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Damage as Gamma-limit of microfractures in linearized elasticity under the non-interpenetration constraint

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DAMAGE AS Γ-LIMIT OF MICROFRACTURES IN LINEARIZED ELASTICITY UNDER THE NON-INTERPENETRATION CONSTRAINT

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Abstract. A homogenization result is given for a material having soft inclusions arranged in a periodic structure, under the requirement that the interpenetration of matter is forbidden. According to the relation between the softness parameter and the size of the microstucture, three different limit models are deduced via Γ-convergence.

Keywords: brittle fracture, damage, non-interpenetration, homogenization, Γ-convergence, integral representation

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1. Introduction

The subject of this paper is a homogenization result for a linearly elastic material presenting soft inclusions arranged in a periodic structure. We assume that cracks can appear only in the inclusions of brittle material and we impose a non-interpenetration constraint between the lips of the fracture.

We denote by $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, the region of the space occupied by the material and by $\varepsilon > 0$ a small parameter. We introduce a periodic structure on Ω, whose cells Q^ε are the ε-homothetic of the unit square $Q := (0,1)^n$, i.e., $Q^\varepsilon = (0,\varepsilon)^n$. We assume that the soft inclusion is well contained in Q^ε. More precisely, it lies in εQ_δ, where $Q_\delta \subset Q$ is the concentric cube $(\delta, 1-\delta)^n$, for $0 < \delta < \frac{1}{2}$.

The starting point of our analysis is the energy associated to a vector valued displacement u of the material. Since we are taking into account the possibility of creating cracks, the displacements are allowed to have discontinuities. Therefore, the natural functional setting for the problem is the space $SBD(\Omega)$ of special functions with bounded deformation. Moreover, the admissible functions $u \in SBD(\Omega)$ are required to satisfy the infinitesimal non-interpenetration condition $[u] \cdot \nu_u \geq 0$ $\mathcal{H}^{n-1}$-a.e. on the jump set J_u, where $[u]$ is the jump of u and ν_u is the normal to the jump set. Physically, this constraint means that the two lips of a fracture cannot interpenetrate. We consider the case in which a homogeneous Dirichlet boundary condition is imposed on $\partial\Omega$, that is, $tr(u) = 0$ on $\partial\Omega$. We denote with $SBD_0(\Omega)$ the subspace of $SBD(\Omega)$ where the boundary datum is attained.

In order to define the energy associated to the displacement u, we introduce some notations. Let $\mathbb{C} = (\mathbb{C}_{ijkl})$ be the elasticity tensor and let $u \in SBD^2_0(\Omega)$ be a displacement. We denote by $\sigma(u) \in \mathbb{M}^{n\times n}_{sym}$ the tensor $\mathbb{C}\mathcal{E}u$, where $\mathcal{E}u$ denotes the absolutely continuous part of the symmetric gradient of u. The energy corresponding to u is given by the functional $\mathcal{F}^\varepsilon$ defined as

$$\mathcal{F}^\varepsilon(u) := \int_\Omega \sigma(u) : \mathcal{E}u \, dx + \int_{J_u} g_{\alpha_\varepsilon}\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big) d\mathcal{H}^{n-1}(x), \tag{1.1}$$

where $g_{\alpha_\varepsilon} : \mathbb{R}^n \times \mathbb{R}^n \times S^{n-1} \to [0,+\infty]$ is a Q-periodic function in the first variable, defined for $y \in Q$, $z \in \mathbb{R}^n$, $\nu \in S^{n-1}$ by

$$g_{\alpha_\varepsilon}(y,z,\nu) := \begin{cases} \alpha_\varepsilon & \text{if } y \in Q_\delta \text{ and } z\cdot\nu \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

and α_ε is a positive parameter depending on ε. The volume term in the expression of $\mathcal{F}^\varepsilon$ represents the elastic energy, while the surface integral describes the energy needed to open a crack. More precisely, the density g_{α_ε} forces the deformation u to have a jump set contained in the fragile part of the material and the lips of the fracture to avoid interpenetration.

The overall properties of the composite material described by the functional $\mathcal{F}^\varepsilon$ can be expressed in terms of a *homogenized* simpler integral, which is given by the Γ-limit of $\mathcal{F}^\varepsilon$, as ε goes to zero. In our case we assume that $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$, and we show that the limit model depends on the behaviour of the ratio $\frac{\alpha_\varepsilon}{\varepsilon}$ as ε goes to zero.

A similar analysis has been developed in a previous work [17], under the assumption of anti-planar shear. We notice that in that case the non-interpenetration constraint is automatically satisfied. As in [17], it turns out that also in the present paper the different limiting models describe an unbreakable material. This means that, even if at scale ε the material has periodically distributed microscopic cracks, when ε goes to zero no macroscopic crack appears. This is due to the fact that in the periodicity cell Q^ε the brittle region is well separated from the boundary ∂Q^ε and this prevents small cracks to glue together into a macroscopic fracture.

In this paper we show that three different limit models can arise, corresponding to the limit $\frac{\alpha_\varepsilon}{\varepsilon}$ being zero (subcritical case), finite (critical case) or $+\infty$ (supercritical case).

In the subcritical case $\alpha_\varepsilon << \varepsilon$, the limit functional is given by

$$\mathcal{F}^0(u) = \begin{cases} \displaystyle\int_\Omega f_0(\mathcal{E}u)dx & \text{in } H^1_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{1.3}$$

The density f_0 is given by the cell formula

$$f_0(\xi) := \inf\left\{ \int_Q \sigma(\xi x + w) : (\xi^s + \mathcal{E}w)dx : w \in SBD^2_\#(Q), J_w \subset Q_\delta,\ [w]\cdot\nu_w \geq 0 \text{ a.e. on } J_w \right\}, \tag{1.4}$$

where $SBD^2_\#(Q)$ denotes the space of $SBD^2(Q)$ functions with periodic boundary conditions on ∂Q and ξ^s denotes the simmetric part of ξ.

An interesting remark is that in general f_0 is not a quadratic form. Indeed, if we assume $\mathbb{C}$ to be isotropic, that is,

$$\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\, Id \otimes Id,$$

where $\lambda, \mu > 0$, $(\mathbb{I})_{ijkl} = \delta_{ik}\delta_{jl}$, and $(Id\otimes Id)_{ijkl} = \delta_{ij}\delta_{kl}$, then it turns out that $f_0(Id) \neq f_0(-Id)$ (see Lemma 5.3).

This is in contrast with the situation in which the non-interpenetration constraint in not assumed. Indeed, in that case, proceeding as in [17], one can prove that the density function $\hat{f}_0$ is defined as

$$\hat{f}_0(\xi) := \inf\left\{ \int_Q \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx : w \in SBD^2_\#(Q), J_w \subset Q_\delta \right\}, \tag{1.5}$$

and is a quadratic form for every choice of the tensor $\mathbb{C}$.

A possible interpretation of this result is the following. For $\xi = Id$ the body is subject to a boundary deformation of pure extension in all directions. In this case, the solutions to (1.4) present discontinuities, since the non-interpenetration constraint is compatible with the boundary conditions and it is energetically convenient to have a nonempty jump set.

On the contrary, when $\xi = -Id$, i.e., in a regime of pure compression, it turns out that the optimal w in (1.4) is $w = 0$. This happens because the minimizers of the problem (1.5) corresponding to $\xi = -Id$ are not admissible for (1.4), since they do not satisfy the non-interpenetration constraint.

Another important remark is that the limit energy density describes a material undergoing a damage process. Indeed for a large class of matrices $\xi \in \mathbb{M}^{n\times n}$ it turns out that $f_0(\xi) \lneqq \mathbb{C}\xi : \xi$, and this means that the elastic properties of the material are reduced by homogenization. Therefore the possible presence of microfractures at scale ε translates into a damage of the material at a macroscopic scale.

In the critical regime, corresponding to $\alpha_\varepsilon = \varepsilon$, the limit functional is

$$\mathcal{F}^{hom}(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\mathcal{E}u)dx & \text{in } H^1_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n), \end{cases} \tag{1.6}$$

where the density f_{hom} is given by the asymptotic cell problem

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,t)^n\big), \\ J_w \subset I_\delta,\ [w]\cdot\nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\},$$

and the set I_δ is defined as

$$I_\delta := (0,t)^n \cap \bigcup_{h\in\mathbb{Z}^n} (Q_\delta + h). \tag{1.7}$$

Since in this case the coefficient α_ε and the size ε of the microstructure are of the same order, there is a competition between the bulk energy and the surface term which gives an intermediate model with respect to the subcritical and the supercritical regimes. Moreover, the limit functional describes a damaged material, as shown in Lemma 6.4.

In the supercritical regime $\alpha_\varepsilon >> \varepsilon$, the limit model is given by the functional

$$\mathcal{F}^\infty(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u\, dx & \text{in } H^1_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{1.8}$$

Therefore, the (possible) presence of cracks in the approximating problems has no effect on the limit. Indeed, as one may expect, in this case the energy penalizes the jumps of the deformations, so that the limit material has the same elastic properties as the original one and no damage occurs.

We want to underline that in this regime the Γ-limit is the same as if the non-interpenetration constraint were not imposed. The feature which makes this case mathematically different from the corresponding one in [17] is the lack of a lower semicontinuity result in SBD when no a priori bound for the L^∞ norm of the deformations is given. Hence, in order to prove the Γ-convergence result for this scaling, we need a modified version of the proof of lower semicontinuity in SBD given in [6], where the assumption of the equiboundedness of the L^∞ norm of the displacements is replaced by the fact that the measure of their jump sets goes to zero (see Lemma 7.2).

The plan of the paper is the following. In Sections 2 and 3 we define the mathematical setting of the problem and we introduce the energy functional, respectively. Section 4 is aimed to show that the limit functional obtained via Γ-convergence admits an integral representation, while Sections 5-7 are devoted to the description of the limit functionals in the subcritical, critical and supercritical cases.

2. Preliminaries

In this section we collect some definitions and results that will be widely used throughout the paper. In order to make precise the mathematical setting, we recall some properties of rectifiable sets and we include a brief presentation of the spaces SBV and SBD. We refer the reader to [3] and to [18] for further details.

A set $\Gamma \subset \mathbb{R}^n$ is rectifiable if there exists $N_0 \subset \Gamma$ with $\mathcal{H}^{n-1}(N_0) = 0$, and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^n$ such that

$$\Gamma \setminus N_0 \subset \bigcup_{i\in\mathbb{N}} M_i.$$

For every $x \in \Gamma \setminus N_0$ we define the normal to Γ at x as $\nu_{M_i}(x)$. It turns out that the normal is well defined (up to the sign) for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

SBV functions. Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $SBV(U)$ as the set of functions $u \in L^1(U)$ such that the distributional derivative Du is a Radon measure which, for every open set $A \subset U$, can be represented as

$$Du(A) = D^a u(A) + D^j u(A) = \int_A \nabla u\, dx + \int_{A\cap S_u} [u](x)\, \nu_u(x)\, d\mathcal{H}^{n-1}(x),$$

where ∇u is the approximate differential of u, S_u is the set of jump of u (which is a rectifiable set), $\nu_u(x)$ is the normal to S_u at x, and $[u](x)$ is the jump of u at x.

For every $p \in]1,+\infty[$ we set

$$SBV^p(U) = \{u \in SBV(U) : \nabla u \in L^p(U;\mathbb{R}^n), \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

If $u \in SBV(U)$ and $\Gamma \subset U$ is rectifiable and oriented by a normal vector field ν, then we can define the traces u^+ and u^- of u on Γ, which are characterized by the relations

$$\lim_{r\to 0} \frac{1}{r^n} \int_{\Omega\cap B_r^{\pm}(x)} |u(y) - u^{\pm}(x)|\, dy = 0 \quad \text{for } \mathcal{H}^{n-1} - \text{a.e. } x \in \Gamma,$$

where $B_r^{\pm}(x) := \{y \in B_r(x) : (y-x)\cdot\nu \gtrless 0\}$ and $B_r(x)$ is the open ball centered in x with radius r.

BD functions. Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $BD(U)$ as the set of functions $u \in L^1(U;\mathbb{R}^n)$ such that the symmetric part of the distributional derivative Du is a bounded Radon measure.

We denote with Eu the symmetric part of Du, that is,

$$Eu := \{(Eu)_{ij}\}, \quad (Eu)_{ij} := \frac{1}{2}(D_i u_j + D_j u_i).$$

We can split the symmetric gradient into its absolutely continuous, jump and Cantor parts, as

$$Eu = E^a u + E^j u + E^c u = \mathcal{E}u\, dx + E^j u + E^c u.$$

Sections of BD functions. Let $u \in BD(U)$, let $\xi \in S^{n-1}$ and let $y \in \mathbb{R}^n$. We denote by π_ξ the hyperplane orthogonal to ξ passing through the origin and by U^ξ the orthogonal projection of U on π_ξ. Moreover the section of U corresponding to y is denoted by U_y^ξ, that is, $U_y^\xi := \{t \in \mathbb{R} : y + t\,\xi \in U\}$.

We can define the section $u_y^\xi : U_y^\xi \to \mathbb{R}$ as $u_y^\xi(t) := u(y + t\,\xi)\cdot\xi$, for every $t \in U_y^\xi$. Then, the following properties are satisfied:

(i) for $\mathcal{H}^{n-1}$-a.e. $y \in U^\xi$ the function u_y^ξ belongs to $BV(U_y^\xi)$;

(ii) $(\mathcal{E}u(y + t\,\xi)\xi, \xi) = \nabla u_y^\xi(t)$;

(iii) $(\mathcal{E}u\xi,\xi) = \int_{U^\xi} \nabla u_y^\xi d\mathcal{H}^{n-1}(y)$, $|(\mathcal{E}u\xi,\xi)| = \int_{U^\xi} |\nabla u_y^\xi|\, d\mathcal{H}^{n-1}(y)$;

(iv) $(E^j u\xi,\xi) = \int_{U^\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y)$, $|(E^j u\xi,\xi)| = \int_{U^\xi} |D^j u_y^\xi|\, d\mathcal{H}^{n-1}(y)$;

(v) $(E^c u\xi,\xi) = \int_{U^\xi} D^c u_y^\xi d\mathcal{H}^{n-1}(y)$, $|(E^c u\xi,\xi)| = \int_{U^\xi} |D^c u_y^\xi|\, d\mathcal{H}^{n-1}(y)$.

$SBD(U)$ functions. We define $SBD(U)$ as the set of functions $u \in L^1(U;\mathbb{R}^n)$ such that the symmetric part of their distributional derivative Du, that is Eu, is a Radon measure which, for every open set $A \subset U$, can be represented as

$$Eu(A) = E^a u(A) + E^j u(A) = \int_A \mathcal{E}u\, dx + \int_{A\cap J_u} [u](x) \odot \nu_u(x) d\mathcal{H}^{n-1}(x),$$

where J_u is the set of jump of u (which is a rectifiable set), $\nu_u(x)$ is the normal to J_u at x, and $[u](x)$ is the jump of u at x. For every $p \in]1,+\infty[$ we set

$$SBD^p(U) = \{u \in SBD(U) : \mathcal{E}u \in L^p(U;\mathbb{M}^{n\times n}_{sym}), \mathcal{H}^{n-1}(J_u) < +\infty\}.$$

We have that if $u \in SBD(U)$, then its sections are in $SBV(U_y^\xi)$ for every $\xi \neq 0$ and for $\mathcal{H}^{n-1}$-a.e. $y \in U^\xi$.

3. Formulation of the problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We assume for simplicity that $\partial\Omega$ is C^2, although this condition may be weakened. In the following we will denote by Q the unit cube $(0,1)^n$ and with Q_ϱ the inner cube $(\varrho, 1-\varrho)^n$, for some $0 < \varrho < \frac{1}{2}$.

For every $\varepsilon > 0$, let us consider the periodic structure in $\mathbb{R}^n$ generated by an ε-homothetic of the basic cell Q. For notational brevity we will use the superscript ε to denote the ε-homothetic of any domain. In particular, $Q^\varepsilon := \varepsilon\, Q$ and $Q_\delta^\varepsilon := \varepsilon\, Q_\delta$, for $0 < \delta < \frac{1}{2}$.

We define the set I_δ^ε of brittle inclusions as

$$I_\delta^\varepsilon := \Omega \cap \bigcup_{h\in\mathbb{Z}^n} \varepsilon(Q_\delta + h), \tag{3.1}$$

and the unbreakable part of the material as

$$\Omega^\varepsilon := \Omega \setminus I_\delta^\varepsilon. \tag{3.2}$$

Notice that we can split $\partial\Omega^\varepsilon = \Gamma^\varepsilon \cup S^\varepsilon$, where

$$\Gamma^\varepsilon := \partial\Omega \cap \overline{\Omega^\varepsilon} \quad \text{and} \quad S^\varepsilon := \partial\Omega^\varepsilon \cap \Omega. \tag{3.3}$$

Let $\mathbb{C} = (\mathbb{C}_{ijkl})$ be the elasticity tensor, considered as a symmetric positive definite linear operator from $\mathbb{M}^{n\times n}_{sym}$ into itself. It turns out that there exists two constants $0 < \vartheta_m \le \vartheta_M$ such that for any $\xi \in \mathbb{M}^{n\times n}_{sym}$, it holds

$$\vartheta_m\,|\xi|^2 \le \mathbb{C}\xi : \xi \le \vartheta_M\,|\xi|^2, \tag{3.4}$$

where $\xi : \eta = \mathrm{trace}(\xi\eta^T) = \xi_{ij}\eta_{ij}$ and $|\xi|^2 = \xi : \xi$ is the standard Euclidean norm. Clearly, the tensor $\mathbb{C}$ is symmetric with respect to any interchange of indices, that is,

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}. \tag{3.5}$$

To every function $u \in SBD^2_0(\Omega)$ we associate the energy

$$\mathcal{F}^\varepsilon(u) = \int_\Omega \sigma(u) : \mathcal{E}u\,dx + \int_{J_u} g_\alpha\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big)\,d\mathcal{H}^{n-1}(x), \tag{3.6}$$

where $\sigma(u) = \mathbb{C}\mathcal{E}u$, $g_\alpha : \mathbb{R}^n \times \mathbb{R}^n \times S^{n-1} \to [0,+\infty]$ is a Q-periodic function defined in Q as

$$g_\alpha(y,z,\nu) = \begin{cases} \alpha & \text{if } y \in Q_\delta \text{ and } z\cdot\nu \ge 0,\\ +\infty & \text{otherwise in } Q, \end{cases} \tag{3.7}$$

and α is a positive parameter. Being g_α Q-periodic, the function

$$x \mapsto g_\alpha\Big(\frac{x}{\varepsilon}, z, \nu\Big) \tag{3.8}$$

turns out to be Q^ε-periodic.
As in [17] we are interested in the case in which δ is fixed and independent of ε, while $\alpha = \alpha_\varepsilon$ depends on ε and goes to zero as $\varepsilon \to 0$. We will study three different cases, i.e.,

$$\begin{array}{lll} \text{1. Subcritical regime} & \dfrac{\alpha_\varepsilon}{\varepsilon} \to 0 & \text{as } \varepsilon \to 0,\\ \text{2. Supercritical regime} & \dfrac{\alpha_\varepsilon}{\varepsilon} \to +\infty & \text{as } \varepsilon \to 0,\\ \text{3. Critical regime} & \dfrac{\alpha_\varepsilon}{\varepsilon} \to c \in (0,+\infty) & \text{as } \varepsilon \to 0. \end{array} \tag{3.9}$$

Before starting the analysis of the different cases we have just described, let us state an extension result that will be often used in the following. For the proof we refer to [16].

Definition 3.1. Let ω be an unbounded domain of $\mathbb{R}^n$ with a Q-periodic structure, where $Q := (0,1)^n$. Assume that the cell of periodicity $\omega \cap Q$ is a domain with a Lipschitz boundary. Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a positive parameter $\varepsilon > 0$, we set $\Omega^\varepsilon := \Omega \cap \varepsilon\,\omega$. Moreover, we set $\Gamma^\varepsilon := \partial\Omega \cap \varepsilon\,\omega$. We define the space $H^1(\Omega^\varepsilon, \Gamma^\varepsilon; \mathbb{R}^n)$ as

$$H^1(\Omega^\varepsilon, \Gamma^\varepsilon; \mathbb{R}^n) := \{v \in H^1(\Omega^\varepsilon; \mathbb{R}^n) : v = 0 \text{ a.e. on } \Gamma^\varepsilon\}. \tag{3.10}$$

Theorem 3.2. *Let Ω_0 be a bounded domain such that $\Omega^\varepsilon \subset \Omega_0$ and dist$(\partial\Omega_0, \Omega) > 1$. Then for every sufficiently small ε there exists a linear extension operator $T^\varepsilon : H^1(\Omega^\varepsilon, \Gamma^\varepsilon; \mathbb{R}^n) \to H^1_0(\Omega_0; \mathbb{R}^n)$ and three constants $k_0, k_1, k_2 > 0$ such that*

$$\begin{aligned} ||T^\varepsilon u||_{(H^1(\Omega_0))^n} &\le k_1 ||u||_{(H^1(\Omega^\varepsilon))^n},\\ ||D(T^\varepsilon u)||_{(L^2(\Omega_0))^{n\times n}} &\le k_2 ||Du||_{(L^2(\Omega^\varepsilon))^{n\times n}},\\ ||E(T^\varepsilon u)||_{(L^2(\Omega_0))^{n\times n}} &\le k_3 ||Eu||_{(L^2(\Omega^\varepsilon))^{n\times n}}, \end{aligned}$$

for any $u \in H^1(\Omega^\varepsilon, \Gamma^\varepsilon; \mathbb{R}^n)$*, where the constants* k_0, k_1, k_2 *do not depend on* ε*.*

Moreover, $(T^\varepsilon u)_{|A} = 0$ *for any open set* A *such that* $\bar{A} \subset \Omega_0 \setminus \Omega$*, if* ε *is sufficiently small.*

4. Integral representation

The purpose of this section is to show that, independently of the convergence rate of α_ε to zero, the sequence $(\mathcal{F}^\varepsilon)$ always admits a Γ-convergent subsequence. Moreover, we will prove that the limit functional can be written in an integral form. This will be done in an abstract setting. The characterization of the limit energy density for the different regimes will be studied in Sections 5-7.

First, we introduce some definitions and results that will be used in the following. For further references see [11].

Definition 4.1. Let $(G^\varepsilon) : L^2(\Omega; \mathbb{R}^n) \to \overline{\mathbb{R}}$ be a sequence of functionals, where the space $L^2(\Omega; \mathbb{R}^n)$ is endowed with the distance induced by the norm. Define the functionals G' and G'' as follows:

$$G' := \Gamma - \liminf_{\varepsilon \to 0} G^\varepsilon \quad \text{and} \quad G'' := \Gamma - \limsup_{\varepsilon \to 0} G^\varepsilon. \tag{4.1}$$

Definition 4.2. Let $\mathcal{A}(\Omega)$ denote the family of the open subsets of Ω. We say that a functional $G : L^2(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty]$ is increasing (on $\mathcal{A}(\Omega)$) if for every $u \in L^2(\Omega; \mathbb{R}^n)$ the set function $G(u, \cdot)$ is increasing on $\mathcal{A}(\Omega)$.

Definition 4.3. Given a functional $G : L^2(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty]$, we define its inner regularization as

$$G_-(u, A) := \sup \big\{ G(u, B) : B \in \mathcal{A}(\Omega), B \subset\subset A \big\}. \tag{4.2}$$

Observe that if G is increasing, then also G_- is increasing.

Definition 4.4. We say that a sequence (G^ε) is $\overline{\Gamma}$-convergent to a functional G whenever

$$G = (G')_- = (G'')_-.$$

We have the following compactness theorem.

Theorem 4.5. *Every sequence of increasing functionals has a* $\overline{\Gamma}$*-convergent subsequence.*

In order to prove the Γ-convergence of a subsequence of $(\mathcal{F}^\varepsilon)$, a crucial step is to show that the functionals $\mathcal{F}^\varepsilon$ satisfy the so-called *fundamental estimate*, independently of the rate of convergence of α_ε.

As a first step, we localize the sequence $(\mathcal{F}^\varepsilon)$, introducing an explicit dependence on the set of integration. That is, for every $u \in L^2(\Omega; \mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$ we define

$$\mathcal{F}^\varepsilon(u, A) := \begin{cases} \displaystyle\int_A \sigma(u) : \mathcal{E}u \, dx + \alpha_\varepsilon \, \mathcal{H}^{n-1}(J_u \cap A) & \text{if } u \in SBD^2(A), J_u \subset I_\delta^\varepsilon \cap A, \\ & [u] \cdot \nu_u \geq 0 \;\; \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^n). \end{cases}$$

For a fixed $u \in L^2(\Omega; \mathbb{R}^n)$ we can extend $(\mathcal{F}^\varepsilon)(u, \cdot)$ to a measure $(\mathcal{F}^\varepsilon)^*(u, \cdot)$ on the class of Borel sets $\mathcal{B}(\Omega)$ in the usual way:

$$(\mathcal{F}^\varepsilon)^*(u, B) := \inf \big\{ \mathcal{F}^\varepsilon(u, A) : A \in \mathcal{A}(\Omega), B \subseteq A \big\}. \tag{4.3}$$

Next theorem provides an extension of the fundamental estimate to the space SBD^2. The proof is obtained by modifying [8, Proposition 3.1], valid for SBV functions.

Theorem 4.6 (Fundamental estimate in SBD^2)**.** *For every* $\eta > 0$ *and for every* A', A'' *and* $B \in \mathcal{A}(\Omega)$*, with* $A' \subset\subset A''$*, there exists a constant* $M > 0$ *with the following property: for every* $\varepsilon > 0$ *and for every* $u \in SBD^2(A'')$ *such that* $J_u \subset I_\delta^\varepsilon \cap A''$ *and* $[u] \cdot \nu_u \geq 0$ $\mathcal{H}^{n-1}$*-a.e. on* J_u*, and for every* $v \in SBD^2(B)$ *such that* $J_v \subset I_\delta^\varepsilon \cap B$ *and* $[v] \cdot \nu_v \geq 0$ $\mathcal{H}^{n-1}$*-a.e. on* J_v*, there exists a function* $\varphi \in C_0^\infty(\Omega)$ *with* $\varphi = 1$ *in a neighborhood of* $\bar{A}'$*,* $spt\,\varphi \subset A''$ *and* $0 \leq \varphi \leq 1$ *such that*

$$\mathcal{F}^\varepsilon(\varphi\, u + (1 - \varphi)\, v, A' \cup B) \leq (1 + \eta)\, \mathcal{F}^\varepsilon(u, A'') + (1 + \eta)\, \mathcal{F}^\varepsilon(v, B) + M \int_T |u - v|^2 dx, \tag{4.4}$$

where $T := (A'' \setminus A') \cap B$*.*

Proof. Let $\eta > 0$, A', A'' and B be as in the statement. Let $A_1, \ldots, A_{k+1}$ be open subsets of $\mathbb{R}^n$ such that $A' \subset\subset A_1 \subset\subset A_2 \subset\subset \cdots \subset\subset A_{k+1} \subset\subset A''$. For $i = 1, \ldots, k$, set $T_i := (A_{i+1} \setminus \bar{A}_i) \cap B$. For every $i = 1, \ldots, k$, let φ_i be a function in $C_0^\infty(\Omega)$ with $\varphi_i = 1$ on a neighborhood of $\bar{A}_i$ and $\operatorname{spt} \varphi \subset A_{i+1}$.

Now, let u and v be as in the statement and define the function w_i on $A' \cup B$ as $w_i := \varphi_i u + (1 - \varphi_i)\, v$ (where u and v are arbitrarily extended outside A'' and B, respectively). We need to verify that w_i belongs to the domain of $\mathcal{F}^\varepsilon(\cdot, A' \cup B)$. By definition we have that $w_i \in SBD^2(A' \cup B)$ and that $J_{w_i} \subset I_\delta^\varepsilon \cap (A' \cup B)$. Hence it remains to check that $[w_i] \cdot \nu_{w_i} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{w_i}. Clearly, for $x \in J_{w_i} \setminus T_i$ the condition is satisfied since it holds true for u and v. Hence we can restrict our attention to the case $x \in T_i \cap (J_u \cap J_w)$. If J_u and J_w intersect tangentially at x, then $\nu_{w_i} = \nu_u = \nu_v$ and the non-interpenetration condition is fulfilled, otherwise the normal ν_{w_i} is not defined at x.

Now we can write, for fixed $\varepsilon > 0$,

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_i, A' \cup B) &= \int_{A' \cup B} \sigma(w_i) : \mathcal{E} w_i \, dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{w_i} \cap (A' \cup B)\big) \\
&= \big(\mathcal{F}^\varepsilon\big)^*(u, (A' \cup B) \cap \bar{A}_i) + \big(\mathcal{F}^\varepsilon\big)^*(v, B \setminus A_{i+1}) + \mathcal{F}^\varepsilon(w_i, T_i) \\
&\leq \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \mathcal{F}^\varepsilon(w_i, T_i).
\end{aligned} \tag{4.5}$$

Let us define $M_k := \max_{1 \leq i \leq k} ||\nabla \varphi_i||^2_{L^\infty}$. Using (3.4), we can estimate the last term in (4.5) as

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_i, T_i) &\leq \vartheta_M \int_{T_i} |\mathcal{E}(\varphi_i u + (1 - \varphi_i)\, v)|^2 dx + \alpha_\varepsilon \, \mathcal{H}^{n-1}\big(J_{w_i} \cap T_i\big) \\
&\leq c \int_{T_i} |\mathcal{E} u|^2 dx + c \int_{T_i} |\mathcal{E} v|^2 dx + c\, M_k \int_{T_i} |u - v|^2 dx \\
&\quad + \alpha_\varepsilon \, \mathcal{H}^{n-1}\big(J_u \cap T_i\big) + \alpha_\varepsilon \, \mathcal{H}^{n-1}\big(J_v \cap T_i\big) \\
&\leq c\, \mathcal{F}^\varepsilon(u, T_i) + c\, \mathcal{F}^\varepsilon(v, T_i) + c\, M_k \int_{T_i} |u - v|^2 dx =: L^\varepsilon(T_i).
\end{aligned} \tag{4.6}$$

Now, let $i_0 \in \{1, \ldots, k\}$ be such that T_{i_0} realizes $\min_{1 \leq i \leq k} L^\varepsilon(T_i)$. Then, being L^ε a measure, we have

$$L^\varepsilon(T_{i_0}) \leq \frac{1}{k} \sum_{i=1}^k L^\varepsilon(T_i) \leq \frac{1}{k} \, L^\varepsilon(T). \tag{4.7}$$

Notice that $i_0 = i_0(\varepsilon)$, it depends on ε.

Combining together (4.5)-(4.7), we get

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_{i_0}, A' \cup B) \leq\ & \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \frac{1}{k} \, L^\varepsilon(T) \\
&= \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \frac{c}{k} \, \mathcal{F}^\varepsilon(u, T) + \frac{c}{k} \, \mathcal{F}^\varepsilon(v, T) + \frac{c}{k} \, M_k \int_T |u - v|^2 dx \\
&\leq \mathcal{F}^\varepsilon(u, A'') + \mathcal{F}^\varepsilon(v, B) + \frac{c}{k} \, \mathcal{F}^\varepsilon(u, A'') + \frac{c}{k} \, \mathcal{F}^\varepsilon(v, B) + \frac{c}{k} \, M_k \int_T |u - v|^2 dx.
\end{aligned} \tag{4.8}$$

Now, since the choice of the number k of the stripes between A' and A'' is completely free, we can assume that k is such that $\frac{c}{k} < \eta$. Hence $k = k(\eta)$. Let us define $\overline{M}_\eta := \frac{c}{k} M_k$; then in (4.8) we have

$$\mathcal{F}^\varepsilon(w_{i_0}, A' \cup B) \leq (1 + \eta)\, \mathcal{F}^\varepsilon(u, A'') + (1 + \eta)\, \mathcal{F}^\varepsilon(v, B) + \overline{M}_\eta \int_T |u - v|^2 dx, \tag{4.9}$$

which is exactly the claim. □

Next theorem shows that the functional $\mathcal{F}' := \Gamma - \liminf_\varepsilon \mathcal{F}^\varepsilon$ is finite only on $H_0^1(\Omega; \mathbb{R}^n)$.

Theorem 4.7. *Let $\mathcal{G} : L^2(\Omega; \mathbb{R}^n) \to [0, +\infty]$ be the functional defined as*

$$\mathcal{G}(u) = \begin{cases} \displaystyle\int_\Omega A_0 \mathcal{E} u : \mathcal{E} u \, dx & \text{in } H_0^1(\Omega; \mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^n), \end{cases} \tag{4.10}$$

where $A_0 = (A_{ijkh})$ is the fourth order tensor with constant coefficients given by the solution of the cell problem

$$A_0\xi : \xi = \min\left\{ \int_{Q\setminus Q_\delta} \sigma(w) : \mathcal{E}w\, dy : w - \xi\, y \in H^1_{\#}(Q;\mathbb{R}^n)\right\}, \tag{4.11}$$

for $\xi \in \mathbb{M}^{n\times n}_{sym}$. Then,

$$\mathcal{F}'(u) \geq \vartheta_m\, \mathcal{G}(u) \quad \textit{for every } u \in L^2(\Omega;\mathbb{R}^n), \tag{4.12}$$

where $\mathcal{F}'$ is defined as in (4.1), with G^ε replaced by $\mathcal{F}^\varepsilon$ and ϑ_m is the constant in (3.4).

Proof. Let $u \in L^2(\Omega;\mathbb{R}^n)$ and let (u^ε) be a sequence converging to u strongly in L^2 and such that $\mathcal{F}^\varepsilon(u^\varepsilon) \leq c < +\infty$.

Let us define the auxiliary functional $\mathcal{G}^\varepsilon : L^2(\Omega;\mathbb{R}^n) \to [0,+\infty]$ as

$$\mathcal{G}^\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\mathcal{E}v|^2 dx & \text{if } v \in H^1(\Omega,\Gamma^\varepsilon;\mathbb{R}^n),\\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n),\end{cases} \tag{4.13}$$

where a is a Q-periodic function given by

$$a(y) = \begin{cases} 0 & \text{for } y \in Q_\delta,\\ 1 & \text{for } y \in Q\setminus Q_\delta.\end{cases} \tag{4.14}$$

It is well known that the sequence $(\mathcal{G}^\varepsilon)$ Γ-converges (with respect to the strong topology of L^2) to the functional $\mathcal{G}$ defined in (4.10). For further details we refer to [10] and [16].

We would like to compare $\mathcal{F}^\varepsilon(u^\varepsilon)$ with the value of $\mathcal{G}^\varepsilon$ on a suitable extension of u^ε. As $\mathcal{F}^\varepsilon(u^\varepsilon) \leq +\infty$ we have in particular that the sequence $(\mathcal{E}u^\varepsilon)$ is equibounded in $L^2(\Omega^\varepsilon;\mathbb{R}^n)$, where Ω^ε is defined as in (3.2). Hence, by Korn inequality we deduce that u^ε is equibounded in $H^1(\Omega^\varepsilon;\mathbb{R}^n)$.

Let $\Omega_0 \supset \Omega$ with $\mathrm{dist}(\Omega,\partial\Omega_0) > 1$ and let us denote with $\tilde{u}^\varepsilon \in H^1_0(\Omega_0;\mathbb{R}^n)$ the extension of u^ε, whose existence is guaranteed by Theorem 3.2. It turns out that $\tilde{u}^\varepsilon$ converges to $\tilde{u}$ weakly in H^1, where $\tilde{u}$ is obtained extending u as zero in $\Omega_0\setminus\Omega$. Hence $\tilde{u} \in H^1(\Omega_0;\mathbb{R}^n)$. Since, by the properties of the extension, for small enough ε $u^\varepsilon \in H^1_0(\Omega';\mathbb{R}^n)$ for every $\overline{\Omega} \subset \Omega' \subset \Omega_0$, then $u \in H^1_0(\Omega;\mathbb{R}^n)$. Moreover, from (3.4) we have

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \vartheta_m\, \mathcal{G}^\varepsilon(\tilde{u}^\varepsilon), \tag{4.15}$$

from which we deduce the bound (4.12). □

Notice that the estimate (4.12) holds true independently of the rate at which α_ε converges to zero and implies that the $\Gamma - \liminf$ of $\mathcal{F}^\varepsilon$ is finite only in $H^1_0(\Omega;\mathbb{R}^n)$.

We can finally state our Γ-convergence result for a subsequence of $(\mathcal{F}^\varepsilon)$.

Theorem 4.8. *Let ε be a sequence converging to zero. Then there exists a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}^\sigma : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that, for every $A \in \mathcal{A}(\Omega)$,*

$$\mathcal{F}^\sigma(\cdot, A) = \Gamma - \lim_{\varepsilon\to 0} \mathcal{F}^{\sigma(\varepsilon)}(\cdot, A) \tag{4.16}$$

in the strong L^2-topology. Moreover, for every $u \in L^2(\Omega;\mathbb{R}^n)$, the set function $\mathcal{F}^\sigma(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω.

Proof. Since for every $\varepsilon > 0$ the functional $\mathcal{F}^\varepsilon$ is increasing, we deduce by Theorem 4.5 that there exists a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}^\sigma : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that $\mathcal{F}^\sigma = \overline{\Gamma}(L^2) - \lim_{\varepsilon\to 0}\mathcal{F}^{\sigma(\varepsilon)}$. We put a superscipt σ in order to underline that the limit functional may depend on the subsequence. Now define the nonnegative increasing functional $K : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ as

$$K(u,A) := \begin{cases} \displaystyle\int_A |\mathcal{E}u|^2 dx & \text{if } u_{|A} \in H^1(A;\mathbb{R}^n),\\ +\infty & \text{otherwise.}\end{cases} \tag{4.17}$$

Clearly, K is a measure with respect to A. Moreover, by (3.4) we have that $0 \leq \mathcal{F}^{\sigma(\varepsilon)} \leq \vartheta_M K$ for every $\varepsilon > 0$ and by Theorem 4.6 the fundamental estimate holds uniformly for the subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$. Therefore, we can proceed as in [11, Proposition 18.6] and we obtain that

$$\mathcal{F}^\sigma(u,A) = (\mathcal{F}^\sigma)'(u,A) = (\mathcal{F}^\sigma)''(u,A) \tag{4.18}$$

for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every $A \in \mathcal{A}(\Omega)$ such that $K(u,A) < +\infty$.

Fix $A \in \mathcal{A}(\Omega)$. We observe that from Theorem 4.7 we have the bound $\mathcal{F}'(\cdot,A) \geq \vartheta_m \mathcal{G}(\cdot,A)$, where we have localized the functional $\mathcal{G}$ defined in (4.10) as in (4.3). Notice that, by definition,

$$\mathcal{F}^\sigma(\cdot,A) = (\mathcal{F}^\sigma)'(\cdot,A) \geq \mathcal{F}'(\cdot,A). \tag{4.19}$$

Hence we deduce that $\mathcal{F}^\sigma(\cdot,A) \geq \vartheta_m \mathcal{G}(\cdot,A)$. This entails in particular that the Γ-limit of $\mathcal{F}^{\sigma(\varepsilon)}(\cdot,A)$ is finite only on $H^1(A;\mathbb{R}^n)$, which is the same domain where $K(\cdot,A)$ is finite, and is given by $\mathcal{F}^\sigma(\cdot,A)$. This proves the stated convergence of a subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$.

Finally, $\mathcal{F}^\varepsilon(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. Then, by Theorem 4.6 and [11, Theorem 18.5] we have that for every $u \in L^2(\Omega;\mathbb{R}^n)$ the set function $\mathcal{F}^\sigma(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. □

We now show general properties for the Γ-limit of $\mathcal{F}^\varepsilon$, even if, so far, we have only proved the convergence of a subsequence. The fact that the whole sequence $(\mathcal{F}^\varepsilon)$ converges will follow from the characterization of the Γ-limit, which will depend only on the symmetric gradient of the deformation and not on the subsequence $\sigma(\varepsilon)$. This will be done separately for the different regimes in Theorems 5.1, 6.2, 7.5, respectively.

In the remaining part of this section we therefore assume that the whole sequence $(\mathcal{F}^\varepsilon)$ converges to a functional that we call $\mathcal{F}$, and we omit the superscript σ.

Lemma 4.9. *The restriction of the functional* $\mathcal{F} : L^2(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0,+\infty]$ *to* $H^1_0(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega)$ *satisfies the following properties: for every* $u, v \in H^1_0(\Omega;\mathbb{R}^n)$ *and for every* $A \in \mathcal{A}(\Omega)$

(a) $\mathcal{F}$ *is local, i.e.,* $\mathcal{F}(u,A) = \mathcal{F}(v,A)$ *whenever* $u_{|A} = v_{|A}$*;*

(b) *the set function* $\mathcal{F}(u,\cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure on* Ω*;*

(c) $\mathcal{F}(\cdot,A)$ *is sequentially weakly lower semicontinuous on* $H^1_0(\Omega;\mathbb{R}^n)$*;*

(d) *for every* $a \in \mathbb{R}^n$ *we have* $\mathcal{F}(u,A) = \mathcal{F}(u+a,A)$*;*

(e) $\mathcal{F}$ *satisfies the bound*

$$0 \leq \mathcal{F}(u,A) \leq \vartheta_M \int_A |\mathcal{E}u|^2 dx. \tag{4.20}$$

Proof. Properties (a) and (c) follow from the fact that $\mathcal{F}(\cdot,A)$ is the Γ-limit of the sequence $\mathcal{F}^\varepsilon(\cdot,A)$, while (b) comes from Theorem 4.8. For property (d) we can proceed as follows. Let $u \in H^1_0(\Omega;\mathbb{R}^n)$, $A \in \mathcal{A}(\Omega)$ and consider a recovery sequence $(u^\varepsilon) \subset L^2(\Omega;\mathbb{R}^n) \cap SBD^2(A)$ satisfying the usual constraints for the jump set, converging to u strongly in $L^2(\Omega;\mathbb{R}^n)$ and such that $\big(\mathcal{F}^\varepsilon(u^\varepsilon,A)\big)$ converges to $\mathcal{F}(u,A)$. Then $(u^\varepsilon + a)$ converges to $u + a$ in $L^2(\Omega;\mathbb{R}^n)$ and

$$\mathcal{F}(u+a,A) \leq \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon + a, A) = \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, A) = \mathcal{F}(u,A). \tag{4.21}$$

On the other hand, $\mathcal{F}(u,A) = \mathcal{F}((u+a)+(-a),A) \leq \mathcal{F}(u+a,A)$, hence (d) is proved. For property (e), we just recall that the Γ-limit of the sequence $\big(\mathcal{F}^\varepsilon\big)$ is bounded from above by the functional $\vartheta_M \int_A |\mathcal{E}u|^2 dx$, by assumption (3.4). □

Next theorem shows that the functional $\mathcal{F}$ admits an integral representation.

Theorem 4.10. *There exists a unique quasi-convex function* $f : \mathbb{M}^{n\times n} \to [0,+\infty[$ *with the following properties:*

(i) $0 \leq f(\xi) \leq \vartheta_M |\xi|^2$ *for every* $\xi \in \mathbb{M}^{n\times n}$*;*

(ii) $\mathcal{F}(u,A) = \displaystyle\int_A f(\nabla u)dx$ *for every* $A \in \mathcal{A}(\Omega)$ *and for every* $u \in H^1(A;\mathbb{R}^n)$.

Proof. Notice that the functional $\mathcal{F}$ satisfies all the assumptions of [11, Theorem 20.1], so thanks to Lemma 4.9 the Carathéodory function $f : \Omega \times \mathbb{M}^{n\times n} \to \mathbb{R}$ defined as

$$f(y,\xi) := \limsup_{\varrho\to 0} \frac{\mathcal{F}(\xi\, x, B_\varrho(y))}{\mathcal{L}^n(B_\varrho(y))} \tag{4.22}$$

provides the integral representation

$$\mathcal{F}(u,A) = \int_A f(x,\nabla u)dx \tag{4.23}$$

for every $A \in \mathcal{A}(\Omega)$ and for every $u \in L^2(\Omega;\mathbb{R}^n)$ such that $u_{|A} \in H^1(A;\mathbb{R}^n)$. Moreover the same theorem ensures that for a.e. $x \in \Omega$ the function $f(x,\cdot)$ is quasi-convex on $\mathbb{M}^{n\times n}$ and that

$$0 \le f(x,\xi) \le \vartheta_M |\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for every } \xi \in \mathbb{M}^{n\times n}. \tag{4.24}$$

It remains to show that f is independent of the first variable and this can be done in the usual way (see for instance [17, Theorem 5.9]). □

In order to distinguish the different regimes, in the next sections we will use a different notation for the limit functional $\mathcal{F}$. It will be denoted by $\mathcal{F}^0$ in the subcritical case, by $\mathcal{F}^{hom}$ in the critical regime, and by $\mathcal{F}^\infty$ in the supercritical case.

5. Subcritical regime: very brittle inclusions

In this section we shall analyze the subcritical case, where the fragility coefficient of the brittle inclusions in the material is much smaller than the size ε of the periodic structure. The energy of the material is thus given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u\, dx + \alpha_\varepsilon\, \mathcal{H}^{n-1}(J_u) & \text{if } u \in SBD^2_0(\Omega), J_u \subset I^\varepsilon_\delta, \\ & [u]\cdot\nu_u \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n), \end{cases} \tag{5.1}$$

with $\frac{\alpha_\varepsilon}{\varepsilon} \to 0$. It is convenient to localize the sequence $(\mathcal{F}^\varepsilon)$ by defining, for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}^\varepsilon(u,A) := \begin{cases} \displaystyle\int_A \sigma(u) : \mathcal{E}u\, dx + \alpha_\varepsilon\, \mathcal{H}^{n-1}(J_u \cap A) & \text{if } u \in SBD^2(A),\ J_u \subset I^\varepsilon_\delta \cap A, \\ & [u]\cdot\nu_u \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_u \cap A, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{5.2}$$

5.1. **Cell formula.** We have already shown that the Γ-limit exists on a subsequence and it admits an integral representation. It remains to characterize the limit density. We shall prove that it is given by a cell problem.

Let $\xi \in \mathbb{M}^{n\times n}$; we will denote with ξ^s its symmetric part, that is,

$$\xi^s := \frac{\xi + \xi^T}{2} \in \mathbb{M}^{n\times n}_{sym}. \tag{5.3}$$

Define the function $f_0 : \mathbb{M}^{n\times n} \to [0,+\infty)$ as

$$f_0(\xi) := \inf\left\{ \int_Q \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx : w \in SBD^2_\#(Q), J_w \subset Q_\delta,\ [w]\cdot\nu_w \ge 0 \text{ a.e. on } J_w \right\}. \tag{5.4}$$

Next theorem shows that the Γ-limit of the sequence $\left(\mathcal{F}^\varepsilon\right)$ can be expressed in terms of the cell formula (5.4).

Theorem 5.1. *The density f of the limit functional $\mathcal{F}$ (see Theorem 4.10) coincides with the function f_0 defined by the cell formula (5.4), i.e., for every $\xi \in \mathbb{M}^{n\times n}$*

$$f(\xi) = f_0(\xi). \tag{5.5}$$

Proof. First step: $f \geq f_0$. Let $\xi \in \mathbb{M}^{n\times n}$ and define $u_\xi(x) := \xi\, x$ for every $x \in \mathbb{R}^n$. By definition of Γ-convergence, there exists a recovery sequence $u^\varepsilon \subset SBD^2(Q)$ with $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}, such that $u^\varepsilon \to u_\xi$ strongly in $L^2(Q;\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) = \mathcal{F}^0(u_\xi, Q) = f(\xi).$$

Let us write $u^\varepsilon =: u_\xi + v^\varepsilon$, where $v^\varepsilon \in SBD^2(Q)$, $J_{v^\varepsilon} \subset I^\varepsilon_\delta$, $[v^\varepsilon]\cdot\nu_{v^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{v^ε} and $v^\varepsilon \to 0$ strongly in $L^2(Q;\mathbb{R}^n)$. Without loss of generality we can assume $v^\varepsilon \in SBD^2_0(Q)$. Hence

$$f(\xi) = \lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u_\xi + v^\varepsilon, Q) = \lim_{\varepsilon\to 0} \left\{ \int_Q \sigma(\xi\, x + v^\varepsilon) : (\xi^s + \mathcal{E}v^\varepsilon)dx + \alpha_\varepsilon \mathcal{H}^{n-1}(J_{v^\varepsilon}) \right\}. \tag{5.6}$$

Now, let us define the function $w^\varepsilon \in SBD^2_0(Q/\varepsilon)$ as

$$v^\varepsilon(x) =: \varepsilon\, w^\varepsilon\Big(\frac{x}{\varepsilon}\Big).$$

Remark that $J_{w^\varepsilon} \subset I_\delta$, where I_δ is defined as

$$I_\delta := \Big(0, \frac{1}{\varepsilon}\Big)^n \cap \bigcup_{h\in\mathbb{Z}^n} (Q_\delta + h). \tag{5.7}$$

Then, rewriting (5.6) in terms of w^ε we obtain

$$\begin{aligned} f(\xi) &= \lim_{\varepsilon\to 0} \varepsilon^n \Big\{ \int_{Q/\varepsilon} \sigma(\xi\, x + w^\varepsilon) : (\xi^s + \mathcal{E}w^\varepsilon)dx + \tfrac{\alpha_\varepsilon}{\varepsilon}\, \mathcal{H}^{n-1}(J_{w^\varepsilon}) \Big\} \\ &\geq \lim_{\varepsilon\to 0} \varepsilon^n \inf \Big\{ \int_{(0,\frac{1}{\varepsilon})^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx : w \in SBD^2_0\big((0,1/\varepsilon)^n\big),\ J_w \subset I_\delta \\ &\qquad\qquad [w]\cdot\nu_w \geq 0\ \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\} \\ &= f_0(\xi), \end{aligned}$$

where the last equality follows by convexity (see [7, Theorem 14.7]). Indeed, the non-interpenetration condition is preserved under convex combinations.

Second step: $f \leq f_0$. Let $\xi \in \mathbb{M}^{n\times n}$ and $l \in \mathbb{N}$; consider a function $w \in SBD^2_0((0,l)^n)$, with $J_w \subset I_\delta$ and $[w]\cdot\nu_w \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_w, such that

$$\begin{aligned} &\int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx \\ &\leq \inf \Big\{ \int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E}v)dx : v \in SBD^2_0((0,l)^n), J_v \subset I_\delta, [v]\cdot\nu_v \geq 0 \text{ a.e. on } J_v \Big\} + 1. \end{aligned} \tag{5.8}$$

Let us define the sequence $u^\varepsilon : Q \to \mathbb{R}^n$ as

$$u^\varepsilon(x) := \xi\, x + \varepsilon\, \widetilde{w}\Big(\frac{x}{\varepsilon}\Big),$$

where $\widetilde{w}$ denotes the function defined in the whole $\mathbb{R}^n$, obtained through a periodic extension of w. We have that $\mathcal{F}^\varepsilon(u^\varepsilon, Q) < +\infty$, being $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}. Moreover u^ε converges to $\xi\, x$ strongly in $L^2(Q;\mathbb{R}^n)$. We can write

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \int_Q \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon\, dx + \alpha_\varepsilon \mathcal{H}^{n-1}(J_{u^\varepsilon}) \tag{5.9}$$

$$= \varepsilon^n \Big\{ \int_{Q/\varepsilon} \sigma(\xi\, x + \widetilde{w}) : (\xi^s + \mathcal{E}\widetilde{w})dx + \tfrac{\alpha_\varepsilon}{\varepsilon}\, \mathcal{H}^{n-1}(J_{\widetilde{w}}) \Big\}. \tag{5.10}$$

Now, in order to use the periodicity of $\widetilde{w}$, we can write the domain Q/ε as union of (suitably translated) periodicity cells $(0,l)^n$. Assume for simplicity that Q/ε is covered exactly by an integer number of these cells, that is by $1/(l\,\varepsilon)^n$ cells. Indeed, in the general case the integral over the remaining part of Q/ε is negligible. Then (5.9) reads as

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \frac{1}{l^n}\Big\{\int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx + \tfrac{\alpha_\varepsilon}{\varepsilon}\,\mathcal{H}^{n-1}(J_w)\Big\}.$$

Passing to the lim sup as $\varepsilon \to 0$ and using the fact that we are in the subcritical regime, (5.1) gives

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) = \frac{1}{l^n}\int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx. \tag{5.11}$$

Then, using (5.8) and (5.11) we get

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \le \frac{1}{l^n}\inf\Big\{\int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E}v)dx : v \in SBD^2_0((0,l)^n),$$
$$J_w \subset I_\delta, [v]\cdot\nu_v \ge 0 \;\; \mathcal{H}^{n-1}\text{-a.e. on } J_v\Big\} + \frac{1}{l^n}.$$

Letting $l \to +\infty$ in the previous expression and using again convexity, we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \le f_0(\xi),$$

hence the claim is proved. □

Remark 5.2. The previous theorem implies in particular that in the subcritical regime the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence.

Moreover, from the cell formula we deduce that $f(\xi) = f(\xi^s)$, that is, the limit density function depends only on the symmetric part of its argument.

When the elasticity tensor $\mathbb{C}$ is isotropic, we can give a more explicit description of the density f_0, as shown in the following lemma.

Lemma 5.3. *Let $\mathbb{C}$ be of the special form $\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\, Id \otimes Id$, $\mu, \lambda > 0$, and let f_0 be the corresponding limit density defined as in (5.4). Then it turns out that $f_0(Id) \neq f_0(-Id)$.*

Proof. By the assumption on $\mathbb{C}$ we have that, for every $w \in SBD^2(Q)$

$$\sigma(w) = 2\,\mu\,\mathcal{E}w + \lambda(\mathcal{E}w : Id)\, Id = 2\,\mu\,\mathcal{E}w + \lambda(\operatorname{div} w)\, Id \in \mathbb{M}^{n\times n}_{sym}. \tag{5.12}$$

First step: $f_0(Id) \lneqq 2\,\mu\, n + \lambda\, n^2$.

First of all, we can notice that f_0 can be rewritten as

$$f_0(\xi) := \inf\Big\{\int_Q \sigma(w) : \mathcal{E}w\, dx : w - \xi\, x \in SBD^2_\#(Q), J_w \subset Q_\delta,\, [w]\cdot\nu_w \ge 0 \;\; \mathcal{H}^{n-1}\text{-a.e. on } J_w\Big\}, \tag{5.13}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$.

For $i = 1, \dots, n$, let us denote with $\{\partial Q^i_{+\delta}, \partial Q^i_{-\delta}\}$ the opposite *hyperfaces* of ∂Q_δ which are orthogonal to the vector e_i. More precisely,

$$\partial Q^i_{\pm\delta} := \big\{x \in \partial Q_\delta : x\cdot e_i \gtrless 0\big\}.$$

Let $\xi \in \mathbb{M}^{n\times n}_{sym}$ and assume that there exists a constant $c_\xi = (c_1, \dots, c_n) \in \mathbb{R}^n$ with the property

$$\max_{x\in\partial Q^i_{-\delta}} \big((\xi\, x)\cdot e_i\big) < c_i < \min_{x\in\partial Q^i_{+\delta}} \big((\xi\, x)\cdot e_i\big) \quad \text{for every } i = 1, \dots, n. \tag{5.14}$$

Then, it turns out that the function w_ξ defined as

$$w_\xi(x) = \begin{cases} \xi\, x & \text{if } x \in Q\setminus Q_\delta, \\ c_\xi & \text{if } x \in Q_\delta, \end{cases}$$

is a competitor in (5.13). Indeed, $w_\xi - \xi\, x \in SBD^2_0(Q) \subset SBD^2_\#(Q)$ and $J_{w_\xi} \subset \partial Q_\delta$. It remains to check the non-interpenetration condition for every $x \in J_{w_\xi}$. Notice that if $\hat{x} \in \partial Q^i_{+\delta}$ for some i, then

$$[w_\xi](\hat{x}) \cdot \nu_{w_\xi}(\hat{x}) = (\xi\,\hat{x} - c_\xi) \cdot e_i \geq \min_{x \in \partial Q^i_{+\delta}} \big((\xi\, x) \cdot e_i\big) - c_i > 0,$$

by (5.14). On the other hand, if $\hat{x} \in \partial Q^i_{-\delta}$ for some i, then

$$[w_\xi](\hat{x}) \cdot \nu_{w_\xi}(\hat{x}) = (\xi\,\hat{x} - c_\xi) \cdot (-e_i) \geq c_i - \max_{x \in \partial Q^i_{-\delta}} \big((\xi\, x) \cdot e_i\big) > 0,$$

again by (5.14). Since w_ξ is a competitor in (5.13), we obtain by comparison that

$$f_0(\xi) \leq \int_Q \sigma(w_\xi) : \mathcal{E} w_\xi\, dx = \mathcal{L}^n(Q \setminus Q_\delta)\big(2\,\mu\,|\xi|^2 + \lambda(\mathrm{tr}\xi)^2\big) \lneqq \big(2\,\mu\,|\xi|^2 + \lambda(\mathrm{tr}\xi)^2\big). \tag{5.15}$$

In particular, since for $\xi = Id$ the property (5.14) is clearly satisfied (it is enough to take $c_i = 0$ for every i), we have by (5.15) that

$$f_0(Id) \lneqq 2\,\mu\, n + \lambda\, n^2.$$

Second step: $f_0(-Id) = 2\,\mu\, n + \lambda\, n^2$.

In order to prove this relation it is more convenient to use the characterization of the density f_0 in the form (5.4). Let us fix $\xi \in \mathbb{M}^{n\times n}_{sym}$. Since $\sigma(\xi\, x) = \mathbb{C}\xi \in \mathbb{M}^{n\times n}_{sym}$, we can assume without loss of generality that $\sigma(\xi\, x)$ is a diagonal matrix. Let us denote with $(\lambda_1, \dots, \lambda_n)$ its eigenvalues. We will derive a necessary and sufficient condition to have $w = 0$ as a minimizer of (5.4).

Let $v \in SBD^2_\#(Q)$ such that $J_v \subset Q_\delta$ and $[v] \cdot \nu_v \geq 0$ $\mathcal{H}^{n-1}$- a.e. on J_v, and let $\eta \geq 0$. We define

$$I(\eta) := \frac{1}{2} \int_Q \sigma(\xi\, x + \eta\, v) : (\xi + \eta\, \mathcal{E} v) dx$$

and we impose that

$$\left(\frac{d}{d\eta} I(\eta)\right)_{|\eta=0} = \frac{1}{2}\left(\frac{d}{d\eta} \int_Q \sigma(\xi\, x + \eta\, v) : (\xi + \eta\, \mathcal{E} v) dx\right)_{|\eta=0} \geq 0 \tag{5.16}$$

for every admissible v.

Since the functional in (5.4) is convex, we have indeed that (5.16) is a necessary and sufficient condition for the minimality of $w = 0$. We notice that condition (5.16) is equivalent to

$$\int_Q \sigma(\xi\, x) : \mathcal{E} v\, dx \geq 0 \tag{5.17}$$

for every admissible v. Integrating by parts and using the fact that $(\sigma(\xi\, x))_{ij} = \lambda_i \delta_{ij}$, the left hand side in the previous expression becomes

$$\int_Q \sigma(\xi\, x) : \mathcal{E} v\, dx = -\sum_{i,j=1}^n \int_{J_v} (\sigma(\xi\, x))_{ij}\, [v_j]\, \nu_{v_i} d\mathcal{H}^{n-1} = -\sum_{i=1}^n \int_{J_v} \lambda_i\, [v_i]\, \nu_{v_i} d\mathcal{H}^{n-1}.$$

Therefore, (5.17) reduces to

$$-\sum_{i=1}^n \int_{J_v} \lambda_i\, [v_i]\, \nu_{v_i} d\mathcal{H}^{n-1} \geq 0 \tag{5.18}$$

for every admissible v. As v satisfies the non-interpenetration condition

$$\sum_{i=1}^n [v_i](x)\, \nu_{v_i}(x) \geq 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_v, \tag{5.19}$$

and is arbitrary, we conclude that the eigenvalues λ_i of $\sigma(\xi\, x)$ are forced to be equal and negative, that is $\lambda_i = -\nu$ for every $i = 1, \dots, n$ and $\nu > 0$. In practice this implies that

$$\frac{1}{2}\left(\frac{d}{d\eta} \int_Q \sigma(\xi\, x + \eta\, v) : (\xi + \eta\, \mathcal{E} v) dx\right)_{|\eta=0} \geq 0 \text{ for every admissible } v \quad \Longleftrightarrow \quad \sigma(\xi\, x) = -\nu\, Id, \tag{5.20}$$

therefore $w = 0$ is minimal in (5.4) if and only if $\sigma(\xi\, x) = -\nu\, Id$, with $\nu > 0$. By (5.12) this condition is fulfilled if and only if

$$2\,\mu\,\xi + \lambda(\mathrm{tr}\,\xi)\,Id = -\nu\,Id, \tag{5.21}$$

i.e., when ξ is a negative multiple of the identity. As $\xi = -\,Id$ clearly satisfies (5.21), we have

$$f_0(-Id) = \int_Q (2\,\mu\,|Id|^2 + \lambda\,(\mathrm{tr}Id)^2)\,dx = 2\,\mu\,n + \lambda\,n^2.$$

□

Remark 5.4. As immediate corollary from the previous lemma we can deduce that, in general, the limit density f_0 is not a quadratic form.

Remark 5.5. Another important consequence of Lemma 5.3 is that the limit functional $\mathcal{F}_0$ describes a material where damage occurs. Indeed it is easy to verify that there exists $\xi \in \mathbb{M}^{n\times n}$ such that $f_0(\xi) \lneqq \mathbb{C}\xi : \xi$. We still restrict our attention to the isotropic case.

Let $\xi \in \mathbb{M}^{n\times n}$ be matrix satisfying the property (5.14) and let w_ξ be the function defined in Lemma 5.3. Since w_ξ is a competitor in (5.13), we obtain by comparison that

$$f_0(\xi) \le \int_Q \sigma(w_\xi) : \mathcal{E}w_\xi\,dx = \mathcal{L}^n(Q\setminus Q_\delta)\mathbb{C}\xi : \xi \lneqq \mathbb{C}\xi : \xi.$$

We notice that the property (5.14) is satisfied by a large class of matrices. In order to prove this, let us restrict to symmetric matrices (with no loss of generality) and let us write them in terms of their eigenvalues $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, so that

$$\mathbb{M}^{n\times n}_{sym} \ni \xi = \sum_{i=1}^n \lambda_i e_i \otimes e_i.$$

It is immediate to verify that if $\lambda_i > 0$ for every i then ξ satisfies (5.14). In particular $\xi = \kappa Id$, with $\kappa > 0$, is a possible choice.

6. Critical regime: intermediate case

In this section we shall analyze the critical case where the fragility coefficient of the inclusions in the material α_ε is of the same order of the size ε of the periodic structure. We can assume, without loss of generality, that $\alpha_\varepsilon = \varepsilon$. The energy of the material is thus given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u\,dx + \varepsilon\,\mathcal{H}^{n-1}(J_u) & \text{if } u \in SBD^2_0(\Omega), J_u \subset I^\varepsilon_\delta, \\ & [u]\cdot\nu_u \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{6.1}$$

We localize the sequence $(\mathcal{F}^\varepsilon)$ by defining, for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}^\varepsilon(u, A) := \begin{cases} \displaystyle\int_A \sigma(u) : \mathcal{E}u\,dx + \varepsilon\,\mathcal{H}^{n-1}(J_u \cap A) & \text{if } u \in SBD^2(A),\ J_u \subset I^\varepsilon_\delta \cap A, \\ & [u]\cdot\nu_u \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_u \cap A, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{6.2}$$

6.1. **Homogenization formula.** We have already shown in Theorem 4.10 that the Γ-limit exists on a subsequence and it admits an integral representation. It remains to characterize the limit density. We shall prove that it is given by an asymptotic cell problem.

Define the function $f_{hom} : \mathbb{M}^{n\times n} \to [0, +\infty)$ as

$$\begin{aligned} f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \sigma(\xi\,x + w) : (\xi^s + \mathcal{E}w)dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,t)^n\big), \\ J_w \subset I_\delta, [w]\cdot\nu_w \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\}, \end{aligned} \tag{6.3}$$

where, according to the notation used so far, we have set

$$I_\delta := (0,t)^n \cap \bigcup_{h\in\mathbb{Z}^n} (Q_\delta + h),$$

Theorem 6.1. *The function f_{hom} in (6.3) is well defined, that is the function*

$$g(t) := \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,t)^n\big), \qquad J_w \subset I_\delta, [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\} \tag{6.4}$$

admits a limit as $t \to +\infty$.

Proof. For the proof we refer to [17]. □

Next theorem shows that the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ can be expressed in terms of the homogenization formula (6.3).

Theorem 6.2. *The density f of the limit functional $\mathcal{F}$ (see Theorem 4.10) coincides with the function f_{hom} defined by the cell formula (6.3), i.e., for every $\xi \in \mathbb{M}^{n\times n}$*

$$f(\xi) = f_{hom}(\xi). \tag{6.5}$$

Proof. First step: $f \geq f_{hom}$. Let $\xi \in \mathbb{M}^{n\times n}$ and define $u_\xi(x) := \xi\, x$ for every $x \in \mathbb{R}^n$. By definition of Γ-convergence, there exists a recovery sequence $u^\varepsilon \subset SBD^2(Q)$ with $J_{u^\varepsilon} \subset I^\varepsilon_\delta$ and $[u^\varepsilon]\cdot\nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}, such that $u^\varepsilon \to u_\xi$ strongly in $L^2(Q;\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) = \mathcal{F}^0(u_\xi, Q) = f(\xi).$$

Let us write $u^\varepsilon =: u_\xi + v^\varepsilon$, where $v^\varepsilon \in SBD^2(Q)$, $J_{v^\varepsilon} \subset I^\varepsilon_\delta$, $[v^\varepsilon]\cdot\nu_{v^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{v^ε} and $v^\varepsilon \to 0$ strongly in $L^2(Q;\mathbb{R}^n)$. Without loss of generality we can assume $v^\varepsilon \in SBD^2_0(Q)$. Hence

$$f(\xi) = \lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u_\xi + v^\varepsilon, Q) = \lim_{\varepsilon\to 0}\Big\{ \int_Q \sigma(\xi\, x + v^\varepsilon) : (\xi^s + \mathcal{E}v^\varepsilon)dx + \varepsilon\,\mathcal{H}^{n-1}(J_{v^\varepsilon})\Big\}. \tag{6.6}$$

Now, let us define the function $w^\varepsilon \in SBD^2_0(Q/\varepsilon)$ as

$$v^\varepsilon(x) =: \varepsilon\, w^\varepsilon\Big(\frac{x}{\varepsilon}\Big).$$

Remark that $J_{w^\varepsilon} \subset I_\delta$. Then, rewriting (6.6) in terms of w^ε we obtain

$$\begin{aligned} f(\xi) &= \lim_{\varepsilon\to 0} \varepsilon^n \Big\{ \int_{Q/\varepsilon} \sigma(\xi\, x + w^\varepsilon) : (\xi^s + \mathcal{E}w^\varepsilon)dx + \mathcal{H}^{n-1}(J_{w^\varepsilon})\Big\} \\ &\geq \lim_{\varepsilon\to 0} \varepsilon^n \inf\Big\{ \int_{(0,\frac{1}{\varepsilon})^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx + \mathcal{H}^{n-1}(J_w) : w \in SBD^2_0\big((0,1/\varepsilon)^n\big), \\ &\qquad J_w \subset I_\delta, [w]\cdot\nu_w \geq 0 \ \ \mathcal{H}^{n-1}\text{-a.e. on } J_w\Big\} \\ &= f_{hom}(\xi). \end{aligned}$$

Second step: $f \leq f_{hom}$. Let $\xi \in \mathbb{M}^{n\times n}$ and $l \in \mathbb{N}$; then, consider a function $w \in SBD^2_0((0,l)^n)$, with $J_w \subset I_\delta$ and $[w]\cdot\nu_w \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_w, such that

$$\int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w)dx + \mathcal{H}^{n-1}(J_w) \leq \inf\Big\{ \int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E}v)dx + \mathcal{H}^{n-1}(J_v) \\ : v \in SBD^2_0((0,l)^n), J_v \subset I_\delta, [v]\cdot\nu_v \geq 0 \text{ a.e. on } J_v\Big\} + 1. \tag{6.7}$$

Let us define the sequence $u^\varepsilon : Q \to \mathbb{R}^n$ as

$$u^\varepsilon(x) := \xi\, x + \varepsilon\, \widetilde{w}\Big(\frac{x}{\varepsilon}\Big),$$

where $\widetilde{w}$ denotes the function defined in the whole $\mathbb{R}^n$, obtained through a periodic extension of w. We have that $\mathcal{F}^\varepsilon(u^\varepsilon, Q) < +\infty$, being $J_{u^\varepsilon} \subset I_\delta^\varepsilon$ and $[u^\varepsilon] \cdot \nu_{u^\varepsilon} \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_{u^ε}. Moreover u^ε converges to $\xi\, x$ strongly in $L^2(Q; \mathbb{R}^n)$. We can write

$$\mathcal{F}^\varepsilon(u^\varepsilon, Q) = \int_Q \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon dx + \varepsilon\, \mathcal{H}^{n-1}(J_{u^\varepsilon}) = \varepsilon^n \left\{ \int_{Q/\varepsilon} \sigma(\xi\, x + \widetilde{w}) : (\xi^s + \mathcal{E}\widetilde{w}) dx + \mathcal{H}^{n-1}(J_{\widetilde{w}}) \right\}. \tag{6.8}$$

Now, in order to use the periodicity of $\widetilde{w}$, we can write the domain Q/ε as union of (suitably translated) periodicity cells $(0,l)^n$. Assume for simplicity that Q/ε is covered exactly by an integer number of these cells, that is by $1/(l\,\varepsilon)^n$ cells. Indeed, in the general case the integral over the remaining part of Q/ε is a negligible term. Then, using (6.7), we get from (6.8)

$$\begin{aligned}\mathcal{F}^\varepsilon(u^\varepsilon, Q) &= \frac{1}{l^n} \left\{ \int_{(0,l)^n} \sigma(\xi\, x + w) : (\xi^s + \mathcal{E}w) dx + \mathcal{H}^{n-1}(J_w) \right\} \\ &\leq \frac{1}{l^n} \inf \left\{ \int_{(0,l)^n} \sigma(\xi\, x + v) : (\xi^s + \mathcal{E}v) dx + \mathcal{H}^{n-1}(J_v) : v \in SBD_0^2((0,l)^n), \right. \\ &\qquad\qquad \left. J_w \subset I_\delta, [v] \cdot \nu_v \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_v \right\} + \frac{1}{l^n}.\end{aligned}$$

Passing to the lim sup as $\varepsilon \to 0$ and then letting $l \to +\infty$ we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon, Q) \leq f_{hom}(\xi),$$

hence the claim is proved. □

Remark 6.3. Notice that from this theorem we deduce that also in the critical case the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence. Moreover, we deduce that $f_{hom}(\xi) = f_{hom}(\xi^s)$, that is the limit density function depends only on the symmetric part of its argument.

Next lemma shows that the limit functional in the critical regime describes a damaged material. We restrict our attention to the isotropic case, i.e., $\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\, Id \otimes Id$ with $\mu, \lambda > 0$.

Lemma 6.4. *There exists $\xi \in \mathbb{M}^{n\times n}$ such that $f_{hom}(\xi) \lneqq \mathbb{C}\xi : \xi$.*

Proof. Let us rewrite the limit energy density in the following way:

$$\begin{aligned} f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf \left\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}w : \mathcal{E}w dx + \mathcal{H}^{n-1}(J_w) : w - \xi x \in SBD^2_\#\big((0,t)^n\big), \right. \\ \left. J_w \subset I_\delta, [w] \cdot \nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \right\}, \end{aligned} \tag{6.9}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$. Let $\xi \in \mathbb{M}^{n\times n}_{sym}$ and assume that there exists a constant $c_\xi = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with the property

$$\max_{x\in\partial Q^i_{-\delta}} \big((\xi\, x) \cdot e_i\big) < c_i < \min_{x\in\partial Q^i_{+\delta}} \big((\xi\, x) \cdot e_i\big) \quad \text{for every } i = 1, \ldots, n, \tag{6.10}$$

as in Lemma 5.3. Let us restrict our attention to the case when in (6.9) $t \in \mathbb{N}$. The general case can be deduced in the same way.

Then, it turns out that the function w_ξ defined as

$$w_\xi(x) = \begin{cases} \xi\, x & \text{if } x \in Q \setminus Q_\delta, \\ c_\xi & \text{if } x \in Q_\delta, \end{cases}$$

and extended by periodicity in $(0,t)^n$ is a competitor in (6.9). Indeed, $w_\xi - \xi\, x \in SBD_0^2((0,t)^n) \subset SBD^2_\#((0,t)^n)$ and $J_{w_\xi} \subset \partial I_\delta$. Moreover, following the proof of Lemma 5.3 one can prove that also the non-interpenetration constraint is satisfied. Therefore, for the class of matrices ξ defined by the condition (6.10) we have

$$f_{hom}(\xi) \leq \lim_{t\to+\infty} \frac{1}{t^n} \left\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}w_\xi : \mathcal{E}w_\xi dx + \mathcal{H}^{n-1}(J_{w_\xi}) \right\} \leq \mathcal{L}^n(Q_\delta)\mathbb{C}\xi : \xi + P(Q_\delta, Q). \tag{6.11}$$

Then, in order to prove the theorem it's sufficient to choose a matrix $\xi \in \mathbb{M}^{n\times n}$ satisfying the property (5.14) and such that

$$\mathcal{L}^n(Q_\delta)\mathbb{C}\xi : \xi + P(Q_\delta, Q) \gneqq \mathbb{C}\xi : \xi.$$

In particular $\xi = \kappa Id$ with $\kappa > 0$ and $\kappa >> 1$ provides a possible choice. □

7. Supercritical regime: stiffer inclusions

In this section we shall analyze the supercritical case, where the fragility coefficient α_ε of the inclusions in the material is bigger than the size ε of the periodic structure.

Before studying this case, we state a technical lemma which will be used in the following. For the proof we refer to [17].

Lemma 7.1. *Let $a_k : \Omega \to \mathbb{R}_+$ be a sequence of measurable functions such that*

$$a_k \to a \quad \textit{in measure.} \tag{7.1}$$

Then, for every $v \in L^2(\Omega; \mathbb{R}^m)$ and for every sequence $(v_k) \subset L^2(\Omega; \mathbb{R}^m)$ such that

$$v_k \rightharpoonup v \quad \textit{weakly in } L^2(\Omega; \mathbb{R}^m),$$

it turns out that

$$\int_\Omega a|v|^2 dx \le \liminf_{k\to+\infty} \int_\Omega a_k |v_k|^2 dx. \tag{7.2}$$

In the sequel we present a proper modification of the argument used in [2] and in [6] to prove compactness and lower semicontinuity in SBD.

Lemma 7.2. *Let $w \in L^2(U; \mathbb{R}^n)$ and let (w_h) be a sequence converging strongly to w in L^2, where $U \subset \mathbb{R}^n$ is an open set. Assume that $||\mathcal{E}w_h||_{L^2(U)} \le c$ and that $\mathcal{H}^{n-1}(J_{w_h}) \to 0$ as $h \to 0$. Then $w \in H^1(U; \mathbb{R}^n)$ and*

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \textit{weakly in } L^2(U; \mathbb{M}^{n\times n}).$$

Proof. First of all, up to subsequences, we can assume that

$$\mathcal{H}^{n-1}(J_{w_h}) \le \frac{1}{h^2}.$$

First step: $w \in H^1(U; \mathbb{R}^n)$.
Let $\xi \in S^{n-1}$, $y \in \Pi^\xi$ and let us define for every $h \in \mathbb{N}$ the section $(w_h)^\xi_y(t) := w_h(y + t\,\xi) \cdot \xi$. It is well known that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the section $(w_h)^\xi_y \in SBV^2(U^\xi_y)$. Moreover, from the fact that $w_h \to w$ strongly in L^2, it follows that, up to subsequences,

$$(w_h)^\xi_y \to w^\xi_y \quad \text{strongly in } L^2(U^\xi_y) \quad \text{for } \mathcal{H}^{n-1} - \text{a.e. } y \in \Pi^\xi.$$

Let us denote with N_1 the set such that for every $y \in \Pi^\xi \setminus N_1$ we have $(w_h)^\xi_y \in SBV^2(U^\xi_y)$ and $(w_h)^\xi_y \to w^\xi_y$ strongly in L^2. As we have already noticed, $\mathcal{H}^{n-1}(N_1) = 0$.
Let us define the set E_h as

$$E_h := \bigcup_{j\ge h} J_{w_j}.$$

From the inequality $\mathcal{H}^{n-1}(J_{w_h}) \le \frac{1}{h^2}$, it turns out that $\mathcal{H}^{n-1}(E_h) \to 0$ as $h \to +\infty$. Hence for every $\vartheta > 0$ there exists $h(\vartheta)$ such that $\mathcal{H}^{n-1}(E_{h(\vartheta)}) < \vartheta$. Clearly, $J_{w_h} \subset E_{h(\vartheta)}$ for every $h \ge h(\vartheta)$.

Let us denote with $(E_{h(\vartheta)})^\xi$ the projection of the set $E_{h(\vartheta)}$ on Π^ξ. By definition, for every $y \in (\Pi^\xi \setminus (E_{h(\vartheta)})^\xi) \setminus N_1$ and for $h \ge h(\vartheta)$, the section $(w_h)^\xi_y \in H^1(U^\xi_y)$. Moreover, the H^1 norm of $(w_h)^\xi_y$ is equibounded.

Indeed, using Fubini we can write

$$\int_U |\mathcal{E}w_h\xi \cdot \xi|^2 dx = \int_U |\nabla w_h \xi \cdot \xi|^2 dx = \int_{\Pi^\xi} \left[\int_{U^\xi_y} |\nabla (w_h)^\xi_y|^2 \, dt \right] d\mathcal{H}^{n-1}(y). \tag{7.3}$$

From the fact that $\xi \in S^{n-1}$, we have

$$\int_U |\mathcal{E}w_h\xi \cdot \xi|^2 dx \le \int_U |\mathcal{E}w_h|^2 dx, \tag{7.4}$$

and the right-hand side of (7.4) is equibounded by assumption. Hence from (7.3) we obtain

$$\int_{\Pi^\xi}\left[\int_{U_y^\xi}|\nabla(w_h)_y^\xi|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c. \tag{7.5}$$

Now, let $w_{k(y)}$ be a subsequence (depending on y) of w_h such that

$$\liminf_{h\to+\infty}\int_{U_y^\xi}|\nabla(w_h)_y^\xi|^2\,dt=\lim_{k(y)\to+\infty}\int_{U_y^\xi}|\nabla(w_{k(y)})_y^\xi|^2\,dt. \tag{7.6}$$

The bound (7.5) guarantees that there exists a function v such that, up to extracting a further subsequence $w_{j(y)}\subset w_{k(y)}$, we have

$$(w_{j(y)})_y^\xi\rightharpoonup v\quad\text{weakly in } H^1\big(U_y^\xi\big), \tag{7.7}$$

for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_{h(\vartheta)})^\xi$. Since for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ the whole sequence $(w_h)_y^\xi$ converges to w_y^ξ strongly in L^2, (7.7) implies that

$$(w_{j(y)})_y^\xi\rightharpoonup w_y^\xi\quad\text{weakly in } H^1\big(U_y^\xi\big). \tag{7.8}$$

By the lower semicontinuity in H^1 and (7.6) we obtain the inequality

$$\int_{U_y^\xi}|\nabla(w_y^\xi)|^2\,dt\le\liminf_{j(y)\to+\infty}\int_{U_y^\xi}|\nabla(w_{j(y)})_y^\xi|^2\,dt=\liminf_{h\to+\infty}\int_{U_y^\xi}|\nabla(w_h)_y^\xi|^2\,dt, \tag{7.9}$$

which holds true for $\mathcal{H}^{n-1}$-a.e. $y\in(\Pi^\xi\setminus(E_{h(\vartheta)})^\xi)$.

Integrating (7.9) with respect to y and using Fatou Lemma we get

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{U_y^\xi}|\nabla(w_y^\xi)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le\liminf_{h\to+\infty}\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{U_y^\xi}|\nabla(w_h)_y^\xi|^2\,dt\right]d\mathcal{H}^{n-1}(y). \tag{7.10}$$

Hence, by (7.5) we obtain

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{U_y^\xi}|\nabla(w_y^\xi)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c, \tag{7.11}$$

where the constant c is independent of ϑ.

The estimate (7.11), together with the fact that $w\in L^2(U;\mathbb{R}^n)$ and that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_{h(\vartheta)})^\xi$ the section $w_y^\xi\in H^1\big(U_y^\xi\big)$, allow us to conclude that $w\in H^1(U;\mathbb{R}^n)$.

Indeed, let us define the sets E_∞ and E_0 as

$$E_\infty:=\cap_h E_h\quad\text{and}\quad E_0:=\lim_h E_h,$$

where the convergence in the definition of E_0 is intended to be almost everywhere with respect to the Hausdorff measure.

From $\mathcal{H}^{n-1}(E_h)\le\frac{1}{h^2}$ and $E_{h+1}\subset E_h$, it turns out that

$$\mathcal{H}^{n-1}(E_\infty)=0=\mathcal{H}^{n-1}(E_0).$$

Now, since $\Pi^\xi\setminus(E_\infty)^\xi$ is contained in $\Pi^\xi\setminus(E_h)^\xi$ for h large enough, we have that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi\setminus(E_\infty)^\xi$ the section $w_y^\xi\in H^1\big(U_y^\xi\big)$. Hence, being $\mathcal{H}^{n-1}(E_\infty)=0$, we conclude that $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ the section $w_y^\xi\in H^1\big(U_y^\xi\big)$. On the other hand, using the monotone convergence in (7.11), we have

$$\lim_{h(\vartheta)\to\infty}\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi}\left[\int_{U_y^\xi}|\nabla(w_y^\xi)|^2\,dt\right]d\mathcal{H}^{n-1}(y)=\int_{\Pi^\xi\setminus(E_0)^\xi}\left[\int_{U_y^\xi}|\nabla(w_y^\xi)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c. \tag{7.12}$$

Again, the fact that $\mathcal{H}^{n-1}(E_0)=0$ implies that

$$\int_{\Pi^\xi}\left[\int_{U_y^\xi}|\nabla(w_y^\xi)|^2\,dt\right]d\mathcal{H}^{n-1}(y)\le c. \tag{7.13}$$

At this point we can apply [3, Proposition 3.105] to conclude that

$$\nabla(w_y^\xi)=D_t[w(y+t\xi)\cdot\xi]\in L^2(U),$$

that is, $Dw\xi \cdot \xi = Ew\xi \cdot \xi \in L^2(U)$, and this is true for every ξ. Using the identity

$$Ew\xi \cdot \eta = \frac{1}{2}[Ew(\xi+\eta)\cdot(\xi+\eta) - Ew\xi\cdot\xi - Ew\eta\cdot\eta] \quad \forall\, \xi, \eta,$$

we conclude that $Ew \in L^2(U;\mathbb{M}^{n\times n})$. Therefore, being $w \in L^2(U;\mathbb{R}^n)$, Korn inequality ensures that $w \in H^1(U;\mathbb{R}^n)$.

Second step: convergence of the symmetric gradient. Let us define, for a given scalar function $v \in L^2(U)$, the functional

$$L^\xi_y(w_h, v) := \int_{U^\xi_y} |\nabla(w_h)^\xi_y - v(t,y)|^2\, dt.$$

Using (7.4) and the fact that $v \in L^2(U)$, we obtain the bound

$$\int_{\Pi^\xi} L^\xi_y(w_h, v) d\mathcal{H}^{n-1}(y) \le \int_U |\mathcal{E}w_h\xi\cdot\xi - v|^2 dx \le c.$$

Now, let $w_{k(y)}$ be a subsequence (depending on y) of w_h such that

$$\liminf_{h\to+\infty} L^\xi_y(w_h, v) = \lim_{k(y)\to+\infty} L^\xi_y(w_{k(y)}, v). \tag{7.14}$$

The bound (7.5) guarantees that, up to extracting a further subsequence $w_{j(y)} \subset w_{k(y)}$, we have

$$(w_{j(y)})^\xi_y \rightharpoonup w^\xi_y \quad \text{weakly in } H^1\big(U^\xi_y\big),$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi \setminus (E_{h(\vartheta)})^\xi$, and in particular

$$\nabla(w_{j(y)})^\xi_y - v \rightharpoonup \nabla w^\xi_y - v \quad \text{weakly in } L^2\big(U^\xi_y\big).$$

Hence, by the lower semicontinuity of the functional L^ξ_y and by (7.14), we obtain

$$L^\xi_y(w, v) \le \liminf_{j(y)\to+\infty} L^\xi_y(w_{j(y)}, v) = \liminf_{h\to+\infty} L^\xi_y(w_h, v).$$

Integrating the previous expression with respect to y leads to

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi} L^\xi_y(w, v)\, d\mathcal{H}^{n-1}(y) \le \liminf_{h\to+\infty} \int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi} L^\xi_y(w_h, v) d\mathcal{H}^{n-1}(y).$$

Being $w \in H^1(U;\mathbb{R}^n)$ we can pass to the limit as $\vartheta \to 0$ in the previous expression and we get

$$\int_U |\mathcal{E}w\xi\cdot\xi - v|^2 dx \le \liminf_{h\to+\infty} \int_U |\mathcal{E}w_h\xi\cdot\xi - v|^2 dx. \tag{7.15}$$

The fact that (7.15) holds true for every $v \in L^2(U)$ implies that, for every $\xi \in S^{n-1}$

$$\mathcal{E}w_h\xi\cdot\xi \rightharpoonup \mathcal{E}w\xi\cdot\xi \quad \text{weakly in } L^2(U). \tag{7.16}$$

Now we consider a basis $\{\xi_1, \dots, \xi_n\}$ of $\mathbb{R}^n$ such that $\xi_i + \xi_j \in S^{n-1}$ for every $i \ne j$, and we specify $\xi = \xi_i + \xi_j$ in (7.16). Then we have

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \text{weakly in } L^2(U;\mathbb{M}^{n\times n}),$$

and this concludes the proof. □

In next lemma we give a Γ-convergence result for an auxiliary functional which will be used in the proof of the main result of this section.

Lemma 7.3. *Let us fix $0 < \bar\delta < \delta < \frac{1}{2}$ so that $Q_\delta \subset\subset Q_{\bar\delta}$. For every $h \in \mathbb{N}$, let $\mathcal{G}^h : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ be the functional defined as*

$$\mathcal{G}^h(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\, dx + \mathcal{H}^{n-1}(J_w) & \text{if } w \in SBD^2(Q_{\bar\delta}),\ J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \le \frac{1}{h^2}, \\ & \quad [w]\cdot\nu_w \ge 0\ \mathcal{H}^{n-1} \text{ a.e. on } J_w, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Then the sequence $(\mathcal{G}^h)$ Γ*-converges with respect to the strong topology of* L^2 *to the functional* $\mathcal{G} : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ *given by*

$$\mathcal{G}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\,dx & \text{if } w \in H^1(Q_{\bar\delta};\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Proof. Let $w \in L^2(Q_{\bar\delta};\mathbb{R}^n)$ and let (w_h) be a sequence converging to w strongly in L^2 and having equibounded energy $\mathcal{G}_h$. Using the bounds (3.4) we can apply the previous lemma with $U = Q_{\bar\delta}$ to obtain that $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$ and that

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \text{weakly in } L^2(Q_{\bar\delta};\mathbb{M}^{n\times n}). \tag{7.17}$$

Hence, by lower semicontinuity we obtain the inequality

$$\mathcal{G}(w) = \int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\,dx \le \liminf_{h\to+\infty} \int_{Q_{\bar\delta}} \sigma(w_h) : \mathcal{E}w_h\,dx,$$

that implies in particular that

$$\mathcal{G}(w) \le \liminf_{h\to+\infty} \mathcal{G}^h(w_h).$$

Finally, the existence of a recovery sequence for a function $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$ follows immediately by taking $w_h = w$ for every $h \in \mathbb{N}$. ☐

Next lemma contains a Γ-convergence result for the same functionals as in Lemma 7.3, but taking into account Dirichlet boundary conditions.

Lemma 7.4. *Let* $(\varphi_h), \varphi \in H^{1/2}(\partial Q_{\bar\delta};\mathbb{R}^n)$ *be such that* $\varphi_h \to \varphi$ *strongly in* $H^{1/2}(\partial Q_{\bar\delta})$*. For every* $h \in \mathbb{N}$*, let* $\mathcal{G}^h_{\varphi_h} : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ *be the functionals defined by*

$$\mathcal{G}^h_{\varphi_h}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\,dx + \mathcal{H}^{n-1}(J_w) & \text{if } w \in SBD^2(Q_{\bar\delta}), J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \le \frac{1}{h^2}, \\ & [w]\cdot\nu_w \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w,\ w = \varphi_h \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases} \tag{7.18}$$

Then the sequence $(\mathcal{G}^h_{\varphi_h})$ Γ*-converges with respect to the strong topology of* L^2 *to the functional* $\mathcal{G}_\varphi : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ *given by*

$$\mathcal{G}_\varphi(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \sigma(w) : \mathcal{E}w\,dx & \text{if } w \in H^1(Q_{\bar\delta};\mathbb{R}^n),\ w = \varphi \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Proof. First step: proof of compactness and liminf. Let $(w_h), w \in L^2(Q_{\bar\delta};\mathbb{R}^n)$ be such that $w_h \to w$ strongly in L^2 and $\mathcal{G}^h_{\varphi_h}(w_h) \le c < +\infty$. From the equality $\mathcal{G}^h_{\varphi_h}(w_h) = \mathcal{G}^h(w_h)$ and the previous lemma we get that $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$; moreover

$$\liminf_{h\to+\infty} \mathcal{G}^h_{\varphi_h}(w_h) = \liminf_{h\to+\infty} \mathcal{G}^h(w_h) \ge \mathcal{G}(w).$$

It remains to show that $w_{|\partial Q_{\bar\delta}} = \varphi$.

From $\mathcal{G}^h_{\varphi_h}(w_h) \le c$, we obtain the equiboundedness of w_h in $H^1(Q_{\bar\delta} \setminus Q_\delta;\mathbb{R}^n)$, and hence the convergence

$$w_h \rightharpoonup w \quad \text{weakly in } H^1(Q_{\bar\delta} \setminus Q_\delta;\mathbb{R}^n).$$

The compactness of the trace operator gives

$$\varphi_h = (w_h)_{|\partial Q_{\bar\delta}} \to w_{|\partial Q_{\bar\delta}} \quad \text{strongly in } L^2(\partial Q_{\bar\delta};\mathbb{R}^n).$$

On the other hand, by assumption, $\varphi_h \to \varphi$ strongly in $H^{1/2}(\partial Q_{\bar\delta};\mathbb{R}^n)$. Therefore, $w_{|\partial Q_{\bar\delta}} = \varphi$.

Second step: limsup.

Let $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$ be such that $w_{|\partial Q_{\bar\delta}} = \varphi$. Let us consider the sequence $(v_h) \subset H^1(Q_{\bar\delta};\mathbb{R}^n)$ such that $(v_h)_{|\partial Q_{\bar\delta}} = \varphi_h - \varphi$; it turns out that $v_h \to 0$ strongly in H^1. We claim that $w_h := v_h + w$ is a recovery sequence. Indeed, $(w_h)_{|\partial Q_{\bar\delta}} = \varphi_h$ and $w_h \to w$ strongly in H^1, hence $\mathcal{E}w_h \to \mathcal{E}w$

strongly in L^2. Since the functional $\mathcal{G}^h_{\varphi_h}$ gives a norm equivalent to the standard L^2-norm, we have the desired convergence. □

Finally we are ready to state and prove the convergence result for the functional $\mathcal{F}^\varepsilon$, in the supercritical regime.

Define the functional $\mathcal{F}^\infty : L^2(\Omega;\mathbb{R}^n) \to [0,+\infty]$ as

$$\mathcal{F}^\infty(u) = \begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathcal{E}u \, dx & \text{in } H^1_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{7.19}$$

Next theorem shows that $\mathcal{F}^\infty$ is the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ in the case $\frac{\alpha_\varepsilon}{\varepsilon} \to +\infty$.

Theorem 7.5 (Γ-convergence). *(i) Let $u \in L^2(\Omega;\mathbb{R}^n)$ and let (u^ε) be a sequence converging to u strongly in L^2 and having equibounded energy $\mathcal{F}^\varepsilon$. Then $u \in H^1_0(\Omega;\mathbb{R}^n)$ and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \mathcal{F}^\infty(u). \tag{7.20}$$

(ii) For every $u \in H^1_0(\Omega;\mathbb{R}^n)$ there exists a sequence (u^ε) such that

- $u^\varepsilon \to u$ *strongly in* $L^2(\Omega;\mathbb{R}^n)$, (7.21)
- $\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = \mathcal{F}^\infty(u)$. (7.22)

Proof. (i) Let us write the domain Ω as union of cubes of side ε:

$$\Omega = \Bigg(\bigcup_{h\in\mathbb{Z}_\varepsilon} \varepsilon(Q+h)\Bigg) \cup R(\varepsilon),$$

where $\mathbb{Z}_\varepsilon := \{h \in \mathbb{Z}^n : \varepsilon(Q+h) \subset \Omega\}$, and $R(\varepsilon)$ is the remaining part of Ω. Notice that the set $R(\varepsilon)$ is defined as

$$R(\varepsilon) = \bigcup_{r\in\mathbb{R}_\varepsilon} \varepsilon(Q+r) \cap \Omega, \tag{7.23}$$

where $\mathbb{R}_\varepsilon := \{r \in \mathbb{Z}^n : \varepsilon(Q+r) \cap \partial\Omega \neq \emptyset\}$. Let $N(\varepsilon)$ be the cardinality of the set $\mathbb{Z}_\varepsilon$; notice that $N(\varepsilon)$ is of order $1/\varepsilon^n$. Notice that the cardinality $N_R(\varepsilon)$ of the set $\mathbb{R}_\varepsilon$ is of order $1/\varepsilon^{n-1}$.

We denote by $\{Q^\varepsilon_k\}_{k=1,\dots,N(\varepsilon)}$ an enumeration of the family of cubes $(Q+h)^\varepsilon$ covering Ω, so that we can rewrite Ω as

$$\Omega = \Bigg(\bigcup_{k=1}^{N(\varepsilon)} Q^\varepsilon_k\Bigg) \cup R(\varepsilon). \tag{7.24}$$

In the same way we can define the sets $\{Q^\varepsilon_{\delta,k}\}_{k=1,\dots,N(\varepsilon)}$.

We now classify the cubes Q^ε_k, with $k = 1,\dots,N(\varepsilon)$, according to the measure of the jump set that they contain. More precisely, let us introduce a parameter $\beta > 0$ that will be chosen later in a suitable way. We say that a cube Q^ε_k is *good* whenever $\mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big) \leq \beta\,\varepsilon^{n-1}$, and *bad* otherwise, and we denote with $N_g(\varepsilon)$ and $N_b(\varepsilon)$ the number of *good* and *bad* cubes, respectively. We can notice that, since the sequence (u^ε) has equibounded energy, there exists a constant $c > 0$ such that $\alpha_\varepsilon \mathcal{H}^{n-1}(J_{u^\varepsilon}) \leq c$. From this we deduce an important bound for the number of bad cubes, that is $N_b(\varepsilon) \leq \dfrac{c}{\alpha_\varepsilon \varepsilon^{n-1}}$. We can write (7.24) in the form

$$\Omega = \Bigg(\bigcup_{k=1}^{N_g(\varepsilon)} Q^\varepsilon_k\Bigg) \cup \Bigg(\bigcup_{k=1}^{N_b(\varepsilon)} Q^\varepsilon_k\Bigg) \cup R(\varepsilon) =: (Q^\varepsilon)^g \cup (Q^\varepsilon)^b \cup R(\varepsilon). \tag{7.25}$$

First step: energy estimate on good cubes. Let Q^ε_k be a good cube and consider

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) = \int_{Q^\varepsilon_k} \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big). \tag{7.26}$$

Define the function v^ε in the unit cube Q_k as $u^\varepsilon(\varepsilon\, y) =: \sqrt{\alpha_\varepsilon \varepsilon}\, v^\varepsilon(y)$. In terms of v^ε, the energy (7.26) can be written as

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q_k^\varepsilon\big) = \alpha_\varepsilon \varepsilon^{n-1}\bigg\{\int_{Q_k} \sigma(v^\varepsilon) : \mathcal{E}v^\varepsilon dx + \mathcal{H}^{n-1}(J_{v^\varepsilon}\cap Q_k)\bigg\}, \tag{7.27}$$

with $\mathcal{H}^{n-1}(J_{v^\varepsilon}\cap Q_k) \le \beta$. Therefore, by means of a change of variables we have reduced the problem to the study of a Mumford-Shah like functional over a fixed domain, with some constraints on the jump set. From now on we will omit the subscript k. Let $\bar\delta, \hat\delta$ be such that $Q_\delta \subset\subset Q_{\bar\delta} \subset\subset Q_{\hat\delta} \subset\subset Q$.

Let us consider the problem of finding local minimizers for the Mumford-Shah like functional under the required conditions, that is

$$\text{(LMin)}\quad \operatorname{loc\,min}\bigg\{\int_{Q_{\hat\delta}} \sigma(w) : \mathcal{E}w\, dx + \mathcal{H}^{n-1}(J_w) : \quad w \in SBD^2(Q_{\hat\delta}), J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \le \beta, \\ [w]\cdot \nu_w \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\bigg\}.$$

According to the definition given in [12], we recall that a local minimizer is a function which minimizes the given functional with respect to all perturbations with compact support. Let us denote by $\mathcal{M}_\beta$ the class of solutions of (LMin). For a given $\hat v \in \mathcal{M}_\beta$, let us consider the function $\tilde v$ solving

$$\text{(Eul)}\qquad \begin{cases} \operatorname{div}\sigma(\tilde v) = 0 & \text{in } Q_{\bar\delta},\\ \tilde v = \hat v & \text{in } Q_{\hat\delta}\setminus Q_{\bar\delta}. \end{cases}$$

We want to prove that for every $\eta > 0$ there exists $\beta > 0$ such that for every $\hat v \in \mathcal{M}_\beta$ and for the corresponding $\tilde v$ we have

$$\int_{Q_{\hat\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v\, dx \le (1+\eta)\int_{Q_{\hat\delta}} \sigma(\hat v) : \mathcal{E}\hat v\, dx. \tag{7.28}$$

Hence we will take such a β in the definition of good and bad cubes.

Let us prove it by contradiction. Suppose (7.28) is false. Then there exists $\eta > 0$ such that for every $\beta > 0$ we can find $\hat v \in \mathcal{M}_\beta$ and a corresponding $\tilde v$ for which

$$\int_{Q_{\hat\delta}} \sigma(\tilde v) : \mathcal{E}\tilde v\, dx > (1+\eta)\int_{Q_{\hat\delta}} \sigma(\hat v) : \mathcal{E}\hat v\, dx. \tag{7.29}$$

In particular (7.29) implies that for every $h > 0$ there exists $\hat v_h \in \mathcal{M}_{\frac{1}{h^2}}$ and $\tilde v_h$ solution of (Eul) for which

$$\int_{Q_{\hat\delta}} \sigma(\tilde v_h) : \mathcal{E}\tilde v_h dx > (1+\eta)\int_{Q_{\hat\delta}} \sigma(\hat v_h) : \mathcal{E}\hat v_h dx. \tag{7.30}$$

Since $Q_{\hat\delta} = \big(Q_{\hat\delta}\setminus Q_{\bar\delta}\big)\cup Q_{\bar\delta}$, we can split the previous integrals and, using the fact that $\tilde v_h = \hat v_h$ in $Q_{\hat\delta}\setminus Q_{\bar\delta}$, we obtain from (7.30)

$$\int_{Q_{\bar\delta}} \sigma(\tilde v_h) : \mathcal{E}\tilde v_h dx > (1+\eta)\int_{Q_{\bar\delta}} \sigma(\hat v_h) : \mathcal{E}\hat v_h dx + \eta\int_{Q_{\hat\delta}\setminus Q_{\bar\delta}} \sigma(\hat v_h) : \mathcal{E}\hat v_h dx. \tag{7.31}$$

Since the problem defining $\tilde v_h$ is linear, we can normalize the left-hand side of (7.31), so that we have

$$1 = \int_{Q_{\bar\delta}} \sigma(\tilde v_h) : \mathcal{E}\tilde v_h dx > (1+\eta)\int_{Q_{\bar\delta}} \sigma(\hat v_h) : \mathcal{E}\hat v_h dx + \eta\int_{Q_{\hat\delta}\setminus Q_{\bar\delta}} \sigma(\hat v_h) : \mathcal{E}\hat v_h dx. \tag{7.32}$$

This means that, in particular,

$$\int_{Q_{\hat\delta}} |\mathcal{E}\hat v_h|^2 dx \le \frac{1}{\eta} < +\infty. \tag{7.33}$$

Without loss of generality we can assume that $\int_{Q_{\hat\delta}\setminus Q_\delta} \hat v_h dx = 0$; therefore, since $J_{\hat v_h} \subset Q_\delta$, (7.33) and Korn inequality imply that $||\hat v_h||_{H^1(Q_{\hat\delta}\setminus Q_\delta)^n} \le c$.

From this bound we deduce that there exists some $\hat{v} \in H^1(Q_{\hat{\delta}} \setminus Q_\delta;\mathbb{R}^n)$ such that $\hat{v}_h \rightharpoonup \hat{v}$ weakly in H^1 and, in particular, strongly in L^2. The local minimality of $\hat{v}_h$ implies that

$$\int_{Q_{\hat{\delta}}\setminus Q_\delta} \sigma(\hat{v}_h) : \mathcal{E}\phi \, dx = 0 \quad \text{for every } \phi \in H^1_0(Q_{\hat{\delta}} \setminus Q_\delta;\mathbb{R}^n). \tag{7.34}$$

Now, if we write (7.34) for a test function $\phi = \psi\,(\hat{v}_h - \hat{v})$, with $\psi \in C^1_0(Q_{\hat{\delta}} \setminus Q_\delta)$, we obtain

$$\int_{Q_{\hat{\delta}}\setminus Q_\delta} \psi\, \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx = \int_{Q_{\hat{\delta}}\setminus Q_\delta} \psi\, \sigma(\hat{v}_h) : \mathcal{E}\hat{v}\, dx - \int_{Q_{\hat{\delta}}\setminus Q_\delta} \sigma(\hat{v}_h) : \big((\hat{v}_h - \hat{v})\nabla\psi\big) dx.$$

Since $\hat{v}_h \rightharpoonup \hat{v}$ weakly in $H^1(Q_{\hat{\delta}} \setminus Q_\delta;\mathbb{R}^n)$, if we let $h \to +\infty$ in the previous equation we get

$$\lim_{h\to+\infty} \int_{Q_{\hat{\delta}}\setminus Q_\delta} \psi\, \sigma(\hat{v}_h) : \mathcal{E}\hat{v}_h dx = \int_{Q_{\hat{\delta}}\setminus Q_\delta} \psi\, \sigma(\hat{v}) : \mathcal{E}\hat{v}\, dx. \tag{7.35}$$

This means in particular that for every $B \subset\subset Q_{\hat{\delta}} \setminus Q_\delta$

$$\mathcal{E}\hat{v}_h \to \mathcal{E}\hat{v} \quad \text{strongly in } L^2(B;\mathbb{M}^{n\times n}_{sym}). \tag{7.36}$$

Indeed, (7.35) together with the weak convergence of the sequence $\hat{v}_h$ in $H^1(Q_{\hat{\delta}} \setminus Q_\delta)$ imply that $\mathcal{E}\hat{v}_h$ converges strongly to $\mathcal{E}\hat{v}$ with respect to the norm induced on L^2 by the tensor $\mathbb{C}$ introduced in (3.4) and (3.5). The equivalence of this norm to the standard L^2 norm gives (7.36). Hence, by the strong convergence of $\hat{v}_h$ to $\hat{v}$ in L^2, (7.36) and Korn inequality, we deduce

$$\hat{v}_h \to \hat{v} \quad \text{strongly in } H^1(B;\mathbb{R}^n).$$

This entails the convergence of the traces of $\hat{v}_h$ on $\partial Q_{\bar{\delta}}$, that is,

$$\varphi_h := (\hat{v}_h)_{|\partial Q_{\bar{\delta}}} \to \varphi := (\hat{v})_{|\partial Q_{\bar{\delta}}} \quad \text{strongly in } H^{1/2}(\partial Q_{\bar{\delta}};\mathbb{R}^n). \tag{7.37}$$

At this point, let us consider the following problems:

$$(\mathrm{Eul})_{\varphi_h} \quad \begin{cases} \operatorname{div}\sigma(w) = 0 & \text{in } Q_{\bar{\delta}} \\ w = \varphi_h & \text{on } \partial Q_{\bar{\delta}}, \end{cases} \qquad (\mathrm{Eul})_{\varphi} \quad \begin{cases} \operatorname{div}\sigma(w) = 0 & \text{in } Q_{\bar{\delta}} \\ w = \varphi & \text{on } \partial Q_{\bar{\delta}}. \end{cases}$$

Clearly, $\tilde{v}_h$ is the solution to $(\mathrm{Eul})_{\varphi_h}$ for every h. Let us call $\tilde{v}$ the solution to $(\mathrm{Eul})_{\varphi}$. From (7.37) it turns out that $\tilde{v}_h \to \tilde{v}$ strongly in $H^1(Q_{\bar{\delta}};\mathbb{R}^n)$, hence,

$$1 = \int_{Q_{\bar{\delta}}} \sigma(\tilde{v}_h) : \mathcal{E}\tilde{v}_h dx \to \int_{Q_{\bar{\delta}}} \sigma(\tilde{v}) : \mathcal{E}\tilde{v}\, dx = 1. \tag{7.38}$$

Notice that since the functions $\hat{v}_h$ are local minimizers for the functional in (LMin), they turn out to be absolute minimizers of the same functional once we fix the boundary data φ_h. Therefore by definition they are absolute minimizers for the functional $\mathcal{G}^h_{\varphi_h}$ defined in (7.18). The Γ-convergence result proved in Lemma 7.4 ensures the L^2 convergence of the sequence $\hat{v}_h$ to the only minimizer of the functional $\mathcal{G}_\varphi$, that is exactly $\tilde{v}$, and the convergence of the energies.

Now, if we let $h \to +\infty$ in (7.32) we obtain

$$1 = \int_{Q_{\bar{\delta}}} \sigma(\tilde{v}) : \mathcal{E}\tilde{v}\, dx \geq (1+\eta) \int_{Q_{\bar{\delta}}} \sigma(\tilde{v}) : \mathcal{E}\tilde{v}\, dx,$$

which gives the contradiction, therefore (7.28) is proved.

Let $\eta > 0$ be fixed; we choose $\beta > 0$ such that the property (7.28) is satisfied and for every $\varepsilon > 0$ we consider the problem

$$(\mathrm{Min}) \ \min\Big\{ \int_{Q_{\hat{\delta},k}} \sigma(w) : \mathcal{E}w\, dx + \mathcal{H}^{n-1}(J_w) \ : w \in SBD^2(Q_{\hat{\delta},k}), J_w \subset Q_{\delta,k}, \mathcal{H}^{n-1}(J_w) \leq \beta, \\ [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w, w = v^\varepsilon \text{ on } \partial Q_{\hat{\delta},k} \Big\}.$$

For a minimizer $\hat{v}^\varepsilon$ in (Min), let us consider the corresponding $\tilde{v}^\varepsilon$ defined by (Eul), with $\hat{v}$ replaced by $\hat{v}^\varepsilon$. We have that, as before,

$$\int_{Q_{\hat{\delta},k}} \sigma(\tilde{v}^\varepsilon) : \mathcal{E}\tilde{v}^\varepsilon dx \le (1+\eta) \int_{Q_{\hat{\delta},k}} \sigma(\hat{v}^\varepsilon) : \mathcal{E}\hat{v}^\varepsilon dx. \tag{7.39}$$

Hence, in particular,

$$\int_{Q_{\hat{\delta},k}} \sigma(v^\varepsilon) : \mathcal{E}v^\varepsilon dx + \mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_{\hat{\delta},k}) \ge \int_{Q_{\hat{\delta},k}} \sigma(\hat{v}^\varepsilon) : \mathcal{E}\hat{v}^\varepsilon dx + \mathcal{H}^{n-1}(J_{\hat{v}^\varepsilon} \cap Q_{\hat{\delta},k}) \tag{7.40}$$

$$\ge \left(1 - \frac{\eta}{1+\eta}\right) \int_{Q_{\hat{\delta},k}} \sigma(\tilde{v}^\varepsilon) : \mathcal{E}\tilde{v}^\varepsilon dx, \tag{7.41}$$

where v^ε is the function in (7.27).
Now we define $\tilde{u}^\varepsilon$ as $\tilde{u}^\varepsilon(\varepsilon y) := \sqrt{\alpha_\varepsilon \varepsilon}\ \tilde{v}^\varepsilon(y)$. By (7.27) and (7.40) we obtain

$$\int_{Q^\varepsilon_{\hat{\delta},k}} \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_{\hat{\delta},k}\big) \ge \left(1 - \frac{\eta}{1+\eta}\right) \int_{Q^\varepsilon_{\hat{\delta},k}} \sigma(\tilde{u}^\varepsilon) : \mathcal{E}\,\tilde{u}^\varepsilon dx. \tag{7.42}$$

Second step: energy estimate on bad cubes. Let Q^ε_k be a bad cube. The idea is to use the trivial inequality

$$\int_{Q^\varepsilon_k} \sigma(u^\varepsilon) : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big) \ge \int_{Q^\varepsilon_k} \chi^\varepsilon_\delta\, \sigma(\hat{u}^\varepsilon) : \mathcal{E}\hat{u}^\varepsilon dx,$$

where χ^ε_δ is the characteristic function of the set $Q^\varepsilon_k \setminus Q^\varepsilon_{\delta,k}$ and the function $\hat{u}^\varepsilon$ coincides with u^ε in $Q^\varepsilon_k \setminus Q^\varepsilon_{\delta,k}$ and is extended to $Q^\varepsilon_{\delta,k}$ in a way that keeps its H^1 norm bounded. We recall also that we have a control on the number of bad cubes, that is, $N_b(\varepsilon) \le \dfrac{c}{\alpha_\varepsilon \varepsilon^{n-1}}$.

Third step: estimate on boundary cubes. Let $\Omega_0 \supset \Omega$ be such that $\mathrm{dist}(\Omega, \partial\Omega_0) > 1$. We still denote with u^ε the extension of u^ε to Ω_0, obtained simply setting $u^\varepsilon = 0$ in $\Omega_0 \setminus \Omega$ and with u its L^2- limit. Notice that, since $\mathrm{tr}(u^\varepsilon) = 0$ on $\partial\Omega$ by assumption, we have that for every $r \in \mathbb{R}_\varepsilon$ $u^\varepsilon_{|((Q\setminus Q_\delta)+r)^\varepsilon} \in H^1(((Q \setminus Q_\delta) + r)^\varepsilon; \mathbb{R}^n)$.

Let $\check{u}^\varepsilon$ be the function in $(\Omega_0 \setminus \Omega) \cup R(\varepsilon)$ obtained extending u^ε in $(Q_\delta + r)^\varepsilon$ for every $r \in \mathbb{R}_\varepsilon$ in a way that keeps its H^1 norm bounded.

Then we have

$$\mathcal{F}^\varepsilon(u^\varepsilon, R(\varepsilon)) = \mathcal{F}^\varepsilon(u^\varepsilon, (\Omega_0 \setminus \Omega) \cup R(\varepsilon)) \ge \int_{(\Omega_0\setminus\Omega)\cup R(\varepsilon)} \chi^\varepsilon_R \sigma(\check{u}^\varepsilon) : \mathcal{E}\check{u}^\varepsilon dx, \tag{7.43}$$

where χ^ε_R is zero in $(Q_\delta + r)^\varepsilon$ for every $r \in \mathbb{R}_\varepsilon$ and 1 otherwise in $(\Omega_0 \setminus \Omega) \cup R(\varepsilon)$.

Fourth step: final estimate. Let us define the new sequence $w^\varepsilon \in SBD^2_0(\Omega_0)$ as

$$w^\varepsilon := \begin{cases} \tilde{u}^\varepsilon & \text{in } \big(Q^\varepsilon_{\hat{\delta}}\big)^g, \\ u^\varepsilon & \text{in } (Q^\varepsilon)^g \setminus \big(Q^\varepsilon_{\hat{\delta}}\big)^g, \\ \hat{u}^\varepsilon & \text{in } (Q^\varepsilon)^b, \\ \check{u}^\varepsilon & \text{in } (\Omega_0 \setminus \Omega) \cup R(\varepsilon), \end{cases}$$

where $(Q^\varepsilon)^g$, $(Q^\varepsilon)^b$ and $R(\varepsilon)$ are given in (7.25) and (7.23), and $\big(Q^\varepsilon_{\hat{\delta}}\big)^g$ denotes the set

$$\big(Q^\varepsilon_{\hat{\delta}}\big)^g := \bigcup_{k=1}^{N_g(\varepsilon)} Q^\varepsilon_{\hat{\delta},k}. \tag{7.44}$$

Notice that $w^\varepsilon \in H^1_0(\Omega_0; \mathbb{R}^n)$ and that $w^\varepsilon \in H^1_0(\Omega'; \mathbb{R}^n)$ for every $\overline{\Omega} \subset \Omega' \subset \Omega_0$.

Define also the function $a^\varepsilon : \Omega_0 \to \mathbb{R}$ as

$$a^\varepsilon(x) := \begin{cases} 0 & \text{in } (Q^\varepsilon_{\hat{\delta}})^b \cup \Big(\bigcup_{r\in\mathbb{R}_\varepsilon} (Q_\delta + r)^\varepsilon\Big), \\ 1 & \text{otherwise in } \Omega_0, \end{cases}$$

where in analogy with (7.44) we defined $\left(Q_\delta^\varepsilon\right)^b$ as

$$\left(Q_\delta^\varepsilon\right)^b := \bigcup_{k=1}^{N_b(\varepsilon)} Q_{\delta,k}^\varepsilon.$$

From what we proved in the previous steps we can write

$$\mathcal{F}^\varepsilon(u^\varepsilon,\Omega) = \mathcal{F}^\varepsilon(u^\varepsilon,\Omega_0) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_{\Omega_0} a^\varepsilon(x)\,\sigma(w^\varepsilon) : \mathcal{E}w^\varepsilon dx. \tag{7.45}$$

It remains to apply Lemma 7.1 to (7.45). First of all we show the convergence of a^ε. We have

$$\int_\Omega |a^\varepsilon - 1|\,dx = \mathcal{L}^n\left((Q_\delta^\varepsilon)^b \cup \left(\bigcup_{r\in\mathbb{R}_\varepsilon} (Q_\delta + r)^\varepsilon \right)\right) = (N_b(\varepsilon) + N_R(\varepsilon))\varepsilon^n \mathcal{L}^n(Q_\delta) \leq c\,\frac{\varepsilon}{\alpha_\varepsilon},$$

hence $a^\varepsilon \to 1$ strongly in $L^1(\Omega_0)$. Once we prove that $w^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega_0;\mathbb{R}^n)$ and that $u_{|\Omega} \in H^1_0(\Omega;\mathbb{R}^n)$, as $u = 0$ in $\Omega_0 \setminus \Omega$, it turns out that

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \left(1 - \frac{\eta}{1+\eta}\right) \int_{\Omega_0} \sigma(u) : \mathcal{E}u\,dx = \left(1 - \frac{\eta}{1+\eta}\right) \int_\Omega \sigma(u) : \mathcal{E}u\,dx,$$

and the thesis follows letting η converge to zero.

Fifth step: convergence of w^ε. First of all it is clear that $||\mathcal{E}w^\varepsilon||_{(L^2(\Omega_0))^{n\times n}} \leq c$. Then, the fact that w^ε and u^ε coincide in a set with positive measure ensures the convergence. Moreover, since $w^\varepsilon \in H^1_0(\Omega';\mathbb{R}^n)$ for every $\overline{\Omega} \subset \Omega' \subset \Omega_0$, then $u \in H^1_0(\Omega;\mathbb{R}^n)$.

(ii) The claim follows trivially by choosing $u^\varepsilon = u$ for every $\varepsilon > 0$. □

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