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Geodesic Flow on Extended Bott-Virasoro  
Group and Generalized Two Component Peakon  
Type Dual Systems

by

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# Geodesic Flow on Extended Bott-Virasoro Group and Generalized Two Component Peakon Type Dual Systems

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## Abstract

This paper discusses several algorithmic ways of constructing integrable evolution equations based on Lie algebraic structure. We derive, in a pedagogical style, a large class of two component peakon type dual systems from their two component soliton equations counter part. We study the essential aspects of Hamiltonian flows on coadjoint orbits of the centrally extended semidirect product group  $\widehat{Diff}(S^1) \ltimes C^\infty(S^1)$  to give a systematic derivation of the dual counter parts of various two component of integrable systems, viz., the dispersive water wave equation, the Kaup-Boussinesq system and the Broer-Kaup system, using moment of inertia operators method and the (frozen) Lie-Poisson structure. This paper essentially gives Lie algebraic explanation of Olver-Rosenau's paper [31].

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**Key Words :** geodesic flow, diffeomorphism, Virasoro orbit, Sobolev norm, dual equation, frozen Lie-Poisson structure.

## 1 Introduction

Recently a 2-component generalization of the Camassa-Holm equation has drawn a lot of interest among scientists. The integrable system group at SISSA, Dubrovin and his coworkers have been working on multi-component analogues, using reciprocal transformations and studying their effect on the Hamiltonian structures, [9, 10, 27]. They show

that the 2-component system cited above admits peakons, albeit of a different shape owing to the difference in the corresponding Green's functions.

It has been shown [22] that a 2-component generalization of the Camassa–Holm equation and its supersymmetric analogue also follow from the geodesic motion with respect to the  $H^1$  metric on the semidirect product space  $Diff(S^1) \ltimes C^\infty(S^1)$  and its supergroup respectively. In fact it is known that numerous coupled KdV equations [19, 20, 21] follow from the geodesic flows of the right invariant  $L^2$  metric on the extended semidirect product group  $Diff(S^1) \widehat{\ltimes} C^\infty(S^1)$  [1, 28]. Since 80's, the coupled KdV systems are considered to be important mathematical models. These set of equations are used in various physical phenomena. In 1981, Fuchssteiner [15] made a detailed study of four coupled KdV equation and formulated the bihamiltonian structure of them. Later Antonowicz and Fordy [2] gave first systematic derivations of a large number of coupled KdV systems.

About ten years ago, Rosenau, [33], introduced a class of solitary waves with compact support as solutions of certain wave equations with nonlinear dispersion. It was found that the solutions of such systems unchanged from collision and were thus called compactons. Later Olver and Rosenau showed [31] that a simple scaling argument shows that most integrable bihamiltonian systems are governed by tri-Hamiltonian structures. They formulated a method of “tri-Hamiltonian duality”, in which a recombination of the Hamiltonian operators leads to integrable hierarchies endowed with nonlinear dispersion that supports *compactons* or *peakons*. A related construction can be found in the contemporaneous paper of Fuchssteiner [16]. The spirit of the Olver and Rosenau method, i.e., algorithmic ways to derive integrable generalizations of the standard integrable systems was given in [17, 12] in the early 1980's. However it was not until these models reappeared in physical problems, and their novel solutions such as compactons and peakons were discovered, that the method achieved recognition. It should be emphasized that a large class of Hamiltonian structures was obtain in [13] using a Backlund transformation, from which it is immediately obvious that the relevant systems are tri-Hamiltonian (without the need for a scaling argument). Among the equations obtained in [13] is the so called Camassa-Holm equation (this is the reason that several authors refer to this equation as the Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation). Furthermore, the ”tri-Hamiltonian” approach was used in [11] when in addition to the FFCH, similar analogues for the NLS and sG were derived. In fact, in a more recent paper Fokas et al [14] discusses several algorithmic ways of constructing integrable evolution equations based on the use of multi-Hamiltonian structures.

The tri-Hamiltonian formalism can be best described through examples. The Korteweg–deVries equation

$$u_t = u_{xxx} + 3uu_x, \tag{1}$$

can be written in bihamiltonian form

$$u_t = J_1 dH_2 = J_2 dH_1$$

using the two compatible Hamiltonian operators

$$J_1 = D, \quad J_2 = D^3 + uD + Du \quad \text{where } D \equiv \frac{d}{dx}$$

and

$$H_1 = \frac{1}{2} \int u^2 dx, \quad H_2 = \frac{1}{2} \int (-u_x^2 + u^3) dx.$$

The tri-Hamiltonian duality construction is implemented as follows:

- A simple scaling argument shows that  $J_2$  is in fact the sum of two compatible Hamiltonian operators, namely  $K_2 = D^3$  and  $K_3 = uD + Du$ , so that  $K_1 = J_1, K_2, K_3$  form a triple of mutually compatible Hamiltonian operators.
- Thus, when we can recombine the Hamiltonian triple as transfer the leading term  $D^3$  from  $J_2$  to  $J_1$ , thereby constructing the Hamiltonian pairs  $\widehat{J}_1 = K_2 \pm K_1 = D^3 \pm D$ . The resulting self-adjoint operator  $S = 1 \pm D^2$  is used to define the new field variable  $\rho = Su = u \pm u_{xx}$ .
- Finally, the second Hamiltonian structure is constructed by replacing  $u$  by  $\rho$  in the remaining part of the original Hamiltonian operator  $K_3$ , so that  $\widehat{J}_2 = \rho D + D\rho$ . Note that this change of variables does not affect  $\widehat{J}_1$ .

As a result of this procedure, we recover the tri-Hamiltonian dual of the KdV equation

$$\rho_t = \widehat{J}_1 \frac{\delta \widehat{H}_2}{\delta \rho} = \widehat{J}_2 \frac{\delta \widehat{H}_1}{\delta \rho}, \quad (2)$$

where

$$\widehat{H}_1 = \frac{1}{2} \int u \rho dx = \frac{1}{2} \int (u^2 \mp u_x^2) dx, \quad \widehat{H}_2 = \frac{1}{2} \int (u^3 \mp uu_x^2) dx.$$

In this case, (2) reduces to the celebrated Camassa–Holm equation [4, 5]:

$$u_t \pm u_{xxt} = 3uu_x \pm (uu_{xx} + \frac{1}{2}u_x^2)_x. \quad (3)$$

Thus the Camassa–Holm (CH) equation is dual to the KdV equation. It is known that the KdV and the CH equations have a geometric derivation and both of them are models of shallow water waves, the two equations have quite different structural properties.

Similarly one can also study the two component KdV equation, one prototypical example is the Ito equation [23],

$$\begin{aligned} u_t &= u_{xxx} + 3uu_x + vv_x, \\ v_t &= (uv)_x, \end{aligned} \quad (4)$$

which is a prototypical example of a two-component KdV equation. The tri-Hamiltonian dual of Ito equation follows from (2) where the first and second Hamiltonian operators for the new equation are given by

$$\begin{aligned} \widehat{J}_1 &= \begin{pmatrix} D \pm D^3 & 0 \\ 0 & D \end{pmatrix} \\ \widehat{J}_2 &= \begin{pmatrix} \rho D + D\rho & vD \\ Dv & 0 \end{pmatrix} \end{aligned}$$

with

$$\widehat{H}_1 = \frac{1}{2} \int (u\rho + v^2) dx \quad \widehat{H}_2 = \frac{1}{2} \int (u^3 + uv^2 \mp uu_x^2) dx$$

The dual system (2) takes the explicit form

$$\begin{aligned} u_t \pm u_{xxt} &= 3uu_x + vv_x + uu_{xx} + \frac{1}{2} (uu_{xx} + \frac{1}{2}u_x^2)_x, \\ v_t &= (uv)_x. \end{aligned} \tag{5}$$

## 1.1 Result and organization

This paper elucidates the Lie algebraic structure of a well known approach for constructing integrable PDEs and employs this approach for constructing a certain two component integrable system. In this article we construct such dual systems from the Lie-Poisson method. We study this rearrangement of Hamiltonian operators via the construction of *moment of inertia* operator. This moment of inertia operator is tacitly connected to  $H^1$ -Sobolev norm, at least for the Ito equation this is readily observable. Using the moment of inertia operators we compute dual variables. We give a systematic method to derive such operators from the frozen Lie-Poisson structure. Using these operators we obtain the dual system for various two component tri-hamiltonian systems. In this algorithmic ways we can derive various new dual extension of known integrable systems, which can not be derived via traditional  $H^1$ -metric approach.

Certainly our method can be thought of a Lie theoretic interpretation of Olver-Rosenau paper [31]. In other words, the duality method of Olver-Rosenau can be manifested in terms of moment of inertial operator related to coadjoint orbit of the centrally extended semidirect product group  $Diff(S^1) \widehat{\times} C^\infty(S^1)$ . The whole paper discusses algorithmic ways of constructing dual systems.

The paper is **organized** as follows. At first we recapitulate the basic definitions of semidirect product and extension of the Bott-Virasoro group. We compute the coadjoint orbit and the Hamiltonian operator in Section 2. We introduce frozen Lie-Poisson structure, moment of inertial operator in Section 3. In Section 4 we construct the generalized two component peakon type systems using the method of moment of inertia operator. We also give several examples of dual equations.

## 2 Semidirect product and extended Bott-Virasoro group

Let  $\rho : G \rightarrow Aut(V)$  denotes a Lie group (left) representation of  $G$  in the vector space  $V$ , and  $\tilde{\rho} : \mathfrak{g} \rightarrow End(V)$  is the induced Lie algebra representation. Let us denote  $G \times V$  the semidirect product group of  $G$  with  $V$  by  $\rho$  with multiplication [8, 30]

$$(g_1, v_1)(g_2, v_2) = (g_1g_2, v_1 + \rho(g_1)v_2).$$

Let  $\mathfrak{g} \times V$  be the Lie algebra of  $G \times V$ . The Lie bracket on  $\mathfrak{g} \times V$  is given by

$$[(\xi_1, u_1), (\xi_2, u_2)] = ([\xi_1, \xi_2], \tilde{\rho}(\xi_1)u_2 - \tilde{\rho}(\xi_2)u_1).$$

An example of a semidirect product structure is when  $\mathfrak{g}$  is the Lie algebra  $so(3)$  associated with the rotation group  $SO(3)$  and  $u$  is  $\mathbf{R}^3$ . Their semidirect product is the algebra of the 6-parameter Galilean group of rotations and translations.

We can build the Lie-Poisson brackets from these algebras. The  $\pm$  Lie-Poisson bracket of  $f, g : (\mathfrak{g} \ltimes V)^* \rightarrow \mathbf{R}$  is given as

$$\{f, g\}_{\pm}(\mu, a) = \pm \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle \pm \left\langle a, \tilde{\rho} \left( \frac{\delta f}{\delta \mu} \right) \cdot \frac{\delta g}{\delta a} \right\rangle \mp \left\langle a, \tilde{\rho} \left( \frac{\delta g}{\delta \mu} \right) \cdot \frac{\delta f}{\delta a} \right\rangle,$$

where  $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$  and  $\frac{\delta f}{\delta a} \in V$ , dual of  $\mu$  under the pairing  $\langle, \rangle : h^* \times h \rightarrow \mathbf{R}$ .

## 2.1 Extension of the Bott-Virasoro group

The Lie algebra of  $Diff(S^1) \ltimes C^\infty(S^1)$  is the semidirect product Lie algebra

$$\mathfrak{g} = Vect(S^1) \ltimes C^\infty(S^1).$$

An element of  $\mathfrak{g}$  is a pair

$$\left( f(x) \frac{d}{dx}, a(x) \right), \quad \text{where} \quad f(x) \frac{d}{dx} \in Vect(S^1), \quad \text{and} \quad a(x) \in C^\infty(S^1).$$

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$\begin{aligned} \omega_1 \left( \left( f(x) \frac{d}{dx}, a(x) \right), \left( g \frac{d}{dx}, b \right) \right) &= \int_{S^1} f'(x) g''(x) dx \\ \omega_2 \left( \left( f(x) \frac{d}{dx}, a(x) \right), \left( g \frac{d}{dx}, b \right) \right) &= \int_{S^1} [f''(x) b(x) - g''(x) a(x)] dx \\ \omega_3 \left( \left( f(x) \frac{d}{dx}, a(x) \right), \left( g \frac{d}{dx}, b \right) \right) &= 2 \int_{S^1} a(x) b'(x) dx. \end{aligned} \tag{6}$$

The first cocycle  $\omega_1$  is the well-known Gelfand–Fuchs cocycle. The Virasoro algebra

$$Vir = Vect(S^1) \oplus \mathbb{R}$$

is the unique non-trivial central extension of  $Vect(S^1)$  based on the Gelfand–Fuchs cocycle. The space  $C^\infty(S^1) \oplus \mathbb{R}$  is identified as *regular part* of the dual space to the Virasoro algebra.

Since the topological dual of the Fréchet space  $Vect(S^1)$  is too big, we restrict our attention to the regular dual  $\mathfrak{g}^*$ , the subspace of  $Vect(S^1)^*$  defined by linear functionals of the form

$$F(u) = \langle u(x), f(x) \rangle = \int_0^{2\pi} u(x) f(x) dx, \quad f(x) \frac{d}{dx} \in Vect(S^1),$$

for some function  $u(x) \in C^\infty(S^1)$ . The regular part of the dual space  $Vect(S^1)^*$  is therefore isomorphic to  $C^\infty(S^1)$  via the  $L^2$  inner product. We say that a smooth real-valued function  $F$  is a regular function if there exists a smooth map [6, 7]  $\delta F : C^\infty(S^1) \rightarrow C^\infty(S^1)$  such that

$$dF(\mu)u = \int_{S^1} u \cdot \delta F(u) dx, \quad \mu, u \in C^\infty(S^1).$$

In other words, the Fréchet derivative  $dF(\mu)$  belongs to the regular dual  $Vect^*(S^1)$  and the mapping  $\mu \rightarrow \delta F(\mu)$  is smooth. Here  $\delta F(\mu)$  stands for variational derivative. For any functional  $f : \mathfrak{g}^* \rightarrow \mathbb{R}$  one can define its variational derivative  $\frac{\delta f}{\delta \mu}$ :

$$\left(w, \frac{\delta f}{\delta \mu}\right) = \frac{d}{ds} f(\mu + sw)|_{s=0}, \quad \mu, w \in \mathfrak{g}^*.$$

In the finite-dimensional situation, variational derivative  $\frac{\delta f}{\delta \mu}$  is always a vector on the Lie algebra  $\mathfrak{g}$ . In the infinite-dimensional case this is not always true, as not any linear operator on the dual algebra  $\mathfrak{g}^*$  can be represented by a smooth vector field from the Lie algebra  $\mathfrak{g}$ .

Other examples are nonlinear polynomial functionals [25]

$$F(u) = \int_{S^1} N(m) dx,$$

where  $N$  is a polynomial in derivatives of  $u$  up to an order  $n$  and the corresponding variational derivative is given by

$$\frac{\delta F}{\delta u} = \sum_{i=0}^n (-1)^i \frac{d^i}{dx^i} \left( \frac{\partial N}{\partial X_i}(u) \right),$$

where  $X_i$  are vector fields generated by the Sobolev  $H^k$ -metric and the operator  $E = \sum_{i=0}^n (-D)^i \frac{\partial}{\partial X_i}$  is also known as the Euler operator.

The pairing between this space and the Virasoro algebra is given by:

$$\left\langle (u(x), \alpha), \left( f(x) \frac{d}{dx}, a \right) \right\rangle = \int_{S^1} u(x) f(x) dx + a\alpha.$$

Similarly we consider the following extension of  $\mathfrak{g}$ ,

$$\widehat{\mathfrak{g}} = Vect^s(S^1) \times C^\infty(S^1) \oplus \mathbb{R}^3. \quad (7)$$

The commutation relation in  $\widehat{\mathfrak{g}}$  is given by

$$\left[ \left( f \frac{d}{dx}, a, \alpha \right), \left( g \frac{d}{dx}, b, \beta \right) \right] := \left( (fg' - f'g) \frac{d}{dx}, fb' - ga', \omega \right) \quad (8)$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ ,  $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ , and where  $\omega = (\omega_1, \omega_2, \omega_3)$  are the cocycles.



Let

$$\widehat{\mathfrak{g}}_{reg}^* = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$$

denote the *regular part* of the dual space  $\widehat{\mathfrak{g}}^*$  to the Lie algebra  $\widehat{\mathfrak{g}}$ , under the following pairing:

$$\langle \widehat{u}, \widehat{f} \rangle = \int_{S^1} [f(x)u(x) + a(x)v(x)] dx + \alpha \cdot \gamma, \quad (9)$$

where  $\widehat{u} = (u(x), v, \gamma) \in \widehat{\mathfrak{g}}_{reg}^*$ ,  $\widehat{f} = (f \frac{d}{dx}, a, \alpha) \in \widehat{\mathfrak{g}}$ . Of particular interest are the coadjoint orbits in  $\widehat{\mathfrak{g}}_{reg}^*$ . In this case, Gelfand, Vershik and Graev, [18], have constructed some of the corresponding representations.

## 2.2 Computation of Hamiltonian operator and coadjoint representation

Let us introduce  $H^1$  inner product on the algebra  $\widehat{\mathfrak{g}}$

$$\langle \widehat{f}, \widehat{g} \rangle_{H^1} = \int_{S^1} [f(x)g(x) + a(x)b(x) + \partial_x f(x)\partial_x g(x)] dx + \alpha \cdot \beta, \quad (10)$$

where

$$\widehat{f} = \left( f \frac{d}{dx}, a, \alpha \right), \quad \widehat{g} = \left( g \frac{d}{dx}, b, \beta \right).$$

Now we compute :

**Lemma 2.1** *The coadjoint operator with respect to the  $H^1$  inner product is given by*

$$\langle ad_{\widehat{f}}^* \begin{pmatrix} u \\ v \end{pmatrix}, \widehat{g} \rangle = \begin{pmatrix} (1 - \partial^2)^{-1} [2f'(x)(1 - \partial_x^2)u(x) + f(x)(1 - \partial_x^2)u'(x) + a'v(x)] + c_1 f''' + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f''(x) + 2c_3 a'(x) \end{pmatrix}. \quad (11)$$

**Proof:** Since we have identified  $\mathfrak{g}$  with  $\mathfrak{g}^*$ , it follows from the definition that

$$\begin{aligned} \langle ad_{\widehat{f}}^* \widehat{u}, \widehat{g} \rangle_{H^1} &= \langle \widehat{u}, [\widehat{f}, \widehat{g}] \rangle_{H^1} \\ &= - \int_{S^1} [(fg' - f'g)u - (fb' - ga')v - \partial_x(fg' - f'g)\partial_x u] dx. \end{aligned}$$

After computing all the terms by integrating by parts and using the fact that the functions  $f(x), g(x), u(x)$  and  $a(x), b(x), v(x)$  are periodic, the right hand side can be expressed as above.

Let us compute now the left hand side:

$$\begin{aligned} ad_{\widehat{f}}^* \begin{pmatrix} u \\ v \end{pmatrix} &= \int_{S^1} [(ad_{\widehat{f}}^* u)g + (ad_{\widehat{f}}^* u)'g' + (ad_{\widehat{f}}^* v)b] dx \\ &= \int_{S^1} [(1 - \partial^2)ad_{\widehat{f}}^* u]g + (ad_{\widehat{f}}^* v)b] dx = \left\langle ((1 - \partial^2)ad_{\widehat{f}}^* u, (ad_{\widehat{f}}^* v)), (g, b) \right\rangle \end{aligned}$$

Thus by equating the the right and left hand sides, we obtain the desired formula.  $\square$

Using standard technique of integrable systems [3] we extract the Hamiltonian operator from the coadjoint action (11).

**Proposition 2.2** *The Hamiltonian operator associated to extended Bott-Virasoro orbit with respect to  $H^1$ -metric is given by*

$$\mathcal{O}_{H^1} = \begin{pmatrix} c_1 D^3 + D\rho + \rho D & vD + c_2 D^2 \\ Dv - c_2 D^2 & 2c_3 D \end{pmatrix}, \quad (12)$$

where  $\rho = (1 - \partial_x^2)u$ .

**Corollary 2.3** *The Hamiltonian operator with respect to right invariant  $L^2$  metric is given by*

$$\mathcal{O}_{L^2} = \begin{pmatrix} c_1 D^3 + Du + uD & vD + c_2 D^2 \\ Dv - c_2 D^2 & 2c_3 D \end{pmatrix} \quad (13)$$

Subsequently we have to restrict on specific hyperplanes for the construction of various types of peakon systems.

## 2.3 Modified Gelfand-Fuchs cocycle

Consider the following “modified” Gelfand-Fuchs cocycle on  $Vect(S^1)$ :

$$\omega_{mGF}(f(x)\frac{d}{dx}, g(x)\frac{d}{dx}) = \int_{S^1} (af'g'' + bf'g)dx. \quad (14)$$

This cocycle is cohomologues to the Gelfand-Fuchs cocycle, hence, the corresponding central-extension is isomorphic to the Virasoro algebra. The additional term in (14) is a coboundary term. It is easy to check that the functional

$$\int_{S^1} f'g dx = \frac{1}{2} \int_{S^1} (f'g - fg')dx$$

depends on the commutator of  $f\frac{d}{dx}, g\frac{d}{dx} \in Vect(S^1)$ .

The Gelfand-Fuchs theorem states that  $H^2(Vect(S^1)) = \mathbf{R}$ , and therefore, every nontrivial cocycle is proportional to the Gelfand-Fuchs cocycle upto a coboundary. Thus one has

$$\tilde{\omega}_1 = \lambda\omega_1 + b,$$

where  $b$  is a coboundary

$$b(f\frac{d}{dx}, g\frac{d}{dx}) = \langle u, [f, g] \rangle$$

for some  $u$  belongs to space quadratic differential form or dual of  $Vect(S^1)$ .

The new coboundary term modified the original Lie-Poisson structure on  $Vect^*(S^1)$ . Thus the new bivector is an affine perturbation of the canonical Lie-Poisson structure on  $Vir^*$ . It is given by

$$\Lambda = \Lambda_0 + \Lambda_1,$$

where  $\Lambda_0$  is the canonical or unperturbed Poisson bivector. The perturbed (constant) bivector  $\Lambda_1$  is itself a Poisson bivector since it satisfies automatically the Schouten-Nijenhuis condition

$$[\Lambda_1, \Lambda_1] = 0.$$

## 2.4 Modified Hamiltonian structure

We now compute the modified Hamiltonian structure corresponding to modified Lie-Poisson structure on the extended Bott-Virasoro group. It is clear that  $Vect(S^1) \times C^\infty(S^1)$  algebra is extended by the non-trivial three 2-cocycles  $(\tilde{\omega}_1, \omega_2, \omega_3)$ . Let us compute the coadjoint action of  $Vect(S^1) \times C^\infty(S^1) \oplus \mathbf{R}^3$  on its dual  $C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3$  and is given by

$$ad_f^* \hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1(af''' + bf') + c_2a'') \\ f'v(x) + f(x)v'(x) - c_2f''(x) + 2c_3a'(x) \\ 0 \end{pmatrix}$$

Thus the modified Hamiltonian structure associated with the coadjoint action in presence of modified cocycle is given by

$$\begin{aligned} \hat{\mathcal{O}}_{L^2} &= \begin{pmatrix} -c_1(aD^3 + bD) + 2uD + u_x & vD + c_2D^2 \\ v_x + vD - c_2D^2 & 2c_3D \end{pmatrix} \\ &\equiv \begin{pmatrix} c_1D^3 + c_4D + 2uD + u_x & vD + c_2D^2 \\ v_x + vD - c_2D^2 & 2c_3D \end{pmatrix}, \end{aligned} \quad (15)$$

where  $c_4$  is a new constant.

## 3 Duality, moment of inertia and equations

In this Section we give an algorithmic construction of the generalized multicomponent Camassa-Holm equation. This method depends directly on the frozen Lie-Poisson structure. We briefly recapitulate frozen Lie-Poisson in the next section.

### 3.1 Frozen Lie-Poisson structure

Consider the dual of the Lie algebra of  $\mathfrak{g}^*$  with a Poisson structure given by the "frozen" Lie-Poisson structure. In other words, we fix some point  $\mu_0 \in \mathfrak{g}^*$  and define a Poisson structure given by

$$\{f, g\}_0(\mu) := \langle [df(\mu), dg(\mu)], \mu_0 \rangle$$

It was shown by Khesin and Misiolek [24] that

**Proposition 3.1** *The brackets  $\{\cdot, \cdot\}_{LP}$  and  $\{\cdot, \cdot\}_0$  are compatible for every "freezing" point  $\mu_0$ .*

**Proof:** Let us take any linear combination

$$\{\cdot, \cdot\}_\lambda := \{\cdot, \cdot\}_{LP} + \lambda\{\cdot, \cdot\}_0$$

is again a Poisson bracket, it is just the translation of the Lie-Poisson bracket from the origin to the point  $-\lambda\mu_0$ .

□

Let us proceed to compute frozen brackets on the dual space of the extended semi-direct product space  $Vect(S^1) \ltimes C^\infty(S^1)$ . In general, given

$$(u_0, v_0, c) \in Vect(\widehat{S^1}) \ltimes C^\infty(S^1)^* \simeq C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3,$$

the frozen bracket is given by

$$\begin{aligned} \{f, g\}(u, v, c) &= \langle (u_0, v_0, c), \left[ \frac{\delta f}{\delta(u, v, c)}, \frac{\delta g}{\delta(u, v, c)} \right] \rangle, \\ &= \langle -ad^*_{\frac{\delta f}{\delta(u, v, c)}}(u_0, v_0, c), \frac{\delta f}{\delta(u, v, c)} \rangle. \end{aligned}$$

Furthermore, recall the corresponding Euler-Poincaré equations of motions are given by

$$\frac{d}{dt}(u, v, c) = -ad^*_{\frac{\delta f}{\delta(u, v, c)}}(u_0, v_0, c).$$

We compute the generalized frozen Hamiltonian structure from equation (15).

**Lemma 3.2** *We consider the frozen Poisson structure at  $u_0 = \mu$ ,  $v_0 = \lambda$ ,  $c_1 = a$ ,  $c_2 = c$  and  $c_3 = d$ , where  $\mu, \lambda, c$  and  $d$  are some constants. It is given by*

$$\mathcal{O}_{gFrozen} = \begin{pmatrix} aD^3 + (c_4 + \mu)D & \lambda D + cD^2 \\ \lambda D - cD^2 & 2dD \end{pmatrix} \equiv \begin{pmatrix} aD^3 + bD & \lambda D + cD^2 \\ \lambda D - cD^2 & 2dD \end{pmatrix} \quad (16)$$

where we assume  $(c_4 + \mu) = b$ .

**Remark on coboundary operator and frozen structure** Every 2-cocycle  $\Gamma$  defines a Lie-Poisson structure on  $\mathfrak{g}^*$ . The vanishing of Schouten-Nijenhuis bracket for Poisson bivector can be recast as a cocycle condition  $\partial\Gamma = 0$ , where  $\partial : \wedge^k \mathfrak{g}^* \rightarrow \wedge^{k+1} \mathfrak{g}^*$ . A special case of Lie-Poisson structure is given by a 2-cocycle  $\Gamma$  which is a coboundary [6, 7]. If  $\Gamma = \partial\mu_0$  for some  $\mu_0 \in \mathfrak{g}^*$ , the expression

$$\{f, g\}_0(\mu) = \mu_0([d_\mu f, d_\mu g])$$

considered to be Lie-Poisson bracket which has been “frozen” at a point  $\mu_0 \in \mathfrak{g}^*$ .

### 3.2 Frozen Hamiltonian structure and moment of inertia operator

Let  $G$  be an arbitrary Lie group,  $\mathfrak{g}$  be its Lie algebra, and  $\mathfrak{g}^*$  be the corresponding dual algebra. Let  $\mathbb{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$  be a positive definite symmetric operator, known as moment of inertia operator, defining ascalar product on the Lie algebra. The moment of inertia

operator defines a left or right-invariant inertia operator  $\mathbb{I}_G$  on the group. This defines a left or right-invariant metric or inner product.

Consider this inner product on the Lie algebra of vector fields  $Vect(S^1)$  on  $S^1$ . If this inner product is local, it is defined via the moment of inertia operator  $\mathbb{I}$

$$\langle \eta \frac{d}{dx}, \beta \frac{d}{dx} \rangle = \int_{S^1} \eta \mathbb{I} \beta dx \quad \eta \frac{d}{dx}, \beta \frac{d}{dx} \in Vect(S^1).$$

We define a quadratic functional, Hamiltonian function

$$H(u) = \frac{1}{2} \int_{S^1} u \mathbb{I}^{-1}(u)$$

on the regular dual  $Vect^*(S^1)$ .

If the metric is left-invariant, then geodesics of this metric are described by the Euler-Poincaré equation

$$\dot{u} = ad_{\mathbb{I}^{-1}u} u \quad u \in \mathfrak{g}^*.$$

Let us generalize this to semidirect product algebra  $Vect(S^1) \ltimes C^\infty(S^1)$ . It must be noticed that in the infinite dimensional case the operator  $\mathbb{I}$  is invertible only on a *regular part* of the dual algebra  $(Vect(S^1) \ltimes C^\infty(S^1))^*$ .

**Definition 3.3** *The generalized moment of inertia  $\mathbb{I}$  maps*

$$\mathbb{I} : \begin{pmatrix} u \frac{d}{dx} \\ v(x) \end{pmatrix} \longrightarrow \begin{pmatrix} m(dx)^2 \\ p(x) \end{pmatrix}, \quad (17)$$

given by

$$\begin{pmatrix} m(x) \\ p(x) \end{pmatrix} = \mathbb{I} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}. \quad (18)$$

Now we state the algorithmic method to compute the generalized moment of inertia operator. This follows directly from the Lie-Poisson structure.

**Recipe to compute generalized moment of inertia operator** The generalized moment of inertia is obtained from the frozen Poisson structure  $\mathcal{O}_{frozen}$ . It is given by

$$\mathcal{O}_{frozen} = \mathbb{I} D \mathbf{I}, \quad (19)$$

where  $\mathbf{I}$  is the identity matrix, and it is given by

$$\mathbb{I}_{2 \times 2} = \begin{pmatrix} aD^2 + b & \lambda + cD \\ \lambda - cD & 2d \end{pmatrix}. \quad (20)$$

This is a symmetric operator, and equation (20) is the most generalized form of moment of inertia associated to the coadjoint orbit of  $\hat{\mathfrak{g}}$ .

### 3.2.1 Examples of moment of inertia operators

We illustrate this phenomena by examples. Let us start with a few special cases.

1. The moment of inertia operator for the KdV equation is a trivial operator.
2. In the case of two component Camassa-Holm equation we choose  $a = -b$ . The moment of inertia operator for the Camassa-Holm equation is  $\mathbb{I} = (1 - \partial^2)$ .
3. The moment of inertia operator for the 2- component Camassa-Holm equation is

$$\mathbb{I}_{2 \times 2} = \begin{pmatrix} b(1 - D^2) & \lambda + cD \\ \lambda - cD & 2d \end{pmatrix}, \quad (21)$$

where we assume  $a = -b$  in equation (20).

**Lemma 3.4** *The moment of inertia operator relates  $(u, v)$  pair to  $(m, p)$  pair for two component Camassa-Holm equation as*

$$\begin{aligned} m(x) &= b(u - u_{xx}) + \lambda v + cv_x \\ p(x) &= \lambda u - cu_x + 2dv. \end{aligned} \quad (22)$$

**Proof:** Using equation (18) we obtain  $m$  and  $p$ .

□

Let  $\hat{\mathfrak{g}} = Vect(S^1) \times C^\infty(S^1) \oplus \mathbf{R}^3$ . The inner product is defined via the moment of inertia operator

$$\mathbb{I}_{2 \times 2} : \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}^*.$$

Thus the moment of inertia  $\mathbb{I}_{2 \times 2}$  plays the role of a metric – it allows us to build a quadratic form from two elements of  $\hat{\mathfrak{g}}^*$ . Thus the Hamiltonian is defined as

$$\begin{aligned} H(m, p) &= \left\langle \begin{pmatrix} m(x) \\ p(x) \end{pmatrix}, \mathbb{I}_{2 \times 2}^{-1} \begin{pmatrix} m(x) \\ p(x) \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} m(x) \\ p(x) \end{pmatrix}, \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\rangle \\ &= \int_{S^1} (mu + pv) dx. \end{aligned} \quad (23)$$

## 4 Moment of inertia, Boussinesq system and duality equation

Let us compute the dual equation associated to the generalized Hamiltonian operator associated to coadjoint orbit of the extended Bott-Virasoro group  $\widehat{Diff}(S^1) \times C^\infty(S^1)$ .

## 4.1 Construction of generalized dual equation associated to the Bott-Virasoro group

The frozen Hamiltonian structure is computed straight away from equation (16) and it is given by

$$\mathcal{O}_{frozen} = \begin{pmatrix} c_1 D^3 + \alpha D & \lambda D + c_2 D^2 \\ -c_2 D^2 + \lambda D & 2c_3 D \end{pmatrix}, \quad (24)$$

where  $\alpha = c_4 + u_0$ .

Thus the generalized moment of inertia operator associated to  $\hat{\mathfrak{g}}$  is given by

$$\mathbb{I}_{gen} = \begin{pmatrix} c_1 D^2 + \alpha & \lambda + c_2 D \\ -c_2 D + \lambda & 2c_3 \end{pmatrix}. \quad (25)$$

Therefore the dual variables are

$$\begin{aligned} m(x) &= c_1 u_{xx} + \alpha u + \lambda v + c_2 v_x \\ p(x) &= -c_2 u_x + \lambda u + 2c_3 v. \end{aligned} \quad (26)$$

The dual variables induces the modified Hamiltonian structure

$$\hat{\mathcal{O}}_{gen} = \begin{pmatrix} Dm + mD & Dp \\ Dv & 0 \end{pmatrix}. \quad (27)$$

The Euler-Poincaré flow on dual space of Lie algebra  $\mathfrak{g}^*$  can be written in the form

$$\begin{pmatrix} m \\ p \end{pmatrix}_t = -\hat{\mathcal{O}}_{gen} \begin{pmatrix} \frac{\delta H_1}{\delta p} \\ \frac{\delta H_1}{\delta p} \end{pmatrix} \quad (28)$$

where the Hamiltonian is given by

$$H = \frac{1}{2} \int_{S^1} mu + pv \, dx.$$

**Proposition 4.1** *The Euler-Poincaré flow associated to the modified Hamiltonian operator yields a flow on  $\hat{\mathfrak{g}}^*$ , given by*

$$\begin{aligned} m_t + \left( mu + \frac{c_1}{2} u_x^2 + \frac{\alpha}{2} u^2 + \lambda uv + c_3 v^2 \right)_x &= 0 \\ p_t + (pu)_x &= 0, \end{aligned} \quad (29)$$

where  $m$  and  $p$  is defined as (26).

**Proof:** By direct computation.  $\square$

Equation (29) is the most general form of peakon version of coupled KdV equation. All other two component peakon type equations are reductions of this equation. Hence we call this equation as the peakon/compacton ( or dual ) version of the Antonowicz-Fordy equation.

**Example** We give a prototypical example of the two component Camassa-Holm equation. The Euler-Poincaré flow of the dual equation yields the generalized two component Camassa-Holm equation (see for example, [10])

$$\begin{aligned} m_t + m_x u + 2m u_x + p v_x &= 0 \\ p_t + (p u)_x &= 0, \end{aligned} \tag{30}$$

where  $m$  and  $p$  satisfy equation (22).

## 4.2 Dual equation of the Boussinesq system

Let us consider the Kupershmidt's version [26] of the Boussinesq system

$$\begin{aligned} u_t &= u u_x + v_x - v_{xx} \\ v_t &= (u v)_x + v_{xx} \end{aligned} \tag{31}$$

The Hamiltonian structure of the Boussinesq system

$$\mathcal{O} = \begin{pmatrix} uD + Du & 2D^2 + vD \\ -2D^2 + Dv & 2D \end{pmatrix} \tag{32}$$

is associated to hyperplane at  $c_1 = 0$ ,  $c_2 = 2$ ,  $c_3 = 1$  in the coadjoint orbit of  $\hat{\mathfrak{g}}$  and corresponding Hamiltonian is given by

$$H = \frac{1}{4} \int_{S^1} (u^2 + v^2) dx.$$

The first Hamiltonian operator is just  $\mathcal{O}_1 = D\mathbf{I}$ , where  $\mathbf{I}$  is the  $2 \times 2$  identity matrix.

We start with the frozen Hamiltonian structure associated to (31). We compute the Hamiltonian structure at  $u_0 = \mu$  and  $v_0 = \lambda$ . Here  $\lambda$  and  $\mu$  are constants. Thus the frozen Hamiltonian structure is

$$\mathcal{O}_{frozen} = \begin{pmatrix} \mu D & 2D^2 + \lambda D \\ -2D^2 + \lambda D & 2D \end{pmatrix}. \tag{33}$$

Thus the moment of inertia operator of the Boussinesq system becomes

$$\mathbb{I}_{Boussinesq} = \begin{pmatrix} \mu & \lambda + 2D \\ \lambda - 2D & 2 \end{pmatrix}. \tag{34}$$

Let us fix  $\mu = \lambda = 1$ . Then  $\mathbb{I}_{Boussinesq}$  yields

$$\begin{pmatrix} m(x) \\ p(x) \end{pmatrix} = \mathbb{I}_{Boussinesq} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} 2u(x) + v - 2v_x \\ u(x) + 2u_x \end{pmatrix}.$$

Once again the moment of inertia transforms the primitive pair  $(u, v)$  to a newer pair  $(m, p)$ , given by

$$\begin{aligned} m(x) &= v + 2v_x \\ p(x) &= u - 2u_x + 2v \end{aligned} \tag{35}$$



Equation (35) transforms the Hamiltonian operator to

$$\begin{pmatrix} uD + Du & vD \\ Dv & 0 \end{pmatrix} \mapsto \begin{pmatrix} mD + Dm & pD \\ Dp & 0 \end{pmatrix}.$$

The Hamiltonian function becomes

$$\tilde{H} = \frac{1}{2} \int_{S^1} (mu + pv) dx. \quad (36)$$

**Proposition 4.2** *The Euler-Poincaré flow with respect to new Hamiltonian and Poisson structure yields the dual equation of Boussinesq system*

$$\begin{aligned} m_t + (mv)_x &= 0 \\ p_t + (vp + u^2 + uv)_x &= 0 \end{aligned} \quad (37)$$

### 4.3 Dual equation of various dispersive water wave equations

We narrate our construction with another prototypical example, the dispersive water waves equation, given by

$$\begin{aligned} u_t &= v_{xxx} + 2(uv)_x, \\ v_t &= u_x + 2vv_x. \end{aligned} \quad (38)$$

This is a geodesic flow on the extension of the Bott-Virasoro group. It is connected to a hyperplane  $c_1 = 1, c_2 = 0, c_3 = \frac{1}{2}$  in the coadjoint orbit of  $\hat{\mathfrak{g}}$  and the flow is given by

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} D^3 + Du + uD & vD \\ Dv & D \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \quad H = \int_{S^1} uv dx. \quad (39)$$

This equation is also known as the Kaup-Boussinesq system. The KB system has a natural two wave structure, which enables one to capture the effects of interaction of unmodular bores or rarefaction waves arising in the decay of a jump discontinuity.

### 4.4 Dual equation of the Kaup-Boussinesq system

The frozen Hamiltonian structure at  $u_0 = \mu$  and  $v_0 = \lambda$  of the Kaup-Boussinesq system is given by

$$\mathcal{O}_{frozen} = \begin{pmatrix} D^3 + \mu D & \lambda D \\ \lambda D & D \end{pmatrix}. \quad (40)$$

It is ready to see that the moment of inertia operator of the KB system  $\mathbb{I}_{KB}$  yields

$$m(x) = u_{xx} + \mu u + \lambda v, \quad p(x) = \lambda u + v.$$

Once again we can normalize  $\mu = \lambda = 2$  The modified Hamiltonian structure of the dual equation is given by

$$\hat{\mathcal{O}} = \begin{pmatrix} Dm + mD & pD \\ Dp & 0 \end{pmatrix} \quad (41)$$

yields the dual equation for the Kaup-Boussinesq system

$$\begin{aligned} m_t + (mu + \frac{1}{2}u_x^2 + u^2 + 2uv)_x &= 0 \\ p_t + (pu)_x &= 0 \end{aligned} \quad (42)$$

for  $\hat{H} = \int_{S^1} (mu + pv) dx$ .

## 4.5 Dual equation of the Broer-Kaup system

Let us study the Broer-Kaup system

$$u_t = -u_{xx} + 2(uv)_x + uu_x, \quad v_t = v_{xx} + 2vv_x - 2u_x$$

is a geodesic flow associated to the hyperplane  $c_1 = 0, c_2 = -1, c_3 = -1$ . Hence the Hamiltonian structure is

$$\mathcal{O}_{BK} = \begin{pmatrix} uD + Du & -D^2 + vD \\ D^2 + Dv & -2D \end{pmatrix}, \quad \text{with } H = \int_{S^1} uv dx.$$

The moment of inertia operator of the Broer-Kaup system computed from the frozen Hamiltonian structure at  $u_0 = v_0 = 1$  and given by

$$\mathbb{I}_{BK} = \begin{pmatrix} 1 & 1 - D \\ D + 1 & -2 \end{pmatrix}$$

Therefore  $\mathbb{I}_{BK}$  yields

$$m(x) = u + v - v_x, \quad p(x) = u + u_x - 2v.$$

Thus the dual equation for the Kaup-Boussinesq system is

$$\begin{aligned} m_t + (mu + uv - v^2)_x &= 0 \\ p_t + (pu)_x &= 0 \end{aligned} \quad (43)$$

for  $\hat{H} = \frac{1}{2} \int_{S^1} (mu + pv) dx$ .

Hence in this paper we present a more generalized formalism to construct two component type Camassa-Holm type equations. We have demonstrated our method with several examples. All these systems obtained appears to be bi-Hamiltonian flow on the coadjoint orbit of  $Diff(S^1) \times C^\infty(S^1)$ .

## 5 Conclusion and outlook

In this paper we have derived various two-component generalization of the Camassa-Holm type systems using extended Bott-Virasoro algebra. Our method is closely related to Olver-Rosenau method. We have given a more Lie algebraic illustration of their construction. It would be rather interesting to generalize this method to supersymmetric dual integrable systems. This would involve extended superconformal group.

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