originally constructed in [Sz1, Sz2]. The ball-type domains are, by definition, diffeomorphic to Euclidean balls such that the sphere-type boundary manifolds are level sets described by equations of the form $\varphi(|X|,|Z|)=0$ resp. $\varphi(|X|,|Z|,t)=0$. These domains are invariant under the action of the orthogonal group $\mathbf{O}(\mathbf{R}^k)\times\mathbf{O}(\mathbf{R}^l)$, thus, they may be called domains of (X,Z)-revolutions. They can be visualized such that there is an X-ball centered at the origin of the X-space considered over the points of which there are Z-balls of radius $R_Z(|X|)$ around the origin of the Z-space considered. Then, the boundary is a level set defined by the equation $\varphi(|X|,|Z|)=|Z|-R_Z(|X|)=0$. By this reason, function φ is called radius function.

Note that radius $R_Z(|X|)$ is constant along a sphere defined by a constant radius $R_X=|X|$ in the X-space. Furthermore, the ball bundle defined by the Z-balls over this X-sphere is trivial. These are the so called *sphere$\times$ball-type manifolds* whose boundaries are *sphere$\times$sphere-type manifolds*. The new examples, not discussed in the earlier papers, are constructed on these domains and surfaces.

An other visualization can be started out with a Z-ball in the Z-space over the points of which there are X-balls of radius $R_X(|Z|)$ considered. Then the boundary is defined by $|X|-R_X(|Z|)=0$. However, this paper proceeds with the first description.

In the solvable case one should consider (Z,t)-balls and (Z,t)-spheres around the origin $(0_Z,1)$ defined for the hyperbolic (Z,t)-space. The base manifold is the same X-sphere as before. Note that a Z-ball $B_{R_Z}(0_Z)$ (resp. Z-sphere $S_{R_Z}(0_Z)$) uniquely extends into a geodesic ball (resp. sphere) of the hyperbolic (Z,t)-space. A sphere$\times$ball-type domain can be described as a hypersurface in a ball-type domain such that the Z-balls (resp. (Z,t)-balls) of the ball type domain are considered only over the points of a sphere S_{R_X} laying in the X-space. Similarly, the sphere$\times$sphere-type manifolds can be regarded as hypersurfaces in the sphere-type manifolds.

The isospectrality will be investigated, first, for the discrete families, $H_l^{(a,b)}$, defined by the same $a+b$ and l. The Laplacian is described then by (3). Comparing with the Zeeman operator (4), this operator involves all the endomorphisms, making the constructions much more difficult. The Laplacians of the members in a family differ from each other just by the last term, $\mathbf{M}$, which is called compound angular momentum operator. The spectral investigation both of $\mathbf{M}$ and Δ is completely missing in the literature. Note that this most intriguing operator, $\mathbf{M}$, commutes with both operators in the rest part of (3).

3.3 Eigenfunctions motivating the intertwining operators.

The eigenfunctions constructed next are not directly used in the isospectrality constructions and the rest part of this paper is understandable without knowing about their actual explicit form described in the second half of this section. However, there are important concepts introduced in the first part which are heavily used later on. The ultimate reason for describing these functions here is that they very clearly suggest the explicit form of the sought intertwining operators.

Since $\mathbf{M}$ commutes with the rest part, $\mathbf{O}$, of Δ, the eigenfunctions of Δ can be sought as common eigenfunctions for both operators $\mathbf{M}$ and $\mathbf{O}$. In the very first step we look for the eigenfunctions of a single angular momentum operator $\mathbf{D}_V\bullet$, defined by a Z-vector V. For a fixed X-vector Q and unit Z-vector $V_u = \frac{1}{|V|}V$, consider the X-function $\Theta_Q(X, V_u) = \langle Q + \mathbf{i}J_{V_u}(Q), X\rangle$ and its conjugate $\overline{\Theta}_Q(X, V_u)$. For vector $V = |V|V_u$, these functions are eigenfunctions of $D_V\bullet$ with eigenvalue $-|V|\mathbf{i}$ resp. $|V|\mathbf{i}$. The higher order eigenfunctions are of the form $\Theta_Q^p\overline{\Theta}_Q^q$ with eigenvalue $(q-p)|V|\mathbf{i}$.

In order to find eigenfunctions of the compound operator $\mathbf{M}$, consider a sphere S_{R_Z} of radius R_Z around the origin in the Z-space. For an appropriate function $\phi(|X|, V)$, depending on $|X|$ and $V \in S_{R_Z}$, define

$$\mathcal{F}_{QpqR_Z}(\phi)(X, Z) = \oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|, V)\Theta_Q^p(X, V_u)\overline{\Theta}_Q^q(X, V_u)dV. \tag{5}$$

By $V_u = V/|V|$, the V_u is considered as a function depending on V. Due to the relation $\mathbf{M}\oint = \oint \mathbf{i}D_V\bullet$, this function is an eigenfunction of $\mathbf{M}$ with the real eigenvalue $(p-q)R_Z$. These functions are eigenfunctions also of Δ_Z with eigenvalue R_Z^2. Also note that these eigenvalues do not change by varying Q, or, if the simple functions $\Theta_Q^p(X, V_u)$ resp. $\overline{\Theta}_Q^q(X, V_u)$ are exchanged for their pluralistic versions $\Theta_{Q_{1\dots p}}(X, V_u) := \Theta_{Q_1}(X, V_u)\dots\Theta_{Q_p}(X, V_u)$ resp. $\overline{\Theta}_{Q_{p+1\dots p+q}}(X, V_u)$.

Functions (5) defined by simple resp. pluralistic functions are said to be one-pole resp. multiple-pole functions with poles Q resp. $\{Q_i\}$. The function space generated for fixed 1-pole (resp. multi-poles) by all possible ϕ is not invariant with respect to the action of Δ_X, thus the eigenfunctions of the complete operator Δ do not appear in this form. In order to find the common eigenfunctions, the homogeneous but non-harmonic 1-pole polynomials $\Theta_Q^p\overline{\Theta}_Q^q$ (resp. the multiple-pole polynomials) of the X-variable should be exchanged for the 1-pole harmonic polynomials $\Pi_X^{(n)}(\Theta_Q^p\overline{\Theta}_Q^q)$, (resp. to the corresponding multiple-pole harmonic polynomials) defined by the projection, $\Pi_X^{(n)}$, onto the space of $n = (p+q)$-order homogeneous harmonic polynomials of the X-variable. These projections are explicitly described in the form

$$\Pi_X^{(n)} = \Delta_X^0 + B_1^{(n)}|X|^2\Delta_X + B_2^{(n)}|X|^4\Delta_X^2 + \dots \tag{6}$$

in [Sz2] (cf. formula (3.14) there), where $\Delta_X^0 = id$ and the constants $B_i^{(n)}$ are determined by a recursion formula. This formula easily implies that also

$$\mathcal{HF}_{QpqR_Z}(\phi)(X,Z) = \oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Pi_X(\Theta_Q^p(X,V_u)\overline{\Theta}_Q^q(X,V_u))dV \tag{7}$$

are eigenfunctions of $\mathbf{M}$ and Δ_Z with eigenvalues belonging to (5). The same statement is true regarding the multiple-pole-cases.

The action of the complete Laplacian (3) is a combination of X-radial differentiation, $\partial_{|X|}$, and multiplications with functions depending just on $|X|$. I. e., the action is completely reduced to X-radial functions. Also this reduced form of the Laplacian is not changing by varying Q, or, switching to multiple-pole functions. The eigenfunctions of Δ can be found by seeking the eigenfunctions of the reduced operator among the X-radial functions. The explicit computations are carried out in [Sz4, Sz5]. Since these details are not used in this paper, we just indicate that the eigenfunctions appear in the form $\oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)F_Q^{(p,q)}(X,V_u))dV$, (resp. in a corresponding multiple-pole version of this function), where $F_Q^{(p,q)}(X,V_u))$ is an eigenfunction of operator $\square_\gamma$ satisfying $|V_\gamma| = |V| = R_Z$. The latter ones are explicitly described in [Sz4, Sz5] in terms of homogeneous harmonic polynomials which are multiplied with radial functions. The eigenfunctions are determined below also by a different method, using the so called Itô polynomials.

Note that this construction is carried out for a fixed 1-pole Q (resp. a fixed multiple-pole, $\{Q_1,\dots,Q_p,Q_{p+1},\dots,Q_{p+q}\}$). An other type of constructions is as follows. For any unit vector V_u of the Z-space, consider a complex orthonormal basis $\{Q_{V_u1},\dots,Q_{V_uk/2}\}$ on the complex X-space defined by the complex structure J_{V_u} such that the vectors in front lay in $\mathbf{v}^{(a)}$ and all the others are in $\mathbf{v}^{(b)}$. Such a basis defines the complex coordinate system $\{z_{V_u1} = \Theta_{Q_{1V_u}},\dots,z_{V_uk/2} = \Theta_{Q_{(k/2)V_u}}\}$ on the X-space. This basis field must be smooth on an everywhere dense open subset of the unit Z-sphere such that it is the complement of a set of 0 measure. For given values $p_1,q_1,\dots,p_{k/2},q_{k/2}$, consider the polynomial $\prod_{i=1}^{k/2} z_{V_ui}^{p_i}\overline{z}_{V_ui}^{q_i}$. Then the functions

$$\oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)\varphi(|X|)\prod_{i=1}^{k/2} z_{V_ui}^{p_i}\overline{z}_{V_ui}^{q_i}dV \tag{8}$$

are eigenfunctions of the compound angular momentum operator $\mathbf{M}$. In order to have an eigenfunction for the complete Laplacian, one can use the above described method of projecting the polynomial into the space of homogeneous harmonic polynomials which have order $(p_1+q_1+\cdots+p_{k/2}+q_{k/2})$. In [Sz4, Sz5], the eigenfunctions of $\Box_\gamma$ are determined also by an other method, seeking them in the form $h^{(p_iq_i)}_{V_u}e^{-R_Z\langle z_{V_u},\overline{z}_{V_u}\rangle}$, where the $h^{(p_iq_i)}_{V_u}$ is an $(p_1+q_1+\cdots+p_{k/2}+q_{k/2})$-order polynomial. Then this function is an eigenfunction of $\Box_\gamma$ satisfying $|Z_\gamma|=R_Z$ if and only if the latter function is an Itô polynomial regarding the complex structure J_{V_u}. The final form of the eigenfunction is

$$\oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)h^{(p_iq_i)}_{V_u}e^{-R_Z z_{V_u}\cdot\overline{z}_{V_u}}dV. \tag{9}$$

Since Itô's polynomials are non-homogeneous, this is a different representation of the eigenfunctions. These explicit descriptions of the eigenfunctions will not be used in the rest part of the paper.

4 Constructing the intertwining operators.

4.1 Constructions on ball-type domains.

The constructions described in this sections are carried out for Heisenberg type groups, $H^{(a,b)}_l$ and $H^{(a',b')}_l$, which are in the same isospectrality family, i.e., $a+b=a'+b'$ holds. From each eigenfunction-construction described above one can derive the corresponding intertwining operator intertwining the corresponding eigenfunctions provided by the construction. Note that the functions appearing there are not of class L^2 regarding the Z-variable thus integral formulas (5), (7), (8) can not be directly used for defining the operator. This is why function $\phi(|X|,V)$, depending on $|X|$ and $V\in S_{R_Z}$ originally, is exchanged for one which an L^2 function of the V-variable, for any fixed $|X|$, and the integral is taken over the whole Z-space $\mathbb{R}^l$. In other words, the Z-Fourier transform on L^2_Z-setting is considered.

Also the order for introducing the various versions of the intertwining operators is an important issue. The first version is defined for a fixed basis, $\mathbf{Q}_F=\{Q_1,\dots,Q_{k/2}\}=\{\mathbf{Q}^{(a)}_F,\mathbf{Q}^{(b)}_F\}$, which does not depend on V_u, where the first $k^{(a)}/2$ number of vectors are in $\mathbf{v}^{(a)}$ and the following $k^{(b)}/2$ number of vectors are in $\mathbf{v}^{(b)}$. If these vectors are chosen such that they form an orthonormal basis regarding a fixed $J_{V_{0u}}$, then they form a complex (in general, non-orthonormal) basis for V_u's which form an everywhere dense open subset on the unit sphere of the Z-space. This operator is defined by means of the polynomials written up in terms of the coordinate functions

$$z_{V_u1}(X) = \Theta_{Q_1}(X, V_u), \dots, z_{V_uk/2}(X) = \Theta_{Q_{k/2}}(X, V_u) \tag{10}$$

resp. $\{z'_{V_u1}, \dots, z'_{V_uk/2}\}$. The denotation indicates that, although the basis is fixed, these coordinate functions depend on V_u.

For constructing the eigenfunctions in the previous section, one is using a fixed Z-sphere and functions, $\phi(V)$, of Dirac-type concentrated on this sphere. Then the eigenfunctions are represented by Z-Fourier transforms of these Dirac-type functions. Whereas, in defining the intertwining operator $\kappa_{\mathbf{Q}_F}$ determined for a constant basis $\mathbf{Q}_F$, one considers Z-Fourier transforms of appropriate L^2_Z-functions. More precisely, for an L^2_Z-function, $\phi(V)$, defined on the Z-space and arbitrary set $\{p_i, q_i\}$, where $i = 1, \dots, k/2$, of natural numbers the intertwining operator is defined by

$$\begin{aligned} \kappa_{\mathbf{Q}_F}: \quad & \mathcal{F}_{\mathbf{Q}_F\{p_iq_i\}}(\phi)(X,Z) = \textstyle\int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)\prod_{i=1}^{k/2} z^{p_i}_{V_ui}(X)\overline{z}^{q_i}_{V_ui}(X)dV \quad (11) \\ & \to \mathcal{F}'_{\mathbf{Q}_F\{p_iq_i\}}(\phi)(X,Z) = \textstyle\int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)\prod_{i=1}^{k/2} z'^{p_i}_{V_ui}(X)\overline{z}'^{q_i}_{V_ui}(X)dV. \end{aligned}$$

I. e., the $\kappa_{\mathbf{Q}_F}$ corresponds to a function, which is defined by the Z-Fourier transform formula in terms of $\phi, z_i^{p_i}, \overline{z}_i^{q_i}$, the function defined by the same expression but which is written up in terms of $\phi, z_i'^{p_i}$, and $\overline{z}_i'^{q_i}$. In these formulas, the $V_u = V/|V|$ is considered as a function depending on V, furthermore, the dependence of the complex coordinate functions on the X-variable is described in (10). Note that in this first version of the operator function ϕ depends just on V and not on $|X|$.

The domain and range of this operator is discussed in the next section. In this section one is focusing on the well-definedness and the intertwining of the Laplacians. Concerning these questions, we have.

Theorem 4.1. *The above $\kappa_{\mathbf{Q}_F}$ is a well defined one-to-one operator.*

Proof. This theorem is well established by proving that the image of a function which is in the domain of $\kappa_{\mathbf{Q}_F}$ and vanishes almost everywhere is a function vanishing almost everywhere. For proving this statement, suppose that function

$$\tilde{\varphi}(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \sum_{\{p_i,q_i\}} \phi_{\{p_i,q_i\}}(V) \prod_{i=1}^{k/2} z^{p_i}_{V_ui}(X)\overline{z}^{q_i}_{V_ui}(X),$$

where the terms of the sum (series) are defined regarding the independent polynomials $\prod_{i=1}^{k/2} z^{p_i}_{V_ui}\overline{z}^{q_i}_{V_ui}$, vanishes almost everywhere. Since the Z-Fourier transform is a one-to-one map on the corresponding L^2-Hilbert space, for any fixed X, function $\varphi(X,V)=\sum_{\{p_i,q_i\}}\phi_{\{p_i,q_i\}}(V)\prod_{i=1}^{k/2}z_{V_ui}^{p_i}(X)\overline{z}_{V_ui}^{q_i}(X)$ (whose Fourier transform is considered) must vanish almost everywhere. By the independence of the polynomials $\prod_{i=1}^{k/2}z_{V_ui}^{p_i}\overline{z}_{V_ui}^{q_i}$, what is satisfied for almost all V_u, a general function φ defined by this formula is non-zero if there is a non-zero L^2-function $\phi_{\{p_i,q_i\}}$ among the component functions. Therefore, due to the assumption of this theorem, all these ϕ's must vanish almost everywhere. Then, also the image $\tilde{\varphi}'$ must vanish almost everywhere. This proves the statement completely.

Actually, one has proved the following stronger statement: Function $\tilde{\varphi}'$ is zero almost everywhere if and only if its preimage $\tilde{\varphi}$ is zero almost everywhere. Thus also the one-to-one property is established completely. $\square$

Much more handy alternative definitions of the very same $\kappa_{\mathbf{Q}_F}$ are established in the following theorem.

Theorem 4.2. *(A) In the definition of $\kappa_{\mathbf{Q}_F}$, function ϕ may depend also on $|X|$, or, even more,on $|X^{(a)}|$ and $|X^{(b)}|$, where $X=X^{(a)}+X^{(b)}$ corresponds to the decomposition $\mathbf{v}=\mathbf{v}^{(a)}\oplus\mathbf{v}^{(b)}$. I. e., an equivalent version is:*

$$\kappa_{\mathbf{Q}_F}:\mathcal{F}_{\mathbf{Q}_F\{p_iq_i\}}(\phi)(X,Z)=\int_{\mathbb{R}^l}e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\prod_{i=1}^{k/2}z_{V_ui}^{p_i}(X)\overline{z}_{V_ui}^{q_i}(X)dV \quad (12)$$
$$\to\mathcal{F}'_{\mathbf{Q}_F\{p_iq_i\}}(\phi)(X,Z)=\int_{\mathbb{R}^l}e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\prod_{i=1}^{k/2}z_{V_ui}^{\prime p_i}(X)\overline{z}_{V_ui}^{\prime q_i}(X)dV.$$

In the more general version, the $\phi(|X|,V)$ is replaced by $\phi(|X^{(a)}|,|X^{(b)}|,V)$.

(B) The $\kappa_{\mathbf{Q}_F}$ intertwines both the Euclidean Laplacian Δ_X and the projections $\Pi_X^{(n)}$, $\Pi_X^{(n_a)}$, $\Pi_X^{(n_b)}$ with themselfs respectively. These projections are described in (6), furthermore, $n=p+q=\sum p_i+\sum q_i$, $n_a=p_a+q_a=\sum_{i=1}^{a}(p_i+q_i)$, $n_b=p_b+q_b=\sum_{i=a+1}^{a+b}(p_i+q_i)$. The operator can be written in the following alternative form:

$$\kappa_{\mathbf{Q}_F}:\mathcal{HF}_{\mathbf{Q}_F\{p_iq_i\}}(\phi)(X,Z)=\int_{\mathbb{R}^l}e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Pi_X^{(n)}(\prod_{i=1}^{k/2}z_{V_ui}^{p_i}\overline{z}_{V_ui}^{q_i})dV \quad (13)$$
$$\to\mathcal{HF}'_{\mathbf{Q}_F\{p_iq_i\}}(\phi)(X,Z)=\int_{\mathbb{R}^l}e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Pi_X^{(n)}(\prod_{i=1}^{k/2}z_{V_ui}^{\prime p_i}\overline{z}_{V_ui}^{\prime q_i})dV.$$

In the more general version, the $\phi(|X|,V)$ resp. $\Pi_X^{(n)}(\prod_{i=1}^{k/2}\dots)$ are replaced by $\phi(|X^{(a)}|,|X^{(b)}|,V)$ resp. $\Pi_X^{(n_a)}(\prod_{i=1}^{a}\dots)\Pi_X^{(n_b)}(\prod_{i=a+1}^{a+b}\dots)$.

(C) These versions for defining $\kappa_{\mathbf{Q}_F}$ allow to introduce its domain in a more precise way. In order to work on L^2 function spaces, one should consider functions of the form $\phi(|X|,V) = e^{-|X|^2}\varphi(|X|,V)$, where φ, for any fixed $|X|$, is of class L^2 with respect to the V-variable, and it is a polynomial with respect to the $|X|$-variable. By plugging them into the Fourier transform formula, they generate a pre-Hilbert space whose closure, regarding the L^2-Hilbert norm, is a Hilbert space. (In the next section, this domain is identified with the standard L^2-Hilbert space defined on $\mathbb{R}^k \oplus \mathbb{R}^l$.) A larger domain can be generated by functions $\phi(|X|,V)$, where, keeping the above assumption regarding the V-variable, function $\phi(|X|,\dot{V})$ depending on variable $|X|$ is of class L^2 for almost all fixed $\dot{V}$ on any interval $0 \leq |X| \leq R$.

Proof. (A) This proof explores that the system $\mathbf{Q}_F$ of independent vectors decomposes into two subsystems, $\mathbf{Q}_F = \{\mathbf{Q}^{(a)}, \mathbf{Q}^{(b)}\}$, consisting vectors from $\mathbf{v}^{(a)}$ resp. $\mathbf{v}^{(b)}$. They form a complex basis, both for J_{V_u} and J'_{V_u}, for an everywhere dense open set of unit vectors V_u. This basis can not be orthonormal for all V_u, even though it is orthonormal for some V_u. For each V_u, let $\mathbf{R}_{V_u} = \{\mathbf{R}^{(a)}_{V_u}, \mathbf{R}^{(b)}_{V_u}\}$ be a complex orthonormal basis, regarding the complex structure J_{V_u}, defining the complex coordinate system $\{z_{\mathbf{R}_{V_u}i}\}$. Then, there exist complex matrix, $(c_{ij}(V_u))$, such that $z_{\mathbf{R}_{V_u}i} = \sum_j c_{ij}(V_u) z_{\mathbf{Q}V_u j}$ hold. The equation expressing the orthonormality is $\sum_{mn} c_{im}\overline{c}_{jn}\langle Q_m, Q_n\rangle = \delta_{ij}$, where matrix with entries $\langle Q_m, Q_n\rangle$ is real. The basis field can be chosen such that it is continuous on an everywhere dense open subset of the unit vectors V_u.

In terms of J'_{V_u}, the above matrix transformation defines an other complex coordinate system, $z'_{\mathbf{R}'_{V_u}i} = \sum_j c_{ij}(V_u) z'_{\mathbf{Q}_F V_u j}$, with the corresponding complex basis $\mathbf{R}'_{V_u}$. Although it is not the same as $\mathbf{R}_{V_u}$, but, due to the relations $J'^{(a)}_{V_u} = J^{(a)}_{V_u}$ and $J'^{(b)}_{V_u} = -J^{(b)}_{V_u}$, this new basis also yields the above orthonormality equation. Therefore, it is orthonormal regarding the complex structure J'_{V_u} and, according to the computations:

$$|X|^2 = \sum_i z_{\mathbf{R}_{V_u}i}\overline{z}_{\mathbf{R}_{V_u}i} = \sum_i(\sum_j c_{ij}(V_u) z_{\mathbf{Q}_F V_u j})(\sum_r \overline{c}_{ir}(V_u)\overline{z}_{\mathbf{Q}_F V_u r}) = \tag{14}$$

$$= \sum_{j,r}(\sum_i c_{ij}(V_u)\overline{c}_{ir}(V_u)) z_{\mathbf{Q}_F V_u j}\overline{z}_{\mathbf{Q}_F V_u r})$$

$$= \sum_i z'_{\mathbf{R}_{V_u}i}\overline{z}'_{\mathbf{R}_{V_u}i} = \sum_{j,r}(\sum_i c_{ij}(V_u)\overline{c}_{ir}(V_u)) z'_{\mathbf{Q}_F V_u j}\overline{z}'_{\mathbf{Q}_F V_u r})$$

the X-radial function $|X|^2$ appears in the same polynomial form regarding both coordinate systems $\{z_{\mathbf{Q}_F V_u j}\}$ and $\{z'_{\mathbf{Q}_F V_u j}\}$. The same statement is true for any power, or, by using power series, for any function $\phi(|X|)$ of the basic radial function. This observation proves (A) completely.

(B) Regarding the Laplacian the same computation yield:

$$\Delta_X = \sum_i \partial_{z_{\mathbf{R}_{V_u} i}} \partial_{\overline{z}_{\mathbf{R}_{V_u} i}} = \sum_i (\sum_j c_{ij}(V_u) \partial_{z_{\mathbf{Q}_F V_u j}})(\sum_r \overline{c}_{ir}(V_u) \partial_{\overline{z}_{\mathbf{Q}_F V_u r}}) = \tag{15}$$
$$= \sum_{j,r} (\sum_i c_{ij}(V_u) \overline{c}_{ir}(V_u)) \partial_{z_{\mathbf{Q}_F V_u j}} \partial_{\overline{z}_{\mathbf{Q}_F V_u r})}$$
$$= \sum_i \partial_{z'_{\mathbf{R}_{V_u} i}} \partial_{\overline{z}'_{\mathbf{R}_{V_u} i}} = \sum_{j,r} (\sum_i c_{ij}(V_u) \overline{c}_{ir}(V_u)) \partial_{z'_{\mathbf{Q}_F V_u j}} \partial_{\overline{z}'_{\mathbf{Q}_F V_u r}}),$$

i. e., also this Laplacian appears regarding both coordinate system in the same form. This proves the invariance of Δ_X under the action of $\kappa_{\mathbf{Q}_F}$. The explicit formula (6) regarding $\Pi_X^{(n)}$ along with (A) and the above statement concerning Δ_X prove (B) completely.

(C) This statement is self-contained. $\square$

Yet an other alternative definition of $\kappa_{\mathbf{Q}_F}$ can be introduced by using 1-pole functions $\Theta_Q^p \overline{\Theta}_Q^q$, where Q is in the real span of the vector-system $\mathbf{Q}_F$, i. e., $Q \in Span_{\mathbb{R}}(\mathbf{Q}_F)$. The version using multi-pole functions, defined by vectors laying in $Span_{\mathbb{R}}(\mathbf{Q}_F)$, adds nothing new to the above polynomial-version, therefore, this case is omitted here. Also note that the multi-pole functions span the same function space spanned by the 1-pole functions, thus, complete analysis can be performed by using only the simple ones.

Theorem 4.3. *The $\kappa_{\mathbf{Q}_F}$ is well defined by each of the following correspondences:*

$$\kappa_{\mathbf{Q}_F}: \quad \mathcal{F}_{Qpq}(\phi)(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \phi(V) \Theta^p_{QV_u}(X) \overline{\Theta}^q_{QV_u}(X) dV \tag{16}$$
$$\to \mathcal{F}'_{Qpq}(\phi)(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \phi(V) \Theta'^p_{QV_u}(X) \overline{\Theta}'^q_{QV_u}(X) dV,$$

$$\kappa_{\mathbf{Q}_F}: \quad \mathcal{F}_{Qpq}(\phi)(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \phi(|X|,V) \Theta^p_{QV_u}(X) \overline{\Theta}^q_{QV_u}(X) dV \tag{17}$$
$$\to \mathcal{F}'_{Qpq}(\phi)(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \phi(|X|,V) \Theta'^p_{QV_u}(X) \overline{\Theta}'^q_{QV_u}(X) dV,$$

$$\kappa_{\mathbf{Q}_F}: \quad \mathcal{HF}_{Qpq}(\phi)(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \phi(|X|,V) \Pi_X^{(p+q)}(\Theta^p_{QV_u} \overline{\Theta}^q_{QV_u}) dV \tag{18}$$
$$\to \mathcal{HF}'_{Qpq}(\phi)(X,Z) = \int_{\mathbb{R}^l} e^{\mathbf{i}\langle Z,V\rangle} \phi(|X|,V) \Pi_X^{(p+q)}(\Theta'^p_{QV_u} \overline{\Theta}'^q_{QV_u}) dV,$$

where $Q \in Span_{\mathbb{R}}(\mathbf{Q}_F)$ is an arbitrary vector and $\Theta_{QV_u}(X) := \Theta_Q(X, V_u)$.

There is a more general version also in this case which corresponds to the exchange of Q for a pair, $(Q^{(a)}, Q^{(b)})$, followed by the exchange of functions $\phi(|X|, V)$, $\Theta^p_{QV_u}\overline{\Theta}^q_{QV_u}$, $\Pi_X^{(p+q)}(..)$ for the following ones

$$\phi(|X^{(a)}|, |X^{(b)}|, V),\ \Theta^{p_a}_{Q^{(a)}V_u}\overline{\Theta}^{q_a}_{Q^{(a)}V_u}\Theta^{p_b}_{Q^{(b)}V_u}\overline{\Theta}^{q_b}_{Q^{(b)}V_u},\ \Pi^{(p_a+q_a)}_{X^{(a)}}(..)\Pi^{(p_b+q_b)}_{X^{(b)}}(..)$$

respectively.

Remark. Although, it is not defined by an appropriate basis, the well-definedness of the operator is not jeopardized in this theorem. It can be defined, however, by constructing a basis as follows.

For fixed Q and natural numbers p and q, let $\mathbf{\Phi}_{Qpq}$ (resp. $\mathbf{\Xi}_{Qpq}$) be the L^2 function space spanned by functions of the form $\mathcal{F}_{Qpq}(\phi)(X, Z)$ (resp. $\mathcal{HF}_{Qpq}(\phi)(X, Z)$), where $\phi(|X|, V)$ can be an arbitrary L^2_Z-function. For fixed p and q, all these spaces sum up to the total spaces $\mathbf{\Phi}_{pq} = \sum_Q \mathbf{\Phi}_{Qpq}$ (resp. $\mathbf{\Xi}_{pq} = \sum_Q \mathbf{\Xi}_{Qpq}$). There exist finite many Q_i such that the total space is the direct sum of the independent subspaces $\mathbf{\Phi}_{Q_ipq}$ (resp. $\mathbf{\Xi}_{Q_ipq}$). For the two type of total spaces these numbers are different. Due to the non-degeneracy of $\kappa_{\mathbf{Q}_F}$, the total spaces $\mathbf{\Phi}'_{pq}$ (resp. $\mathbf{\Xi}'_{pq}$) are the direct sums of the independent subspaces $\mathbf{\Phi}'_{Q_ipq}$ (resp. $\mathbf{\Xi}'_{Q_ipq}$). The $\kappa_{\mathbf{Q}_F}$ can be defined just by its actions $\kappa_{\mathbf{Q}_F} : \mathbf{\Phi}_{Q_ipq} \to \mathbf{\Phi}'_{Q_ipq}$ (resp. $\kappa_{\mathbf{Q}_F} : \mathbf{\Xi}_{Q_ipq} \to \mathbf{\Xi}'_{Q_ipq}$) on these subspaces. This construction method, which will not be used in this paper, can be applied for explicit spectral computations. □

Above, six versions of the very same intertwining operator defined by a constant complex basis, $\mathbf{Q}_F$, were introduced. A changing orthonormal complex basis field, $\mathbf{Q}(V_u) = \{\mathbf{Q}^{(a)}(V_u), \mathbf{Q}^{(b)}(V_u)\}$, which is supposed to be continuous on an everywhere dense open subset of the unit vectors V_u, also defines an intertwining operator. Unlike the constant field, which can be orthonormal only for V_u's of zero measure, the changing field is supposed to be orthonormal almost everywhere. Then, by $J'^{(a)}_{V_u} = J^{(a)}_{V_u}$ and $J'^{(b)}_{V_u} = -J^{(b)}_{V_u}$, also the basis $\mathbf{Q}'(V_u)$ is orthonormal regarding the complex structure J'_{V_u}. The starting version of the intertwining operator $\kappa_{\mathbf{Q}(V_u)}$ defined by a changing complex orthonormal basis is introduced by formula (11), where the complex coordinates, z_{V_ui}, are defined by $\mathbf{Q}(V_u)$. Now we have

Theorem 4.4. *Operator*

$$\kappa_{\mathbf{Q}(V_u)}: \quad \mathcal{F}_{Q(V_u)\{p_i,q_i\}}(\phi)(X, Z) \to \mathcal{F}'_{Q(V_u)\{p_i,q_i\}}(\phi)(X, Z), \tag{19}$$

defined for all sets, $\{p_i, q_i\}$, of natural numbers and L^2-functions $\phi(V)$, is a well-defined one-to-one map leaving the X-radial functions, the Laplacian

Δ_X and the projections $\Pi_X^{(n)}$ invariant. Thus, this operator can be defined in the alternative ways

$$\kappa_{\mathbf{Q}(V_u)}: \quad \mathcal{F}_{Q(V_u)\{p_i,q_i\}}(\phi)(X,Z) \to \mathcal{F}'_{Q(V_u)\{p_i,q_i\}}(\phi)(X,Z), \tag{20}$$

$$\kappa_{\mathbf{Q}(V_u)}: \quad \mathcal{HF}_{Q(V_u)\{p_i,q_i\}}(\phi)(X,Z) \to \mathcal{HF}'_{Q(V_u)\{p_i,q_i\}}(\phi)(X,Z), \tag{21}$$

where the $\phi(|X|,V)$ may depend also on $|X|$, or, $|X^{(a)}|$ and $|X^{(b)}|$.

The proof is the same as for the constant basis case. Since both basis', $\mathbf{Q}(V_u)$ and $\mathbf{Q}'(V_u)$, are orthonormal, the proof of the invariance of the radial functions, Euclidean Laplacian Δ_X, and projections $\Pi_X^{(n)}$ is even simpler as in the previous case. Let it also be mentioned that versions (16)-(18) can not be introduced for the changing basis case because the Q's must be in the intersection of all real subspaces $span_{\mathbb{R}}\mathbf{Q}(V_u)$.

Now we are ready to prove the first main theorem in this paper.

Theorem 4.5. *Both in the constant and the changing basis cases, the $\kappa_{\mathbf{Q}}$ intertwines the complete Laplacians Δ and Δ'.*

Proof. There is proved above that both the Laplacian Δ_X and operators defined by multiplication with radial functions are intertwined by the $\kappa_{\mathbf{Q}}$. The other parts of the Laplacians are also intertwined because of the following identities.

$$\begin{aligned}\kappa_{\mathbf{Q}}: \quad & \mathbf{M}\mathcal{F}_{\mathbf{Q}(V_u)\{p_i,q_i\}}(\phi) = \mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}((q-p)|V|\phi) \\ & \to \mathcal{F}'_{\mathbf{Q}\{p_i,q_i\}}((q-p)|V|\phi) = \mathbf{M}'\mathcal{F}'_{\mathbf{Q}\{p_i,q_i\}}(\phi),\end{aligned} \tag{22}$$

$$\begin{aligned}\kappa_{\mathbf{Q}}: \quad & \Delta_Z\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(\phi) = \mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(-|V|^2\phi) \\ & \to \mathcal{F}'_{\mathbf{Q}\{p_i,q_i\}}(-|V|^2\phi) = \Delta_Z\mathcal{F}'_{\mathbf{Q}\{p_i,q_i\}}(\phi)\end{aligned} \tag{23}$$

where $p = \sum p_i$ and $q = \sum q_i$. These formulas remain true if $\mathcal{F}$ is exchanged for $\mathcal{HF}$. ∎

4.2 Constructions on sphere×ball-type domains.

For introducing the intertwining operators on sphere×ball-type domains, one can start with version (13), where it is defined for a constant basis in terms of homogeneous harmonic polynomials of the X-variable defined on the ambient space. It is pointed out there, that the operator is well defined without using a basis, but now, for each n, consider functions of the form $\Pi_X^{(n)}(\prod_{i=1}^{k/2} z_{V_u i}^{p_i}\overline{z}_{V_u i}^{q_i}) = |X|^{2n}\sigma_{V_u\{p_i,q_i\}}$ such that, for a fixed V_u, functions $\sigma_{V_u\{p_i,q_i\}}$ form a basis among the corresponding spherical harmonics defined on the unit sphere of the X-space. Note that the dimension of n^{th}-order harmonic polynomials is less than the dimension of n^{th}-order homogeneous polynomials, thus, not all polynomials from the latter set are subjugated to the projection. Anyhow, such choices for such basis' exist. All those V_u's satisfying this property form an everywhere dense open subset of the unit Z-vectors. The functions whose Fourier transforms are considered in an arbitrary version of the intertwining operator have unique expansions, $\sum_{\{p_i,q_i\}}\phi_{\{p_i,q_i\}}(|X|,V)\sigma_{V_u\{p_i,q_i\}}$, by these spherical harmonics and the intertwining operator defined by this representation is the same as for the original representation. These spherical harmonics can be pulled back from the unit X-sphere to the considered X-sphere by the central projection $\pi : X \to X_u = X/|X|$. Then, for any function $c(V)$, function $c(Y)\pi^*(\sigma_{V_u\{p_i,q_i\}})$ defined on the sphere$\times$ball-type manifold is the restriction exactly of those functions $\phi_{\{p_i,q_i\}}(|X|,V)\sigma_{V_u\{p_i,q_i\}}$ for which $\phi_{\{p_i,q_i\}}(R_X,V) = c(V)$ holds. Thus we have

Theorem 4.6. *Both in the fixed and changing basis cases, intertwining operator $\kappa_{\mathbf{Q}}$ induces a well defined action on functions defined by restrictions from the ambient manifolds onto the sphere$\times$ball submanifolds. This induced operator, $\tilde{\kappa}_{\mathbf{Q}}$, is well defined for all versions and is the same as the operator constructed by a basis of the space of spherical harmonics.*

This induced operator intertwines the Laplacians defined on the sphere$\times$-ball-type submanifolds.

The proof of well-definedness for a basis of the space of spherical harmonics is the same as on the ambient manifold. The Laplacian on the submanifold differs from (3) just by the terms Δ_X and $|X|$ which should be exchanged for Δ_{S_X} (which is the Laplacian on the X-sphere) and R_X (which is the radius of the X-sphere), respectively. Note that the X-directional derivatives included into $\mathbf{M}$ concern directions tangent to the sphere. Thus this term is the same as for the ambient space. In other words, in order to have the Laplacian on the submanifold, just the radial Laplacian, Δ_r, of the X-space should be dropped from (3). Since both the radial and the complete Laplacians are invariant under the action of the ambient intertwining operator, also the Laplacian on the submanifold is invariant.

5 Domain and range of $\kappa_{\mathbf{Q}}$.

The domain and range of the intertwining operators is determined by a function transformation which is noteworthy also without this application.

5.1 The dual Radon transform.

This transform was first investigated in [Sz6], pages 264-266, where it is called *boomerang transform*. The results provided there include also an inversion formula, which, by a new proof, was reestablished by Á. Kurusa [Ku1, Ku2]. He called the operator itself *dual Radon transform* which name better describes the area this transform belongs to. We adopt this name, however, the following review proceeds with the author's original ideas.

Let $g_\theta(r)$ be a half-line parameterized by arc-length r which has its end-point, corresponding to $r = 0$, at the origin O of $\mathbb{R}^l$ and which is pointing to the point $\theta \in S_0^{l-1}(1)$ of the unit sphere $S_0^{l-1}(1) \subset \mathbb{R}^l$ around the origin O. Then (θ, r) serve as polar coordinates for the points, Z, of $\mathbb{R}^l$. These denotations indicate that this transform will be used on the Z-space of H-type groups. If $f(\theta, r)$ is a continuous function defined on $\mathbb{R}^l$, then, for each fixed θ_0, it determines a cylindrical function $f^c_{\theta_0}(Z)$ defined on the unique half-space whose perpendicular projection onto the line spanned by $g_{\theta_0}(r)$ is equal to this half-line. If the projection of Z is the point having the polar coordinates (θ_0, r), then, by definition, $f^c_{\theta_0}(Z) = f(\theta_0, r)$. By considering this construction for each θ, one can define the function-valued function $\theta \to f^c_\theta$. The dual Radon transform, $f \to f_\tau$, is defined by the integral

$$f_\tau := \int_{S^{l-1}} f^c_\theta d\theta, \tag{24}$$

which can be written also in the form

$$f_\tau(Z) := \int_{\langle\theta,Z\rangle\geq 0} f(\theta, \langle\theta, Z\rangle)d\theta. \tag{25}$$

Apply Thales' theorem to the last formula to see that the transform is defined by the integral of f on the sphere of diameter $[0, Z]$ by the measure $d\theta$. Note that this measure differs from the canonical measure of the Thales sphere. By (24) and Fubini's theorem we have:

Lemma 5.1. *Let $f(Z)$ be an arbitrary continuous function and let μ be a continuous function with compact support in $\mathbb{R}^l$. Then the integral formula*

$$\int_{\mathbb{R}^l} f_\tau(Z)\mu(Z)dZ = \int_{S^{l-1}} \int_0^\infty f(\theta, r)\mu_R(\theta, r)drd\theta \tag{26}$$

holds, where $\mu_R(\theta, r)$ is the Radon transform of μ defined by the integrals of this function on the hyperspaces intersecting g_θ at the points having the polar coordinates (θ, r) perpendicularly.

Formula (26) reveals that the considered transform is dual to the Radon transform, indeed. The main result in this section is:

Theorem 5.2. *Let $f_\tau(\theta, r)$ be an arbitrary function of class C^{2m} with compact support in $\mathbb{R}^l$, where $l = 2m+1$ is odd or $l = 2m$ is even. Then f_τ has an inverse, f, regarding the dual Radon transform, which is of the form*

$$f = \frac{(-1)^m}{(2\pi)^{2m}}((f_\tau)_R)^{(2m)}, \quad \text{if} \quad l = 2m+1, \tag{27}$$

where $(2m)$ means the $2m$th derivative of the functions with respect to r, resp.

$$f = \frac{(-1)^m (l-1)!}{(2\pi)^{2m}}((f_\tau)_R)^{[2m]}, \quad \text{if} \quad l = 2m, \tag{28}$$

where $\varphi^{[2m]}(r)$ is defined for a function $\varphi(t)$ on $\mathbb{R}$ by

$$\varphi^{[2m]}(r) := \int_0^\infty \frac{1}{t^{2m}}\big(\varphi(r+t) + \varphi(r-t) \tag{29}$$
$$-2\big[\varphi(r) + \frac{t^2}{2!}\varphi''(r) + \cdots + \frac{t^{2m-2}}{(2m-2)!}\varphi^{(2m-2)}(r)\big]\big)dt.$$

Proof. Let μ be a function of class C^{2m} with compact support in $\mathbb{R}^{2m+1}$. From (26) we get

$$\int_{\mathbb{R}^l} [((f_\tau)_R)^{(2m)}]_\tau(Z)\mu(Z)dZ = \tag{30}$$
$$\int_{S^{l-1}} \int_0^\infty \frac{(-1)^m}{(2\pi)^{2m}}((f_\tau)_R)^{(2m)}(\theta, r)\mu_R(\theta, r)drd\theta = \tag{31}$$
$$\int_{S^{l-1}} \int_0^\infty (f_\tau)_R(\theta, r)\big(\frac{(-1)^m}{(2\pi)^{2m}}(\mu)_R^{(2m)}(\theta, r)\big)drd\theta = \tag{32}$$
$$\int_{\mathbb{R}^l} f_\tau(Z)\big(\frac{(-1)^m}{(2\pi)^{2m}}(\mu)_R^{(2m)}\big)_\tau(Z)dZ = \int_{\mathbb{R}^l} f_\tau(Z)\mu(Z)dZ.$$

In the last step the well known Radon inverse formula is used. Since μ is arbitrary, formula (27) is established. Formula (28) can be established in the same way. ☐

Let it be mentioned that all non-trivial isospectrality examples constructed in this paper arise from odd dimensional Z-spaces. Thus, only formula (27) applies to these cases.

5.2 The domain of the intertwining operators.

The above theorem is used to prove that the function space generated by functions of the form $\mathcal{F}_{\mathbf{Q}\{p_iq_i\}}(\phi)(X,Z)$ contains all functions $P(X)f(Z)$, where $P(X)$ is a complex valued polynomial and $f(Z)$ is a smooth function of compact support on the Z-space. Thus, by using appropriate limiting procedures, the whole standard $L^2_{\mathbb{C}}$-Hilbert space on the (X,Z)-space can be generated in this way. Note that the Z-Fourier transform of ϕ is "twisted" with the polynomials appearing in the formula which depend, beside X, also on V_u. By this reason, it is called also *twisted Z-Fourier transform.* It is this feature what makes the constructions of the above functions highly non-trivial.

The proof of the above statement needs some preparations. For a fixed unit Z-vector V_u^0 and positive number δ, let $T_\delta(V_u^0)$ be the tube of radius δ around the half-line $g_{V_u^0}(r)$. By the standard definition, it is the union of those discs, $D_\delta^{(l-2)}(r)$, of radius δ about the points of the half-line which intersect the half-line perpendicularly. The characteristic functions of this tube and the half-line, defined on the whole Z-space resp. line determined by $g_{V_u^0}(r)$, are denoted by $\chi_{\delta V_u^0}(Z)$ and $\chi_{g_{V_u^0}}(t)$ respectively. Then,

$$\lim_{\delta\to 0}\frac{1}{Vol(D_\delta^{(l-2)})}\mathcal{F}_{\mathbf{Q}\{p_iq_i\}}(\chi_{\delta V_u^0}\phi)(X,Z)=\prod_{i=1}^{k/2}z'^{p_i}_{V_u^0 i}(X)\overline{z}'^{q_i}_{V_u^0 i}(X)\mathbb{L}^c_{V_u}(\phi_{V_u^0})(Z),$$

where function $L_{V_u}(\phi_{V_u^0})(t)=\chi_{g_{V_u^0}}(t)Fou_{\pm V_u}(\chi_{g_{V_u^0}}\phi_{V_u^0})(t)$, defined on the whole line spanned by V_u^0 and parameterized by t satisfying $t(V_u)=1$, vanishes for $t<0$ and it is the Laplace transform of $\phi_{V_u^0}(r)$ defined on the half-line $g_{V_u^0}(r)$. In the latter formula, this function is described in terms of the 1-dimensional Fourier transform $Fou_{\pm V_u^0}$ defined on the whole line spanned by V_u^0.

Note that the above function does appear as a product of X- and Z-depending functions. In the next averaging process they are used to construct $P(X)f(Z)$ such that one considers the same polynomial $P(X)$ for all V_u and, in the end, the f appears as the dual Radon transform of an appropriate function defined on the Z-space.

For a complex valued polynomial, $P(X)$, let $P_{V_u^0}(X)$ be a representation of the polynomial in terms of the complex structure $J_{V_u^0}$. Furthermore, let $P_{V_u^0}(X,V_u)$ be the function defined by replacing V_u^0 with general V_u in $P_{V_u^0}(X)$. Therefore, $P_{V_u^0}(X)=P_{V_u^0}(X,V_u^0)$ holds, but for other V_u's there are other polynomials defined. Denotation $\mathcal{F}_{\mathbf{Q}P_{V_u^0}}$ means that

$\prod_{i=1}^{k/2} z'^{p_i}_{V_u^0 i}(X)\overline{z}'^{q_i}_{V_u^0 i}(X)$ is replaced by $P_{V_u^0}(X, V_u)$ in the above formulas. Then we have:

$$\lim_{\delta\to 0}\int_{S^{l-1}} \frac{1}{Vol(D_\delta^{(l-2)})}\mathcal{F}_{\mathbf{Q}P_{V_u^0}}(\chi_{\delta V_u^0}\phi)(X,Z)dV_u^0 = \tag{33}$$

$$P(X)\int_{S^{l-1}} \mathbb{L}^c_{V_u^0}(\phi_{V_u^0})(Z)dV_u^0 = P(X)(\mathbb{L}_{\pm V_u}(\phi))_\tau(Z),$$

where $\mathbb{L}_{\pm V_u}(\phi)$ denotes the function defined on the whole line spanned by $\pm V_u$ by the Laplace transforms of functions $\phi_{V_u}(r)$ resp. $\phi_{-V_u}(r)$. Thus we have:

Theorem 5.3. *For given polynomial $P(X)$ and smooth function $f(Z)$ of compact support the product $P(X)f(Z)$ is limit of convergent sequences of functions belonging to the domain of an intertwining operator. This sequence is constructed by the above method, where function $\phi(V) = \mathbb{L}^{-1}_{\pm V_u}(f_{\tau^{-1}})(V)$ is derived from ϕ by the inverse dual-Radon resp. 1-dimensional Laplace transforms defined above on the corresponding half-lines. (The inverse formula for the Laplace transform is called Mellin's formula. An alternative version is the so called Post's formula.)*

The same proof yield those versions of the theorem when function f is of the form $f(|X|, Z)$ such that, for any fixed $|X_0|$, function $f(|X_0|, Z)$ is of compact support on the Z-space, or, when this problem is considered on a sphere$\times$ball-type domain and $P(X)$ is replaced by its restriction, $\tilde{P}(X)$, onto the sphere and $f(Z)$ is the same function as before. In these cases $P(X)f(|X|, Z)$ resp. $\tilde{P}(X)f(Z)$ are in the domain of the intertwining operator. In both cases, functions $\phi(|X|, V)$ resp. $\phi(V)$ can be found by the same inverse operations defined on the Z-space.

6 Intertwining of the boundary conditions.

The most important tool applied in establishing the intertwining of the boundary conditions is a theory developed for one- and two-pole functions.

6.1 Formulas for one- and two-pole functions.

For a unit Z-vector Z_0 and $Q \in \mathbb{R}^k$, denotation X_{QZ_0} means that this X-vector is in the subspace spanned by Q and $J_{Z_0}(Q)$. On the plane $P(Q, J_{Z_0}(Q))$ spanned by these two vectors the polar coordinates $(|X_{QZ_0}|, \alpha)$ are defined such that $\alpha(Q) = 0, \alpha(J_{Z_0}(Q)) = \pi/2$ hold. By the restriction

$\alpha \leq \pi$ imposed for all Z_0, one has a spherical coordinate system on the $(l+1)$-dimensional space, S_Q, spanned by Q and all $J_{Z_0}(Q)$. Thus these α parameter lines are half circles running in the half-plane $P^+(Q, J_{Z_0}(Q)) \subset P(Q, J_{Z_0}(Q))$ bounded by $\mathbb{R}Q$ and containing $J_{Z_0}(Q)$.

Function Θ_Q can be described by this coordinate system as follows. If the orthogonal projection, X_Q, of X onto S_Q is in $P^+(Q, J_{Z_0}(Q))$, then

$$\Theta_Q(X, V_u) = \langle Q, X_Q\rangle + \mathbf{i}\langle [Q, X_Q], V_u\rangle = |X_Q|(\cos\alpha + \mathbf{i}\langle Z_0, V_u\rangle \sin\alpha). \quad (34)$$

Powering performed in $\Theta_Q^p \overline{\Theta}_Q^q$ yield:

Theorem 6.1. *On those vectors, X, whose projections onto S_Q fall onto a fixed α-half-circle around the origin of the half-plane $P^+(Q, J_{Z_0}(Q))$, a 1-pole function $\mathcal{F}_{Qpq}(\phi)$ has the form:*

$$\mathcal{F}_{Qpq}(\phi)(X, Z) = \int_{\mathbb{R}^l} \phi(|X|, V)\Theta_Q^p(X, V_u)\overline{\Theta}_Q^q(X, V_u)e^{\mathbf{i}\langle Z, V\rangle}dV = \quad (35)$$

$$\sum_{s=0}^{p+q} |X_Q|^{p+q}\cos^{p+q-s}\alpha \sin^s\alpha \int_{\mathbb{R}^l} A_{spq}\langle Z_0, V_u\rangle^s \phi(|X|, V)e^{\mathbf{i}\langle Z, V\rangle}dV = \quad (36)$$

$$\sum_{s=0}^{p+q} |X_Q|^{p+q}\cos^{p+q-s}\alpha \sin^s\alpha (A_{spq}(-\mathbf{i})^s\partial_{Z_0}^s) \int_{\mathbb{R}^l} \phi(|X|, V)|V|^{-s}e^{\mathbf{i}\langle Z, V\rangle}dV. \quad (37)$$

Function $\phi_s(X, V) = \phi(|X|, V)|V|^{-s}$, whose Z-Fourier transform, $\tilde{\phi}_s$, appears as the last integral term of (36), is derived from ϕ such that it depends just on $|X|$ and the Z-variable. Term behind $\sin^s\alpha$ is denoted by $\tilde{A}_{spq}(|X|, Z_0, Z)$. If both $|X|$ and Z are fixed, then $\tilde{A}_{spq}$ is constant for those X's which project onto the half circle determined for Z_0 by the parameter-range $\alpha \leq \pi$. On S_Q, whose points are denoted by X_Q, this function appears in the form

$$\sum_{s=0}^{p+q} |X_Q|^{p+q}\cos^{p+q-s}\alpha \sin^s\alpha\, \tilde{A}_{spq}(|X_Q|, Z_0, Z), \quad (38)$$

where, for fixed values of $|X|$ and Z, the $\tilde{A}_{spq}$ is an s^{th}-order polynomial which can be described in terms of the unit vectors $J_{Z_0}(Q_u)$, where $Q_u = Q/|Q|$, as follows.

Originally, this polynomial can explicitly be determined on the Z-space by the expansion $\langle Z_0, V_u\rangle^s = \sum_{j=0}^s B_j\sigma_{Z_0}^{s-j}(V_u)$ *in terms of the spherical harmonics* $\sigma_{Z_0}^{s-j}(V_u)$*. Since, for any fixed* Z_0*, function* $\langle Z_0, V_u\rangle^s$ *defined on the unit Z-sphere is radial about the center* Z_0*, thus also the spherical harmonics are radial about* Z_0 *and the convolutions with them are nothing but the projections onto the corresponding subspaces of spherical harmonics. Thus,*

$$\tilde{A}_{spq}(|X|, Z_0, Z) = \sum_j B_j\varphi^{(s-j)}(|X|, Z_0, Z),$$

where $\varphi^{(s-j)}$ *is the corresponding spherical harmonics appearing in the expansion of* $\varphi(|X|, Z, V_u) = \int_0^\infty A_{spq}\phi_s(|X|, V_u, r)e^{r\mathbf{i}\langle Z,V_u\rangle}dr$ *which function is defined, for fixed* Z *and* $|X|$*, by integrals with respect to* dr *defined for the polar coordinate system* (V_u, r)*.*

But this function depends on X*. In its final form, it can be viewed such that the function determined on the Z-space defines, first, a 0-homogeneous function on the equator plane* E_{Q_u} *spanned by the X-vectors* $J_{Z_0}(Q_u)$*. Then, it extends onto* S_Q *such that, on an* X_Q*, it takes the value determined by* Z_0 *if and only if* $X_Q \in P^+(Q, J_{Z_0}(Q))$*.*

For other points, which are outside of S_Q*, the function is determined by this function and projection onto* S_Q*. Note that the* $\tilde{A}_{spq}$ *is defined for* X *and not for* X_Q*, meaning that, instead of* $|X_Q|$*, function* ϕ *involves* $|X|$ *to this term. The latter parameters stand in front of the formula and are in connection with the trigonometric polynomials.*

Such formulas can be established also for $\mathcal{HF}_{Qpq}(\phi)$. Functions $\overline{\Theta}_Q^q$ resp. Θ_Q^p are homogeneous harmonic polynomials of the X-variable, thus in cases satisfying $p = 0$ or $q = 0$, the function in (35) is nothing but $\mathcal{HF}_{Qpq}(\phi)$. If $pq \neq 0$, there are new terms, $|X|^{2r}\Theta_Q^{p-r}\overline{\Theta}_Q^{q-r}$, appearing in the X-harmonic polynomial $\Pi_X(\Theta_Q^p\overline{\Theta}_Q^q)$. For each r, an additional sum shows up both in (36) and (37). Comparing the first and the r^{th} sums, the $p+q$ is exchanged for $p+q-2r$, which is the greatest possible value for s_r in the sum. Such a new sum can be combined with the first one, where $r = 0$, by multiplying the r^{th} sum by $1 = (\cos^2\alpha + \sin^2\alpha)^r$. Thus one gets trigonometric polynomials appearing in the first sum. By collecting the terms belonging to the same trigonometric polynomial, the first term behind the integral sign of (36) is exchanged for a polynomial of the form $P_{spq}(|X|, \langle Z_0, V_u\rangle) = \sum_{r=0}^s A_{spq}^{(s-r)}|X|^r\langle Z_0, V_u\rangle^{s-r}$, resulting the integral terms

$$\tilde{P}_{spq}(|X|, Z_0, Z) = \int_{\mathbb{R}^l} P_{spq}(|X|, Z_0, V_u)\phi(|X|, V)e^{\mathbf{i}\langle Z,V\rangle}dV, \tag{39}$$
$$\tilde{P}_{spq}(|X|, \partial_{Z_0}^{s-r}, \tilde{\phi}_{s-r}) = \sum_{r=0}^{s} A_{spq}^{(s-r)}(-\mathbf{i})^{s-r}|X|^r\partial_{Z_0}^{s-r}\tilde{\phi}_{s-r}$$

behind $\sin^s\alpha$ in formulas (36) resp. (37). Note that constants $A_{spq}^{(s-r)}$ are built up by but not equal to the constants A_{spq}. Thus we have:

Theorem 6.2. *On S_Q, function $\mathcal{HF}_{Qpq}(\phi)$ appears in the form*

$$\sum_{s=0}^{p+q} |X_Q|^{p+q}\cos^{p+q-s}\alpha\sin^s\alpha\,\tilde{P}_{spq}(|X_Q|, Z_0, Z), \tag{40}$$

where P_{spq} is explicitly described in (39). For fixed values of $|X|$ and Z also this term is an X-depending s^{th}-order polynomial which appears in the same form as $\tilde{A}_{spq}$ does. But this one has also lower order terms, $\langle Z_0, V_u\rangle^{s-r}$, beneath the main term. For other points not being on S_Q also this function is determined by projections onto S_Q, in which case it is defined in terms of $|X|$ and not $|X_Q|$.

The above constructions restricted onto spheres S_{R_X} provide the formulas on sphere$\times$ball-type domains. In this case function $|X|$ is constant, thus functions $\tilde{\phi}_{s-r}$ depend just on Z and Z_0. Let it be pointed out again that the latter variable is involved by the assumption $X_Q \in P^+(Q, J_{Z_0}(Q))$, i. e., it is determined by X over which the Fourier transform in the Z-space is performed. In other words, it is an X-depending function whose precise denotation would be $Z_0(X)$.

Since the Z-balls, $B_{R_Z}(X)$, where $X \in S_{R_X}$, are naturally identified on this trivial ball-bundle, they determine the same functions in the Z-space for all those X's which project onto the half-plane $P^+(Q, J_{Z_0}(Q))$. Thus functions $\mathcal{F}_{Qpq}(\phi)$ resp. $\mathcal{HF}_{Qpq}(\phi)$ appear in the form (38) resp. (40) such that $|X| = R_X$ is constant in this case.

Later on, we need these functions described also on circles, $C = P_2 \cap S_{R_X}$, which are represented as intersections of 2-dimensional linear subspaces, P_2, with S_{R_X}. If $Q \in P_2$, these functions are perfectly described by the above formulas also on these circles. Therefore, we suppose $Q \notin P_2$. First also suppose that $P_2 \subset S_Q$ holds, in which case the computations below are carried out on the 3-space, S_{Q3}, spanned by P_2 and $\mathbb{R}Q$. This space intersects S_{R_X} at the 2-sphere denoted by S_{R_X2}. The north-pole, O, of this sphere is cut out by the ray $\mathbb{R}_+Q$. The north-pole, O_C, on the circle is defined by the closest point to O. Let α_C be the angle parameterization of C with origin O_C such that on both sides $0 \leq \alpha_C \leq \pi$ hold. The angle between C and the great circle CE_O with center O (equator) on the 2-sphere $S_{R_X}2$ is denoted by β_C. It is uniquely determined by the assumption $0 \leq \beta_C \leq \pi/2$. For a point $P \in C$ satisfying $0 \leq \alpha_C \leq \pi/2$, let $\tilde{C}_P$ be the great circle connecting O and P, which intersects CE_O at a point N perpendicularly. If $M_C = C \cap CE_O$, then the spherical sine theorem applied to the right spherical triangle PNM_C yields

$$\frac{\sin(\frac{\pi}{2} - \alpha(P))}{\sin \beta_C} = \frac{\sin(\frac{\pi}{2} - \alpha_C(P))}{\sin \frac{\pi}{2}} \;\Rightarrow\; \cos \alpha(P) = \sin \beta_C \cos \alpha_C(P) \quad (41)$$

This equation along with $\sin \alpha(P) = \sqrt{1 - \sin^2 \beta_C \cos^2 \alpha_C(P)}$ imply that functions (38) and (40) restricted onto C can be expressed in terms of $\cos \alpha_C(P)$.

It is a very important issue to understand the precise appearance of functions $\tilde{A}_{spq}$ and $\tilde{P}_{spq}$ on these circles. They appear as polynomials on the equator circle CE_O but on C they appear as functions which are pulled back from the equator to the C by the central projectivity $\tau_O : C \to CE_O$. This τ_O can be explicitly computed as follows.

Parameterize both P_2 and E_O by complex numbers z and z' respectively such that these coordinate systems have common imaginary axis $\mathbb{R}_+\mathbf{i} \subset P_2 \cup E_O$ and $z = 1$ and $z' = 1$ correspond to $P = O_C$ and $N_P = \tilde{C}_P \cap CE_O$ respectively, where $\tilde{C}_P$ is the great circle connecting O and P. This circle intersects CE_O at N_P perpendicularly. Pick up also such unit complex numbers, u and u', between the units and the imaginary numbers which are corresponded to each other by the τ_O.

Actually, the τ_O is a real projectivity between the two projective lines $\tilde{C}$ and $\tilde{CE_O}$ defined by identifying the antipodal points on the great circles C and CE_O. Thus, it can be described in terms of the real cross ratio defined on these projective lines. But this real one is the same as the complex cross ratio defined on the complex planes if the points are laying on the same half-circle. (This statement is well known in conform geometry of 2-spheres.) Also note that the common imaginary numbers cut the great circles into half-circles which are corresponded to each other by the τ_O, therefore, this projectivity can be described in terms of the complex cross ratio by the relation $(z' = \tau_O(z), 1', \mathbf{i}', u') = (z, 1, \mathbf{i}, u)$. This equation describes the τ_O as a fractional linear function (Möbius transform) of the form $z'(z) = (az + b)/(cz + d)$. Since such a function preserves the circles and the three corresponding points

$(1', \mathbf{i}', u')$ and $(1, \mathbf{i}, u)$ are on CE_O and C, the transformation is really defined between the two circles. Thus we have:

Lemma 6.3. *The τ_O is a fractional linear function (Möbius transform) between the two great circle which pulls back a trigonometric function $\cos_u \alpha = (1/2)(uz + \overline{uz})$, defined on the complex plane E_0 by a fixed complex unit u, to the trigonometric rational function $(a\cos_u + b)/(c\cos_u + d)$ defined on C. All trigonometric polynomials can be generated on CE_0 by the functions $\cos_u$, therefore, all trigonometric polynomials are pulled back to a trigonometric rational function defined on the other great circle. Functions $\tilde{A}_{spq}$ and $\tilde{P}_{spq}$ can be described by the trigonometric functions $\varphi^{(s-j)}$ defined on the unit sphere of E_0. Thus they are trigonometric polynomials on CE_0 which pull back to trigonometric rational functions defined on C.*

For a P_2 which is not subspace of S_Q these functions can be determined by projecting it into S_Q. Almost every plane projects to a plane of S_Q. If P_2' is the projected plane, then the sought functions on P_2 are the pull-back's of the corresponding functions defined for P_2'. Note that $\cos\alpha_C'$ arises from a linear function, therefore, so does the pull-back function whose kernel is the pull-back of the line (linear subspace) connecting the points $M_{C'}$ and $-M_{C'}$ defined above for C'. Then the closer midpoint, O_C, to the O, which is between the pull-back-points M_C and $-M_C$ on C, is called the north-pole on C. This point determines the parameterization α_C. The functions restricted onto C are described in terms of this parameter. Thus we have

Theorem 6.4. *When the constructions are restricted onto a fixed sphere S_{R_X}, functions $\mathcal{F}_{Qpq}(\phi)$ resp. $\mathcal{HF}_{Qpq}(\phi)$ appear in the form (38) resp. (40) such that functions $\tilde{A}_{spq}$ and $\tilde{P}_{spq}$ involve the constant $|X| = R_X$. In the following statement denotation $\tilde{R}_{spq}$ can be replaced by any of these two functions.*

On a circle, $C = P_2 \cap S_{R_X}$, represented by intersection of a 2-dimensional linear subspace P_2 with S_Q, these functions are of the form

$$\sum_{s=0}^{p+q} K_C^{p+q} (\sin\beta_{C'} \cos\alpha_C)^{p+q-s} (1 - \sin^2\beta_{C'} \cos^2\alpha_C)^{\frac{s}{2}} \tilde{R}_{spq}(Z_0, Z), \qquad (42)$$

where C' is the projected circle cut out by the projected 2-space P_2' which intersects the equator E_O' on the projected space $S_{Q3'}$ at angle $\beta_{C'}$. Constant K_C^{p+q} is due to the fact that the pulled back linear functions are restricted to a circle in this process. For circles in S_Q this constant is $K_C = R_X$.

For any fixed Z and $|X|$, function $\tilde{R}_{spq}(Z_0, Z)$ defines a trigonometric rational function on C which is the pull back of an s^{th}-order polynomial *with such a combined map, where the first map, τ'_O, is a Möbius transform between C' to CE'_O and the second one takes C onto C' by a projection.*

We need these theorems in the more general case when, instead of Q, one considers a pair, $(Q^{(a)}, Q^{(b)})$, of vectors and functions $\phi(|X|, V)$, $\Theta^p_{QV_u}\overline{\Theta}^q_{QV_u}$, $\Pi_X^{(p+q)}(..)$ are exchanged for the following ones

$$\phi(|X^{(a)}|, |X^{(b)}|, V),\ \Theta^{p_a}_{Q^{(a)}V_u}\overline{\Theta}^{q_a}_{Q^{(a)}V_u}\Theta^{p_b}_{Q^{(b)}V_u}\overline{\Theta}^{q_b}_{Q^{(b)}V_u},\ \Pi_{X^{(a)}}^{(p_a+q_a)}(..)\Pi_{X^{(b)}}^{(p_b+q_b)}(..),$$

respectively. Such functions are called 2-pole functions which can be investigated in two ways. They can be considered either on subsets $(X_F^{(a)}, X^{(b)})$ defined by a fixed $X_F^{(a)}$, or, on the similarly defined subsets $(X^{(a)}, X_F^{(b)})$. Because of the exact similarities, only the first case should be described, when, functions

$$\phi(|X_F^{(a)}|, |X^{(b)}|, V),\ \Theta^{p_a}_{Q^{(a)}V_u}\overline{\Theta}^{q_a}_{Q^{(a)}V_u},\ \Pi_{X^{(a)}}^{(p_a+q_a)}(..) \tag{43}$$

depend (non-trivially) just on $|X^{(b)}|$ and V. On a circle $(X_F^{(a)}, C^{(b)})$ the considered functions appear in the form (42) where the last function is defined by those listed in (43) and the other functions are defined on $\mathbf{v}^{(b)}$.

6.2 Intertwining of the Dirichlet conditions.

The Dirichlet Intertwining Theorem will be established, first, for a constant basis, $\mathbf{Q}_F$. The changing basis case will be traced back to this first one.

Observe that functions $\cos^{p+q-s}(\alpha)\sin^s(\alpha)$, satisfying $0 \leq s \leq p+q$ are linearly independent, furthermore, for any fixed Z_0 and Z, function $\tilde{A}_{spq}$ is constant on the α-parameter line determined by Z_0. These two statements yield the following theorem obviously.

Theorem 6.5. *A function $\mathcal{F}_{Qpq}(\phi)$ satisfies the Dirichlet condition at the boundary points (X, Z) if and only if functions $\tilde{A}_{spq}$ vanish on the sphere S_{R_Z}, for all $Z_0(X)$ and $0 \leq s \leq p+q$. Regarding $\mathcal{HF}_{Qpq}(\phi)$, this condition is $\tilde{P}_{spq} = 0$, for all $0 \leq s \leq p+q$ and $Z_0(X)$ at any boundary point $Z \in S_{R_Z}$.*

For fixed Q and natural numbers p and q, function spaces $\mathbf{\Phi}_{Qpq}$ resp. $\mathbf{\Xi}_{Qpq}$ are defined by the L^2 function spaces spanned by functions of the form $\mathcal{F}_{Qpq}(\phi)(X, Z)$ resp. $\mathcal{HF}_{Qpq}(\phi)(X, Z)$, where $\phi(|X|, V)$ (which depends, non-trivially, just on V on sphere$\times$sphere-type manifolds) can be an arbitrary L^2-function. For fixed Q but running p and q, all these spaces sum up to the total space $\mathbf{\Phi}_Q = \sum_{p,q}\mathbf{\Phi}_{Qpq} = \mathbf{\Xi}_Q = \sum_{p,q}\mathbf{\Xi}_{Qpq}$. Then, for *functions φ_Q resp. $\varphi'_Q = \kappa_{\mathbf{Q}_F}(\varphi_Q)$ from $\mathbf{\Phi}_Q$ resp. $\mathbf{\Phi}'_Q$ the Dirichlet condition is satisfied always simultaneously. Actually, the intertwining operator $\kappa_{\mathbf{Q}_F} : \mathbf{\Phi}_Q \to \mathbf{\Phi}'_Q$ between these total spaces is induced by a point transformation of the form $(T_Q(X), id_Z)$, where the T_Q is an orthogonal transformation on the X-space, depending on Q.*

Such simple proof can be given only for total spaces defined by a fixed pole Q. The proof is much more difficult on the complete L^2-Hilbert space, which can be represented both as $\sum_{Q\in span_{\mathbb{R}}(\mathbf{Q}_F)} \mathbf{\Phi}_Q$, and $\sum_{Q\in span_{\mathbb{R}}(\mathbf{Q}_F)} \mathbf{\Phi}'_Q$, i. e., by the sums of all functions defined by all poles, Q, which are in the real span of $\mathbf{Q}_F$. The main idea of such an extension is as follows.

Suppose that a function $\varphi(X,Z) = \sum_{Q_i} \varphi_{Q_i}(X,Z)$ satisfies the Dirichlet condition. Decompose each Θ_Q in the form $\Theta_Q(X,V_u) = \langle Q,X\rangle + \mathbf{i}\langle J_{V_u}(Q),X\rangle$ and $\overline{\Theta}_Q(X,V_u)\langle Q,X\rangle - \mathbf{i}\langle J_{V_u}(Q),X\rangle$, which, after multiplications, result the decomposition

$$\varphi = \varphi_{evn_J} + \varphi_{odd_J} = \sum_{Q_i} \varphi_{Q_i evn_J} + \sum_{Q_i} \varphi_{Q_i odd_J}, \tag{44}$$

where the first function involves all terms having even number of J_{V_u} while for the other one this number is odd. For a fixed boundary point Z, these functions depend just on X. One can prove, by formula (42), that these two functions are in two completely independent subspaces of functions. The proof will be based on the fact that, on a circle C, the first function appears as a trigonometric rational function depending on $\sin^2\beta_{C'}\cos^2\alpha_C$ while the other function depends irrationally on these terms. This independence implies then that both $\varphi_{evn_J}(X,Z)$ and $\varphi_{odd_J}(X,Z)$ must satisfy the Dirichlet condition. In the next step one decomposes these functions in the form

$$\varphi_{evn_J}(X,Z) = \varphi_{evn_J evn_{J^{(b)}}}(X,Z) + \varphi_{evn_J odd_{J^{(b)}}}(X,Z), \tag{45}$$

$$\varphi_{odd_J}(X,Z) = \varphi_{odd_J evn_{J^{(b)}}}(X,Z) + \varphi_{odd_J odd_{J^{(b)}}}(X,Z), \tag{46}$$

where the first term of each function involves all terms having even number of $J^{(b)}_{V_u}$, while, for the second one, this number is odd. Now using the double pole version of (42), one can see that all 4 functions in this final decomposition fall in completely independent subspaces. But, then, all these 4 functions must satisfy the Dirichlet condition. Now we observe that

$$\kappa_{\mathbf{Q}_F} : \quad \varphi_{evn_J evn_{J^{(b)}}}(X,Z) \quad \to \quad \varphi_{evn_J evn_{J^{(b)}}}(X,Z), \tag{47}$$

$$\kappa_{\mathbf{Q}_F} : \quad \varphi_{evn_J odd_{J^{(b)}}}(X,Z) \quad \to \quad -\varphi_{evn_J odd_{J^{(b)}}}(X,Z), \tag{48}$$

$$\kappa_{\mathbf{Q}_F} : \quad \varphi_{odd_J evn_{J^{(b)}}}(X,Z) \quad \to \quad \varphi_{odd_J evn_{J^{(b)}}}(X,Z), \tag{49}$$

$$\kappa_{\mathbf{Q}_F} : \quad \varphi_{odd_J odd_{J^{(b)}}}(X,Z) \quad \to \quad -\varphi_{odd_J odd_{J^{(b)}}}(X,Z), \tag{50}$$

which relations are due to $J^{(a)\prime} = J^{(a)}, J^{(b)\prime} = -J^{(b)}$. Thus φ and φ' satisfies the Dirichlet condition simultaneously.

For completing this proof only the above mentioned *Independence Theorems* should be established. Suppose the contrary, there is a function φ which can be represented as linear combinations both of evn_J- and odd_J-type functions:

$$\sum_{\tilde{Q}_r} \varphi_{\tilde{Q}_r evn_J} = \varphi = \sum_{Q_i} \varphi_{Q_i odd_J}. \tag{51}$$

We may suppose that the trigonometric polynomials $\cos^{p_j+q_j-s}\sin^s$ are of the same order $n = p_j + q_j$ and there are only finite linear combinations in this expression. Consider a circle C on which each term in the linear combinations appears in the form (42) with the corresponding constant $\sin\beta_{C'_j}$ and origin (north pole) O_{Cj} on C. (The origin (pole), O_C, on C is constructed above Theorem 6.4.)

Next we work just on the right side of (51), involving only the odd_J-type functions. They can be sorted out into classes according their parameters $\sin\beta_{C'_j}$ and O_{Cj}. Re-numerate these classes with m in the form $\{(\beta_{C'}m, O_{Cq})|m = 1, 2, \dots, d\}$ such that at least one of the parameters is different for two distinct m's. From the partial sum determined by a class factor out $(1-\sin^2\beta_{C'm}\cos^2\alpha_{Cm})^{\frac{1}{2}}$. Thus, it appears in the one-term-form $\tilde{S}_m(1-\sin^2\beta_{C'm}\cos^2\alpha_{Cm})^{\frac{1}{2}}$, where $\tilde{S}_m$ is a rational trigonometric function. Furthermore,

$$\varphi = \sum_{m=1}^{d} \tilde{S}_m\sqrt{1-\sin^2\beta_{C'm}\cos^2\alpha_{Cm}} = \sum_{m,r=1}^{d,\infty} A_r\tilde{S}_m\sin^{2r}\beta_{C'm}\cos^{2r}\alpha_{Cm}, \tag{52}$$

where $\sum_{r=1}^{\infty} A_r x^r$ is the Taylor expansion of $\sqrt{1-x}$ about $x = 0$. All the coefficients A_r are non-zeros. Since the left side of (51) is a trigonometric rational functions, all terms

$$\sum_{m=1}^{d} \tilde{S}_m\sin^{2r}\beta_{C'm}\cos^{2r}\alpha_{Cm}, \tag{53}$$

considered for a fixed r, vanish for big numbers r, say, if $r > N$.

Now observe that there exist an everywhere dense open subset, $B \subset C$, such that, for any fixed point $P \in B$ the function values $\sin\beta_{C'm}\cos\alpha_{Cm}(P)$, considered for all $m = 1, \dots, d$ are distinct and, therefore, the $d \times d$-matrix $(T_{mk} = \sin^{2(k+N)}\beta_{C'm}\cos^{2(k+N)}\alpha_{Cm}(P)$, where $k = 1, \dots, d$, is non-degenerated. Thus equations $\sum_m T_{mk}\tilde{S}_m(P) = 0, \forall k$, implies $\tilde{S}_m(P) = 0$ on the whole circle C. This argument can be repeated in those cases when $\tilde{S}_m$ is substituted by $\tilde{S}_m \sin^{2r}\beta_{C'm}\cos^{2r}\alpha_{Cm}, \forall r \geq 0$. Therefore, $\varphi = 0$, which proves the independence of the considered function spaces completely.

This argument repeated for 2-pole functions proves the desired independence, first, on the subsets $(X_F^{(a)}, X^{(b)})$. But this statement obviously implies the independence on the whole X-space.

Both independence theorems can be formulated in terms of polynomials such that, after considering the decompositions, the evn_J- resp. odd_J-type subspaces are spanned by functions involving even resp. odd number of J's. This observation leads to a simple establishment of the independence and Dirichlet-intertwining theorems regarding changing basis cases as follows. First consider the complex matrix $c_{ij}(V_u)$ transforming the fixed basis $\mathbf{Q}_F$ to the changing one, $\mathbf{Q}(V_u)$. It is obvious that, both in the 1-pole and 2-pole cases, both type of subspaces regarding the two systems are transformed to each other by the non-degenerated map

$$\omega : \quad \mathcal{F}_{\mathbf{Q}_F\{p_i,q_i\}} \to \mathcal{F}_{\mathbf{Q}(V_u)\{p_i,q_i\}} \quad , \quad \mathcal{HF}_{\mathbf{Q}_F\{p_i,q_i\}} \to \mathcal{HF}_{\mathbf{Q}(V_u)\{p_i,q_i\}} \tag{54}$$

induced by the basis-transformation. This proves both independence theorems for the changing basis case immediately. Thus we have:

Theorem 6.6. *The $\kappa_{\mathbf{Q}}$ intertwines the Dirichlet condition both in the fixed, $\mathbf{Q}_F$, and the changing basis, $\mathbf{Q}(V_u)$, cases.*

The proof is based on the **Independence Theorem** *stating that the total space*

$$\mathcal{F}_{\mathbf{Q}_F,n} = \sum_{\{n=\sum(p_i+q_i)\}} \mathcal{F}_{\mathbf{Q}_F\{p_i,q_i\}},$$

defined for a fixed n and Z, is the direct sum of the independent subspaces $\mathcal{F}_{\mathbf{Q}_F,n,evn_J}$ and $\mathcal{F}_{\mathbf{Q}_F,n,odd_J}$, where, after implementing the above described natural decomposition, the functions from the first resp. second space contain even resp. odd number of J's. Both of these subspaces further decompose into the independent subspaces $\mathcal{F}_{\mathbf{Q}_F,n,par_J,evn_{J^{(b)}}}$ and $\mathcal{F}_{\mathbf{Q}_F,n,par_J,odd_{J^{(b)}}}$ defined by the options par = evn or odd given for the parities of the number of $J^{(b)}$'s in the expressions. Vector Z should not be the same for the participating functions but it can be chosen individually and independently both for the even- and odd-type functions.

The independence guaranties that all 4 component functions of a φ and φ itself satisfy the Dirichlet condition always simultaneously. The same statements hold for the function spaces $\mathcal{HF}_{\mathbf{Q}_F,n}$ as well as for both versions of function spaces defined by changing basis fields.

7 Intertwining of the Neumann conditions.

The Neumann conditions create a new more complicated situation which requires reformulations of the proofs given for the Dirichlet conditions at several points. The Z-Neumann conditions, however, which require the vanishing of the derivatives of functions taken from the Z-radial directions at the boundary points, can be strait-forwardly traced back to the Dirichlet conditions. Let it also be mentioned that the proofs on the boundary manifolds exploit only the intertwining of the Dirichlet and Z-Neumann conditions. By this reason, the Z-Neumann conditions are considered first.

In the following computations the integral defining the Z-Fourier transform is considered on the polar coordinate system. The computations are carried out by formulas $\partial_{|Z|}e^{\mathbf{i}\langle Z,V\rangle} = \mathbf{i}\langle Z_u, V_u\rangle|V|e^{\mathbf{i}\langle Z,V\rangle} = |Z|^{-1}|V|\partial_{|V|}e^{\mathbf{i}\langle Z,V\rangle}$ combined with integration by parts. Without loosing the generality, one can suppose that the test-function, ϕ, vanishes at the infinity. Then, in terms of $\phi' := \partial_{|Z|}\phi$, we have

$$\partial_{|Z|}\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(\phi) = -|Z|(\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(|V|\phi') + l\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(\phi)). \tag{55}$$

Therefore, a function $\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(\phi)$ satisfies the Z-Neumann condition if an only if $\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(|V|\phi') + l\mathcal{F}_{\mathbf{Q}\{p_i,q_i\}}(\phi)$ satisfies the Dirichlet condition. Since the Dirichlet condition is intertwined in all cases, we have

Theorem 7.1. *The Z-Neumann condition is intertwined both in the fixed and changing basis cases.*

From now on, the standard Neumann condition is scrutinized. By formulas (3.2) and (3.7) of [Sz2], the normal vector at a boundary point (X, Z) and the Laplacian on the boundary manifolds are of the form

$$\mu = A(|X|, |Z|)X_u + B(|X|, |Z|)J_Z(X) + C(|X|, |Z|)Z_u, \tag{56}$$

$$\tilde{\Delta} = \Delta_{S_X(Z)} + (1 + \frac{1}{4}|X|^2)\Delta_{S_Z(X)} + \sum_{\alpha=1}^{l-1}\partial_\alpha D_\alpha \bullet, \tag{57}$$

where $S_X(Z)$ is the X-sphere over Z and $S_Z(X)$ is the Z-sphere over X, furthermore, $\{\partial_1, \dots, \partial_{l-1}\}$ is an orthonormal basis in the tangent space of the Z-sphere $S_Z(X)$ at Z. On sphere×ball-type manifolds the first term of (56) should be omitted.

The standard Neumann condition is considered, first, for one pole functions. The derivative with respect to the normal direction is built up by X- resp. Z-radial-derivatives and $J_Z(X)\bullet$. If the foot of perpendicular through X to S_Q is X_Q, and thus $X = X_Q + X_Q^\perp$ holds, then $J_Z(X)\bullet = J_Z(X_Q)\bullet + J_Z(X_Q^\perp)\bullet$. First $J_Z(X_Q)\bullet$ is considered. If $X_Q = |X_Q|(\cos(\beta)Q + \sin(\beta)J_{Z_0}(Q) \in P(Q, Z_0)$, then

$$J_Z(X_Q) = |X_Q|(\cos(\beta)J_Z(Q) - \sin(\beta)\langle Z_0, Z\rangle Q + \sin(\beta)J_{Z^\perp}J_{Z_0}(Q)), \quad (58)$$

where $Z^\perp$ is the perpendicular component of Z to Z_0. Therefore, the $J_{Z^\perp}$ and J_{Z_0} are anti-commuting and the $J_{Z^\perp}J_{Z_0}$ is a skew endomorphism. Thus,

$$\langle Q, J_Z(X_Q)\rangle = -|X_Q|\sin\beta\langle Z_0, Z\rangle\,, \quad (59)$$
$$\langle J_{Z_0}(Q), J_Z(X_Q)\rangle = |X_Q|\cos\beta\langle Z_0, Z\rangle, \quad (60)$$
$$J_Z(X_Q)\bullet\cos\alpha = -|X_Q|\sin\beta\langle Z_0, Z\rangle\,, \quad (61)$$
$$J_Z(X_Q)\bullet\sin\alpha = |X_Q|\cos\beta\langle Z_0, Z\rangle, \quad (62)$$
$$J_Z(X_Q)\bullet\mathcal{HF}_{Qpq}(\phi)(X,Z) = \sum |X_Q|^{p+q}\cos^{p+q-s}(\beta)\sin^s(\beta)\tilde{S}^T_{spq}, \quad (63)$$
$$where\ \tilde{S}^T_{spq} = -|X_Q|\langle Z_0, Z\rangle((p+q-s+1)\tilde{P}_{(s-1)pq} - (s+1)\tilde{P}_{(s+1)pq}). \quad (64)$$

The computations with $J_Z(X_Q^\perp)\bullet$ are based on

$$\langle Q, J_Z(X_Q^\perp)\rangle = 0 \quad , \quad \langle J_{Z_0}(Q), J_Z(X_Q^\perp)\rangle = -\langle J_{Z^\perp}J_{Z_0}(Q), X_Q^\perp\rangle. \quad (65)$$

Since differentiation $J_Z(X_Q^\perp)\bullet$ acts, non-trivially, on the considered functions only by its contribution $\langle J_{Z_0}(Q), J_Z(X_Q^\perp)\rangle(\cos\beta\partial_\alpha + \sin\alpha\partial_r)$ to the ∂_α- and the radial ∂_r-direction, therefore:

$$J_Z(X_Q^\perp)\bullet\cos\alpha = \langle J_{Z^\perp}J_{Z_0}(Q), X_Q^\perp\rangle\cos\beta\sin\beta, \quad (66)$$
$$J_Z(X_Q^\perp)\bullet\sin\alpha = -\langle J_{Z^\perp}J_{Z_0}(Q), X_Q^\perp\rangle\cos\beta\cos\beta, \quad (67)$$
$$J_Z(X_Q^\perp)\bullet\mathcal{HF}_{Qpq}(\phi)(X,Z) = |X_Q|^{p+q}\sum_s\cos^{p+q-s}(\beta)\sin^s(\beta)\tilde{S}^\perp_{spq}, \quad (68)$$
$$\tilde{S}^\perp_{spq} = -\langle J_{Z^\perp}J_{Z_0}(Q), X_Q^\perp\rangle((p+q)|X_Q|^{-1}\sin\beta\tilde{P}_{spq} - \cos\beta\tilde{D}_{spq}), \quad (69)$$
$$\tilde{D}_{spq} = (p+q-s+1)\tilde{P}_{(s-1)pq} - (s+1)\tilde{P}_{(s+1)pq}. \quad (70)$$

A preliminary version of the standard Neumann condition for a one-pole function can be stated in the following form. A ϕ-generated one-pole function satisfies the Neumann condition if and only if

$$\tilde{R}_{spq} = (A\partial_{|X|} + C\partial_{|Z|})\tilde{P}_{spq} + B(\tilde{S}^T_{spq} + \tilde{S}^\perp_{spq}) = 0 \quad (71)$$

holds at the boundary points, for all $Z_0(X)$ and $0 \leq s \leq p+q$, where coefficients A, B and C are defined by the normal vector μ. Already this version reveals that only the intertwining regarding the first two terms in this condition can be traced back to the intertwining of the Dirichlet conditions. The other two terms involve functions such as $\langle Z_0, Z\rangle$ and $\langle J_{Z_0}(Q), J_Z(X_Q^\perp)\rangle = |X_Q^\perp|\langle J_{Z_0}(Q), J_Z(X_{Q0}^\perp)\rangle$ which appear outside of the integral terms. It is noteworthy that the second function is zero on spaces $H_3^{(a,b)}$ and $H_7^{(a,b)}$. This is due to the fact that the irreducible components $H_3^{(1,0)}$ and $H_7^{(1,0)}$, yield the well known J^2-condition, meaning, that for any product $J_{Z^\perp}J_{Z_0}$ there exist $J_{\tilde{Z}}$ such that $J_{Z^\perp}J_{Z_0} = J_{\tilde{Z}}$ holds. On arbitrary H-type groups, for fixed Z and unit vector $X_{Q0}^\perp$, all these terms define polynomials which are suitable to establish the independence theorems seen for the Dirichlet condition also for the Neumann condition. Similar formulas can be established also for the two-pole functions which also yield the corresponding independence theorem. Finally we get:

Theorem 7.2. *The $\kappa_{\mathbf{Q}}$ intertwines the standard Neumann conditions both in the fixed, $\mathbf{Q}_F$, and the changing basis, $\mathbf{Q}(V_u)$, cases.*

The proof is based on observing that the total space

$$\mu \bullet \mathcal{HF}_{\mathbf{Q}_F,n} = \mu \bullet \sum_{\{p_i,q_i\}} \mathcal{HF}_{\mathbf{Q}_F\{p_i,q_i\}},$$

defined for fixed Z, n and running $\{p_i, q_i\}$ satisfying $n = \sum(p_i + q_i)$, is a direct sum of independent subspaces $\mathcal{HF}_{\mathbf{Q}_F,n,evn_J}$ and $\mathcal{HF}_{\mathbf{Q}_F,n,odd_J}$, which are defined such that the functions from the first resp. second space contain even resp. odd number of J's (i. e. , $\sin\beta$'s, according to the above decomposition). Both subspaces further decompose into the independent subspaces $\mu \bullet \mathcal{F}_{\mathbf{Q}_F,n,par_J,evn_{J^{(b)}}}$ and $\mu \bullet \mathcal{F}_{\mathbf{Q}_F,n,par_J,odd_{J^{(b)}}}$ defined by the options par = evn or par = odd, available for the parity of number of $J^{(b)}$'s in the expressions. The independence guaranties that all the 4 component functions of a φ along with φ satisfy the standard Neumann condition always simultaneously. The same statements hold for the function spaces $\mathcal{F}_{\mathbf{Q}_F,n}$ as well as for both versions of function spaces defined by changing basis fields.

8 Intertwining on the boundary manifolds.

Since the intertwining operator preserves the Dirichlet condition, by restrictions, it induces a well defined bijection between the L^2 spaces defined on the boundaries. Since each smooth function on the boundary extends to ones satisfying the Z-Neumann condition, furthermore, this condition is also preserved by the operator, it is enough to represent the functions on the boundary by restrictions of those satisfying the Z-Neumann condition.

If $\partial_l = \partial_{|Z|}$ is the Z-partial derivative with respect to the normal direction Z_u, then the angular momentum operator $\mathbf{M}$ (resp. $\tilde{\mathbf{M}}$) on the ambient (resp. boundary) manifold differ from each other just by $\partial_{|Z|} D_{Z_u}\bullet$. This operator vanishes on functions satisfying the Z-Neumann condition, thus $\mathbf{M}$ and $\tilde{\mathbf{M}}$ acting on these functions provide the same results. This argument proofs that not just $\mathbf{M}$ and $\mathbf{M}'$ but also $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{M}}'$ are intertwined by the operator. This is the most crucial part in the proof of the pursued theorem. The intertwining regarding $\Delta_{S_X(Z)}$ has already been established on the ambient manifold, thus one should consider only $\Delta_{S_Z(X)}$. Since the intertwining regarding Δ_Z is established on the ambient manifold, only the intertwining of the radial Laplacian $\Delta_{|Z|}$ should be established. Since this operator acts on functions satisfying the Z-Neumann condition, the question is if $\partial^2_{|Z|^2}$ is invariant under the action of the operator. This statement immediately follows from the following computations where the integral defining the Z-Fourier transform is considered on the polar coordinate system. The computations start out with $\partial^2_{|Z|^2} e^{\mathbf{i}\langle Z,V\rangle} = -\langle Z_u, V_u\rangle^2 |V|^2 e^{\mathbf{i}\langle Z,V\rangle} = |Z|^{-2}|V|^2 \partial^2_{|V|^2} e^{\mathbf{i}\langle Z,V\rangle}$ and are completed by integration by parts. Then, in terms of $\phi' = \partial_{|Z|}\phi$, we have

$$\partial^2_{|Z|^2}\mathcal{F}_{Qpq}(\phi) = |Z|^{-2}(\mathcal{F}_{Qpq}(|V|^2\phi'') + 2(l+1)\mathcal{F}_{Qpq}(|V|\phi') + l\mathcal{F}_{Qpq}(\phi)), \quad (72)$$

where the $|Z| = R_Z$ is a constant. Like in case of the Z-Neumann condition, this formula establishes the desired intertwining property for $\partial^2_{|Z|^2}$.

9 Intertwining on solvable extensions.

The isospectrality theorems naturally extend to the solvable extensions. The Laplacians on the ambient- and boundary-manifolds, furthermore, the normal vectors at the boundaries are described in formulas (1.12), (3.30), and (3.29) of [Sz2]. The generator functions are of the form $\phi(|X|, t, V)$ in this case, till, the intertwining operator is defined by the same Z-Fourier transform like on H-type groups. I. e., the intertwining on the solvable group is completely determined by its action induced on the nilpotent group. The details in [Sz2] show that the terms due to the t-variable, which are in a certain combination with the terms of the Laplacian defined on the nilpotent group, make no effect on proving the intertwining for the solvable groups in the same way as for the H-type groups.

10 The new striking examples.

There is a subgroup, $\mathbf{Sp}(a)\times\mathbf{Sp}(b)$, of isometries on a Heisenberg-type group $H_3^{(a,b)}$ which acts as the identity on the Z-space. Note that these isometries act transitively on the X-spheres of $H_3^{(a+b,0)}$.

The complete isotropy group of isometries fixing the origin is $(\mathbf{Sp}(a)\times\mathbf{Sp}(b))\cdot SO(3)$, where the action of $SO(3)$, described in terms of unit quaternions q by $\alpha_q(X_1,\dots,X_{a+b},Z)=(qX_1\overline{q},\dots,qX_{a+b}\overline{q},qZ\overline{q})$, is transitive on the Z-sphere. The elements of this isotropy group induce isometries on the sphere×sphere-type submanifolds, furthermore, there is proved in the Extension Theorem of [Sz2] that these are the only isometries on these submanifolds. Note that these isospectral manifolds in a family have non-isomorphic isometry groups of different dimensions such that they are homogeneous in $H_3^{(a+b,0)}\cong H_3^{(0,a+b)}$, while the other members are locally inhomogeneous.

The sphere×sphere-type submanifolds of the solvable extensions of H-type groups is defined such that, over each point of a sphere in the X-space, one considers the same geodesic sphere around the origin $(0,1)$ of the hyperbolic (Z,t)-space. The isotropy group of isometries acting on $SH_3^{(a,b)}$, where $ab\neq 0$, is $(\mathbf{Sp}(a)\times\mathbf{Sp}(b))\cdot SO(3)$, while it is $\mathbf{Sp}(a+b)\cdot\mathbf{Sp}(1)$ on $SH_3^{(a+b,0)}\cong SH_3^{(0,a+b)}$. The above statements extend also to these groups.

These are the new striking examples brought by the reconstructed intertwining operator. The sphere-type striking examples discussed in the earlier papers can be explained similarly. They are the geodesic spheres defined by the same radius for the family $SH_3^{(a,b)}$. In this case the metric on $SH_3^{(a+b,0)}$ is two-point homogeneous, therefore, having homogeneous geodesic spheres. The geodesic spheres on the other groups are locally inhomogeneous.

It should be mentioned that Schüth [Sch1] constructed the literature's first isospectral metrics defined on simply connected manifolds on Cartesian products of spheres. Among them there are also sphere×sphere-type manifolds. Her construction arises from a completely different setting, however, and all provided metrics are locally inhomogeneous.

11 Isospectralities for σ-equivalent metrics.

A σ-deformation of an endomorphism space $J_{\mathbf{z}}$ is defined by an involutive orthogonal transformation, σ, of the X-space which commutes with all endomorphisms of $J_{\mathbf{z}}$. The σ-deformed endomorphism space consists of endomorphisms σJ_Z. By using irreducible decomposition regarding the orthogonal Lie algebra generated by the elements of $J_{\mathbf{z}}$, for any σ-deformation, there exist a decomposition $\mathbf{v} = \mathbf{v}^{(a)} \oplus \mathbf{v}^{(b)}$ with components invariant under the action of the endomorphisms such that for their restriction onto the components the relations $\sigma J_Z^{(a)} = J_Z^{(a)}$ and $\sigma J_Z^{(b)} = -J_Z^{(b)}$ hold. Note that the family, $J_l^{(a,b)}$, of Cliffordian endomorphism spaces defined by the same $a+b$ and l consists of σ-equivalent endomorphism spaces. In papers [Sz1, Sz2] the isospectrality is stated on the ball- and sphere-type domains of such σ-equivalent 2-step nilpotent Lie groups and their solvable extensions whose endomorphism spaces contain at least one anticommutator. Thus the extension of the isospectrality theorems to σ-equivalent 2-step nilpotent Lie groups and their solvable extensions provide plenty additional examples to those produced by the anticommutator technique. For the sake of simplicity, we consider such two step nilpotent Lie groups whose endomorphism spaces contain at least one non-degenerated endomorphism. Then almost all endomorphisms must be non-degenerated acting on an even dimensional X-space.

First, we look for the necessary modifications which make the techniques developed for H-type groups working also for σ-deformations. The Laplacians in these general cases differ from the Laplacians of H-type groups just by the term $(1/4)|X|^2\Delta_Z$, what is now $(1/4)\sum\langle J_\alpha(X), J_\beta(X)\rangle\partial^2_{\alpha\beta}$. Also the intertwining operator, defined for the changing basis case, must be modified as follows. Let $(Q_1(V_u),\dots,Q_{k/2}(V_u)) = (\mathbf{Q}^{(a)},\mathbf{Q}^{(b)})$ be an appropriate orthonormal basis field such that each $Q_{V_u i}$ is an eigenvector of $J^2_{V_u}$ with eigenvalue $-\lambda_i^2(V_u)$. Let $\tilde{J}_{V_u}$ be the normalized endomorphism which has the same kernel as J_{V_u} and is defined by $(1/\lambda_i(V_u))J_{V_u}$ on the maximal eigensubspaces belonging to $\lambda_i > 0$. Note that the kernel is trivial for an everywhere dense open subset of the unit vectors V_u, furthermore, this endomorphism may not be in $J_{\mathbf{z}}$. Then, by definition, $\Theta_{Q(V_u)}(X,V_u) = \langle Q + \mathbf{i}\tilde{J}_{V_u}(Q), X\rangle$. The complex coordinate system defined by this basis for non-degenerated endomorphisms is denoted by $\{z_{V_u 1},\dots,z_{V_u k/2}\}$. Then the intertwining is defined by these changing complex coordinates such that functions ϕ should be of the form $\phi(|X_{V1}|,\dots,|X_{Vr}|,V)$, where $X = \sum X_{Vi}$ is the decomposition regarding the eigenspaces of J_V^2. This requirement is slightly different from that what is considered on H-type groups, but one can reach to them in the same way: Start with functions depending just on V, first. Then, it turns out that the same operator is defined by the above more complicated functions. Furthermore, the domain is the largest possible, containing the complete L^2- function space. Also Theorem 4.4 concerning the intertwining of the Euclidean Laplacian and radial functions on the X-space, remains true for σ-deformations. Therefore, by $J'^2_V = \sigma J_V \sigma J_V = \sigma^2 J_V^2 = J_V^2$ and formulas (22), (23), which also extend to σ-deformations, this is indeed an operator intertwining the Laplacians Δ and Δ' term by term.

One should check out also the intertwining of the boundary conditions. First note that the technique developed for H-type groups works out straitforwardly only on groups where the intertwining can be established also with a fixed basis $\mathbf{Q}_F$, therefore, one can use one- and two-pole functions with poles being in $span_{\mathbb{R}}(\mathbf{Q}_F)$. Also well-defined polar coordinate systems established on $\mathbb{R}Q \oplus \tilde{J}_{\mathbf{z}}(Q)$, where $\tilde{J}_{\mathbf{z}}$ is spanned by endomorphisms of the form $\tilde{J}_{V_u}$, are necessary conditions for this technique. All these requirements are satisfied only in those cases where, for all unit pole $Q \in span_{\mathbb{R}}(\mathbf{Q}_F)$, the unit vectors $\tilde{J}_{\mathbf{z}}(Q)$ form an everywhere dense open subset of the unit sphere of the space spanned by these vectors (equator). Then, any such vector is connected with the pole by an α-parameter circle defined for $0 \leq \alpha \leq \pi$ such that it has the parameter $\pi/2$. The possible missing circles, which are due to the degenerated endomorphisms, can be implemented by limiting. The extension to this cases works out after other additional modifications.

First we check on formula (34) of Section 6.1, where, on a parameter-circle, the corresponding Z_0 should be exchanged for $Z_*(Z_0)$ defined by the dual of the functional $\varphi(Z) = \langle \tilde{J}_{Z_0}(Q), J_Z(Q)\rangle$. Yet, the $Z_*(Z_0)$ is a polynomial function of Z_0, implying that functions $\tilde{R}_{spq}$, introduced in (42), will be polynomials on the above equator. Thus the computations can be processed in the same way as earlier. Since for σ-deformations the relations $\varphi(Z) = \varphi'(Z)\,, Z_* = Z'_*$ hold, the proof regarding the Dirichlet and Z-Neumann condition can be completed by the same argument seen for H-type groups.

Regarding the Neumann condition the Z_0 inside of the integral term should be exchanged for Z_*, while terms $\langle Z_0, Z\rangle$ resp. $\langle J_{Z^\perp}\tilde{J}_{Z_0}(Q), X_Q^\perp\rangle$ outside of the integral should be exchanged for much more complicated expressions. Even so, they provide polynomial functions and the modified computations concerning formulas (59)-(64) along with $\langle J_{Z^*}\tilde{J}_{Z_0}(Q), Q^*\rangle = \langle J'_{Z^*}\tilde{J}'_{Z_0}(Q), Q^*\rangle$ yield the intertwining also of the Neumann conditions for the σ-deformations whose endomorphism spaces satisfy the above conditions. In these cases, the theorem extends also to the boundary manifolds and the solvable extensions.

Fortunately, one should not go through the steps of this complicated construction which is incomplete without scrutinizing the question of the existence. Basically, what the above proof exploits is that the operator introduced by the changing-basis-technique for σ-deformations intertwines both the Laplacians and boundary conditions if the manifold satisfies the independence theorems. But this theorems are certainly yielded on groups having endomorphism spaces which are the results of "small" perturbations performed on the endomorphism space of H-type groups such that, by choosing a new endomorphism space close to the Clifordians, one perturbs the endomorphism space acting on the irreducible space $\mathbb{R}^{r(l)}$. This defines uniquely determined perturbations for the reducible endomorphisms (see more details in the end of Introduction). If the perturbation is "small" which changes the endomorphisms just slightly, the subspaces in the independence theorems (which are closed in the ambient Hilbert space) keep being independent. Even the conditions for the existence of polar coordinate systems are satisfied on manifolds defined by "smaller" perturbations. However, the above construction can be left out completely because the idea of perturbation provides more examples than those provided by the above process. Also the non-isometry proofs, guaranteeing that the considered isospectral metrics have different local geometries, are inherited from the metrics being "slightly" perturbed. For the latter metrics the non-isometry proofs are completely established in [Sz1, Sz2]. Thus we have:

Theorem 11.1. *The intertwining operator introduced for σ-equivalent groups by the changing-basis-technique always intertwines the Laplacians both on ball$\times$ball- and sphere$\times$ball-type manifolds. Furthermore, there exist an open neighborhood, U, in the space of l-dimensional endomorphisms spaces with endomorphisms acting on the irreducible space $\mathbb{R}^{r(l)}$ of a given Clifford endomorphism space, $J_{\mathbf{z}}$ such that the latter one is in U and all 2-step nilpotent groups and their solvable extensions constructed by endomorphism spaces belonging to U satisfy the independence theorems. Therefore, for these groups, the intertwining operator intertwines also the boundary conditions. In other words, small perturbations performed on the endomorphism spaces of H-type groups provide a wide range of σ-equivalent groups which are isospectral on the corresponding ball$\times$ball- and sphere$\times$ball-type submanifolds.*

This theorem extends to solvable groups as well as to the boundary manifolds, both in the nilpotent and solvable cases. For dimensions satisfying $l = 4n + 3$ also such U exist for which the corresponding metrics in an isospectrality family have different local geometries.

Remark. The key point in the above process dealing with one-pole functions is that the multi-linear function

$$H(X, X^*, Z, Z^*) := \langle J_Z(X), J_{Z^*}(X^*)\rangle \tag{73}$$

does not change during σ-deformations. The Ricci tensor can be described in terms of this function by

$$R(X,X^*) = -(1/2)\sum H(X,X^*,Z_\alpha,Z_\alpha)\,, \tag{74}$$

$$R(Z,Z^*) = (1/4)\sum H(E_i,E_i,Z,Z^*) \quad , \quad R(X,Z) = 0 \tag{75}$$

(cf. formula (1.9) of [Sz1]), thus also this tensor is not changing during σ-deformations.

The Gordon-Wilson [GW] isospectrality examples were constructed on ball$\times$torus-type manifolds by spectrally equivalent endomorphism spaces, meaning the existence of orthogonal transformations associating isospectral endomorphisms to each other. More precisely, they constructed continuous families of metrics which are isospectral on functions. These metrics are not isospectral on 1-forms, however, due to the fact that the norm of the Ricci tensor is changing during these deformations [Sch2]. The question arises if the domains investigated in this paper are isospectral on the Gordon-Wilson examples.

Even though the changing Ricci tensor strongly suggests the negative answer, the question is more complicated. Indeed, the intertwining operator, constructed on ball$\times$torus-type manifolds such that the globally defined operator is the direct sum of operators constructed on the invariant subspaces W_γ by the single endomorphism J_γ separately, provides an operator intertwining the Laplacians also for the ball$\times$ball- and sphere$\times$ball-type manifolds which are in the ball$\times$torus-type manifold. Actually, this is a discrete version of those constructions where one is using changing basis.

There are many problems arising when one tries to establish the intertwining of the boundary conditions for this operator on the Gordon-Wilson examples. First of all, one can not pass to a fixed basis and involve one- and two-pole functions because no fixed basis exists which is transformed to a well defined fixed basis by all those point transformations which define the intertwining operator for the subspaces W_γ. The metrics in the Gordon-Wilson examples are out of the touch also of the perturbation technique. Even though the independence theorems were guaranteed in an other way, they would work together just with the σ deformations for which the decomposition $\mathbf{v} = \mathbf{v}^{(a)} \oplus \mathbf{v}^{(b)}$ holds. (This argument shows that the complete isospectrality can not be directly established by this discrete version of the intertwining operators even in case of σ-deformations, because,it is necessary to involve 1-and 2-pole functions defined by a constant basis.) In short, there is no way to prove the intertwining of the boundary conditions by our technique.

On the other hand, by the above considerations, a non-changing Ricci tensor is always a necessary condition for intertwining even the Dirichlet conditions on ball×ball-type manifolds. Therefore, these particular continuous operators change the Dirichlet conditions along with the Dirichlet spectra. Thus, they can not induce operators transforming functions defined on the boundaries either. In other words, operators defined by restrictions onto the boundary manifolds are not well defined regarding the GW-deformations. Even though they were introduced by suitable reductions, they would change the spectra also on the boundary manifolds. This phenomena strongly suggests that no other suitable operators exist and the spectra of the ball×ball- and sphere×ball-type manifolds change during these continuous deformations.

Acknowledgements. The author is indebted for the hospitality and excellent working conditions provided by the Max Planck Institute for Mathematics in the Sciences, Leipzig, in the academic year 2007/2008, where important key ideas of this paper were found.

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