Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The Hijazi inequality on conformally parabolic manifolds

by

Nadine Große

The Hijazi inequality on conformally parabolic manifolds

Nadine Große

Abstract

We prove the Hijazi inequality, an estimate for Dirac eigenvalues, for complete manifolds of finite volume. Under some additional assumptions on the dimension and the scalar curvature, this inequality is also valid for elements of the essential spectrum. This allows to prove the conformal version of the Hijazi inequality on conformally parabolic manifolds if the spin analog to the Yamabe invariant is positive.

1   Introduction

On a closed n-dimensional Riemannian spin manifold (M, g, σ) with scalar curvature s_g, Friedrich [6, Thm. A] gave an estimate for an eigenvalue λ of the classical Dirac operator D_g:

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M s_g.$$

This inequality was improved by Hijazi [9] for dimension $n \geq 3$

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu,$$

where μ is the smallest eigenvalue of the conformal Laplacian $L_g = 4\frac{n-1}{n-2}\Delta_g + s_g$.

On closed manifolds, there is a conformal version of the Hijazi inequality that relates the corresponding conformal quantities, that means the Yamabe invariant

$$Q(M, g) = \inf \left\{ \int_M v L_g v \mathrm{dvol}_g \ \Big| \ \|v\|_{\frac{2n}{n-2}} = 1, v \in C_c^\infty(M) \right\}$$

with the λ_{min}^+-invariant

$$\lambda_{min}^+(M, g, \sigma) = \inf_{g_0 \in [g], \ \mathrm{vol}(M,g_0) < \infty} \lambda_1^+(M, g_0, \sigma) \mathrm{vol}(M, g_0)^{\frac{1}{n}}$$

where

$$\lambda_1^+(M,g,\sigma) = \inf\left\{\left.\frac{\|D_g\phi\|^2}{(D_g\phi,\phi)}\,\right|\,0 < (D_g\phi,\phi),\ \phi \in C_c^\infty(M,S)\right\}$$

and $[g]$ is the set of all metrics conformal to g. Furthermore, $C_c^\infty(M,S)$ denotes the compactly supported smooth spinors on (M,g,σ) and $\|.\| := \|.\|_{L^2}$.
The conformal Hijazi inequality reads

$$\lambda_{min}^+(M,g,\sigma)^2 \geq \frac{n}{4(n-1)}Q(M,g).$$

This can be seen immediately since on closed manifolds λ_1^+ is just the lowest positive Dirac eigenvalue, and for $Q \geq 0$ we have

$$Q(M,g) = \inf_{g_0 \in [g],\ \mathrm{vol}(M,g_0)<\infty} \mu(g_0)\mathrm{vol}(M,g_0)^{\frac{2}{n}} \tag{1}$$

where $\mu(g_0)$ is the infimum of the spectrum of L_{g_0}.

We note, that the λ_{min}^+-invariant can also be defined as a variational problem similar to the Yamabe invariant [2].

Since both the Yamabe and the λ_{min}^+-invariant can also be considered on open manifolds, cf. [10], [8], it is interesting to know whether the conformal Hijazi inequality also holds on these manifolds.
In this paper, we examine this question for conformally parabolic manifolds, i.e. those that admit a complete metric of finite volume in their conformal class.

At first, we obtain an Hijazi equality for Riemannian manifolds equipped with a complete metric of finite volume.

Theorem 1.1. *Let (M,g,σ) be a complete Riemannian spin manifold of finite volume and dimension $n > 2$. Moreover, let λ be an eigenvalue of its Dirac operator D_g, and let μ be the infimum of the spectrum of the conformal Laplacian. Then the following inequality holds:*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

If equality is attained, the manifold admits a real Killing spinor and has to be Einstein and closed.

On complete manifolds, the Dirac operator is essentially self-adjoint and, in general, its spectrum consists of eigenvalues and the essential spectrum. For elements of the essential spectrum, we also obtain an Hijazi-type inequality:

Theorem 1.2. *Let (M,g,σ) be a complete Riemannian spin manifold of dimension $n \geq 5$ with finite volume. Furthermore, let the scalar curvature of (M,g) be bounded from below. If λ is in the essential spectrum of the Dirac operator $\sigma_{ess}(D_g)$, then*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

These two Hijazi inequalities allow to prove the conformal version:

Theorem 1.3. *Let (M, g, σ) be a conformally parabolic Riemannian spin manifold of dimension $n > 2$. Let one of the following assumptions be fulfilled:*

(i) There is a complete conformal metric $\bar{g}$ of finite volume such that $0 \notin \sigma_{ess}(D_{\bar{g}})$.

(ii) The dimension is $n \geq 5$, and there is a complete conformal metric $\bar{g}$ of finite volume whose scalar curvature is bounded from below.

Then the conformal Hijazi inequality holds:

$$\lambda_{min}^+(M, g, \sigma)^2 \geq \frac{n}{4(n-1)} Q(M, g).$$

A manifold that does not fulfill the assumption (i) possesses a vanishing λ_{min}^+-invariant, cf. Lem. 3.3.iii. Thus, we obtain

Corollary 1.4. *Let (M, g, σ) be a conformally parabolic Riemannian spin manifold of dimension $n > 2$ and with $\lambda_{min}^+ > 0$. Then the conformal Hijazi inequality is valid.*

We give a brief outline of the paper: In Sect. 2, we review some notations for the identification of spinor bundles of conformally equivalent metrics. Furthermore, we give a refined Kato inequality that will be used in the proof of Theorem 1.2. In Sect. 3, we give some properties of λ_1^+ and λ_{min}^+ on conformally parabolic manifolds. With these preparations, the Theorems 1.1, 1.2 and 1.3 can be shown in Sect. 4.

2 Preliminaries

In this section, we first review the identification of spinor bundles of conformally equivalent metrics to fix notations. Then we give the refined Kato inequality that we use to prove Theorem 1.2.

Spinor bundles of conformally related metrics

Let $\bar{g} = f^2 g$ with $0 < f \in C^\infty(M)$. Having fixed a spin structure σ on (M, g) with corresponding spinor bundle S_g, there always exists a corresponding spinor bundle $S_{\bar{g}}$ on $(M, \bar{g})$ and a vector bundle isomorphism

$$A: \ S_g \to S_{\bar{g}}, \ \psi \mapsto \overline{\psi} := A(\psi)$$

that is fibrewise an isometry [9, Sect. 4.1], i.e. $< \overline{\psi}, \overline{\psi} >_{\bar{g}} = < \psi, \psi >_g$. In the following, we write for both spinor bundles just S.

Using this isometry, it is possible to compare the corresponding Dirac operators $D := D_g$ and $\overline{D} := D_{\bar{g}}$ [9, Prop. 4.3.1]:

$$\overline{D}(f^{-\frac{n-1}{2}} \overline{\psi}) = f^{-\frac{n+1}{2}} \overline{D\psi}.$$

Refined Kato inequalities

The Kato inequality states that for any section ϕ of a Riemannian or Hermitian vector bundle E endowed with a metric connection ∇ on a Riemannian manifold (M, g) we have

$$2|\phi||d|\phi|| = |d|\phi|^2| = 2| < \nabla\phi, \phi > | \leq 2|\phi||\nabla\phi|, \tag{2}$$

3

i.e. $|d|\phi|| \leq |\nabla\phi|$ away from the zero set of ϕ. For this estimate it is used that $< \nabla_X\phi, \phi >\in \mathbb{R}$ for all $X \in TM$.

In [5], refined Kato inequalities were obtained for sections in the kernel of first-order elliptic differential operators. They are of the form

$$|d|\phi|| \leq k_P|\nabla\phi|$$

where k_P is a constant depending on the operator P.

We sketch the set-up used in [5]: Let E be an irreducible natural vector bundle E over an n-dimensional Riemannian (spin) manifold (M, g) with scalar product $< ., . >$ and a metric connection ∇. Irreducible natural means that the vector bundle is obtained either from the orthonormal frame bundle of M or from the spinor frame bundle with an irreducible representation of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$ on a vector space V. We will denote this representation by λ. Further, let τ be the standard representation of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$ on $\mathbb{R}^n$. Then the real tensor product $\tau \otimes \lambda$ splits into irreducible components as

$$\tau \otimes \lambda = \bigoplus_{j=1}^{N} \mu^j, \qquad \mathbb{R}^n \otimes V = \bigoplus_{j=1}^{N} W_j.$$

This induces a decomposition of $T^*M \otimes E$ into irreducible subbundles F_j associated to μ^j. Further, let Π_j denote the projection onto the jth summand of $\mathbb{R}^n \otimes V$ and $T^*M \otimes E$, respectively.

Let P be a first-order linear differential operator of the form $P = \sum_{i \in I} \Pi_i \circ \nabla$ where $I \subseteq \{1, \ldots, N\}$. Moreover, we denote $\Pi_I := \sum_{i \in I} \Pi_i$ and $\hat{I} := \{1, \ldots, N\} \setminus I$.

Following the ansatz for the refined Kato inequalities we obtain the estimate:

Lemma 2.1. *Let P be an operator as defined above. Then we have away from the zero set of ϕ*

$$|d|\phi|| \leq |P\phi| + k_P|\nabla\phi|$$

where $k_P := \sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha \otimes v)|$.

The proof is done analogously to the one of [5] without the assumption that $\phi \in \ker P$. That's why the additional summand $|P\phi|$ appears and why the constant k_P remains the same.

For the shifted (classical) Dirac operator $D - \lambda$ we have $k = \sqrt{\frac{n-1}{n}}$ [5, (3.9)].

3   The invariant on conformally parabolic manifolds

Firstly, we give a characterization of conformally parabolic manifolds and consider the example of the Euclidean space.

In the rest of this section, we provide some properties of λ_1^+ for complete metrics with finite volume.

Definition 3.1. [11, Sect. 3] A Riemannian manifold is conformally parabolic if and only if its conformal class contains a complete metric of finite volume.

Example 3.2. Let (M^m, g_M) be a closed m-dimensional Riemannian manifold. Then $(M \times \mathbb{R}, \ g = g_M + dt^2)$ is conformally parabolic, since the conformal metric $\overline{g} = \frac{1}{t^2}g$ is complete and of finite volume.
Furthermore, for the new metric and for dimension $n > 2$ the scalar curvature is calculated as (where $h = t^{-\frac{n-2}{2}}$ and $n = m + 1$)

$$
\begin{aligned}
s_{\overline{g}} &= 4\frac{n-1}{n-2}h^{-\frac{n+2}{n-2}}\Delta_g h + s_g h^{-\frac{4}{n-2}} \\
&= -4\frac{n-1}{n-2}t^{\frac{n+2}{2}}\left(1 - \frac{n}{2}\right)\left(-\frac{n}{2}\right)t^{-\frac{n+2}{2}} + s_M t^2 \\
&= -(n-1)n + s_M t^2.
\end{aligned}
$$

Next we give some properties of λ_1^+:

Lemma 3.3.
i) If $\lambda_1^+(M, g, \sigma) = 0$ and $\operatorname{vol}(M, g) < \infty$, then $\lambda_{min}^+(M, g, \sigma) = 0$.
ii) If (M, g) is complete and $\lambda > 0$ is an eigenvalue of D or an element of its essential spectrum, then $\lambda_1^+(M, g, \sigma) \leq \lambda$.
iii) A complete Riemannian spin manifold of finite volume for which there exists a $\lambda > 0$ in the essential spectrum of its Dirac operator has a vanishing λ_{min}^+-invariant.

Proof. i) is seen immediately from the definition of λ_{min}^+.
ii) There exists a sequence $\phi_i \in C_c^\infty(M, S)$ with $\|D\phi_i - \lambda\phi_i\| \to 0$ and $\|\phi_i\| \to 1$: If λ is in the essential spectrum, this is obvious. If λ is an eigenvalue with eigenspinor $\phi \in C^\infty(M, S) \cap L^2(M, S)$, we choose $\phi_i = \eta_i\phi$ where η_i is a smooth cut-off function such that $\eta_i \equiv 1$ on $B_i(p)$ ($p \in M$ fixed), $\eta_i \equiv 0$ on $M \setminus B_{2i}(p)$ and in between $|\nabla\eta_i| \leq \frac{2}{i}$. This is always possible since (M, g) is complete. Then ϕ_i is the sequence in demand since $\|(D - \lambda)\phi_i\| = \|\nabla\eta_i \cdot \phi\| \leq \frac{2}{i}\|\phi\|$. Thus, in both cases

$$\frac{\|D\phi_i\|^2}{(D\phi_i, \phi_i)} \to \lambda$$

which proves the claim.
iii) Since the essential spectrum is a property of the manifold at infinity, see [4, Prop. 1], there is a sequence $\phi_i \in C_c^\infty(M \setminus B_i(p), S)$ ($p \in M$ fixed) with $\|(D - \lambda)\phi_i\| \to 0$ and $\|\phi_i\| = 1$. Thus, as in ii) we find

$$\lambda_{min}^+(M \setminus B_r(p), g, \sigma) \leq \lambda \operatorname{vol}(M \setminus B_r(p), g) \to 0$$

for $r \to \infty$. With $\lambda_{min}^+(M, g, \sigma) \leq \lambda_{min}^+(M \setminus B_r(p), g, \sigma)$ [8, Lem. 2.1], we have $\lambda_{min}^+(M, g, \sigma) = 0$. □

Lemma 3.4. Let (M, g, σ) be a complete Riemannian spin manifold. Then

$$\lambda_1^+(M, g, \sigma) = \inf\{\sigma(D) \cap (0, \infty)\}$$

where $\sigma(D)$ denotes the Dirac spectrum.

Proof. Since (M,g) is complete, D is essentially self-adjoint and has no residual spectrum, cf. [7, Chapt. 4]. By the spectral theorem for unbounded self-adjoint operators, we obtain that for every $\phi \in C_c^\infty(M,S)$ with $(D\phi,\phi) > 0$

$$\frac{\|D\phi\|^2}{(D\phi,\phi)} = \frac{\int_{\sigma(D)} \lambda^2 \, d < E_\lambda\phi, \phi >}{\int_{\sigma(D)} \lambda \, d < E_\lambda\phi, \phi >} \geq \frac{\int_{\sigma(D)\cap(0,\infty)} \lambda^2 \, d < E_\lambda\phi, \phi >}{\int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda\phi, \phi >}$$

$$\geq \frac{\lambda_0 \int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda\phi, \phi >}{\int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda\phi, \phi >} = \lambda_0$$

where $\lambda_0 = \inf\{\sigma(D) \cap (0,\infty)\}$. Note that $(D\phi,\phi) > 0$ and, thus, the denominator $\int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda\phi, \phi >$ is always positive. Hence, we have $\lambda_1^+ \geq \inf\{\sigma(D) \cap (0,\infty)\}$.
The converse inequality is obtained by Lemma 3.3.ii. $\qquad\square$

From Lemma 3.3.iii and Lemma 3.4, we have

Corollary 3.5. *Let (M,g,σ) be a complete Riemannian spin manifold of finite volume with $\lambda_{min}^+ > 0$. Then $\sigma(D) \cap (0,\infty)$ consists only of eigenvalues.*

The next Lemma shows that for defining the λ_{min}^+-invariant on conformally parabolic manifolds we do not need the infimum over all conformal metrics.

Lemma 3.6. *Let (M,g,σ) be a conformally parabolic Riemannian spin manifold. Then there exists a sequence of complete conformal metrics g_i of unit volume such that $\lambda_1^+(g_i) \to \lambda_{min}^+(g)$ and $g_i \equiv g_1$ near infinity, i.e.*

$$\lambda_{min}^+(M,g,\sigma) = \inf\{\lambda_1^+(M,\overline{g},\sigma) \mid \overline{g} \equiv g_1 \text{ near infinity, } \mathrm{vol}(M,\overline{g}) = 1\},$$

where "near infinity" refers to the existence of a compact subset $U \subset M$ such that $f \equiv 1$ on $M \setminus U$.

Proof. Assume that $g = g_1$ is already complete and of unit volume. Let $g_i = f_i^2 g$ be a sequence of conformal metrics of unit volume with $\lambda_1^+(g_i) \to \lambda_{min}^+$ for $i \to \infty$. Thus, there is a sequence $\phi_i \in C_c^\infty(M,S)$ such that

$$F(\phi_i,g_i) := \frac{\| D_{g_i}\phi_i \|_{g_i}^2}{(D_{g_i}\phi_i, \phi_i)_{g_i}} \to \lambda_{min}^+$$

Now, we choose the conformal factor h_i such that h_i is equal to f_i on the support of ϕ_i, $h_i = 1$ near infinity and $\int_M h_i^n \mathrm{dvol}_g = 1$. Then, $F(\phi_i, h_i^2 g) = F(\phi_i, g_i) \to \lambda_{min}^+$, and the metrics $h_i^2 g$ are complete, since g is complete, and they have unit volume. $\qquad\square$

4 Proof of Hijazi inequalities

Firstly, we follow the main idea of the proof of the original Hijazi inequality, but we fix the used conformal factor with the help of an eigenspinor. This results in a conformal metric on the manifold without the zero-set of the eigenspinor and we have to use cut-off functions near this zero-set and near infinity to obtain compactly supported test functions.

Theorem 1.1. Let $\psi \in C^\infty(M, S) \cap L^2(M, S)$ be an eigenspinor satisfying $D\psi = \lambda\psi$ and $\|\psi\| = 1$. Its zero-set Ω is closed and contained in a closed countable union of smooth $(n-2)$-dimensional submanifolds which has locally finite $(n-2)$-dimensional Hausdorff measure [3, p. 189].

We fix a point $p \in M$. Since M is complete, there exists a cut-off function $\eta_i : M \to [0, 1]$ which is zero on $M \setminus B_{2i}(p)$ and one on $B_i(p)$. In between the function is chosen such that $|\nabla \eta_i| \le \frac{4}{i}$ and $\eta_i \in C_c^\infty(M)$.
While η_i cuts off ψ at infinity, we define another cut-off near the zeros of ψ. For this purpose, we can assume without loss of generality that Ω is itself the countable union of $(n-2)$-submanifolds described above.

Let now $\rho_{a,\epsilon}$ be defined as

$$
\rho_{a,\epsilon}(x) = \begin{cases} 0 & \text{for } r < a\epsilon \\ 1 - \delta \ln \frac{\epsilon}{r} & \text{for } a\epsilon \le r \le \epsilon \\ 1 & \text{for } \epsilon < r \end{cases}
$$

where $r = d(x, \Omega)$ is the distance from x to Ω. The constant $a < 1$ is chosen such that $\rho_{a,\epsilon}(a\epsilon) = 0$, i.e. $a = e^{-\frac{1}{\delta}}$. Then $\rho_{a,\epsilon}$ is continuous, constant outside a compact set and Lipschitz. Hence, for $\phi \in C^\infty(M, S)$ the spinor $\rho_{a,\epsilon}\phi$ is an element in $H_1^r(M, S)$ for all $1 \le r \le \infty$.

Now, let $\psi_{ia} := \eta_i \rho_{a,\epsilon}\psi \in H_1^r(M, S)$ be defined. These spinors are compactly supported on $M \setminus \Omega$. Furthermore, $\bar{g} = e^{2u}g = h^{\frac{4}{n-2}}g$ with $h = |\psi|^{\frac{n-2}{n-1}}$ is a metric on $M \setminus \Omega$. Setting $\overline{\phi_{ia}} := e^{-\frac{n-1}{2}u}\overline{\psi_{ia}}$ ($\phi = e^{-\frac{n-1}{2}u}\psi$), the Lichnerowicz-type formula [9, (5.4)] implies

$$
\|(\overline{D} - \lambda e^{-u})\overline{\phi_{ia}}\|_{\bar{g}}^2 = \|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\bar{g}}^2 + \int_{M \setminus \Omega} \left(\frac{\bar{s}}{4} - \frac{n-1}{n}\lambda^2 e^{-2u} \right) |\overline{\phi_{ia}}|^2 \mathrm{dvol}_{\bar{g}}
$$

$$
- \frac{n-1}{n}(2\lambda e^{-u}(\overline{D} - \lambda e^{-u})\overline{\phi_{ia}} + \lambda e^{-u}\overline{\mathrm{grad}\,e^{-u} \cdot \phi_{ia}}, \overline{\phi_{ia}})_{\bar{g}}
$$

$$
= \|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\bar{g}}^2 + \int_M \left(\frac{\bar{s}}{4} - \frac{n-1}{n}\lambda^2 e^{-2u} \right) e^u |\psi_{ia}|^2 \mathrm{dvol}_g
$$

$$
- 2\frac{n-1}{n}((D - \lambda)\psi_{ia}, \lambda e^{-u}\psi_{ia})_g
$$

$$
= \|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\bar{g}}^2 + \frac{1}{4}\int_M h^{-1}Lh\, e^{-u}|\psi_{ia}|^2 \mathrm{dvol}_g
$$

$$
- \frac{n-1}{n}\lambda^2 \int_M e^{-u}|\psi_{ia}|^2 \mathrm{dvol}_g - 2\frac{n-1}{n}((D - \lambda)\psi_{ia}, \lambda e^{-u}\psi_{ia})_g,
$$

where $\nabla_X^f \phi := \nabla_X \phi + \frac{f}{n}X \cdot \phi$ for $f = \lambda e^{-u} \in C^\infty(M)$ is the Friedrich connection. For the second line we used $|\overline{\phi_{ia}}|^2 \mathrm{dvol}_{\bar{g}} = e^u|\psi_{ia}|^2 \mathrm{dvol}_g$, and the term $(\lambda e^{-u}\overline{\mathrm{grad}\,e^{-u} \cdot \phi_{ia}}, \overline{\phi_{ia}})_{\bar{g}}$ vanishes since $< \nabla f \cdot \phi, \phi > \in i\mathbb{R}$, cf. [9, Lem. 3.1]. The last line is obtained by replacing $\bar{s}e^{2u} = h^{-1}Lh$.

With $D\psi = \lambda\psi$ and $<\nabla f \cdot \psi, \psi > \in i\mathbb{R}$, we obtain

$$((D - \lambda)\psi_{ia}, \lambda e^{-u}\psi_{ia})_g = (\nabla(\eta_i\rho_{a,\epsilon})\psi, \lambda e^{-u}\eta_i\rho_{a,\epsilon}\psi)_g = 0.$$

Inserting this result, $\overline{D}\,\overline{\phi} = \lambda e^{-u}\overline{\phi}$ and $\|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi}_{ia}\|^2_{\overline{g}} \geq 0$ into the formula from above we further have

$$\|\overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}\|^2_{\overline{g}} \geq \int\limits_M \left(\frac{1}{4}\eta_i^2\rho_{a,\epsilon}^2|\psi|^{\frac{n-2}{n-1}}L|\psi|^{\frac{n-2}{n-1}} - \frac{n-1}{n}\lambda^2\eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\right)\mathrm{dvol}_g.$$

Moreover, we have

$$\|\overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}\|^2_{\overline{g}} = \int\limits_M |e^{-u}\overline{\nabla(\eta_i\rho_{a,\epsilon}) \cdot \phi}|^2\mathrm{dvol}_{\overline{g}} = \int\limits_M |\nabla(\eta_i\rho_{a,\epsilon}) \cdot \psi|^2 e^{-u}\mathrm{dvol}_g.$$

Thus, with $e^u = |\psi|^{\frac{2}{n-1}}$ the above inequality reads

$$\int\limits_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \geq \frac{1}{4}\int\limits_M \eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}}L(\eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}})\mathrm{dvol}_g$$

$$-\frac{n-1}{n-2}\int\limits_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g - \frac{n-1}{n}\lambda^2\int\limits_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g.$$

Hence, we obtain

$$\frac{2n-3}{n-2}\int\limits_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \geq \left(\frac{\mu}{4} - \frac{n-1}{n}\lambda^2\right)\int\limits_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g,$$

where μ is the infimum of the spectrum of the conformal Laplacian. With $(a + b)^2 \leq 2a^2 + 2b^2$ we have

$$k\int\limits_M (\eta_i^2|\nabla\rho_{a,\epsilon}|^2 + \rho_{a,\epsilon}^2|\nabla\eta_i|^2)|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \geq \left(\frac{\mu}{4} - \frac{n-1}{n}\lambda^2\right)\|\eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}}\|^2$$

where $k = 2\frac{2n-3}{n-2}$.

Next, we want a tend to zero:
Recall that $\Omega \cap \overline{B_{2i}(p)}$ is bounded, closed, $(n-2)$-C^∞-rectifiable and has still locally finite $(n-2)$-dimensional Hausdorff measure. For fixed i we estimate

$$\int\limits_M |\nabla\rho_{a,\epsilon}|^2\eta_i^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \leq \sup_{B_{2i}(p)}|\psi|^{2\frac{n-2}{n-1}}\int\limits_{B_{2i}(p)} |\nabla\rho_{a,\epsilon}|^2\mathrm{dvol}_g.$$

Furthermore, we set $B_\epsilon^2(p) := \{x \in B_\epsilon \mid d(x, p) = d(x, \Omega)\}$ with $B_\epsilon := \{x \in M \mid d(x, \Omega) \leq \epsilon\}$. For ϵ sufficiently small each $B_\epsilon^2(p)$ is star shaped. Moreover, there is an inclusion $B_\epsilon^2(p) \hookrightarrow B_\epsilon(0) \subset \mathbb{R}^2$ via the normal exponential map.

Then we can calculate

$$\int\limits_{B_\epsilon \cap B_{2i}(p)} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_g \leq \mathrm{vol}_{n-2}(\Omega \cap B_{2i}(p)) \sup_{x \in \Omega \cap B_{2i}(p)} \int\limits_{B_\epsilon^2(x) \setminus B_{a\epsilon}^2(x)} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_{g_2}$$

$$\leq c\,\mathrm{vol}_{n-2}(\Omega \cap B_{2i}(p)) \int\limits_{B_\epsilon(0) \setminus B_{a\epsilon}(0)} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_{g_E}$$

$$\leq c' \int\limits_{a\epsilon}^{\epsilon} \frac{\delta^2}{r} dr = -c'\delta^2 \ln a = c'\delta \to 0 \quad \text{for } a \to 0$$

where vol_{n-2} denotes the $(n-2)$-dimensional volume and $g_2 = g_{|_{B_\epsilon^2(p)}}$. The positive constants c and c' arise from $\mathrm{vol}_{n-2}(\Omega \cap B_{2i}(p))$ and the comparison of dvol_{g_2} with the volume element of the Euclidean metric.

Furthermore, by the monotone convergence theorem, we obtain

$$\int\limits_{B_{2i}(p)} \rho_{a,\epsilon}^2 |\nabla \eta_i|^2 |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g \to \int\limits_{B_{2i}(p)} |\nabla \eta_i|^2 |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g$$

as $a \to 0$ and, thus,

$$k \int\limits_M |\nabla \eta_i|^2 |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g \geq \left(\frac{\mu}{4} - \frac{n-1}{n} \lambda^2\right) \int\limits_M \eta_i^2 |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g.$$

Next we want to establish the limit for $i \to \infty$:
Since M has finite volume and $\|\psi\| = 1$, the Hölder inequality ensures that $\int\limits_M |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g$ is bounded. With $|\nabla \eta_i| \leq \frac{4}{i}$ we get

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu.$$

Equality is attained if and only if $\|\overline{\nabla}^{\lambda e^{-u}} \overline{\phi_{ia}}\|_{\overline{g}}^2 \to 0$ for $i \to \infty$ and $a \to 0$. We have

$$0 \leftarrow \|\overline{\nabla}^{\lambda e^{-u}} \overline{\phi_{ia}}\|_{\overline{g}} = \|\eta_i \rho_{a,\epsilon} \overline{\nabla}^{\lambda e^{-u}} \overline{\phi} + \overline{\nabla}(\eta_i \rho_{a,\epsilon}) \overline{\phi}\|_{\overline{g}}$$

$$\geq \|\eta_i \rho_{a,\epsilon} \overline{\nabla}^{\lambda e^{-u}} \overline{\phi}\|_{\overline{g}} - \|\overline{\nabla}(\eta_i \rho_{a,\epsilon}) \overline{\phi}\|_{\overline{g}}.$$

With $\|\overline{\nabla}(\eta_i \rho_{a,\epsilon}) \overline{\phi}\|_{\overline{g}} \to 0$, see above, $\overline{\nabla}^{\lambda e^{-u}} \overline{\phi}$ has to vanish on $M \setminus \Omega$. With [9, Cor. 3.6] this implies that e^{-u} is constant. Thus, (M, g) is Einstein and possesses a real Killing spinor, cf. [7, p. 118]. Furthermore, its Einstein constant is positive. Thus, the Ricci curvature is a positive constant and, hence, due to the Theorem of Bonnet-Myers M is already closed. $\qquad\square$

Next, we want to prove Theorem 1.2 using the refined Kato inequality:

Theorem 1.2. We may assume $\mathrm{vol}(M, g) = 1$. If λ is in the essential spectrum of D, then 0 is in the essential spectrum of $D - \lambda$ and of $(D - \lambda)^2$. Thus, there is a

sequence $\phi_i \in C_c^\infty(M, S)$ such that $\|(D-\lambda)^2\phi_i\| \to 0$ and $\|(D-\lambda)\phi_i\| \to 0$ while $\|\phi_i\| = 1$. We may assume that $|\phi_i| \in C_c^\infty(M)$. That can always be achieved by a small perturbation.

Now let $\frac{1}{2} \le \beta \le 1$. Then $|\phi_i|^\beta \in H_1^2(M)$. Firstly, we will show that the sequence $\||d|\phi_i|^\beta\|$ is bounded:
By the Hölder inequality we have

$$
0 \leftarrow \|\phi_i\|^{2\beta-1}\|(D-\lambda)^2\phi_i\| \ge \||\phi_i|^{2\beta-1}\|_{\{|\phi_i|\neq 0\}}\|(D-\lambda)^2\phi_i\|
$$
$$
\ge \Big| \int\limits_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2} <(D-\lambda)^2\phi_i, \phi_i> \mathrm{dvol}_g \Big|.
$$

Using the Lichnerowicz formula [9, (5.4)], we obtain

$$
\|(D-\lambda)^2\phi_i\| \ge \Big| \int\limits_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2} <\Delta^\lambda\phi_i, \phi_i> \mathrm{dvol}_g
$$
$$
+ \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g
$$
$$
- 2\frac{n-1}{n} \int\limits_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2} <(D-\lambda)\phi_i, \lambda\phi_i> \mathrm{dvol}_g \Big|
$$
$$
\ge \int\limits_{|\phi_i|\neq 0}|\phi_i|^{2\beta-2}|\nabla^\lambda\phi_i|^2\mathrm{dvol}_g + 2(\beta-1)\int\limits_{|\phi_i|\neq 0}|\phi_i|^{2\beta-3}<d|\phi_i|\cdot\phi_i, \nabla^\lambda\phi_i>\mathrm{dvol}_g
$$
$$
+ \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g - 2\frac{n-1}{n}\lambda\||\phi_i|^{2\beta-1}\|_{\{|\phi_i|\neq 0\}}\|(D-\lambda)\phi_i\|
$$

With the Hölder inequality (recall that $\beta \le 1$) and the Kato inequality for the connection ∇^λ, see (2), we have

$$
0 \leftarrow \|(D-\lambda)^2\phi_i\|
$$
$$
\ge (2\beta-1)\int\limits_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2}|d|\phi_i|||\nabla^\lambda\phi_i|\mathrm{dvol}_g + \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g
$$
$$
- 2\frac{n-1}{n}\lambda\|\phi_i\|^{2\beta-1}\|(D-\lambda)\phi_i\|
$$
$$
\ge (2\beta-1)\int\limits_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2}|d|\phi_i||^2\mathrm{dvol}_g + \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g
$$
$$
- 2\frac{n-1}{n}\lambda\|(D-\lambda)\phi_i\|
$$
$$
\ge (2\beta-1)\frac{1}{\beta^2}\int\limits_{|\phi_i|\neq 0} |d|\phi_i|^\beta|^2\mathrm{dvol}_g + \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g
$$
$$
- 2\frac{n-1}{n}\lambda\|(D-\lambda)\phi_i\|
$$

Since s is bounded from below, $\int s|\phi_i|^{2\beta}\mathrm{dvol}_g \ge \inf s \, \|\phi_i\|_{2\beta}^{2\beta} \ge \min\{\inf s, 0\}$ is also bounded. Thus, with $\|(D-\lambda)\phi_i\| \to 0$ we see that $\||d|\phi_i|^\beta\|$ is also bounded.

Next we fix $\alpha = \frac{n-2}{n-1}$ and obtain

$$\frac{\mu}{4} - \frac{n-1}{n}\lambda^2 \leq \left(\frac{\mu}{4} - \frac{n-1}{n}\lambda^2\right) \||\phi_i|^\alpha\|^2$$

$$\leq \frac{1}{4}\int |\phi_i|^\alpha L|\phi_i|^\alpha \mathrm{dvol}_g - \frac{n-1}{n}\lambda^2\||\phi_i|^\alpha\|^2$$

$$= \int |\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2 + \frac{1}{2}d^*d|\phi_i|^2 + \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right)|\phi_i|^2\right)\mathrm{dvol}_g$$

where we used the definition of μ as infimum of the spectrum of $L = 4\frac{n-1}{n-2}\Delta + s$.
The third line is obtained from

$$|\phi_i|^\alpha d^*d|\phi_i|^\alpha = \frac{\alpha}{2}|\phi_i|^{2\alpha-2}d^*d|\phi_i|^2 - \alpha(\alpha-2)|\phi_i|^{2\alpha-2}|d|\phi_i||^2.$$

Next, using

$$\frac{1}{2}d^*d < \phi_i, \phi_i > = < \nabla^*\nabla\phi_i, \phi_i > -|\nabla\phi_i|^2 = < D^2\phi_i, \phi_i > -\frac{s}{4}|\phi_i|^2 - |\nabla\phi_i|^2$$

and

$$|\nabla^\lambda\phi_i|^2 = |\nabla\phi_i|^2 - 2\mathrm{Re}\frac{\lambda}{n} < (D-\lambda)\phi_i, \phi_i > -\frac{\lambda^2}{n}|\phi_i|^2,$$

we have

$$\frac{\mu}{4} - \frac{n-1}{n}\lambda^2 \leq \int |\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2 - |\nabla^\lambda\phi_i|^2\right)\mathrm{dvol}_g$$

$$+ \int |\phi_i|^{2\frac{n-2}{n-1}-2} < (D^2-\lambda^2)\phi_i, \phi_i > \mathrm{dvol}_g$$

$$- \int 2|\phi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re}\frac{\lambda}{n} < (D-\lambda)\phi_i, \phi_i > \mathrm{dvol}_g$$

$$\leq \int |\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2 - |\nabla^\lambda\phi_i|^2\right)\mathrm{dvol}_g$$

$$+ \int |\phi_i|^{2\frac{n-2}{n-1}-2} < (D-\lambda)^2\phi_i, \phi_i > \mathrm{dvol}_g$$

$$+ \int 2\left(1 - \frac{1}{n}\right)\lambda|\phi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re} < (D-\lambda)\phi_i, \phi_i > \mathrm{dvol}_g.$$

The last two summands vanish in the limit since

$$\left|\int |\phi_i|^{2\frac{n-2}{n-1}-2} < (D-\lambda)^2\phi_i, \phi_i > \mathrm{dvol}_g\right| \leq \|(D-\lambda)^2\phi_i\| \, \| \, |\phi_i|^{\frac{n-3}{n-1}}\| \to 0$$

and

$$\left|\int |\phi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re} < (D-\lambda)\phi_i, \phi_i > \mathrm{dvol}_g\right| \leq \|(D-\lambda)\phi_i\| \, \| \, |\phi_i|^{\frac{n-3}{n-1}}\| \to 0.$$

For the other summand we use the Kato-type inequality of Lemma 2.1

$$|d|\psi|| \leq |(D-\lambda)\psi| + k|\nabla^\lambda\psi|$$

11

which holds outside the zero set of ψ. Due to [5, (3.9)] we have $k = \sqrt{\frac{n-1}{n}}$. Thus, for $n \geq 5$ we can estimate

$$\int |\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2 - |\nabla^\lambda \phi_i|^2\right) \mathrm{dvol}_g$$

$$= \int |\phi_i|^{2\frac{n-2}{n-1}-2}(k^{-1}|d|\phi_i|| - |\nabla^\lambda \phi_i|)(k^{-1}|d|\phi_i|| + |\nabla^\lambda \phi_i|)\mathrm{dvol}_g$$

$$\leq k^{-1}\int\limits_{\{|d|\phi_i||\geq k|\nabla^\lambda \phi_i|\}} |\phi_i|^{2\frac{n-2}{n-1}-2}|(D-\lambda)\phi_i|(k^{-1}|d|\phi_i|| + |\nabla^\lambda \phi_i|)\mathrm{dvol}_g$$

$$\leq 2k^{-2}\int\limits_{\{|d|\phi_i||\geq k|\nabla^\lambda \phi_i|\}} |\phi_i|^{2\frac{n-2}{n-1}-2}|(D-\lambda)\phi_i||d|\phi_i||\mathrm{dvol}_g$$

$$\leq 2k^{-2}\int \left(2\frac{n-2}{n-1}-1\right)^{-1}|(D-\lambda)\phi_i||d|\phi_i|^{2\frac{n-2}{n-1}-1}|\mathrm{dvol}_g$$

$$\leq 2k^{-2}\frac{n-1}{n-3}\|(D-\lambda)\phi_i\|\,\||d|\phi_i|^{\frac{n-3}{n-1}}\|.$$

For $n \geq 5$ we have $1 \geq \frac{n-3}{n-1} \geq \frac{1}{2}$ and, thus, $\||d|\phi_i|^{\frac{n-3}{n-1}}\|$ is bounded. Together with $\|(D-\lambda)\phi_i\| \to 0$ we obtain the following: For all $\epsilon > 0$ there is an i_0 such that for all $i \geq i_0$ we have

$$\int |\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2 - |\nabla^\lambda \phi_i|^2\right) \mathrm{dvol}_g \leq \epsilon.$$

Hence, we have $\frac{\mu}{4} \leq \frac{n-1}{n}\lambda^2$. $\qquad\square$

With these Hijazi inequalities, we can now prove the conformal Hijazi inequality:

Theorem 1.3. For $Q < 0$ the inequality is trivially satisfied. Thus, we restrict ourselves to the case $Q \geq 0$:
We may assume that g is itself a complete metric of finite volume satisfying the condition (i): $0 \notin \sigma_{ess}(D_g)$. Due to Lemma 3.6 there exists a sequence g_i of complete metrics of unit volume with $g_i \equiv g$ near infinity and $\lambda_1^+(g_i) \to \lambda_{min}^+$.
We first consider the case that there is an infinite subsequence g_{i_j} such that $\lambda_1^+(g_{i_j})$ is an eigenvalue of $D_{g_{i_j}}$. Then we can apply Theorem 1.1 and equality (1) and obtain

$$\lambda_1^+(M, g_{i_j}, \sigma)^2 \geq \frac{n}{4(n-1)}\mu(M, g_{i_j}) \geq \frac{n}{4(n-1)}Q(M, g).$$

Thus, for $j \to \infty$ we obtain the conformal Hijazi inequality.
Now we consider the remaining case – only finitely many $\lambda_1^+(g_i)$ are eigenvalues. Thus, from Lemma 3.4 we know that then there is an infinite subsequence g_{i_j} such that $\lambda_1^+(g_{i_j}) \in \sigma_{ess}(D_{g_{i_j}})$. But if for two metrics g_i and g_k we have $\sigma_{ess}(D_{g_i}) \ni \lambda_1^+(g_i) \geq \lambda_1^+(g_k) \in \sigma_{ess}(D_{g_k})$, then $\lambda_1^+(g_i)$ already equals $\lambda_1^+(g_k)$ since $g_k \equiv g_i$ near infinity and the essential spectrum only depends on the manifold at infinity. Hence, there has to exist a constant subsequence $\lambda_{min}^+ = \lambda_1^+(g_{i_j}) \in \sigma_{ess}(D_{g_{i_j}}) = \sigma_{ess}(D_g)$. Lemma 3.3.iii then gives $\lambda_{min}^+ = 0$ and, thus, $0 \in \sigma_{ess}(D_g)$. This is a contradiction to the assumption.

So we assume now that $0 \in \sigma_{ess}(D)$. Then condition (ii) has to be fulfilled and Theorem 1.2 implies $\mu \leq 0$ and, thus, $Q \leq 0$. $\qquad\square$

Example 4.1. We consider the n-dimensional Riemannian manifold $(M \times \mathbb{R}, g_M + dt^2)$ where (M, g_M) is closed, spin and with positive scalar curvature. Due Example 3.2 $M \times \mathbb{R}$ is conformally parabolic and the scalar curvature of $\overline{g} = \frac{1}{t^2} g$ is bounded from below. Hence, with Theorem 1.3 we know that at least for $n \geq 5$ the conformal Hijazi inequality is valid Furthermore, for $n = 3$ and $n = 4$ if this inequality turns out to be wrong, λ_{min}^+ has to vanish. Note, that $Q(M \times \mathbb{R}, g_M + dt^2) > 0$ for M having positive scalar curvature [1, Prop. 5.7].

References

[1] K. Akutagawa and B. Botvinnik. Yamabe metrics on cylindrical manifolds. *Geom. Funct. Anal.*, 13(2):259–333, 2003.

[2] Bernd Ammann. The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions. *arXiv: math.DG/0309061 v2.*

[3] Christian Bär. Zero sets of solutions to semilinear elliptic systems of first order. *Invent. Math.*, 138(1):183–202, 1999.

[4] Christian Bär. The Dirac operator on hyperbolic manifolds of finite volume. *J. Differential Geom.*, 54(3):439–488, 2000.

[5] David M. J. Calderbank, Paul Gauduchon, and Marc Herzlich. Refined Kato inequalities and conformal weights in Riemannian geometry. *J. Funct. Anal.*, 173(1):214–255, 2000.

[6] Thomas Friedrich. Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung. *Math. Nachr.*, 97:117–146, 1980.

[7] Thomas Friedrich. *Dirac operators in Riemannian geometry*, volume 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[8] Nadine Große. On a conformal invariant of the Dirac operator on noncompact manifolds. *Ann. Global Anal. Geom.*, 30(4):407–416, 2006.

[9] Oussama Hijazi. A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors. *Comm. Math. Phys.*, 104(1):151–162, 1986.

[10] R. Schoen and S.-T. Yau. Conformally flat manifolds, Kleinian groups and scalar curvature. *Invent. Math.*, 92(1):47–71, 1988.

[11] V. A. Zorich and V. M. Kesel'man. On the conformal type of a Riemannian manifold. *Funktsional. Anal. i Prilozhen.*, 30(2):40–55, 96, 1996.

MPIMiS, Inselstr. 22, Leipzig 04013, Germany; ngrosse@mis.mpg.de