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Regularity for weakly Dirac-harmonic maps to
hypersurfaces

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Abstract

We prove that a weakly Dirac-harmonic map from a Riemann spin surface to a compact hypersurface $N \subset \mathbb{R}^{d+1}$ is smooth.

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1. Introduction

As critical points of an action functional, which is motivated by the supersymmetric nonlinear sigma model from quantum field theory [3], Dirac-harmonic maps are defined as solutions of a system of nonlinear elliptic equations which couples harmonic-type equations and Dirac-type equations in a natural way [2].

It is observed in [2] that the conformal invariance of harmonic maps in two dimensions and Dirac operators in arbitrary finite dimensions are preserved in the case of two-dimensional Dirac-harmonic maps. This brings the study of Dirac-harmonic maps into the framework of two-dimensional conformally invariant variational problems. As is typical in calculus of variations, the regularity issue for weak solutions is of particular importance. For the regularity of weakly harmonic maps, see for instance [4], [5].

Chen et al. [1] studied the regularity problem for Dirac-harmonic maps and proved that any weakly Dirac-harmonic map from a spin surface to the standard sphere \mathbb{S}^d is actually smooth. They [1] observed a Jacobian structure of the Dirac-harmonic map equation for a spherical target manifold and proved the regularity property by applying Wente's lemma [9]. However, when the target is a general compact Riemannian manifold, the spinor terms in the harmonic-type equations for Dirac-harmonic maps cannot be written as a Jacobian form and thus Wente's lemma cannot be applied to obtain the regularity.

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Recently, Tristan Rivière [8] discovered the existence of a conservation law for weak solutions of the Euler-Lagrange equations of conformally invariant variational integrals in two dimensions. He succeeded in writing two-dimensional conformally invariant nonlinear elliptic PDEs (e.g., harmonic map equations, prescribed mean curvature equations, etc.) in divergence form, from which one can recover all the classical regularity results for such weak solutions and obtain new results.

In this paper, we observe that, when the target is a compact hypersurface $N \subset \mathbb{R}^{d+1}$, the harmonic-type equations for Dirac-harmonic maps can be written in a special form to which Rivière's regularity results can be applied. Our result is the following:

Theorem 1.1. *A weakly Dirac-harmonic map (ϕ, ψ) from a Riemann spin surface M to a compact hypersurface $N \subset \mathbb{R}^{d+1}$ is smooth.*

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2. Dirac-harmonic maps

Let (M, h) be an oriented, compact Riemannian surface with a fixed spin structure. Let ΣM be the spinor bundle over M with a hermitian metric $\langle \cdot, \cdot \rangle_{\Sigma M}$ and a compatible spin connection ∇ . For $X \in \Gamma(TM)$, $\psi \in \Gamma(\Sigma M)$, we denote by $X \cdot \psi$ the Clifford multiplication, which satisfies the following relations:

$$\langle X \cdot \psi, \varphi \rangle = -\langle \psi, X \cdot \varphi \rangle, \quad X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2h(X, Y)\psi,$$

for $X, Y \in \Gamma(TM)$, $\psi, \varphi \in \Gamma(\Sigma M)$. The Dirac operator \not{D} is defined by $\not{D}\psi := e_\alpha \cdot \nabla_{e_\alpha} \psi$ for a local orthonormal frame $\{e_1, e_2\}$ of TM and $\psi \in \Gamma(\Sigma M)$. The summation convention will be used throughout the paper. We refer to [7] for more spin geometric materials.

Let (N, g) be a compact Riemannian manifold of dimension $d \geq 2$, and let its metric in local coordinates be given by g_{ij} , with Christoffel symbols Γ_{jk}^i and Riemannian curvature tensor R_{lij}^m . Let ϕ be a smooth map from M to N and $\phi^{-1}TN$ the pull-back bundle of TN under ϕ . Consider the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ with the induced metric $\langle \cdot, \cdot \rangle_{\Sigma M \otimes \phi^{-1}TN}$ and the induced connection $\tilde{\nabla} := \nabla \otimes 1 + 1 \otimes \nabla^{\phi^{-1}TN}$. We write the section ψ of $\Sigma M \otimes \phi^{-1}TN$ locally as follows

$$\psi(x) = \psi^i(x) \otimes \partial_{y^i}(\phi(x)),$$

where ψ^i is a spinor and $\{\partial_{y^i}\}$ is a local basis on N . Then, $\tilde{\nabla}$ can be expressed by

$$\tilde{\nabla} \psi = \nabla \psi^i(x) \otimes \partial_{y^i}(\phi(x)) + \Gamma_{jk}^i d\phi^j(x) \psi^k(x) \otimes \partial_{y^i}(\phi(x)).$$

The Dirac operator along the map ϕ is defined by

$$\begin{aligned}\mathcal{D}\psi &:= e_\alpha \cdot \tilde{\nabla}_{e_\alpha} \psi \\ &= \not\partial \psi^i(x) \otimes \partial_{y^i}(\phi(x)) + \Gamma_{jk}^i \nabla_{e_\alpha} \phi^j(x) e_\alpha \cdot \psi^k(x) \otimes \partial_{y^i}(\phi(x)).\end{aligned}$$

Set

$$\chi(M, N) := \{(\phi, \psi) | \phi \in C^\infty(M, N), \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)\}.$$

We consider the following functional defined on $\chi(M, N)$:

$$\begin{aligned}L(\phi, \psi) &:= \int_M (|d\phi|^2 + \langle \psi, \mathcal{D}\psi \rangle) \\ &= \int_M \left(g_{ij}(\phi) h^{\alpha\beta} \frac{\partial \phi^i}{\partial x_\alpha} \frac{\partial \phi^j}{\partial x_\beta} + g_{ij}(\phi) \langle \psi^i, \mathcal{D}\psi^j \rangle_{\Sigma M} \right) \sqrt{\det(h_{\alpha\beta})} dx^1 dx^2.\end{aligned}$$

The corresponding Euler-Lagrange equations are

$$\Delta \phi^m + \Gamma_{jk}^m(\phi) h^{\alpha\beta} \partial_\alpha \phi^j \partial_\beta \phi^k = \frac{1}{2} R_{lij}^m(\phi) \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle, \quad (2.1)$$

$$\not\partial \psi^i + \Gamma_{jk}^i(\phi) \partial_\alpha \phi^j e_\alpha \cdot \psi^k = 0. \quad (2.2)$$

Here $\nabla \phi^l := \partial_\alpha \phi^l e_\alpha = \phi^l e_\alpha$ and “ \cdot ” denotes the Clifford multiplication. Set

$$\mathcal{R}(\phi, \psi)(x) := \frac{1}{2} R_{lij}^m(\phi(x)) \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \partial_{y^m}(\phi(x)).$$

Then, (2.1) and (2.2) can be written in the following global form

$$\tau(\phi) = \mathcal{R}(\phi, \psi), \quad (2.3)$$

$$\not\partial \psi = 0, \quad (2.4)$$

where $\tau(\phi) \in \Gamma(\phi^{-1}TN)$ is the tension field of the map ϕ . Solutions (ϕ, ψ) of (2.1), (2.2) are called *Dirac-harmonic maps* from M to N .

Note that (2.1) and (2.2) are the intrinsic versions of the Dirac-harmonic map equations. Now we shall give the extrinsic version of these equations. By the Nash-Moser embedding theorem, we embed N into some \mathbb{R}^K . Let A be the second fundamental form of N in \mathbb{R}^K and P the shape operator. It is not difficult to verify the following relation:

$$\langle P(\xi; X), Y \rangle_{TN} = \langle A(X, Y), \xi \rangle_{\mathbb{R}^K} \quad (2.5)$$

for any $X, Y \in \Gamma(TN), \xi \in \Gamma(T^\perp N)$, where $T^\perp N$ is the normal bundle to N . Then, by the Gauss equation (see [6]), we have

$$\begin{aligned}
R_{lij}^m \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle &= g^{mk} \langle A(\partial_{y^k}, \partial_{y^i}), A(\partial_{y^l}, \partial_{y^j}) \rangle \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \\
&\quad - \langle A(\partial_{y^k}, \partial_{y^j}), A(\partial_{y^l}, \partial_{y^i}) \rangle \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \\
&= g^{mk} \langle A(\partial_{y^k}, \partial_{y^i}), A(\partial_{y^l}, \partial_{y^j}) \rangle \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \\
&\quad + g^{mk} \langle A(\partial_{y^k}, \partial_{y^i}), A(\partial_{y^l}, \partial_{y^j}) \rangle \overline{\langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle} \\
&= 2\text{Re } g^{mk} \langle A(\partial_{y^k}, \partial_{y^i}), A(\partial_{y^l}, \partial_{y^j}) \rangle \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \\
&= 2\text{Re } g^{mk} \langle P(A(\partial_{y^l}, \partial_{y^j}); A(\partial_{y^i}), \partial_{y^k}) \langle \psi^i, e_\alpha \cdot \psi^j \rangle \phi_\alpha^l, \quad (2.6)
\end{aligned}$$

where in the last step we used (2.5). Set

$$\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) := \phi_\alpha^i e_\alpha \cdot \psi^j \otimes A(\partial_{y^i}, \partial_{y^j}), \quad (2.7)$$

$$\mathcal{P}(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi) := P(A(\partial_{y^l}, \partial_{y^j}); \partial_{y^i} \langle \psi^i, e_\alpha \cdot \psi^j \rangle \phi_\alpha^l). \quad (2.8)$$

Then, equations (2.1) and (2.2) become the following form:

$$-\Delta \phi = A(d\phi(e_\alpha), d\phi(e_\alpha)) + \text{Re } \mathcal{P}(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi), \quad (2.9)$$

$$\not\partial \psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi). \quad (2.10)$$

Here, ϕ is a map from M to \mathbb{R}^K with

$$\phi(x) \in N \quad (2.11)$$

for any $x \in M$, and the spinor field ψ along the map ϕ is a K -tuple of spinors $(\psi^1, \psi^2, \dots, \psi^K)$ satisfying

$$\sum_i \nu_i \psi^i = 0, \quad \text{for any normal vector } \nu = \sum_{i=1}^K \nu_i E_i \text{ at } \phi(x), \quad (2.12)$$

where $\{E_i, i = 1, 2, \dots, K\}$ is the standard basis of \mathbb{R}^K . Set

$$\chi_{1,4/3}^{1,2}(M, N) := \{(\phi, \psi) \in W^{1,2} \times W^{1,4/3} \text{ with (2.11) and (2.12) a.e.}\}.$$

Then the functional $L(\phi, \psi)$ is well-defined for $(\phi, \psi) \in \chi_{1,4/3}^{1,2}(M, N)$. Critical points (ϕ, ψ) of the functional $L(\phi, \psi)$ in $\chi_{1,4/3}^{1,2}(M, N)$ are called *weakly Dirac-harmonic maps* from M to N (see [1]).

Note that \mathcal{A}, \mathcal{P} in (2.7), (2.8) are defined in terms of a fixed local coordinates of N . This requires the continuity of the map ϕ . For weakly Dirac-harmonic maps, we need an extrinsic version of the equations, where everything is written down in coordinates of the ambient space \mathbb{R}^K . To this end, we fix a canonical coordinates (y^1, y^2, \dots, y^K) of \mathbb{R}^K . Let $\nu_l, l = d+1, \dots, K$ be an orthonormal frame field for the normal bundle

$T^\perp N$ to N . We still denote by ν_l the corresponding unit normal vector field along the map ϕ . Then, for $X = X^i \partial_{y^i} \in T_y N, Y = Y^j \partial_{y^j} \in T_y N, Z = Z^l \partial_{y^l} \in T_y N$, we have $\nabla_Y \nu_k = \sum_{i=1}^K Y^i \frac{\partial \nu_k}{\partial y^i}$, and the following holds (c.f. [5], Chapter. 1):

$$A(y)(X, Y) = \langle X, \nabla_Y \nu_l \rangle \nu_l = \langle X, Y^j \frac{\partial \nu_l}{\partial y^j} \rangle \nu_l = X^i Y^j \frac{\partial \nu_l^i}{\partial y^j} \nu_l, \quad (2.13)$$

$$P(A(X, Y); Z) = (\nabla_Z \nu_l)^\top X^i Y^j \frac{\partial \nu_l^i}{\partial y^j} = \left(\frac{\partial \nu_l}{\partial y^k} \right)^\top X^i Y^j Z^k \frac{\partial \nu_l^i}{\partial y^j}, \quad (2.14)$$

where \top denotes the projection $:\mathbb{R}^K \rightarrow T_y N$. In view of (2.7), (2.8), we can write \mathcal{A}, \mathcal{P} as follows, for $X = X^i \partial_{y^i} \in T_y N, \psi = \psi^j \partial_{y^j} \in \Sigma M \otimes T_y N, \varphi = \varphi^k \partial_{y^k} \in \Sigma M \otimes T_y N$,

$$\mathcal{A}(X, \psi) = X^i \psi^j \otimes \frac{\partial \nu_l^i}{\partial y^j} \nu_l \quad (2.15)$$

$$\mathcal{P}(\mathcal{A}(X, \psi); \varphi) = \left(\frac{\partial \nu_l}{\partial y^k} \right)^\top X^i \langle \varphi^k, \psi^j \rangle \frac{\partial \nu_l^i}{\partial y^j}. \quad (2.16)$$

By the symmetry of $A(X, Y)$ with respect to X and Y , we have

$$X^i \psi^j \otimes \frac{\partial \nu_l^i}{\partial y^j} \nu_l = X^j \psi^i \otimes \frac{\partial \nu_l^i}{\partial y^j} \nu_l \quad (2.17)$$

$$\left(\frac{\partial \nu_l}{\partial y^k} \right)^\top X^i \langle \varphi^k, \psi^j \rangle \frac{\partial \nu_l^i}{\partial y^j} = \left(\frac{\partial \nu_l}{\partial y^k} \right)^\top X^j \langle \varphi^k, \psi^i \rangle \frac{\partial \nu_l^i}{\partial y^j}. \quad (2.18)$$

Consider a weakly Dirac-harmonic map $(\phi, \psi) \in \chi_{1,4/3}^{1,2}(M, N)$. Let D be a simply connected domain of M . Choose local isothermal coordinates $z = x_1 + ix_2$ on D and let $e_1 = \partial_{x_1}, e_2 = \partial_{x_2}$. We write

$$\phi = \phi^i \partial y^i, \quad \psi = \psi^j \otimes \partial y^j,$$

and denote $\phi_1 := \nabla_{e_1} \phi = \phi_{x_1}, \phi_2 := \nabla_{e_2} \phi = \phi_{x_2}$. Then, we have

$$A(\phi_\alpha, \phi_\alpha) = \phi_\alpha^i \phi_\alpha^j \frac{\partial \nu_l^i}{\partial y^j} \nu_l \quad (2.19)$$

$$\mathcal{A}(\phi_\alpha, e_\alpha \cdot \psi) = \phi_\alpha^i e_\alpha \cdot \psi^j \otimes \frac{\partial \nu_l^i}{\partial y^j} \nu_l \quad (2.20)$$

$$\mathcal{P}(\mathcal{A}(\phi_\alpha, e_\alpha \cdot \psi); \psi) = \left(\frac{\partial \nu_l}{\partial y^k} \right)^\top \phi_\alpha^i \langle \psi^k, e_\alpha \cdot \psi^j \rangle \frac{\partial \nu_l^i}{\partial y^j}. \quad (2.21)$$

Note that $\phi_\alpha \in TN$ and $\nu_l \in T^\perp N$, hence,

$$\phi_\alpha^i \nu_l^i = 0, \quad \forall \alpha, l. \quad (2.22)$$

It follows that

$$A^m(\phi_\alpha, \phi_\alpha) = \phi_\alpha^i \phi_\alpha^j \frac{\partial \nu_l^i}{\partial y^j} \nu_l^m - \phi_\alpha^i \phi_\alpha^j \frac{\partial \nu_l^m}{\partial y^j} \nu_l^i = \phi_\alpha^i (\phi_\alpha^j \frac{\partial \nu_l^i}{\partial y^j} \nu_l^m - \phi_\alpha^j \frac{\partial \nu_l^m}{\partial y^j} \nu_l^i). \quad (2.23)$$

On the other hand, we have

$$\mathcal{A}^m(\phi_\alpha, e_\alpha \cdot \psi) = (\phi_\alpha^i e_\alpha \cdot \psi^j \otimes \frac{\partial \nu_l^i}{\partial y^j} \nu_l)^m = \frac{\partial \nu_l^i}{\partial y^j} \nu_l^m \phi_\alpha^i e_\alpha \cdot \psi^j \quad (2.24)$$

and

$$\begin{aligned} \operatorname{Re} \mathcal{P}^m(\mathcal{A}(\phi_\alpha, e_\alpha \cdot \psi); \psi) &= \operatorname{Re} \phi_\alpha^i \langle \psi^k, e_\alpha \cdot \psi^j \rangle \frac{\partial \nu_l^i}{\partial y^j} \left(\frac{\partial \nu_l}{\partial y^k} \right)^{\top, m} \\ &= \phi_\alpha^i \left(\langle \psi^k, e_\alpha \cdot \psi^j \rangle + \overline{\langle \psi^k, e_\alpha \cdot \psi^j \rangle} \right) \frac{\partial \nu_l^i}{\partial y^j} \left(\frac{\partial \nu_l}{\partial y^k} \right)^{\top, m} \\ &= \phi_\alpha^i (\langle \psi^k, e_\alpha \cdot \psi^j \rangle + \langle e_\alpha \cdot \psi^j, \psi^k \rangle) \frac{\partial \nu_l^i}{\partial y^j} \left(\frac{\partial \nu_l}{\partial y^k} \right)^{\top, m} \\ &= \phi_\alpha^i (\langle \psi^k, e_\alpha \cdot \psi^j \rangle - \langle \psi^j, e_\alpha \cdot \psi^k \rangle) \frac{\partial \nu_l^i}{\partial y^j} \left(\frac{\partial \nu_l}{\partial y^k} \right)^{\top, m} \\ &= \phi_\alpha^i \langle \psi^k, e_\alpha \cdot \psi^j \rangle \left(\frac{\partial \nu_l^i}{\partial y^j} \left(\frac{\partial \nu_l}{\partial y^k} \right)^{\top, m} - \frac{\partial \nu_l^i}{\partial y^k} \left(\frac{\partial \nu_l}{\partial y^j} \right)^{\top, m} \right), \end{aligned} \quad (2.25)$$

where $(\cdot)^m$ denotes the m -th component of a vector of \mathbb{R}^K . Thus, we can write (2.9) and (2.10) in the following extrinsic form in terms of the orthonormal frame field ν_l , $l = d+1, \dots, K$, for $T^\perp N$.

$$-\Delta \phi^m = \phi_\alpha^i (\phi_\alpha^j \frac{\partial \nu_l^i}{\partial y^j} \nu_l^m - \phi_\alpha^j \frac{\partial \nu_l^m}{\partial y^j} \nu_l^i) + \phi_\alpha^i \langle \psi^k, e_\alpha \cdot \psi^j \rangle \left(\frac{\partial \nu_l^i}{\partial y^j} \left(\frac{\partial \nu_l}{\partial y^k} \right)^{\top, m} - \frac{\partial \nu_l^i}{\partial y^k} \left(\frac{\partial \nu_l}{\partial y^j} \right)^{\top, m} \right), \quad (2.26)$$

$$\oint \psi^m = \frac{\partial \nu_l^i}{\partial y^j} \nu_l^m \phi_\alpha^i e_\alpha \cdot \psi^j. \quad (2.27)$$

3. Regularity for weakly Dirac-harmonic maps

When the target N is the standard sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, Chen et al. [1] observed a Jacobian structure for equation (2.9), namely,

Proposition 3.1. *Let M be a Riemann surface with a fixed spin structure and $(\phi, \psi) \in \chi_{1,4/3}^{1,2}(M, \mathbb{S}^d)$ a weakly Dirac-harmonic map from M to \mathbb{S}^d . Let D be a simply connected domain of M . Then there exists $\Omega = (\Omega^{ij}) \in W^2(D, \mathbb{R}^{(d+1) \times (d+1)})$ such that*

$$-\Delta \phi = \frac{\partial \Omega}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \Omega}{\partial y} \frac{\partial \phi}{\partial x}.$$

Moreover, they [1] proved that

Theorem 3.1. *Let $(\phi, \psi) : (D, \delta_{\alpha\beta}) \rightarrow (N, g_{ij})$ be a weakly Dirac-harmonic map. If ϕ is continuous, then (ϕ, ψ) is smooth.*

Combining Proposition 3.1, Theorem 3.1 and Wentz's lemma [9], the following regularity theorem was then proved in [1].

Theorem 3.2. *Let M be a Riemann surface with a fixed spin structure. Suppose that $(\phi, \psi) \in \chi_{1,4/3}^{1,2}(M, \mathbb{S}^d)$ is a weakly Dirac-harmonic map from M to \mathbb{S}^d . Then $\phi \in C^0$, and hence (ϕ, ψ) is smooth.*

Before we consider the case of general target manifolds, let us recall Tristan Rivière's main result in [8].

Theorem 3.3. *Let $K \in \mathbb{N}$, and let D be the unit disk of \mathbb{R}^2 . For every $\Omega = (\Omega_j^i)_{1 \leq i, j \leq K}$ in $L^2(D, so(K) \otimes \mathbb{R}^2)$ (i.e., $\forall i, j \in 1, \dots, K, \Omega_j^i \in L^2(D, \mathbb{R}^2)$ and $\Omega_j^i = -\Omega_i^j$), every $u \in W^{1,2}(D, \mathbb{R}^K)$ solving*

$$-\Delta u = \Omega \cdot \nabla u, \quad (3.1)$$

is continuous, where the contracted notation in the above equation stands for $\forall i = 1, \dots, K, -\Delta u^i = \sum_{j=1}^K \Omega_j^i \cdot \nabla u^j$ in coordinates.

We observe now that when the target $N \subset \mathbb{R}^{d+1}$ is a compact hypersurface, (2.26) has a structure similar to (3.1).

Proposition 3.2. *Let M be a Riemann surface with a spin structure and $N \subset \mathbb{R}^{d+1}$ be a compact hypersurface. Let $(\phi, \psi) \in \chi_{1,4/3}^{1,2}(M, N)$ be a weakly Dirac-harmonic map from M to N . Let D be a small domain of M . Then there exists $\Omega = (\Omega_i^m)_{1 \leq i, m \leq d+1}$ in $L^2(D, so(d+1) \otimes \mathbb{R}^2)$ such that*

$$-\Delta \phi = \Omega \cdot \nabla \phi. \quad (3.2)$$

Proof. Let $N \subset \mathbb{R}^{d+1}$ be a compact hypersurface with normal ν , then it follows from differentiating the equation $\nu \cdot \nu = 1$ that $\nabla \nu \in TN$, hence

$$\left(\frac{\partial \nu}{\partial y^k}\right)^{\top, m} = \left(\frac{\partial \nu}{\partial y^k}\right)^m = \left(\frac{\partial \nu^m}{\partial y^k}\right), \quad k, m = 1, 2, \dots, d+1. \quad (3.3)$$

By (2.25), we get

$$\begin{aligned} \operatorname{Re} \mathcal{P}^m(\mathcal{A}(\phi_\alpha, e_\alpha \cdot \psi); \psi) &= \phi_\alpha^i \langle \psi^k, e_\alpha \cdot \psi^j \rangle \left(\frac{\partial \nu^i}{\partial y^j} \left(\frac{\partial \nu}{\partial y^k} \right)^{\top, m} - \frac{\partial \nu^i}{\partial y^k} \left(\frac{\partial \nu}{\partial y^j} \right)^{\top, m} \right) \\ &= \phi_\alpha^i \langle \psi^k, e_\alpha \cdot \psi^j \rangle \left(\frac{\partial \nu^i}{\partial y^j} \frac{\partial \nu^m}{\partial y^k} - \frac{\partial \nu^i}{\partial y^k} \frac{\partial \nu^m}{\partial y^j} \right). \end{aligned} \quad (3.4)$$

On the other hand, from (2.23), we have

$$A^m(\phi_\alpha, \phi_\alpha) = \phi_\alpha^i (\phi_\alpha^j \frac{\partial \nu^i}{\partial y^j} \nu^m - \phi_\alpha^j \frac{\partial \nu^m}{\partial y^j} \nu^i). \quad (3.5)$$

Denote

$$\Omega_i^m := (\lambda_i^m, \mu_i^m), \quad i, m = 1, 2, \dots, d+1, \quad (3.6)$$

where

$$\lambda_i^m := \left(\frac{\partial \nu^i}{\partial y^j} \nu^m - \frac{\partial \nu^m}{\partial y^j} \nu^i \right) \phi_1^j + \left(\frac{\partial \nu^i}{\partial y^j} \frac{\partial \nu^m}{\partial y^k} - \frac{\partial \nu^i}{\partial y^k} \frac{\partial \nu^m}{\partial y^j} \right) \langle \psi^k, e_1 \cdot \psi^j \rangle, \quad (3.7)$$

$$\mu_i^m := \left(\frac{\partial \nu^i}{\partial y^j} \nu^m - \frac{\partial \nu^m}{\partial y^j} \nu^i \right) \phi_2^j + \left(\frac{\partial \nu^i}{\partial y^j} \frac{\partial \nu^m}{\partial y^k} - \frac{\partial \nu^i}{\partial y^k} \frac{\partial \nu^m}{\partial y^j} \right) \langle \psi^k, e_2 \cdot \psi^j \rangle. \quad (3.8)$$

Then we can write (2.26) in the following form

$$-\Delta \phi^m = \Omega_i^m \cdot \nabla \phi^i. \quad (3.9)$$

Now it remains to show that $\Omega = (\Omega_i^m)_{1 \leq i, m \leq d+1} \in L^2(D, so(d+1) \otimes \mathbb{R}^2)$.

On the one hand, since $(\phi, \psi) \in \chi_{1,4/3}^{1,2}(M, N)$ is a weakly Dirac-harmonic map, we must have $\phi \in W^{1,2}$ and $\psi \in W^{1,4/3} \subset L^4$ (by the Sobolev embedding theorem). In view of (3.6), (3.7) and (3.8), we have

$$\Omega_i^m \in L^2(D, \mathbb{R}^2), \quad \forall 1 \leq i, m \leq d+1.$$

On the other hand, it is easy to see from (3.7) and (3.8) that both λ_i^m and μ_i^m are real valued and are skew-symmetric with respect to the indices i and m . Thus,

$$\Omega = (\Omega_i^m)_{1 \leq i, m \leq d+1} \in L^2(D, so(d+1) \otimes \mathbb{R}^2).$$

This completes the proof. \square

Proof of Theorem 1.1. The result follows from combining Proposition 3.2, Theorem 3.1 and Theorem 3.3. \square

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