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Classification of Bipartite and Tripartite Qutrit
Entanglement under SLOCC

by

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Abstract: We classify biqutrit and triqutrit pure states under stochastic local operations and classical communication. By investigating the right singular vector spaces of the coefficient matrices of the states, we obtain explicitly two inequivalent classes of biqutrit states and twelve inequivalent classes of triqutrit states respectively.

1   Introduction

It is well known that quantum entanglement plays very important roles in quantum information theory. One of the main tasks in quantum information theory is to find out how many different ways multipartite pure states can be entangled. It has been shown that entangled qudits are less affected by noise than entangled qubits [1, 2]. In quantum cryptography it is more secure against evesdropping attacks using entangled qutrits or qudits than using qubits [3, 4, 5, 6]. These facts motivate our interest in multi-dimensional entangled states.

As the concept of entanglement is related to the nonlocal properties of a state, local quantum operations can not affect the intrinsic nature of entanglement[7]. It is natural and meaningful to classify pure states in terms of stochastic local operations and classical communication (SLOCC). In [8] it was shown that SLOCC equivalent pure states can carry out the same quantum-informational tasks with non-null possibly different possibilities, and two N-partite states Ψ and Φ are equivalent under SLOCC if and only if there exist invertible local operations (ILOS) $F^{[1]}, \cdots, F^{[N]}$ such that $\Psi = F^{[1]} \otimes F^{[2]} \otimes \cdots \otimes F^{[N]}\Phi$.

In recent years, a lot of efforts have been made on classification of multipartite entanglement under SLOCC [3][9-18]. In [15] an inductive method of classifying n-qubit entanglement under SLOCC has been presented, from which the entanglement classification of three and four qubits have been obtained. In the classification of biqutrit pure

states some entanglement measures have been also used [3]. In [9] a range criterion has been used to judge the equivalence of two states under SLOCC. The classification of entanglement in $2 \times m \times n$ systems is investigated. In [10] the complete SLOCC classification of multipartite entanglement in $2 \times 2 \times n$ cases has been studied in two different ways. It has been proved that a pure state of four qubits can be transformed into nine families by determinant one SLOCC operations [13]. In [14] 3-qubit states under SLOCC on the basis of the canonical forms and on the local unitary operator polynomial invariants have been classified.

In this paper we study the classification of biqutrit and triqutrit pure states under SLOCC by using the method introduced in [15]. According to the dimensions of the right singular vector spaces of the coefficient matrices of the states, we obtain explicitly all the inequivalent classifications of biqutrit states and triqutrit states under SLOCC.

2   Classification of biqutrit entanglement

Let $\{e_i\}_{i=1,\ldots,m}$ and $\{f_j\}_{j=1,\ldots,n}$ denote the bases in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Any bipartite state $\Psi \in C^m \otimes C^n$ can be written as

$$\Psi = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, e_i \otimes f_j, \tag{1}$$

$c_{ij} \in \mathbb{C}$. We denote $C = (c_{ij})_{m,n}$ the coefficient matrix of the state Ψ.

According to the singular value decomposition, an $m \times n$ matrix C can always be decomposed as $C = V\Sigma W^\dagger$, where V, W are unitary matrices and Σ is a diagonal matrix with non-negative entries (singular values), $\Sigma_{ij} = \sigma_i \delta_{ij}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$ and $\sigma_k \geq 0$ for all k. The columns v_i of $V = [v_1\ v_2\ \ldots\ v_m]$ (resp. w_i of $W = [w_1\ w_2\ \ldots\ w_n]$) are called left (resp. right) singular vectors of C.

Since the relevant singular vectors will be those associated to non-null singular values, in the following we refer as singular vectors only to those v_k and w_k for which $\sigma_k > 0$. We denote by Γ (resp. Π) the subspace generated by the left (resp. right) singular vectors, i.e. $\Gamma = \mathrm{span}\{v_1,\ldots,v_k\}$ (resp. $\Pi = \mathrm{span}\{w_1,\ldots,w_k\}$). From the Schmidt decomposition of bipartite pure states, state (1) is separable if and only if $\dim W = 1$ (or $\dim V = 1$) [15].

Let $\Psi, \bar{\Psi} \in \mathbb{C}^m \otimes \mathbb{C}^n$ denote two bipartite states related by SLOCC, i.e.

$$\bar{\Psi} = F^{[1]} \otimes F^{[2]}(\Psi), \tag{2}$$

where $F^{[1]}$ and $F^{[2]}$ are non-singular operators upon $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Similar to the two-qubit case [15], in terms of singular decomposition one can prove that the

corresponding coefficient matrices C of Ψ, $\bar{C}$ of $\bar{\Psi}$ in an arbitrary product basis are related through

$$\bar{C} = (F^{[1]^T}V)\Sigma(F^{[2]\dagger}W)^\dagger, \tag{3}$$

i.e. if v_j and w_j are the left and right singular vectors of the coefficient matrix C respectively, then the new left and right singular vectors with respect to $\bar{C}$ will be $F^{[1]^T}(v_j)$ and $F^{[2]\dagger}(w_j)$ respectively.

For simplicity in stead of (2), we write $\bar{\Psi} = F^{[1]^T} \otimes F^{[2]\dagger}(\Psi)$ in the following. We first consider the biqutrit ($n = m = 3$) case. In this case, the coefficient matrix of an arbitrary pure state in $\mathbb{C}^3 \otimes \mathbb{C}^3$ can be expressed as

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

The dimensions of the right singular subspaces Π could be either 1, 2 or 3.

If $dim\,\Pi = 1$, we can choose ILOS $F^{[1]}$ and $F^{[2]}$ such that

$$F^{[1]}(v_1) = \frac{1}{\sigma_1}e_1, \quad F^{[2]}(w_1) = e_1,$$

where e_i, $i = 1, 2, 3$, denote the bases of $\mathbb{C}^3$. The new coefficient matrix $\bar{C}$ is then of the form

$$\bar{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which correspond to the product state $\Psi_0 = |00\rangle$ (we denote $e_1 = |0\rangle$, $e_2 = |1\rangle$ and $e_3 = |2\rangle$ as usual in the following).

In the case $dim\,\Pi = 2$, we choose ILOS $F^{[1]}$ and $F^{[2]}$ such that

$$F^{[1]}(v_1) = \frac{1}{\sigma_1}e_1, \quad F^{[1]}(v_2) = \frac{1}{\sigma_2}e_2, \quad F^{[2]}(w_1) = e_1, \quad F^{[2]}(w_2) = e_2.$$

The new coefficient matrix will turn to be

$$\bar{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which correspond to the state $\Psi_1 = |00\rangle + |11\rangle$.

For the case $dim\,\Pi = 3$, we choose ILOS $F^{[1]}$ and $F^{[2]}$ such that

$$F^{[1]}(v_1) = \frac{1}{\sigma_1}e_1, \quad F^{[1]}(v_2) = \frac{1}{\sigma_2}e_2, \quad F^{[1]}(v_3) = \frac{1}{\sigma_3}e_3,$$

$$F^{[2]}(w_1) = e_1, \quad F^{[2]}(w_2) = e_2, \quad F^{[2]}(w_3) = e_3.$$

Then $\bar{C}$ turns out to be a 3×3 identity matrix, and the corresponding state is $\Psi_2 = |00\rangle + |11\rangle + |22\rangle$.

Therefore for biqutrit case, states can be entangled in two inequivalent ways (Ψ_1 and Ψ_2) under SLOCC. While in [3] biqutrit entangled states are classified into three types:

$$|I\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle), \quad |II\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |00\rangle + |-1-1\rangle),$$

$$|III\rangle = \frac{1}{\sqrt{6}}(|11\rangle + |-1-1\rangle + |10\rangle + |01\rangle + |0-1\rangle + |-10\rangle).$$

We find that the type $|II\rangle$ is in fact equivalent to the type $|III\rangle$ under SLOCC: $|III\rangle$ can written as

$$|III\rangle = \frac{1}{\sqrt{3}}[|1\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) + |0\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle) + |-1\rangle \otimes \frac{1}{\sqrt{2}}(|-1\rangle + |0\rangle)],$$

by choosing the ILOS

$$F^{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{[2]} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

we get $F^{[1]} \otimes F^{[2]}|III\rangle = |II\rangle$. This result can be also obtained by using the method of Schmidt decomposition provided in [8].

3 Classification of triqutrit entanglement

An arbitrary triqutrit pure state $\Psi \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ has the form

$$\Psi = \sum_{i,j,k=1}^{3} c_{ijk}\, e_i \otimes e_j \otimes e_k. \tag{4}$$

If we view the system in a bipartite decomposition: the first system and the rest systems, the coefficient matrix of Ψ has the form

$$C = C_{1|23} = \begin{pmatrix} C_{111} & C_{112} & C_{113} & C_{121} & C_{122} & C_{123} & C_{131} & C_{132} & C_{133} \\ C_{211} & C_{212} & C_{213} & C_{221} & C_{222} & C_{223} & C_{231} & C_{232} & C_{233} \\ C_{311} & C_{312} & C_{313} & C_{321} & C_{322} & C_{323} & C_{331} & C_{332} & C_{333} \end{pmatrix}.$$

There are also two other ways to write the coefficient matrix of Ψ: $C_{2|13}$ and $C_{3|12}$. Without loss of generality, we use $C_{1|23}$ in the following.

The classification of triqutrit pure states is to choose the ILOS $F^{[1]}$, $F^{[2]}$ and $F^{[3]}$ such that the final coefficient matrix reduces to a canonical one. In order to do so, we have to find all possible structures of the space Π.

Let $\Psi, \bar{\Psi} \in \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$ denote two tri-partite states that are equivalent under SLOCC, i.e.

$$\bar{\Psi} = F^{[1]^T} \otimes (F^{[2]} \otimes F^{[3]})^\dagger (\Psi), \tag{5}$$

where $F^{[1]}$, $F^{[2]}$ and $F^{[3]}$ are non-singular operators upon $\mathbb{C}^m$, $\mathbb{C}^n$ and $\mathbb{C}^l$, respectively. Similarly we can prove that the coefficient matrices C and $\bar{C}$ in an arbitrary product basis are related through

$$\bar{C} = (F^{[1]}V)\Sigma(F^{[2]} \otimes F^{[3]}W)^\dagger. \tag{6}$$

Concerning the dimensions of the right singular subspace Π, for triqutrit there are again three possibilities: $dim\,\Pi = 1$, $dim\,\Pi = 2$ and $dim\,\Pi = 3$.

3.1   The case of $\dim \Pi = 1$

1. $\Pi = \text{span}\{\Psi_0\}$, where Ψ_0 is the product state defined in the last section. In this case, w_1 is of the form, $w_1 = \phi \otimes \psi$. We can choose ILOS $F^{[1]}$, $F^{[2]}$ and $F^{[3]}$ such that

$$F^{[1]}(v_1) = \frac{1}{\sigma_1}e_1, \quad F^{[2]}(\phi) = e_1, \quad F^{[3]}(\psi) = e_1.$$

Then the new coefficient matrix is of the form

$$
\bar{C} = \begin{pmatrix} \frac{1}{\sigma_1} & . & . \\ 0 & . & . \\ 0 & . & . \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

which corresponds to the state $e_1 \otimes e_1 \otimes e_1 \equiv |000\rangle$. The irrelevant characters in the above matrix entries have been omitted.

2. $\Pi = \text{span}\{\Psi_1\}$. In this case, the vector $w_1 = \phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2$. Using the same strategy above, we obtain the state $e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_2$, which corresponds to the canonical vector: $|000\rangle + |011\rangle$.

3. $\Pi = \text{span}\{\Psi_2\} = \text{span}\{\phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2 + \phi_3 \otimes \psi_3\}$. We obtain the state $e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_1 \otimes e_3 \otimes e_3$, which corresponds to the canonical vector: $|000\rangle + |011\rangle + |022\rangle$.

3.2 The case of $\dim \Pi = 2$

We first consider $\Pi = \mathrm{span}\{\Psi_0, \Psi_0\}$. One of the three possible cases in this class is $\Pi = \mathrm{span}\{\phi \otimes \psi_1, \phi \otimes \psi_2\}$. In this case, $w_1 = u_{11}\phi \otimes \psi_1 + u_{12}\phi \otimes \psi_2$ and $w_2 = u_{21}\phi \otimes \psi_1 + u_{22}\phi \otimes \psi_2$, where the matrix with entries u_{ij} has rank two. Since w_1 and w_2 are linearly independent, we choose the ILOS $F^{[1]}, F^{[2]}$ and $F^{[3]}$ such that

$$F_1^{[1]}(v_1) = \frac{1}{\sigma_1}e_1, \quad F_1^{[1]}(v_2) = \frac{1}{\sigma_2}e_2, \quad F_2^{[1]} = \left(\begin{array}{cc} \left(\begin{array}{cc} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{array} \right)^{-1} & 0 \\ 0 & 1 \end{array} \right),$$

$$F^{[1]} = F_2^{[1]}F_1^{[1]}, \quad F^{[2]}(\phi) = e_1, \quad F^{[3]}(\psi_1) = e_1, \quad F^{[3]}(\psi_2) = e_2.$$

The new coefficient matrix will be

$$
\begin{aligned}
\bar{C} &= F_2^{[1]} \left(\begin{array}{ccc} \frac{1}{\sigma_1} & 0 & . \\ 0 & \frac{1}{\sigma_2} & . \\ 0 & 0 & . \end{array} \right) \left(\begin{array}{ccc} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{ccccccccc} u_{11}^* & u_{12}^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{21}^* & u_{22}^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \end{array} \right) \\
&= \left(\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),
\end{aligned}
$$

which corresponds to the state $|000\rangle + |101\rangle$.

The other two cases can be dealt with similarly. Here one needs not to consider the case $\Pi = \mathrm{span}\{\phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2, \phi_1 \otimes \psi_2 + \phi_2 \otimes \psi_1\}$, because $(\phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2) + (\phi_1 \otimes \psi_2 + \phi_2 \otimes \psi_1) = (\phi_1 + \phi_2) \otimes (\psi_1 + \psi_2)$ is a product state.

Using the same strategy, all together we obtain the following classifications:

Class	*Canonical vector*	
$\mathrm{span}\{\Psi_0, \Psi_0\}$	$|000\rangle + |101\rangle$	
	$|000\rangle + |110\rangle$	
	$|000\rangle + |111\rangle$	
$\mathrm{span}\{\Psi_1, \Psi_1\}$	$|000\rangle + |011\rangle + |101\rangle + |112\rangle$	$|000\rangle + |011\rangle + |112\rangle + |120\rangle$
	$|000\rangle + |011\rangle + |120\rangle + |101\rangle$	$|000\rangle + |011\rangle + |120\rangle + |102\rangle$
$\mathrm{span}\{\Psi_0, \Psi_1\}$	$|000\rangle + |011\rangle + |101\rangle$	$|000\rangle + |011\rangle + |112\rangle$
	$|000\rangle + |011\rangle + |120\rangle$	$|000\rangle + |011\rangle + |122\rangle$
$\mathrm{span}\{\Psi_0, \Psi_2\}$	$|000\rangle + |011\rangle + |022\rangle + |101\rangle$	
$\mathrm{span}\{\Psi_1, \Psi_2\}$	$|000\rangle + |011\rangle + |022\rangle + |101\rangle + |112\rangle$	
	$|000\rangle + |011\rangle + |022\rangle + |112\rangle + |120\rangle$	
	$|000\rangle + |011\rangle + |022\rangle + |120\rangle + |101\rangle$	

We did not consider the case $\Pi = \text{span}\{\Psi_2, \Psi_2\}$. This is due to that any two-dimensional subspace in $\mathbb{C}^3 \otimes \mathbb{C}^3$ contains at least one product vector Ψ_0 or one entangled vector Ψ_1 with coefficient matrix rank two. This can be seen in this way: Let V be a two-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$. Without loss of generality two rank-3 entangled vectors can be chosen as generators of V with coefficient matrices given by $C_1 = I$ and C_2 being an arbitrary rank-3 matrix in the product canonical basis. Then it is always possible to find non-null complex numbers α and β such that $\alpha I + \beta C_2$ has rank one or two: because $-\beta/\alpha$ must be chosen to be an eigenvalues of C_2, and if two eigenvalue of C_2 is the same, $\alpha I + \beta C_2$ will have rank one.

3.3 The case of $\dim \Pi = 3$

We first consider $\Pi = \text{span}\{\Psi_0, \Psi_0, \Psi_0\}$. One of the subcases is $\Pi = \text{span}\{\phi \otimes \psi_1, \phi \otimes \psi_2, \phi \otimes \psi_3\}$. In this case, $w_1 = u_{11}\phi \otimes \psi_1 + u_{12}\phi \otimes \psi_2 + u_{13}\phi \otimes \psi_3$, $w_2 = u_{21}\phi \otimes \psi_1 + u_{22}\phi \otimes \psi_2 + u_{23}\phi \otimes \psi_3$ and $w_3 = u_{31}\phi \otimes \psi_1 + u_{32}\phi \otimes \psi_2 + u_{33}\phi \otimes \psi_3$, with the matrix of entries u_{ij} rank three. Since w_1, w_2 and w_3 are linearly independent, we choose the ILOS $F^{[1]}, F^{[2]}$ and $F^{[3]}$ such that

$$F_1^{[1]}(v_1) = \frac{1}{\sigma_1}e_1, \quad F_1^{[1]}(v_2) = \frac{1}{\sigma_2}e_2, \quad F_1^{[1]}(v_3) = \frac{1}{\sigma_3}e_3, \quad F_2^{[1]} = \begin{pmatrix} u_{11}^* & u_{12}^* & u_{13}^* \\ u_{21}^* & u_{22}^* & u_{23}^* \\ u_{31}^* & u_{32}^* & u_{33}^* \end{pmatrix}^{-1},$$

$$F^{[1]} = F_2^{[1]} F_1^{[1]}, \quad F^{[2]}(\phi) = e_1, \quad F^{[3]}(\psi_1) = e_1, \quad F^{[3]}(\psi_2) = e_2, \quad F^{[3]}(\psi_2) = e_2.$$

Then the new coefficient matrix is then

$$\bar{C} = F_2^{[1]} \begin{pmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} u_{11}^* & u_{12}^* & u_{13}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{21}^* & u_{22}^* & u_{23}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{31}^* & u_{32}^* & u_{33}^* & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which corresponds to the state $|000\rangle + |101\rangle + |202\rangle$.

By investigating the rest cases similarly, all together we have the following canonical states under SLOCC in this class:

Class	Canonical vector
$\text{span}\{\Psi_0, \Psi_0, \Psi_0\}$	$\|000\rangle + \|101\rangle + \|202\rangle$
	$\|000\rangle + \|110\rangle + \|220\rangle$
	$\|000\rangle + \|111\rangle + \|202\rangle$
	$\|000\rangle + \|111\rangle + \|220\rangle$
	$\|000\rangle + \|111\rangle + \|201\rangle$
	$\|000\rangle + \|111\rangle + \|222\rangle$
$\text{span}\{\Psi_0, \Psi_0, \Psi_1\}$	$\|000\rangle + \|011\rangle + \|1\phi\varphi\rangle + \|2\chi\psi\rangle$
$\text{span}\{\Psi_0, \Psi_0, \Psi_2\}$	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|202\rangle$
	$\|000\rangle + \|011\rangle + \|022\rangle + \|110\rangle + \|220\rangle$
	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|212\rangle$
$\text{span}\{\Psi_1, \Psi_1, \Psi_0\}$	$\|000\rangle + \|011\rangle + \|101\rangle + \|112\rangle + \|2\phi\varphi\rangle$
	$\|000\rangle + \|011\rangle + \|112\rangle + \|120\rangle + \|2\phi\varphi\rangle$
	$\|000\rangle + \|011\rangle + \|120\rangle + \|101\rangle + \|2\phi\varphi\rangle$
$\text{span}\{\Psi_1, \Psi_1, \Psi_1\}$	$\|000\rangle + \|011\rangle + \|101\rangle + \|112\rangle + \|202\rangle + \|221\rangle$
	$\|000\rangle + \|011\rangle + \|101\rangle + \|112\rangle + \|210\rangle + \|202\rangle$
	$\|000\rangle + \|011\rangle + \|101\rangle + \|112\rangle + \|221\rangle + \|210\rangle$
	$\|000\rangle + \|011\rangle + \|112\rangle + \|120\rangle + \|202\rangle + \|221\rangle$
	$\|000\rangle + \|011\rangle + \|112\rangle + \|120\rangle + \|221\rangle + \|210\rangle$
	$\|000\rangle + \|011\rangle + \|120\rangle + \|101\rangle + \|221\rangle + \|210\rangle$
$\text{span}\{\Psi_1, \Psi_1, \Psi_2\}$	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|112\rangle + \|202\rangle + \|221\rangle$
	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|112\rangle + \|210\rangle + \|202\rangle$
	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|112\rangle + \|221\rangle + \|210\rangle$
$\text{span}\{\Psi_2, \Psi_1, \Psi_0\}$	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|112\rangle + \|202\rangle$
	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|112\rangle + \|220\rangle$
	$\|000\rangle + \|011\rangle + \|022\rangle + \|101\rangle + \|112\rangle + \|221\rangle$

where ϕ, φ, χ, ψ are pure states in $\mathbb{C}^3$.

According to structures of the space Π, we have got that three qutrits can be entangled in twelve inequivalent ways under SLOCC, where the states in the class of dim $\Pi = 1$, the first two states in $\text{span}\{\Psi_0, \Psi_0\}$ and $\text{span}\{\Psi_0, \Psi_0, \Psi_0\}$ are either fully or bi-separable.

4   Conclusion and Remarks

We have shown that two and three-qutrit entangled states can be classified into two and twelve inequivalent types respectively under stochastic local operations and classical

communication, based upon the analysis of the structure of the right singular subspace of the coefficient matrix of the states in an arbitrary conical product base.

The range criterion [9] to judge whether two pure states are inequivalent under SLOCC, classifies multipartite entanglement by analyzing the structure of the ranges of the states. In fact to study the range of a state is equivalent to study the right singular subspace of the state. As the ways of entanglement are concerned, the result of theorem 2 in [9] is included in our result (in section 3.2).

The way of classifying pure states under SLOCC can be generalized to high-dimensional case by investigating the structures of the space Π. It is easily seen that there are n types of pure states in the space $\mathbb{C}^n \otimes \mathbb{C}^n$ under SLOCC. For the pure stats in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, one can also deal with their classification according to the dimensions of right singular subspace: If the dimensions of the subspace is 2, there will be $C_n^2 + (n-1)$ families of entanglement; If the dimension of the subspace is 3, then $C_n^3 + 2(C_n^2 - 1) + (n-1)$ families; ...; If the dimension of the subspace is $n-1$, then $C_n^{n-1} + (n-2)(C_n^{n-2} - 1) + ... + 2(C_n^2 - 1) + (n-1)$ families. All together the number of the classification is $(n-1)^2 + \sum_{i=2}^{n}[(1 + i(n-i))C_n^{n-i} - i(n-i)]$, where $C_n^{n-i} = n!/i!(n-i)!$. For instance for $n = 2$, we have that, a result of [8], two qubits can be entangled in two inequivalent ways.

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