Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 49 2008

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It is pleasure to dedicate this paper to Professor M. Lakshmanan on his 60th birthday

Abstract

We establish a close relation between higher order Riccati equations and Faá di Bruno polynomial respectively Ramanujan's differential equations connected to modular forms.

Keywords and phrases Eisenstein series, Ramanujan's differential equation, Riccati equation, Faá di Bruno polynomial.

Mathematics Subject Classification Primary 58C20; Secondary 11C08.

1 Introduction

The ordinary Riccati equation plays a very important role in the solution of nonlinear integrable partial differential equations. Higher order Riccati equations can be obtained by reduction of the Matrix Riccati equation. All the higher order Riccati equations can be linearlized via a Cole-Hopf transformation to linear differential equations. It is known, that the higher order Riccati equations play the role of Bäcklund transformations for integrable partial differential equations of higher order than the KdV equation. The Riccati chain without potential is naturally associated to Faá di Bruno polynomials. The Faá di Bruno polynomials appear in several branches of mathematics and physics and can be introduced in several ways [14, 15]. The generalized Riccati equation can be written in a compact form in terms of Faá di Bruno polynomials. It plays a very important role in integrable systems.

The Chazy equation is deeply connected to special automorphic functions. In 1909, Chazy [6] in his study of Painlevé type equations of third order, considered the following nonlinear differential equation

$$Y''' = 2YY'' - 3Y'^2.$$

It is known that a particular solution of the Chazy equation is given in terms of weight 2 Eisenstein series of the full modular group $SL(2, \mathbf{Z})$.

In 1916, Ramanujan [16, 17] (see also [3]) introduced the functions $P(q), Q(q)$ and $R(q)$ defined for $|q| < 1$ by

$$P(q) := 1-24\sum_{i=1}^{\infty}\sigma_1(n)q^n, \qquad Q(q) := 1+240\sum_{i=1}^{\infty}\sigma_3(n)q^n, \qquad R(q) := 1-504\sum_{i=1}^{\infty}\sigma_5(n)q^n.$$

We relate the third order Riccati equation in terms of modified Ramanujan differential equations for certain functions $\tilde{P}, \tilde{Q}$ and $\tilde{R}$ which should be somehow related to Ramanujan's Eisenstein series P, Q and R.

In this report we illucidate two things. At first we study higher Riccati equations and its relations to Faá di Bruno polynomials. Inspired by the work of Ablowitz et. al [2] in this article we study further connections between generalized Ramanujan differential equations and Faá di Bruno polynomials through the Chazy equations. In fact Faá di Bruno polynomials play an important role in Ramanujan's Eisenstein series [4, 11].

2 The Riccati chain, Faá di Bruno polynomials and Chazy systems

It is known that by using the Cole-Hopf transformation one obtains a whole class of nonlinear ODEs, which possesses the same kind of properties as the Riccati equation, known as Riccati chain ([10], references therein). Let L be a following differential operator

$$L = \frac{d}{dx} + v(x). \tag{1}$$

For n any integer the nth–order equation of the Riccati chain is given by the following formula

$$L^n v(x) + \sum_{j=1}^{n-1} \alpha_j(x)\big(L^{j-1} v(x)\big) + \alpha_0(x) = 0, \tag{2}$$

where $\alpha_j(x)$, $j = 0, 1, \cdots N$, are arbitrary functions.

The lowest-order equations for $n = 2$ and $n = 3$ in this chain after the ordinary Riccati equation have the explicit form:

$$v_{xx} + 3v(x)v_x + v^3(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0 \tag{3}$$

$$v_{xxx} + 4vv_{xx} + 3v_x^2 + 6v^2 v_x + \alpha_2(x)v_x + +v^4(x) + \alpha_2(x)v^2(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \tag{4}$$

2.1 The Restricted Riccati chain and Faá di Bruno polynomials

In the following we study the restricted Riccati chain, when all the coefficients α_i vanish, and its relation to the Faá di Bruno formula.

We consider the sequence of derivatives $f^{(j)} = (L^j v)f$, $j = 0, 1, 2, 3, \cdots$ with the operator L in (1) and $f = f(x)$ an arbitrary function. Thereby one arrives at the following functions depending on an increasing number of arguments

$$f^{(0)} \equiv f^{(0)}(x, 0) = v(x)f(x)$$

$$f^{(1)} \equiv f^{(1)}(x, v, v_x) = (\frac{d}{dx} + v)f^{(0)} = (v_x + v^2)f$$

$$f^{(2)} \equiv f^{(2)}(x, v, v_x, v_{xx}) = (\frac{d}{dx} + v)f^{(1)} = (v^3 + 3vv_x + v_{xx})f$$

and so on.

Then the restricted Riccati chain is closely related to the famous *Faá di Bruno polynomials* defined by

$$f^{(j+1)} = (\partial_x + v)f^{(j)} \tag{5}$$

The ordinary Riccati equation $v_x + v^2 + u = 0$ for instance can be written also in the form

$$f^{(1)}(x, v, v_x) + uf = 0, \tag{6}$$

where u is a potential and f an arbitrary function.

2.2 Faá di Bruno's formula and the Riccati chain

In this section we discuss the Faá di Bruno formula and the nature of the differential polynomials appearing in this formula.

If g and h are arbitrary functions with a sufficient number of derivatives, then

$$\frac{d^n}{dt^n} g(h(t)) = \sum \frac{n!}{b_1! b_2! \cdots b_n!} g^{(k)}(h(t)) \big(\frac{h'(t)}{1!}\big)^{b_1} \cdots \big(\frac{h^{(n)}(t)}{n!}\big)^{b_n}$$

where the sum is over all different solutions in nonnegative integers $b_1, \cdots, b_n$ of $b_1 + 2b_2 + \cdots + nb_n = n$ and $k := b_1 + \cdots + b_n$. For example, when $n = 3$, Faá di Bruno's formula reads

$$\frac{d^3}{dt^3} g(h(t)) = g'(h(t))h''' + 3g''h'(t)h''(t) + g'''((h(t))\big(h'(t)\big)^3.$$

$$\Longrightarrow \qquad \sum \frac{n!}{b_1!b_2!b_3!} g^{(k)}(h(t))\big(\frac{h'(t)}{1!}\big)^{b_1}\big(\frac{h''(t)}{2!}\big)^{b_2}\big(\frac{h'''(t)}{3!}\big)^{b_3}$$
$$= g'(h(t))h''' + 3g''h'(t)h''(t) + g'''((h(t))\big(h'(t)\big)^3,$$

where $b_1 + 2b_2 + 3b_3 = 3$ and $k = b_1 + b_2 + b_3$.

One can associate partitions of the sets $\{1, 2, 3, .., n\}$ to the n-th derivative of composite functions $g(h(t))$ in the following way: for $n = 1$ we associate $\{1\}$ with the term $g'(h(t))h'(t)$, for $n = 2$ there are two partitions, namely $\{1, 2\}$ and $\{1\}$, $\{2\}$. To them we associate the term $g'(h(t))h''(t)$ respectively the term $g''(h(t))(h'(t))^2$. Di Bruno's formula can then be reformulated as follows [**?**]:

Proposition 1 *(Faá di Bruno) If g and h are functions with a sufficient number of derivatives, then*

$$\frac{d^n}{dt^n} g(h(t)) = \sum g^{(k)}(h(t)\big(h'(t)\big)^{b_1}\big(h''(t)\big)^{b_2} \cdots \big(h^{(n)}(t)\big)^{b_n} \tag{7}$$

where the sum is over all partitions of $\{1, 2, \cdots, n\}$, and, for each partition, k is its number of blocks and b_i is the number of blocks with exactly i elements.

There is a direct relation of this formula with the n-th order equation $L^n v = 0$ of the Riccati chain (2) for vanishing α's:

Corollary 1 *For L the operator $L = \frac{d}{dx} + v$, the following formula holds*

$$L^n v(x) = \sum_{\substack{\mathcal{A}=\{A:\, A\subset\{1,..,n+1\},\, \cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:\, |A|=i\}\}}} (v)^{b_1}(v')^{b_2} \cdots (v^{(n)})^{b_{n+1}}.$$

where the sum is over all partitions $\mathcal{A}$ of the set $\{1, .., n+1\}$ into non empty different subsets A.

Proof The proof is by induction on n and follows similar arguments in [**?**]. For $n = 0$ the formula is trivial. Let's assume the formula holds for n. Then one gets

$$L^{n+1}v = LL^n v = L \sum_{\substack{\mathcal{A}=\{A:\, \cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1}(v')^{b_2} \cdots (v^{(n)})^{b_{n+1}}.$$

Hence inserting expression (1) we get

$$L^{n+1}v = \sum_{\substack{\mathcal{A}=\{A:\,\cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1+1}(v')^{b_2}\cdots(v^{(n)})^{b_{n+1}} +$$

$$+ \sum_{i=1}^{n} \sum_{\substack{\mathcal{A}=\{A:\,\cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1}(v')^{b_2}\cdots b_i(v^{(i-1)})^{b_i-1}(v^{(i)})^{b_{i+1}+1}(v^{(n)})^{b_{n+1}} + \quad (8)$$

$$+ \sum_{\substack{\mathcal{A}=\{A:\,\cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1}(v')^{b_2}\cdots(v^{(n)})^{b_{n+1}-1}b_{n+1}v^{(n+1)}.$$

But given a partition $\mathcal{A}=\{A\}$ of the set of numbers $\{1,...,n+1\}$ then $\mathcal{A}'=\mathcal{A}\cup\{n+2\}$ defines a partition of the set of numbers $\{1,..,n+2\}$ with $b_1'=b_1+1$ and $b_i'=b_i$, $i=2,..,n+1$ respectively $b_{n+2}=0$. This corresponds just to the first term on the right hand side of eq. (8). Furthermore, for any $1\le i\le n$ and any element $B\in\mathcal{A}$ with $|B|=i$ the set $\mathcal{A}'=\mathcal{A}_B=\{A'\}$, where $A'=\{B\,\cup\{n+2\}\}$ and $A'=A$ for all $A\in\mathcal{A}, A\neq B$, defines a partition $\mathcal{A}'$ of the set $\{1,...,n+2\}$ with $b_i'=b_i-1$ and $b_{i+1}'=b_{i+1}+1$ respectively $b_j'=b_j$ for $j\neq i,i+1$. There are exactly b_i such partitions of the set of numbers $\{1,..,n+2\}$. Obviously they give rise to the middle term in eq. (8). The last term in this equation however corresponds to the partition $\mathcal{A}'=\{1,...,n+1\}$ with $b_i'=0$, $1\le i\le n+1$ and $b_{n+2}=1$. This concludes the proof of the Corollary.
Hence the n-th order Ricatti equation can be directly expressed with Faá di Bruno's formula.

2.3 Connections to Chazy systems

Chazy [6] attempted to generalize the work of Painlevé to third order differential equations. In an attempt to classify the third-order ODEs $y'''=F(x,y,y',y''$ with F polynomial in y, y', y'' having the Painlevé property, Chazy introduced 13 classes of reduced equations. Chazy's work is closely related to the theory of modular functions. Modular functions are an important family of special functions that satisfy a third order differential equation. These functions also appear in integrable systems [12, 18]. Thus Chazy equations are always fascinating equations and can be considered to be a close analogue of Painlevé equations of third order differential equations. In fact, Chazy's third order equation has a special solution that is also related to the sixth Painlevé equation. These special solutions are known as Picard solutions of the sixth Painlevé equation. Later, Bureau [5] extended Painlevés first objective, and gave a partial classification of fourth-order equations. In recent years Cosgrove [8, 9] presented at first in a superb paper the results on the Painlevé classification of the fourth- and fifth-order ODEs of the reduced forms $y^{(4)}=Ayy''+B(y')^2+Cy^3$ and $y^{(5)}=Ay'''+By'y''+Cy^2y'$.

We show that the generalized Chazy equation is a third order Riccati equation in disguise.

Proposition 2 *The third order restricted Riccati equation (4) (or third Faá di Bruno* *polynomial) is equivalent to*

$$v_{xxx} - 2vv_{xx} + 3v_x^2 = \frac{1}{8}(6v_x - v^2)^2. \tag{9}$$

Proof: The proof follows from rescaling $v \to v/2$ and $x \to -x$.
□

Finally, we show that another Chazy equation, namely, Chazy -IV is connected to our programme.

Proposition 3 *The Chazy -IV equation*

$$v_{xxx} = -3vv_{xx} - 3v_x^2 - 3v^2 v_x \tag{10}$$

is a derivative of second Faá di Bruno polynomial.

3 Eisenstein series and Ramanujan differential equations

Let τ be a complex number with strictly positive imaginary part. In contemporary notation, the Eisenstein series $G_{2k}(\tau)$ of weight $2k$ on the full modular group $\Gamma(1)$, where $k \geq 1$, are defined by

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}} \qquad m, n \in \mathbf{Z}.$$

Suppose $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ then for $k > 1$

$$G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_{2k}(\tau)$$

and G_{2k} is therefore a modular form of weight $2k$ for $k \geq 2$.

Let us define $q = e^{2\pi i\tau}$. Then the Fourier series of the Eisenstein series is

$$G_{2k}(\tau) = 2\zeta(2k)\left(1 + c_{2k}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n\right),$$

where the Fourier coefficients c_{2k} are given as

$$c_{2k} = \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)} = \frac{-4k}{B_{2k}},$$

with $\sigma_\nu(n)$ the sum of the ν-th powers of the divisors of n.

This allows us to define the *normalized Eisenstein series* $E_{2k}(\tau)$, with $E_{2k}(i\infty) = 1$ as

$$E_{2k}(\tau) := \frac{G_{2k}(\tau)}{2\zeta(2k)}. \tag{11}$$

In Ramanujan's notation, the three most relevant Eisenstein series are defined for $|q| < 1$ by

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\ Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},\ R(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

In contemporary notation we have

$$P = E_2, \qquad Q = E_4 \qquad R = E_6.$$

Then Ramanujan's famous differential equations are given as

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12} \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3} \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{12}$$

Let us introduce the variable y as $q = e^{2y}$. Then Ramanujan's differential equations become

$$\frac{dP}{dy} = \frac{1}{6}(P^2 - Q) \qquad \frac{dQ}{dy} = \frac{2}{3}(PQ - R) \qquad \frac{dR}{dy} = (PR - Q^2),$$

which lead to a third order differential equation for $P(y) = E_2(e^{2y})$:

$$P_{yyy} - 2PP_{yy} + 3P_y^2 = 0. \tag{13}$$

This equation is called the Chazy III equation. It arises in the study of third order ordinary differential equations having the " Painleve property", that means, their solutions have only poles as moveable singularities.

3.1 Faá di Bruno to Ramanujan's equation

We establish a direct connection between Faá di Bruno polynomials and Ramanujan-like differential equations.

Proposition 4 *The third order Riccati equation (4) (or Faá di Bruno polynomial $f^{(3)}$)*

$$P_{yyy} - 2PP_{yy} + 3P_y^2 = \frac{1}{8}(6P_y - P^2)^2$$

is equivalent to some modified Ramanujan's equations

$$\frac{dP}{dy} = \frac{1}{6}(P^2 - Q) \qquad \frac{dQ}{dy} = \frac{2}{3}(PQ - R) \qquad \frac{dR}{dy} = (PR + \frac{1}{8}Q^2),$$

which differ from the ordinary equations of Ramanujan only by the factor $1/8$ *in the equation for* R*.*

Proof: From the first equation we obtain $Q = P^2 - 6P_y$. This yields

$$R = 9P_{yy} - 9PP_y + P^3$$

from the second equation. The last equation can also be rewritten as $R_y = (PR - Q^2(1 - 9/8))$ Substituting the values of R and Q in the last equation we obtain our desired result. If on the other hand P solves the Chazy equation then Q and R are determined by the modified Ramanujan equations. □

It must be noticed that R can also be expressed in terms of second Faá di Bruno polynomial $f^{(2)}(P)$ of the variable P. In fact if P solves the second Riccati equation, then Q and R are given by the Faá di Bruno polynomials determined by P. These are the underlying structures also of the ordinary Ramanujan differential equations. It would be interesting to clarify the nature of these functions P, Q and R and especially their relation to the functions of Ramanujan.

4 Abel equation and the modified Ramanujan differential equations

By replacing y by $-\frac{1}{2}y$, we find that the above Ramanujan-like differential equations become a special case of the generalized Ramanujan equations

$$P_y = -\frac{1}{12}(\delta P^2 - Q), \qquad Q_y = -\frac{1}{3}(PQ - R), \qquad R_y = -\frac{1}{2}(PR - \gamma Q^2). \tag{14}$$

Changing $\{P, Q, R\}$ to a new set of variables $\{P_1, Q_1, R_1\}$ such that the vector fields $X_1 = P\frac{\partial}{\partial P}$, $X_2 = Q\frac{\partial}{\partial Q}$ and $X_3 = R\frac{\partial}{\partial R}$ become three translation generators (Straightening-out theorem) which generates a one-parameter Lie group of point symmetries of equations (14) [for details, [13]]. Therefore the differential equations are invariant under the simple one-parameter group of transformations

$$y_1 = e^{\kappa} y, \qquad P_1 = e^{-\kappa} P, \qquad Q_1 = e^{-2\kappa} Q, \qquad R_1 = e^{-3\kappa} R. \tag{15}$$

The invariance of each equation follows from a straightforward calculation.

Definition 1 *Consider a new set of variables defined in terms of P, Q and R by*

$$a = \frac{R}{Q^{3/2}}, \qquad b = \frac{Q^{1/2}}{P}, \qquad c = yQ^{1/2}, \tag{16}$$

where $Q^{1/n}$ denotes the principal nth root of Q.

Proposition 5 *In terms of these new set of variables, the generalized Ramanujan equations become*

$$y\frac{da}{dy} = -\frac{c}{2}(a^2 - \gamma), \tag{17}$$

$$y\frac{db}{dy} = -\frac{c}{12}(b^2 - 2ab + \delta), \tag{18}$$

$$y\frac{dc}{dy} = c\Big(1 + \frac{c}{6}(a - \frac{1}{b})\Big) \tag{19}$$

Proof: Straightforward computation.
□

Proposition 6 *The generalized Ramanujan differential equations are equivalent to a Painlevé-type differential equation of the third order in the polynomial class given by*

$$P_{xxx} = \frac{1}{9}\Big((15+3\delta)PP_{xx} + (6-36\gamma+3\delta)P_x^2 + 6(2\delta\gamma-\delta-1)P^2P_x + \delta(1-\delta\gamma)P^4\Big). \quad (20)$$

This differential equation is associated with the Bureau symbol $P1$ when not all of the constants are zero.

Proof: Setting $x = -2y$ we obtain from the first equation in (14)
$Q = \delta P^2 - 6P_x$, and this yields

$$R = \delta P^3 - (6+3\delta)PP_x + 9P_{xx}$$

from the second equation. Substituting these into the third equation yields equation (20).
□

Remark Chazy and Bureau [6, 5] have determined all the cases where equations (20) demonstrate the Painlevé property. Chazy listed twelve canonical reduced equations, denoted by $I--XII$, and Bureau listed eleven because he choose to absorb Chazy-III into $XIII$. Equation (20) can not be reduced to Chazy classes except for two cases: (a) $\delta = \gamma = 1$ and (b) $\delta = 1, \gamma = 1 + K$. For these two cases equation [6] reduces to the Chazy III and Chazy XII equations.

Acknowledgements : We express grateful thanks to Professor Bruce Berndt for numerous correspondences and enlighting discussions.

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