Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Regularity results for flat minimizers of the Willmore functional

(revised version: July 2009)

by

Peter Hornung

Preprint no.: 52 2008

Regularity results for flat minimizers of the Willmore functional

Peter Hornung*

Abstract

Let $S \subset \mathbb{R}^2$ be a bounded domain with boundary of class C^∞ and let $g_{ij} = \delta_{ij}$ denote the flat metric on $\mathbb{R}^2$. Let u be a minimizer of the Willmore functional within a subclass (defined by prescribing boundary conditions on parts of ∂S) of all $W^{2,2}$ isometric immersions of the Riemannian manifold (S, g) into $\mathbb{R}^3$. In this article we study the regularity properties of such u. Our main result roughly states that minimizers u are C^∞ away from three kinds of line segments: Segments which intersect ∂S tangentially, segments which bound regions on which ∇u is locally constant and segments for which $\nabla^2 u$ diverges near one endpoint. At segments of the third kind, we prove that u is precisely C^3 (in the interior), and we obtain sharp estimates for the size of its derivatives. Our main motivation to study this problem comes from nonlinear elasticity: On isometric immersions, the Willmore functional agrees with Kirchhoff's energy functional for thin elastic plates.

1 Introduction

The Willmore functional for surfaces $u : \Sigma \to \mathbb{R}^3$ immersed in $\mathbb{R}^3$ is given by

$$\mathcal{W}(u) = \frac{1}{4} \int_\Sigma |H|^2 \, d\mu_g,$$

where H denotes the mean curvature of the immersion u, the pull-back metric is $g = (\nabla u)^T (\nabla u)$ and μ_g denotes the induced area measure. There has been considerable interest in the properties of critical points of this functional, the so-called Willmore surfaces (cf. e.g. [25, 22, 19, 15, 16, 21] and the references cited therein). Recently, there has been growing interest in costrained

*Institut für Angewandte Mathematik, Universität Bonn, peter.hornung@uni-bonn.de

versions of the Willmore functional, see e.g. [1] and work in progress by Kuwert and Schätzle, where constraints are imposed on the conformal class of the admissible surfaces. Constrained Willmore functionals are also interesting for applications, e.g. in the modelling of biological membranes (Helfrich model, cf. [6]) or in nonlinear elasticity (Kirchhoff's plate theory, cf. [3]).
In the present article we study critical points of the Willmore functional within all possible realizations (i.e. isometric immersions) in $\mathbb{R}^3$ of the same Riemannian surface. Motivated by applications in three dimensional nonlinear elasticity, we consider the case of flat surfaces, i.e. $u : S \to \mathbb{R}^3$ with $S \subset \mathbb{R}^2$ such that the pull-back metric $(\nabla u)^T(\nabla u)$ agrees with the standard metric in $\mathbb{R}^2$. This is a very strong constraint, and the problem is highly degenerate. Therefore, it is a priori not even clear whether there exist enough variations to obtain informative Euler-Lagrange equations. (Indeed, prescribing boundary data on too large a set leads to an ill-posed problem where no variations are possible at all.) In [11] certain variations were found which satisfy the natural local boundary conditions. Their corresponding Euler-Lagrange equation, the constrained Willmore equation, was derived. However, these variations are quite artificial and it is therefore not obvious that their Euler-Lagrange equations carry enough information to derive good regularity properties from them. As usual, the most difficult task is to obtain some low regularity in an initial step. (Indeed, arbitrary competitors can display a very pathological behaviour, as shown in the appendix to [10].) Higher regularity is then obtained by standard arguments.
Apart from being very relevant for applications (cf. [3, 23, 24]), the problem studied here is also interesting because, on the one hand, it is particularly accessible, whereas, on the other hand, it displays a nontrivial (and unexpected) regularity. It is accessible because, at least partially, it can be reduced to the analysis of a system of ordinary differential equations (the Euler-Lagrange equations derived in [11]). It is nontrivial because this description is not global and, more importantly, because the Euler-Lagrange system exhibits two kinds of degeneracies. One of them is a vestige of the fact that the original problem is a problem on surfaces and not just on curves. It is this degeneracy which leads to an unexpected failure of regularity even in the interior of the so-called "developable" regions on which the Euler-Lagrange system is satisfied. Nonetheless, one can perform a detailed analysis of the surface at this singular set. It reveals, among other details, an interesting scaling behaviour of the mean curvature.

Let us describe the main results of this article more precisely. By the direct method, one easily finds minimizers of the Willmore functional among isometric immersions of S into $\mathbb{R}^3$ satisfying the boundary conditions mentioned earlier, cf. [11]. The natural class in which minimizers exist is the set

$$W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) = \{u \in W^{2,2}(S;\mathbb{R}^3) : (\nabla u)^T(\nabla u) = Id \ \text{ almost everywhere}\} \quad (1)$$

of $W^{2,2}$ isometric immersions (cf. [11]). This class is natural because it consists exactly of those isometries which have finite Willmore energy. Indeed, if $u : S \to \mathbb{R}^3$ is an isometric immersion then, up to a constant prefactor, $\mathcal{W}(u)$ agrees with

$$\mathcal{E}(u; S) := \int_S |\nabla^2 u(x)|^2 \, dx. \tag{2}$$

In nonlinear elasticity, $\mathcal{E}$ is called Kirchhoff's plate functional, and $W^{2,2}_{\text{iso}}$ is its natural domain of definition. $\mathcal{E}$ models the behaviour of unstrechable thin elastic films. This modern formulation as well as the relation between Kirchhoff's plate theory and nonlinear three dimensional elasticity was obtained in [3]. It motivated the present analysis.

While simple counterexamples show that $W^{2,2}_{\text{iso}}(S; \mathbb{R}^3) \not\subset C^2(S; \mathbb{R}^3)$, it turns out that all mappings $u \in W^{2,2}_{\text{iso}}(S; \mathbb{R}^3)$ are C^1 even up to the boundary provided that the latter is smooth enough, cf. [17, 14]. (This regularity, however, is too low to get started in the regularity analysis below.) Another fundamental property shared by all $W^{2,2}$ isometric immersions is that they are developable surfaces, cf. [20, 5, 14, 18, 13]). More precisely, they consist of planar regions and of "developable" regions.

In the interior of their planar region, the surfaces are trivially smooth. In the interior of their developable region, critical points satisfy the Euler-Lagrange equations from [11]. The boundary of the developable region belongs to the singular set. This part of the singular set, as all others, consists of straight line segments with endpoints on ∂S on which ∇u is constant. Under extra regularity conditions on the boundary values one can prove that it consists of only finitely many segments. In general, it is not empty and minimizers indeed fail to be smooth there. (This and other examples can be found in [12].) The second part of the singular set, denoted Σ_τ, consists of lines which intersect ∂S tangentially. The third, quite unexpected, kind of singular line segments is the one mentioned at the beginning. We denote it by Σ_0. It is contained in the developable region. It consists of at most countably many lines which can only accumulate near $\Sigma_\tau \cup \partial S$. This local finiteness is a key regularity result since, for arbitrary $W^{2,2}$ isometric immersions, the set Σ_0 can be Cantor-like and can accumulate anywhere. Its proof is based on the discovery of a monotone quantity that is linked to the mean curvature via the Euler-Lagrange equations. Once local finiteness is established, one can prove that the surface is precisely C^3 (in the interior of the domain) at lines in Σ_0. Moreover, the energy density diverges at one endpoint of each such line $Y \subset \Sigma_0$. Near Y (and uniformly away from ∂S), the mean curvature $H(x)$ scales like $F(\text{dist}_Y(x))$, where F is the inverse of $t \mapsto t \log t$.

Our main positive result is that minimizers are C^∞ away from the singular sets just described. The regularity results obtained in this paper are optimal: In forthcoming work ([12]) we provide examples of boundary data leading to

minimizers for which the first two kinds of singular sets indeed occur and which fail to be better than C^3 at any of these sets. Moreover, there is very strong numerical evidence that, in general, $\Sigma_0 \neq \emptyset$, cf. [23] and other physics literature about the Möbius strip.
This article is organized as follows. In Section 2 we recall some basic definitions a properties of $W^{2,2}_{\text{iso}}$, mainly taken from [9, 10], and the Euler-Lagrange equations from [11]. Then we state our main results, Theorem 2.4 and Theorem 2.7. In Section 3 we derive some basic consequences of the Euler-Lagrange equations and deduce a first partial regularity result. In Section 4 we derive our main regularity result and the scalings alluded to above. In Section 5 we use this result to deduce C^∞ regularity at planar lines which do not belong to Σ_0. In Section 6 we analyze the singular set and provide an instance of sufficient conditions on the boundary data ensuring that it consists of only finitely many line segments.
The main results of this article were announced in [8].

Notation. S will always denote a bounded domain in $\mathbb{R}^2$. Unless stated otherwise, its boundary is assumed to be of class C^∞. The curves Γ are always parametrized by arclength. Statements involving pointwise properties of L^1_{loc}-functions refer to their precise representatives as defined e.g. in [4]. We write $*$ for $+$ or $-$, and if $* = +$ then we set $\bar{*} = -$ and viceversa. The letters C and c will denote positive constants which may vary from line to line. Often we abbreviate $\{t \in [-T, T] : f(t) = c\}$ by $\{f = c\}$. We abbreviate $f \circ g$ by $f(g)$.
By e_i we denote standard unit vectors in $\mathbb{R}^n$ and by $\mathcal{L}^k$, $\mathcal{H}^k$ the k-dimensional Lebesgue and Hausdorff measures, respectively. If f, g are real-valued functions on an interval then we write "$f \sim g$ near t_0" to denote that there is $c > 0$ such that $c|g(t)| \leq |f(t)| \leq \frac{1}{c}|g(t)|$ for all $t \in (t_0 - c, t_0 + c)$. The equality $u = (\Gamma, \kappa_n)$ is to be understood as equality up to a rigid motion, i. e. there exist $Q \in SO(3)$ and $d \in \mathbb{R}^3$ such that $d + Qu(x) = (\Gamma, \kappa_n)(x)$ for all $x \in [\Gamma(-T, T)]$.

2 Preliminaries and main results

2.1. Some definitions. We briefly recall some definitions introduced in [8, 9, 10, 11]. For details and proofs, we refer to these articles. Unless stated otherwise, throughout this article $S \subset \mathbb{R}^2$ denotes a bounded C^∞ domain and $\partial_c S \subset \partial S$ is closed. For $\mu \in \mathbb{S}^1$ and $x \in S$ we denote by $[x]_\mu$ the connected component of $(x + \text{Span}\, \mu) \cap S$ which contains x. For $x \in S$ and $\mu \in \mathbb{R}^2 \setminus \{0\}$ we define $\nu(x, \mu) = \inf\{\theta > 0 : x + \theta\mu \notin S\}$. We also set $\nu_1(x, \mu) \cdot e_i = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(\nu(x + \varepsilon e_i, \mu) - \nu(x, \mu))$ whenever this limit exists. A simple implicit function argument (cf. e.g. [9]) shows that if the boundary of S is C^k, $k \geq 1$, and if $\overline{[x]_\mu}$ intersects ∂S transversally (i.e. μ is not perpendicular

to the outer unit normal to ∂S at any of the two points in $\overline{[x]_\mu} \cap \partial S$) then ν is C^k in a neighbourhood of (x, μ).
Let $X \subset S$. A mapping $q : X \to \mathbb{S}^1$ is called an S-ruling on X if $[x]_{q_f(x)} \cap [y]_{q_f(y)} \neq \emptyset$ implies $[x]_{q_f(x)} = [y]_{q_f(y)}$ whenever $x, y \in X$. This nonintersection property ensures that q is locally Lipschitz when considered as a mapping into the projective space $\mathbb{P}^1$, so locally one can choose antipodal points such that q is Lipschitz into $\mathbb{S}^1$. A mapping $f : X \to \mathbb{R}^P$, $P \geq 1$, is called S-developable on X if there exists an S-ruling for f, i.e. an S-ruling $q_f : X \to \mathbb{S}^1$ such that f is constant on $X \cap [x]_{q_f(x)}$ for all $x \in X$.
Let $u \in W^{2,2}_{\text{iso}}(S; \mathbb{R}^3)$. We set

$$C_{\nabla u} = \{x \in S : \nabla u \text{ is constant in a neighbourhood of } x\}.$$

It is shown in [14, 18, 17] that $\nabla u \in C^0(S; \mathbb{R}^{3\times 2})$ and that ∇u is S-developable on $S \setminus C_{\nabla u}$, cf. also [20, 5, 13]. Regarded as a mapping into $\mathbb{P}^1$, the mapping $q_{\nabla u} : S \setminus C_{\nabla u} \to \mathbb{S}^1$ is uniquely determined by u, cf. Remark 2.2.1 in [9]. The notation $[x]$ will always mean $[x]_{q_{\nabla u}(x)}$.
In [9] it was shown that $C_{\nabla u}$ consists of countably many connected components U, and each U has finite perimeter and there is a countable set $Z_U \subset S \setminus C_{\nabla u}$ such that $S \cap \partial U = \bigcup_{x \in Z_U} [x]$ and such that for $x, y \in Z_U$ the segments $[x]$ and $[y]$ only intersect when $x = y$. We denote by $\hat{C}_{\nabla u}$ the union of those U for which the cardinality of Z_U is at least three, i.e. $S \cap \partial U$ consists of at least three connected components. We set

$$D_{\nabla u} = \{x \in S : \nabla u \text{ is } S\text{-developable in a neighbourhood of } x\}.$$

If $x_0 \in D_{\nabla u}$ then there is a ∇u-integral curve passing through x_0, i.e. there is $T > 0$ and a solution $\Gamma \in W^{2,\infty}([-T, T]; S)$ of the ODE system

$$\Gamma'(t) = -\Big(q_{\nabla u}(\Gamma(t))\Big)^{\perp} \text{ for all } t \in (-T, T) \text{ and } \Gamma(0) = x_0. \tag{3}$$

For arbitrary arclength parametrized (we will tacitly assume this for all curves Γ occuring from now on) $\Gamma \in W^{2,\infty}([-T, T]; S)$, we introduce the following definitions:

- For $J \subset [-T, T]$ we set $[\Gamma(J)] := \bigcup \{[\Gamma(t)]_{N(t)} : t \in J\}$.
- We define $s^{\pm}_{\Gamma} : [-T, T] \to \mathbb{R}$ by setting $s^{*}_{\Gamma}(t) := *\nu(\Gamma(t), N(t))$ for $* = +, -$ (in the sequel we will often omit the index Γ).
- Γ is said to be admissible if $[\Gamma(t_1)]_{N(t_1)} \cap [\Gamma(t_2)]_{N(t_2)} \neq \emptyset$ implies that $t_1 = t_2$ for any $t_1, t_2 \in [-T, T]$.
- Γ is said to be locally admissible if $1 - s^{\pm}_{\Gamma}\kappa(t) \geq 0$ for almost every $t \in (-T, T)$.

- Γ is said to be transversal on $J \subset [-T, T]$ if $\overline{[\Gamma(t)]}$ intersects ∂S transversally for all $t \in J$.

In what follows we will, with a slight abuse of notation, mostly write $[\Gamma(t)]$ instead of $[\Gamma(t)]_{N(t)}$.
If Γ solves (3) then Γ is admissible, provided that T is small enough (cf. Lemma 3.2.3 in [9]).
For admissible $\Gamma \in W^{2,\infty}([-T, T]; S)$ and $\kappa_n \in L^2(-T, T)$ we define a mapping $(\Gamma, \kappa_n) : [\Gamma(-T, T)] \to \mathbb{R}^3$ as follows. Let $N = (\Gamma')^\perp$ and $\kappa = \Gamma'' \cdot N$, and let $r : [-T, T] \to SO(3)$ be the unique $W^{1,2}$-solution to the ODE

$$r' = \begin{pmatrix} 0 & \kappa & \kappa_n \\ -\kappa & 0 & 0 \\ -\kappa_n & 0 & 0 \end{pmatrix} r \text{ and } r(0) = Id. \tag{4}$$

Define γ', v, n to be the first, second and third row of r and set $\gamma(t) = \int_0^t \gamma'$. Then we define

$$(\Gamma, \kappa_n)(\Gamma(t) + sN(t)) = \gamma(t) + sv(t), \tag{5}$$

where and $t \in [-T, T]$, and $s \in (s_\Gamma^-(t), s_\Gamma^+(t))$. The mapping (Γ, κ_n) obtained in (5) is a well defined element of $W^{2,2}_{\text{loc, iso}}([\Gamma(-T, T)]; \mathbb{R}^3)$, provided that Γ is admissible, cf. Proposition 2.3 in [10]. Moreover, we have

$$(s, t) \mapsto \frac{\kappa_n^2(t)}{1 - s\kappa(t)} \in L^1(M_{s_\Gamma^\pm}), \tag{6}$$

if and only if $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(-T, T)]; \mathbb{R}^3)$, see again Proposition 2.3 in [10] and Remark 2.6 in [11]. Here we have introduced $M_{s_\Gamma^\pm} = \bigcup_{t \in (-T,T)} (s_\Gamma^-(t), s_\Gamma^+(t)) \times \{t\}$.
If $\Gamma : [-T, T] \to S$ is a ∇u-integral curve then there is $\kappa_n \in L^2(-T, T)$ such that $u = (\Gamma, \kappa_n)$ on $[\Gamma(-T, T)]$, and we have $\kappa \in L^\infty(-T, T)$. (Here and below the equality $u = (\Gamma, \kappa_n)$ is to be understood up a rigid motion. When studying the local behaviour of u, it constitutes no loss of generality to assume that it is an equality. A true equality is obtained by choosing appropriate values of $r(0)$ and $\gamma(0)$.) Moreover, γ is a line of curvature on the surface $u\big([\Gamma(-T, T)]\big)$ with normal curvature κ_n and geodesic curvature κ. And r is the Darboux frame along γ. In addition, we have (6), and

$$\int_{[\Gamma(-T,T)]} |\nabla^2(\Gamma, \kappa_n)(x)|^2 \, dx = \int_{-T}^{T} \kappa_n^2(t) g(s^\pm(t), \kappa(t)) \, dt. \tag{7}$$

Here $g(s^\pm, x) = \int_{s^-}^{s^+} \frac{1}{1-sx} \, ds$, and in the integrand on the right-hand side of (7) we define $0 \cdot \infty := 0$. Notice that (6) implies the important fact that $\kappa_n = 0$ almost everywhere on the set

$$I_0^\Gamma := \Big\{ t \in [-T, T] : \kappa(t) \in \{ \frac{1}{s^-(t)}, \frac{1}{s^+(t)} \} \Big\}.$$

For $\eta > 0$ we define

$$I_\eta^\Gamma := \{t \in [-T,T] : 1 - s^+(t)\kappa(t) \geq \eta \text{ and } 1 - s^-(t)\kappa(t) \geq \eta\}.$$

In what follows we will omit the index Γ. We recall that κ refers to the precise representative, so the above sets are well defined.
For $s^- < 0 < s^+$ and $x \in (\frac{1}{s^-}, \frac{1}{s^+})$ and $* \in \{-,+\}$ we define

$$g_*(s^\pm, x) = *\frac{1}{1 - s^* x} \tag{8}$$

$$g_2(s^\pm, x) = -\int_{s^-}^{s^+} \frac{1}{(1-sx)^2}\, ds \tag{9}$$

$$g_3(s^\pm, x) = \int_{s^-}^{s^+} \frac{s}{(1-sx)^2}\, ds. \tag{10}$$

We set $\chi_* = \chi_{\{*\kappa>0\}}$ and $\sigma = \sum_* \chi_* s^*$. Finally, we recall from [11] the following definitions

$$h = \kappa \sum_* *\chi_* \nu_1(\Gamma, *N) \cdot \Gamma' \tag{11}$$

$$F_1 = \sum_* \frac{\nu_1(\Gamma, *N) \cdot \Gamma'}{1 - s^*\kappa} + h g_2(s^\pm, \kappa) \tag{12}$$

$$F_2 = \sum_* \frac{s^* \nu_1(\Gamma, *N) \cdot \Gamma'}{1 - s^*\kappa} + \sigma h g_2(s^\pm, \kappa). \tag{13}$$

2.2. Definition. *A pair (Γ, κ_n) with $\Gamma \in W^{2,\infty}((-T,T);S)$ locally admissible and transversal and $\kappa_n \in L^2(-T,T)$ is said to satisfy the Euler-Lagrange equations if there exist $\lambda_1, \lambda_2 \in \mathbb{R}^3$ and $\lambda_3, \lambda_4 \in \mathbb{R}$ such that the following equations are satisfied for almost every $t \in (-T,T)$:*

$$2(1 - \chi_{I_0}(t))\kappa_n(t) g(s_\Gamma^\pm(t), \kappa(t)) = -v(t) \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma') \tag{14}$$

$$(1 - \chi_{I_0}(t))\kappa_n^2(t) g_2(s_\Gamma^\pm(t), \kappa(t)) = (1 - \chi_{I_0}(t))\Omega_2(t) \tag{15}$$

$$(1 - \chi_{I_0}(t))\kappa_n^2(t) g_3(s_\Gamma^\pm(t), \kappa(t)) = \Omega_3(t) + \chi_{I_0}(t)\frac{\Omega_2(t)}{\kappa(t)} \tag{16}$$

Here, Ω_2 and Ω_3 are the unique Lipschitz continuous solutions to the terminal value problems

$$\Omega_2' = -h\Omega_2 + \kappa_n(\lambda_1 \cdot n) + \kappa_n^2 F_1 \text{ and } \Omega_2(T) = \lambda_3 + \lambda_1 \cdot \gamma'(T) \tag{17}$$

$$\Omega_3' = h\sigma\Omega_2 - \kappa_n \gamma' \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma') - \kappa_n^2 F_2 \text{ and } \Omega_3(T) = \lambda_4 + \lambda_2 \cdot n(T). \tag{18}$$

2.3. Remarks.

(i) If Γ is transversal on $[-T,T]$ then $\nu_1(\Gamma,*N) \in L^\infty(-T,T)$, cf. e.g. Proposition 3.1.11 in [9]. So in this case

$$h, F_1, F_2 \in L^\infty(-T,T). \tag{19}$$

For h this follows from the definition. For the F_i it is a consequence of the equalities

$$F_1 = \Big(\chi_{\{\kappa=0\}} + \sum_* \frac{\chi_*}{1-s^*\kappa}\Big)(\nu_1(\Gamma,N) + \nu_1(\Gamma,-N)) \cdot \Gamma', \tag{20}$$

$$F_2 = \Big(\chi_{\{\kappa=0\}} + \sum_* \frac{\chi_*}{1-s^*\kappa}\Big)(s^+\nu_1(\Gamma,N) + s^-\nu_1(\Gamma,-N)) \cdot \Gamma'. \tag{21}$$

The simple proof of (20, 21) is given below the formulae (59, 60) in [11].

(ii) For solutions of the Euler-Lagrange equations with $\mathcal{L}^1(I_0) = 0$ (we will see that this is satisfied in the cases of interest), we have $\Omega_2 = \kappa_n^2 g_2(s^\pm,\kappa)$ almost everywhere by (15) and the differential equations (17, 18) are equivalent to

$$\Omega_2' = \kappa_n(\lambda_1 \cdot n) + \kappa_n^2 \sum_* \frac{\nu(\Gamma,*N)\cdot\Gamma'}{1-s^*\kappa} \tag{22}$$

$$\Omega_3' = -\kappa_n\gamma' \cdot (\lambda_2 - \lambda_1 \wedge \int_t^T \gamma') - \kappa_n^2 \sum_* \frac{s^*\nu_1(\Gamma,*N)\cdot\Gamma'}{1-s^*\kappa}. \tag{23}$$

Our main regularity result for solutions to the Euler-Lagrange equations reads as follows.

2.4. Theorem. *Let $S \subset \mathbb{R}^2$ be a bounded C^∞-domain, let $T > 0$ and let $\Gamma \in W^{2,\infty}([-T,T];S)$ be locally admissible and transversal. Let $\kappa_n \in L^2(-T,T)$ be such that (Γ,κ_n) solve the Euler-Lagrange equations in the sense of Definition 2.1, assume that (6) holds and assume that $\mathcal{L}^1(\{t \in (-T,T) : \kappa_n(t) \neq 0\}) > 0$. Then*

$$\kappa_n \in C^0([-T,T]) \text{ and } \kappa, \kappa_n \in C^\infty\big([-T,T] \setminus \partial\{t \in [-T,T] : \kappa_n(t) = 0\}\big),$$

and the set I_0 has empty interior.
Assume, in addition, that the set $\{t \in (-T,T) : \kappa_n(t) = 0\}$ has empty interior. Then κ_n only has finitely many zeroes and it changes its sign at

each of them. Moreover, $\kappa \in C^\infty([-T,T] \setminus I_0) \cap C^2([-T,T])$ *and* $\kappa_n \in C^\infty([-T,T]\setminus I_0)\cap C^1([-T,T])$. *If there exists a* $\delta > 0$ *such that* $\kappa \in C^{2,\delta}(-T,T)$ *or* $\kappa_n \in C^{1,\delta}(-T,T)$ *then* $I_0 = \emptyset$.

2.5. Remarks.

(i) As seen above, if Γ is admissible and (6) holds, then the pair (Γ, κ_n) induces a mapping $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(-T,T)];\mathbb{R}^3)$ via (5). If $\kappa_n = 0$ almost everywhere on $(-T,T)$ then (Γ, κ_n) is affine on $[\Gamma(-T,T)]$, so the assumption $\mathcal{L}^1(\{\kappa_n \neq 0\}) > 0$ is not restrictive on the level of surfaces, i.e. if one is interested in the regularity of the surface (Γ, κ_n).

(ii) Emptyness of $\operatorname{int} I_0$ is a key regularity result. In the situation of interest when Γ is admissible and the induced surface (Γ, κ_n) is a critical point of $\mathcal{E}$ (under its own boundary conditions on $[\Gamma(0)] \cup [\Gamma(T)]$), Theorem 2.4 in fact implies that either the interior of $\{\kappa_n = 0\}$ is empty or $\kappa_n = 0$ almost everywhere.
Indeed, if $\operatorname{int}\{\kappa_n = 0\}$ is nonempty, then there exists a pair $(\hat{\Gamma}, \hat{\kappa}_n)$ such that $\operatorname{int} I_0^{\hat{\Gamma}}$ is nonempty but the induced surfaces still agree, i.e. $(\Gamma, \kappa_n) = (\hat{\Gamma}, \hat{\kappa}_n)$. This is proven in Lemma 7.4. Thus $(\hat{\Gamma}, \hat{\kappa}_n)$ is critical as well, and so it solves the Euler-Lagrange equations by the results in [11]. Hence Theorem 2.4 implies that $\hat{\kappa}_n = 0$ almost everywhere, i.e. $(\hat{\Gamma}, \hat{\kappa}_n)$ affine. Thus (Γ, κ_n) is affine, i.e. $\kappa_n = 0$ almost everywhere.
Once $\{\kappa_n = 0\}$ has empty interior, the second part of Theorem 2.4 implies that it is finite. Thus by (6) also I_0 is finite. This is a key regularity result, and it is clearly relevant for applications as well. It sharply contrasts e.g. with the example in the appendix to [10]. There it is shown that for arbitrary $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}$ the set I_0 can be open and dense (while still $\mathcal{L}^1(\{\kappa_n \neq 0\}) > 0$).

(iii) In Proposition 4.1 below we obtain a good picture of the behaviour of Γ and κ_n near the set I_0: Suppose that $t_0 \in I_0$, denote by $* \in \{+,-\}$ the sign of $\kappa(t_0)$ and set $\alpha^* = 1 - s^*\kappa$. Then, in a neighbourhood of t_0, we have $\sqrt{\alpha^*(t)}|\log \alpha^*(t)| \sim |t - t_0|$. Moreover,

$$|\kappa - \kappa(t_0)| \sim \alpha^*, \ |\kappa'| \sim \frac{\sqrt{\alpha^*}}{|\log \alpha^*|} \text{ and } |\kappa''| \leq C(\log \alpha^*)^{-2}.$$

In addition,

$$|\kappa_n| \sim \sqrt{\alpha^*} \text{ and } |\kappa_n'| \sim |\log \alpha^*|^{-1}. \tag{24}$$

In particular, κ', κ'' and κ_n' are continuous and zero at t_0. Moreover, using e.g. (4) in [10] together with the above scalings, we have

$$|\nabla^2(\Gamma,\kappa_n)(\Gamma(t)+s^*(t)N(t))| \geq C\frac{|\kappa_n(t)|}{1-s^*(t)\kappa(t)} \geq C\frac{1}{\sqrt{\alpha^*(t)}}.$$

So $\nabla^2(\Gamma,\kappa_n)$ diverges near $\Gamma(t_0)+s^*(t_0)N(t_0)\in\partial S$.
If $(\Gamma,\kappa_n)\in C^2$ then one can define a line of striction of the developable surface $(\Gamma,\kappa_n)([\Gamma(-T,T)])$, cf. [2]. The line of striction does not intersect this surface. But it intersects the boundary of the surface at the point $(\Gamma,\kappa_n)(\Gamma(t_0)+s^*(t_0)N(t_0)) = \gamma(t_0)+s^*(t_0)v(t_0)$ precisely if t_0 is as above.

(iv) The last statement of Theorem 2.4 shows that the regularity of κ and κ_n stated in the theorem are optimal. It could only be improved by showing that $I_0=\emptyset$. There is strong numerical evidence that in general $I_0\neq\emptyset$, cf. [23].

(v) If $\Gamma\in C^\infty([-T,T];S)$ is admissible and transversal, and if $I_0=\emptyset$ and $\kappa_n\in C^\infty([-T,T])$, then (Γ,κ_n) is C^∞ up to the boundary of the domain $[\Gamma(-T,T)]$. It can even be extended as a C^∞ isometric immersion to a domain containing $\overline{[\Gamma(-T,T)]}$. This is proven in Proposition 7.3 in the appendix.

2.6. For given $u\in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ and a closed subset $\partial_c S\subset\partial S$ we set

$$\mathcal{A}_u(S,\partial_c S) = \{\tilde{u}\in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) : \tilde{u}=u \text{ on } \partial_c S \text{ and } \nabla\tilde{u}=\nabla u \text{ on } \partial_c S\}. \quad (25)$$

In (25) the boundary values of the derivatives are understood in the trace sense. (Of course this definition makes sense even if S is only Lipschitz.) To formulate our main result in terms of the surface u we introduce three kinds of line segments in S:

$$\Sigma_\tau^u = \{x\in S\setminus C_{\nabla u} : [x] \text{ intersects } \partial S \text{ tangentially}\}$$
$$\Sigma_c^u = S\cap\overline{\{x\in S\setminus C_{\nabla u} : [x] \text{ intersects } \partial_c S\}}$$
$$\Sigma_0^u = \{x\in S\setminus(\overline{C}_{\nabla u}\cup\Sigma_\tau^u\cup\Sigma_c^u) : \text{ there is a } \nabla u\text{-integral curve } \Gamma \text{ and } t_0\in I_0 \text{ such that } [x]=[\Gamma(t_0)]_{N(t_0)}\}.$$

These sets are well defined by the uniqueness of $q_{\nabla u}$ on $S\setminus C_{\nabla u}$. In what follows we omit the index u.

2.7. Theorem. *Let $S\subset\mathbb{R}^2$ be a bounded C^∞-domain, let $\partial_c S\subset\partial S$ be closed and let u be a minimizer of $\mathcal{E}(\cdot;S)$ within the class $\mathcal{A}_u(S,\partial_c S)$. Then*

$$u\in C^3(S\setminus(\Sigma_\tau\cup\Sigma_c\cup\partial\hat{C}_{\nabla u});\mathbb{R}^3) \quad (26)$$

and

$$u \in C^\infty(S \setminus (\Sigma_0 \cup \Sigma_\tau \cup \Sigma_c \cup \partial\hat{C}_{\nabla u}); \mathbb{R}^3).$$

2.8. Remarks.

(i) The existence of minimizers is easy to prove, cf. [11].

(ii) The set Σ_c belongs to the singular set because the regularity of u on Σ_c is determined by the boundary conditions, which are not assumed to be regular.

(iii) The set $S \cap \partial\hat{C}_{\nabla u}$ consists of line segments on which ∇u is constant. In general, however, there can be uncountably many such segments (but of course $\operatorname{int} \partial\hat{C}_{\nabla u} = \emptyset$), cf. Section 6. The reason why $\hat{C}_{\nabla u}$ rather than $C_{\nabla u}$ occurs in Theorem 2.7 is that $C_{\nabla u} \setminus \hat{C}_{\nabla u} \subset D_{\nabla u}$. Notice also that $\partial\hat{C}_{\nabla u} \subset C_{\nabla u}$, cf. again Section 6.
Under certain assumptions on the boundary conditions one can prove that $\partial\hat{C}_{\nabla u}$ consists of only finitely many segments, cf. Proposition 6.4 and, for more sophisticated results, cf. [12]. In general, the boundary conditions imposed on $\partial_c S$ force $\hat{C}_{\nabla u} \neq \emptyset$ whenever $u \in W^{2,2}_{\text{iso}}(S; \mathbb{R}^3)$ satisfies them, and minimizers are not smooth at $\partial\hat{C}_{\nabla u}$, cf. again [12].

(iv) The set Σ_τ is closed and has empty interior, cf. Section 6. It consists of straight line segments intersecting ∂S tangentially at one end, and ∇u is constant on each such line segment. If S is convex, then Σ_τ is empty. In [12] we give an example of a domain S and boundary data such that for every minimizer u the set Σ_τ is nonempty and u fails to be smooth at Σ_τ.

(v) A related issue is the role of the regularity of ∂S. From the Euler-Lagrange equations and the arguments in this paper it is easy to see that u cannot be smooth if ∂S is not. As at Σ_τ, it is failure of regularity of the directed distance ν which determines that of u.

(vi) Since $u = (\Gamma, \kappa_n)$ locally on $D_{\nabla u}$, Theorem 2.4 implies the following facts (cf. also Proposition 6.3):

- We have $\nabla^2 u = 0$ on Σ_0. (By (26) this makes sense pointwise.)
- The set Σ_0 consists of countably many line segments. They can accumulate only at $\partial S \cup \Sigma_c \cup \Sigma_\tau$.
- The regularity at Σ_0 is optimal.

- If $x_0 \in \Sigma_0$ then there is $x_1 \in \partial S \cap \overline{[x_0]}$ (i.e. x_1 is one of the endpoints of $[x_0]$) such that $\nabla^2 u(y_n)$ diverges as $y_n \to x_1$ with $|y_n - x_1| \sim \mathrm{dist}_{[x_0]}(y_n)$ (cf. Remark 2.5). This explains the energy concentration found by numerical simulations for the Möbius strip in [23]. Remark 2.5 describes the local behaviour of the surface near such lines.
- Since the Gauss curvature of u is zero, its mean curvature H agrees with the normal curvature κ_n. More precisely, $H(\Gamma(t) + sN(t)) = \frac{\kappa_n(t)}{1-s\kappa(t)}$ whenever Γ is a ∇u-integral curve, i.e. (the preimage of) a nontrivial line of curvature. From this and from (24) one obtains the scaling of $H(x)$ with respect to the distance of x from a given segment Y in Σ_0 that was mentioned in the introduction. For each $\delta > 0$, the scaling constants can be chosen uniformly on $\{\mathrm{dist}_{\mathbb{R}^2\setminus S} \geq \delta\}$.

3 Partial regularity

3.1. Some definitions. We fix $T > 0$, $\Gamma \in W^{2,\infty}([-T,T]; S)$ locally admissible and transversal and $\kappa_n \in L^2(-T,T)$ such that (Γ, κ_n) solve the Euler-Lagrange equations in the sense of Definition 2.1. In addition, we assume that (6) holds. This implies that $\kappa_n = 0$ almost everywhere on I_0.
We define

$$\begin{aligned}
\Lambda_1 &= \gamma' \cdot \left(\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'(s)\ ds\right)\\
\Lambda_2 &= v \cdot \left(\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'(s)\ ds\right)\\
\Lambda_3 &= n \cdot \left(\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'(s)\ ds\right)\\
m &= \lambda_1 \cdot n.
\end{aligned}$$

The following equalities are obtained directly from the definitions and using the ODE (4):

$$\Lambda_1' = \kappa\Lambda_2 + \kappa_n\Lambda_3 \tag{27}$$
$$\Lambda_2' = -\kappa\Lambda_1 + \lambda_1 \cdot n \tag{28}$$
$$\Lambda_3' = -\kappa_n\Lambda_1 - \lambda_1 \cdot v \tag{29}$$

Since Γ is continuous, $\tilde\eta := \frac{1}{2}\inf \mathrm{dist}_{\partial S}(\Gamma([-T,T])) > 0$. We fix this $\tilde\eta$ for the rest of this section. For $\eta \geq 0$ we define the bounded domain

$$M_\eta = \{(s^-, s^+, x) \in \mathbb{R}^3 : |s^\pm| \in (\tilde\eta, 2\,\mathrm{diam}\, S) \text{ and } 1 - s^\pm x > \eta\}. \tag{30}$$

Since Γ is locally admissible, we have $(s^+(t), s^-(t), \kappa(t)) \in \bar{M}_0$ for all $t \in [-T, T]$. For $s^- < 0 < s^+$ and $x \in [\frac{1}{s^-}, \frac{1}{s^+}]$ we define the function

$$Z(s^\pm, x) = \begin{cases} -\frac{g_3(s^\pm,x)}{g_2(s^\pm,x)} & \text{if } x \in (\frac{1}{s^-}, \frac{1}{s^+}) \\ \frac{1}{x} & \text{if } x \in \{\frac{1}{s^-}, \frac{1}{s^+}\}. \end{cases} \tag{31}$$

For $z \in [s^-, s^+]$ we define $Q(s^\pm, z)$ implicitly by setting $Z(s^\pm, Q(s^\pm, z)) = z$. By Lemma 7.6, this gives a well-defined function which is continuous up to the boundary of M_0'. The behaviour of $Z(s^\pm, x)$ as $s^\pm x \to 1$ is studied in Lemma 7.6.

The following facts about g (defined in §2.1) and the functions related to it are easy to check, and they will often be used implicitly:

- The functions g and g_2 and g_3 are in $C^\infty(M_0)$ and in $C^\infty(\bar{M}_\eta)$ for any $\eta > 0$.
- There is $c > 0$, depending only on $\tilde{\eta}$, such that $g(s^\pm, x) \geq c$ and $g_2(s^\pm, x) \leq -c$ for all $(s^\pm, x) \in M_0$. This follows from the definitions and from the trivial estimate $0 < 1 - sx \leq 1 + |s||x| \leq C(\operatorname{diam} S, \tilde{\eta})$.
- For $(s^\pm, x) \in M_0$ we have $-xg_2(s^\pm, x) = \sum_* g_*(s^\pm, x)$ and $g(s^\pm, x) = -g_2(s^\pm, x) - xg_3(s^\pm, x)$.

3.2. Basic consequences of the Euler-Lagrange equations. We will make frequent use of the estimate

$$|\kappa_n| + |m'| + |\Lambda_1'| \leq C|\Lambda_2|. \tag{32}$$

To prove it, notice that $|\kappa_n| \leq C\Lambda_2$ by (14) because $g(s^\pm, \kappa) \geq c > 0$. This estimate implies the others because $m' = -\kappa_n \lambda_1 \cdot \gamma'$ (by (4)) and because of the above expression for Λ_1'.

Since $g(s^\pm, \kappa) \geq c$, (14) implies that, up to a set of measure zero (and in fact this null set is empty, see Proposition 3.4 below),

$$\{t \in [-T, T] : \Lambda_2(t) = 0\} = \{t \in [-T, T] : \kappa_n(t) = 0\}. \tag{33}$$

This fact will be used implicitly throughout the paper. We claim that

$$\{\Lambda_2 = 0\} \cap \{\Omega_2 \neq 0\} \subset I_0 \subset \{\Lambda_2 = 0\} \text{ and } \{\Omega_2 = 0\} \subset \{\Omega_3 = 0\} \cap \{\Lambda_2 = 0\}. \tag{34}$$

In fact, on the complement of I_0 we have $\Omega_2 = \kappa_n^2 g_2(s^\pm, \kappa)$ (by (15)) and $\Lambda_2 = -2\kappa_n g(s^\pm, \kappa)$ (by (14)). So $\{\Omega_2 = 0\} \setminus I_0 = \{\Lambda_2 = 0\} \setminus I_0$ (up to a null set) because $|g_2(s^\pm, \kappa)|$ and $g(s^\pm, \kappa)$ are never zero. This proves $\{\Lambda_2 = 0\} \cap \{\Omega_2 \neq 0\} \subset I_0$. By (33) we have $\Lambda_2 = 0$ on I_0. Regarding the second part of (34),

notice that on I_0 we have $\Omega_3 = -\Omega_2\sigma$ (by (16)). Since σ is not zero on I_0, we have $\{\Omega_2 = 0\} \cap I_0 = \{\Omega_3 = 0\} \cap I_0 \subset \{\Lambda_2 = 0\}$. Since $\Omega_3 = 0$ on $\{\Lambda_2 = 0\} \setminus I_0$ (by (16) and (33), we are done because $\{\Omega_2 = 0\} \setminus I_0 = \{\Lambda_2 = 0\} \setminus I_0$.
Assume that κ_n does not vanish identically, i.e. $\mathcal{L}^1(\{t \in (-T, T) : \kappa_n(t) \neq 0\}) > 0$. Then we have:

$$\{t \in [-T, T] : m(t) = \Lambda_2(t) = \Lambda_1(t) = 0\} = \emptyset \tag{35}$$

and

$$\{t \in [-T, T] : m(t) = \Omega_2(t) = 0\} = \emptyset. \tag{36}$$

To prove (35), assume that there exists a t_0' with $m(t_0') = \Lambda_2(t_0') = \Lambda_1(t_0') = 0$. Then there is $t_0 \in \{m = \Lambda_1 = 0\} \cap \partial\{\Lambda_2 \neq 0\}$ because $\Lambda_1' = m' = 0$ almost everywhere on $\{\Lambda_2 = 0\}$ (by (33) and by $\Lambda_1' = \kappa_n\Lambda_3 + \kappa\Lambda_2$, see §3.1) and since $\{\Lambda_2 \neq 0\} \neq \emptyset$ by the hypothesis and by (33). Let us assume that $(t_0, t_0 + \varepsilon) \cap \{\Lambda_2 \neq 0\} \neq \emptyset$ for all $\varepsilon > 0$. Then $|\Lambda_2(t)| \leq C\int_{t_0}^t |m'| + C\int_{t_0}^t |\Lambda_1'| \leq C\int_{t_0}^t |\Lambda_2|$ by (28, 32). Thus Gronwall's inequality implies that $\Lambda_2 = 0$ on $(t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$, contradicting the choice of t_0. The case when $(t_0 - \varepsilon, t_0) \cap \{\Lambda_2 \neq 0\} \neq \emptyset$ is similar.
To prove (36), assume the contrary. Let us first consider the situation that there is $t_0 \in \partial\{\Omega_2 \neq 0\}$ with $m(t_0) = \Omega_2(t_0) = 0$. Without loss of generality we assume $t_0 = 0$ and that $(0, \delta) \cap \{\Omega_2 \neq 0\} \neq \emptyset$ for all $\delta > 0$; the case $(-\delta, 0) \cap \{\Omega_2 \neq 0\} \neq \emptyset$ is similar. From (17) we deduce, since $h \in L^\infty$ by transversality,

$$|\Omega_2(t)| \leq C\big((\int_0^t |\kappa_n|)^2 + \int_0^t |\Omega_2| + \int_0^t \kappa_n^2\big). \tag{37}$$

By Jensen's inequality $(\int_0^t |\kappa_n|)^2 \leq t\int_0^t \kappa_n^2$. But since $|g_2| \geq c > 0$, from (15) we also have $\kappa_n^2 \leq C\kappa_n^2|g_2(s^\pm, \kappa)| \leq C|\Omega_2|$. Thus (37) implies $|\Omega_2| \leq C\int_0^t |\Omega_2|$. Hence by Gronwall's inequality we find $\delta > 0$ such that $\Omega_2 = 0$ on $(0, \delta)$, a contradiction.
Now let $t_0 \in \operatorname{int}\{\Omega_2 = 0\} \subset (-T, T)$ and suppose that $m(t_0) = 0$. Denoting by (t_1, t_2) the maximal interval in $\{\Omega_2 = 0\}$ containing t_0 we find that $m(t_2) = m(t_1) = m(t_0) = 0$ because $m' = -(\lambda_1 \cdot \gamma')\kappa_n = 0$ on $\{\Omega_2 = 0\}$. But unless $\Omega_2 = 0$ on all of $[-T, T]$ (which would imply $\kappa_n = 0$ almost everywhere, contradicting the hypothesis), we have that $t_i \in (-T, T) \cap \partial\{\Omega_2 \neq 0\}$ for some $i \in \{1, 2\}$. Now we can apply the first part of the proof with t_i instead of t_0 to find the desired contradiction. This concludes the proof of (36).

Again assume that κ_n does not vanish identically. Then, for every interval J in $\{\Lambda_2 = 0\}$ we have:

$$\Lambda_1 \text{ and } m \text{ are constant on } J \text{ with } \Lambda_1 \neq 0, \text{ and } \kappa = \tfrac{m}{\Lambda_1} \text{ on } \operatorname{int} J. \tag{38}$$

To prove this, let J be a nondegenerate maximal (therefore closed) interval in the closed set $\{\Lambda_2 = 0\}$. Then

$$\kappa_n = 0 \text{ and } 0 = \Lambda_2' = -\kappa\Lambda_1 + m \text{ almost everywhere on } J. \tag{39}$$

So $\Lambda_1' = \kappa\Lambda_2 + \kappa_n\Lambda_3 = 0$ (see §3.1) and $m' = -(\lambda_1 \cdot \gamma')\kappa_n = 0$ on J as well. If $\Lambda_1 = 0$ then (39) implies $m = 0$ on J, contradicting (35). If $\Lambda_1 = c \neq 0$ on int J then by continuity $\Lambda_1 = c$ on J, and from (39) we deduce that the precise representative of κ equals the constant $\frac{m}{\Lambda_1}$ everywhere on int J.

3.3. For all $t \in [-T, T]$ we define

$$\zeta(t) = \begin{cases} -\frac{\Omega_3(t)}{\Omega_2(t)} & \text{if } \Omega_2(t) \neq 0 \\ \frac{\Lambda_1(t)}{m(t)} & \text{if } \Omega_2(t) = 0. \end{cases}$$

In the relevant situation (namely when κ_n does not vanish identically), it is well defined by (36). Moreover, we have

$$\zeta = Z(s^{\pm}, \kappa) \text{ almost everywhere on } \{t \in [-T, T] : \Omega_2(t) \neq 0\}. \tag{40}$$

In fact, by (16) we have $\zeta = -\frac{\Omega_3}{\Omega_2} = \sigma = Z(s^{\pm}, \kappa)$ on $I_0 \cap \{\Omega_2 \neq 0\}$. But on the complement of I_0, by (15, 16) we have $\kappa_n^2 g_2(s^{\pm}, \kappa) = \Omega_2$ and $\kappa_n^2 g_3(s^{\pm}, \kappa) = \Omega_3$, so $\zeta = Z(s^{\pm}, \kappa)$ also on $\{\Omega_2 \neq 0\} \setminus I_0$. This proves (40).
Notice that (40) together with Lemma 7.6 imply that

$$\zeta(t) \in [s^-(t), s^+(t)] \text{ for all } t \in \{\Omega_2 \neq 0\}. \tag{41}$$

Let us note the useful estimate

$$|\Omega_3| \leq C|\Omega_2|. \tag{42}$$

To prove it, note that $\Omega_3 = \sigma\Omega_2$ on I_0 by (16). Outside I_0 we have $|\Omega_3| = |\Omega_2||Z(s^{\pm}, \kappa)|$ by (15, 16). And $|Z(s^{\pm}, \kappa)| \in [s^-, s^+]$ by Lemma 7.6.

The next proposition provides a first regularity result for κ and κ_n. It is not very useful at the moment due to poor control over the sets $\{\Omega_2 \neq 0\}$ and $\{\kappa_n = 0\}$.

3.4. Proposition. *We have $\kappa_n \in C^0([-T, T])$ and $\kappa \in C^0(\{t \in [-T, T] : \Omega_2(t) \neq 0\})$. Moreover, $\kappa, \kappa_n \in C^\infty\Big([-T, T] \setminus \partial\{t \in [-T, T] : \kappa_n(t) = 0\}\Big)$.*

Proof. We claim that

$$\kappa, \kappa_n \in C^\infty(\{\Lambda_2 \neq 0\}) \cap C^0(\{\Omega_2 \neq 0\}). \tag{43}$$

To prove this let $t' \in \{\Omega_2 \neq 0\}$. By §3.3 and the definition of Q we have $\kappa(t) = Q(s^{\pm}(t), \zeta(t))$ for all t in a relatively open interval $J \subset [-T, T]$ containing t',

since $\{\Omega_2 \neq 0\}$ is relatively open. Hence $\kappa \in C^0(J)$ because $\Omega_2, \Omega_3, s^\pm$ are continuous, because $\zeta \in [s^-, s^+]$ on $\{\Omega_2 \neq 0\}$ and because $Q \in C^0(\bar{M}_0')$, cf. Lemma 7.6. But $\kappa_n(t) = -\frac{\Lambda_2(t)}{2g(s^\pm(t),\kappa(t))}$ by (14). Hence $\kappa_n \in C^0(J)$ because $\frac{1}{g} \in C^0(\bar{M}_0)$. This proves continuity on $\{\Omega_2 \neq 0\}$. To prove smoothness on $\{\Lambda_2 \neq 0\}$ assume that $t' \in J$, where $J \subset [-T, T]$ is a relatively open interval with $\bar{J} \subset \{\Lambda_2 \neq 0\}$. Then κ, $\kappa_n \in C^0(\bar{J})$ because $\{\Lambda_2 \neq 0\} \subset \{\Omega_2 \neq 0\}$ by (34). But if k is a nonnegative integer and $\kappa, \kappa_n \in C^k(J)$ then $\Omega_2, \Omega_3, s^\pm \in C^{k+1}(J)$. On the other hand, we have $\bar{J} \cap I_0 = \emptyset$, since $I_0 \subset \{\Lambda_2 = 0\}$ (see 34). So $(s^-(t), s^+(t), \kappa(t)) \in M_0$ for all $t \in \bar{J}$. But $g \in C^\infty(M_0)$ and $Q \in C^\infty(M_0')$ by Lemma 7.6. So $\kappa = Q(s^\pm, \zeta)$ is in C^{k+1}. Hence so is $\kappa_n = -\frac{\Lambda_2}{2g(s^\pm,\kappa)}$, and (43) follows inductively.
Let us next prove continuity of κ_n on $[-T, T]$. Since $\kappa_n = 0$ almost everywhere on $\{\Lambda_2 = 0\}$, by what was shown above we have $\kappa_n \in C^\infty\Big(\{\Lambda_2 \neq 0\} \cup \mathrm{int}\{\Lambda_2 = 0\}\Big)$. But $g(s^\pm, \kappa)$ is bounded from below by a positive constant. So from (14) we deduce that $\lim_{t\to t_0} |\kappa_n(t)| \le C \lim_{t\to t_0} |\Lambda_2(t)|$ for every $t_0 \in \partial\{t \in [-T, T] : \Lambda_2(t) = 0\}$. So κ_n is continuous and $\{\Lambda_2 = 0\} = \{\kappa_n = 0\}$. Hence (43) and (38) imply the claim. □

4 Main regularity results

For $* \in \{-, +\}$ and $t \in [-T, T]$ we define

$$\alpha^*(t) = 1 - s^*(t)\kappa(t). \tag{44}$$

In this section we will prove the following key result:

4.1. Proposition. *Assume that (6) holds, that $\mathcal{L}^1(\{t \in (-T, T) : \kappa_n(t) \neq 0\}) > 0$ and that $\{t \in (-T, T) : \kappa_n(t) = 0\}$ has empty interior. Then the following hold:*

(i) $\kappa \in C^0([-T, T])$.

(ii) The set $\{t \in [-T, T] : \kappa_n(t) = 0\}$ is finite. In particular, I_0 is finite. Moreover, if $\kappa_n(t_0) = 0$ then κ_n changes its sign at t_0 and $|\Lambda_2(t)| \geq c|t - t_0|$ for all t near t_0. In particular, $|\kappa_n(t)| \geq c|t - t_0|$ whenever $\kappa_n(t_0) = 0$ and $t_0 \notin I_0$.

(iii) The set I_0 agrees with $\{t \in [-T, T] : \Lambda_2(t) = 0$ and $\Omega_2(t) \neq 0\}$.

(iv) If $t_0 \in I_0$ and $*$ denotes the sign of $\kappa(t_0)$, then there are $c, C > 0$ such that for all t in a neighbourhood of t_0:

$$c|t-t_0| \le \sqrt{\alpha^*(t)}|\log \alpha^*(t)| \le C|t-t_0| \tag{45}$$

$$c\alpha^*(t) \le |\kappa(t)-\kappa(t_0)| \le C\alpha^*(t) \tag{46}$$

$$c\frac{\sqrt{\alpha^*(t)}}{|\log \alpha^*(t)|} \le |\kappa'(t)| \le C\frac{\sqrt{\alpha^*(t)}}{|\log \alpha^*(t)|} \tag{47}$$

$$|\kappa''(t)| \le C\big(\log \alpha^*(t)\big)^{-2} \tag{48}$$

$$c\sqrt{\alpha^*(t)} \le |\kappa_n(t)| \le C\sqrt{\alpha^*(t)}. \tag{49}$$

$$c|\log \alpha^*(t)|^{-1} \le |\kappa_n'(t)| \le C|\log \alpha^*(t)|^{-1}. \tag{50}$$

In particular, κ', κ'' and κ_n' are continuous and zero at t_0.

(v) If J is relatively open in $[-T,T]$ and $\kappa \in C^{2,\varepsilon}(J)$ or $\kappa_n \in C^{1,\varepsilon}(J)$ for some $\varepsilon > 0$ then $I_0 \cap J = \emptyset$.

Remark and Definition. In Proposition 4.1 the hypothesis $\mathrm{int}\{t \in (-T,T) : \kappa_n(t) = 0\} = \emptyset$ can be replaced by the following weaker assumption (A):
We say that (Γ, κ_n) satisfies condition (A) if the following holds: If J is a non-degenerate maximal interval in $\{t \in [-T,T] : \kappa_n(t) = 0\}$ with $\mathcal{L}^1(J \cap I_0) = 0$ then κ is not constant on J (i.e. there is no $\kappa_0 \in \mathbb{R}$ such that $\kappa(t) = \kappa_0$ for $\mathcal{L}^1$ almost every $t \in J$).

The remainder of this section is devoted to the proof of Proposition 4.1. In addition to the standing assumptions, we also assume that κ_n is nonzero on a set of positive measure. However, we do not assume that $\mathrm{int}\{\kappa_n = 0\} = \emptyset$ (nor the nondegeneracy assumption (A)) unless stated explicitly. Condition (A) is only assumed for the second half of Corollary 4.10 below. (And this is then used in the proof of Proposition 4.1.)
In view of Proposition 3.4, in order to prove global continuity of κ, we must understand the behaviour at and of the zeroes of Ω_2. The main result in this direction is the following proposition. Recall from (36) that $m(t_0) \neq 0$ whenever $\Omega_2(t_0) = 0$.

4.2. Proposition. *Suppose that $\Omega_2(t_0) = 0$. Then two cases can occur:*

(i) Assume that $\frac{\Lambda_1(t_0)}{m(t_0)} \notin \{s^-(t_0), s^+(t_0)\}$. Then $\frac{\Lambda_1(t_0)}{m(t_0)} \in (s^-(t_0), s^+(t_0))$ and there is $c_1 > 0$ such that either $\Omega_2 = 0$ on $(t_0, t_0 + c_1)$ or $\Omega_2 \neq 0$ on $(t_0, t_0 + c_1)$. In the latter case, there are $\eta > 0$ and $* \in \{-,+\}$ such

that $(t_0, t_0 + c_1) \subset I_\eta$, $*\kappa_n(t) \geq \eta|t - t_0|$ *for all* $t \in (t_0, t_0 + c_1)$, *and* $\lim_{t\downarrow t_0} \zeta(t) = \frac{\Lambda_1(t_0)}{m(t_0)}$. *A similar alternative applies on* $(t_0 - c_1, t_0)$.

(ii) Assume that $\frac{\Lambda_1(t_0)}{m(t_0)} \in \{s^-(t_0), s^+(t_0)\}$. *Then* $t_0 \in \operatorname{int} I_0$.

The remaining zeroes of κ_n, i.e. the set $\{\Omega_2 \neq 0\} \cap \{\Lambda_2 = 0\}$, lie in I_0 by (34). They are handled in the following proposition.

4.3. Proposition. *Suppose that* $\Lambda_2(t_0) = 0$ *and* $\Omega_2(t_0) \neq 0$. *Then either* $t_0 \in \operatorname{int} I_0$ *or there is* $\varepsilon > 0$ *such that* $|\Lambda_2(t)| \geq \varepsilon|t - t_0|$ *for all* $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

In the proof of Propositions 4.2 and 4.3 we will assume without loss of generality that $t_0 = 0$. So the following lemmas are only stated and proven for this case. We define

$$K_n(t) = \int_0^t \kappa_n(r)\, dr.$$

Proposition 4.2 (i) will follow from the next two lemmas. Lemma 4.4 states that $\Omega_2 = mK_n$ and $\Omega_3 = -\Lambda_1 K_n$, plus error terms of the order $\int_0^t |K_n|$. Lemma 4.5 shows that these error terms are indeed negligible.

4.4. Lemma. *Assume that* $\Omega_2(0) = 0$. *Then* $\Omega_3(0) = 0$ *and there is* $C_1 > 0$ *such that for all* $t \in [-T, T]$ *we have*

$$|\Omega_2(t) - m(t)K_n(t)| \leq C_1 \int_0^t |K_n(r)|\, dr \tag{51}$$

$$|\Omega_3(t) + \Lambda_1(t)K_n(t)| \leq C_1 \int_0^t |K_n(r)|\, dr. \tag{52}$$

Proof. Firstly, $\Omega_3(0) = 0$ by (42). By a partial integration, from $\Omega_2' = -h\Omega_2 + \kappa_n m + \kappa_n^2 F_1$ we find $\Omega_2 = mK_n + \int_0^t \big((\lambda_1 \cdot \gamma')\kappa_n K_n - h\Omega_2 + \kappa_n^2 F_1\big)$. So

$$\begin{aligned} |\Omega_2 - mK_n| &\leq \int_0^t \big(|(\lambda_1 \cdot \gamma')\kappa_n||K_n| + (|h| + \frac{|F_1|}{|g_2(s^\pm, \kappa)|})|\Omega_2|\big) \\ &\leq \int_0^t \big(|(\lambda_1 \cdot \gamma')\kappa_n||K_n| + (|h| + \frac{|F_1|}{|g_2(s^\pm, \kappa)|})|mK_n|\big) \\ &\quad + \int_0^t (|h| + \frac{|F_1|}{|g_2(s^\pm, \kappa)|})|\Omega_2 - mK_n| \\ &\leq C \int_0^t |K_n| + C \int_0^t |\Omega_2 - mK_n|. \end{aligned} \tag{53}$$

Here we used that $\kappa_n^2 \le \left|\frac{\Omega_2}{g_2(s^\pm,\kappa)}\right|$ everywhere. This estimate is trivially true on $\{\kappa_n = 0\}$, and on $\{\kappa_n \neq 0\}$ we have $\kappa_n^2 = \frac{\Omega_2}{g_2(s^\pm,\kappa)}$ by (15). In the last step leading to (53) we used that, due to transversality, $h, F_1 \in L^\infty$ (cf. (19)) and that $|g_2(s^\pm,\kappa)| \ge c > 0$. From (53) and Gronwall's inequality we deduce (51). The proof for (52) is similar, and now we can use $\kappa_n^2(t) \le C|\Omega_2(t)| \le C|K_n(t)| + C\int_0^t |K_n|$. $\square$

4.5. Lemma. *Assume that $\Omega_2(0) = 0$, that $\frac{\Lambda_1(0)}{m(0)} \notin \{s^-(0), s^+(0)\}$ and that $(0,\varepsilon) \cap \{\Omega_2 \neq 0\} \neq \emptyset$ for all $\varepsilon > 0$. Then there is $\eta > 0$ such that $\kappa_n \neq 0$ on $(0,\eta)$ and $(0,\eta) \subset I_\eta$. In particular, $\int_0^t |K_n| \le t|K_n(t)|$ on $(0,\eta)$. A similar result applies to left neighbourhoods.*

Proof. We begin by remarking that $\int_0^t |K_n|$ is strictly positive for all $t > 0$. In fact, otherwise K_n would vanish identically in a right neighbourhood of zero. This would imply the same for Ω_2 (e.g. by Lemma 4.4), which would contradict the hypotheses.

Claim #1. There are positive constants $c_1^\#$, C_2, δ and η such that

$$t' \in (0, c_1^\#) \text{ and } |K_n(t')| \ge C_2 \int_0^{t'} |K_n| \tag{54}$$

implies that $\kappa_n(t') \neq 0$, that $t' \in I_\eta$ and that $|\Omega_2(t')| \ge \delta |K_n(t')| \neq 0$.

To prove the claim, notice that since $m(0) \neq 0$ and $\frac{\Lambda_1(0)}{m(0)} \neq s^*(0)$, by continuity there are $\delta, c_1^\# \in (0,1)$ such that

$$|m(t)| \ge 2\delta \text{ and } |\frac{\Lambda_1(t)}{m(t)} - s^*(t)| \ge 2\delta \text{ for all } t \in [0, c_1^\#] \text{ and } * \in \{+,-\}. \tag{55}$$

Now let C_2 be such that, with C_1 from the conclusion of Lemma 4.4,

$$\frac{C_1}{C_2} \le \delta \text{ and } \frac{C_1}{C_2} \le \frac{\delta^2}{1 + |\frac{\Lambda_1(t)}{m(t)}|} \text{ for all } t \in [0, c_1^\#]. \tag{56}$$

Then by Lemma 4.4 and (56) we have, for all t' satisfying (54),

$$\begin{aligned} |\Omega_2(t')| &\ge (|m(t')| - C_1 \frac{\int_0^{t'} |K_n|}{|K_n(t')|})|K_n(t')| \\ &\ge (|m(t')| - \frac{C_1}{C_2})|K_n(t')| \ge \delta |K_n(t')|. \end{aligned} \tag{57}$$

And clearly $\delta|K_n(t')| \ge \delta C_2 \int_0^{t'} |K_n| \neq 0$.
We claim that

$$|\zeta(t') - \frac{\Lambda_1(t')}{m(t')}| \le \delta. \tag{58}$$

In fact, since $\Omega_2(t') \neq 0$, by definition we have $\zeta(t') = -\frac{\Omega_3(t')}{\Omega_2(t')}$. And $\frac{\Omega_3}{\Omega_2} + \frac{\Lambda_1}{m} = \frac{-\Lambda_1 + \tilde{\Omega}_3}{m + \tilde{\Omega}_2} + \frac{\Lambda_1}{m}$, where $|\tilde{\Omega}_2(t)|, |\tilde{\Omega}_3(t)| \leq C_1 |K_n(t)|^{-1} \int_0^t |K_n|$ by Lemma 4.4. Hence by (54) we have $|\tilde{\Omega}_2(t')|, |\tilde{\Omega}_3(t')| \leq \frac{C_1}{C_2}$. Thus since $|m(t')| - \frac{C_1}{C_2} \geq |m(t')| - \delta \geq \delta$ we conclude

$$\Big|\frac{\Omega_3(t')}{\Omega_2(t')} + \frac{\Lambda_1(t')}{m(t')}\Big| \leq \frac{1}{|m(t')| - |\tilde{\Omega}_2(t')|}\Big(\frac{|\Lambda_1(t')|}{|m(t')|} \cdot |\tilde{\Omega}_2(t')| + |\tilde{\Omega}_3(t')|\Big)$$
$$\leq \frac{1}{\delta}\Big(1 + \frac{|\Lambda_1(t')|}{|m(t')|}\Big)\frac{C_1}{C_2} \leq \delta \tag{59}$$

by (56). This proves (58).
Since $\Omega_2(t') \neq 0$, by (40) (and since κ is continuous in a neighbourhood of t' by Proposition 3.4) we have

$$|Z(s^{\pm}(t'), \kappa(t')) - s^*(t')| \geq |\frac{\Lambda_1}{m}(t') - s^*(t')| - |\zeta(t') - \frac{\Lambda_1}{m}(t')|.$$

By (55) and (58) this implies $|Z(s^{\pm}(t'), \kappa(t')) - s^*(t')| \geq \delta$ for $* \in \{+, -\}$ and for t' as above. By Lemma 7.6 this implies that $t' \in I_\eta$ for some $\eta > 0$ depending only on δ.

Claim #2. There is $c_1 \in (0, c_1^{\#})$ with the property that whenever $t' \in (0, c_1)$ is such that $|K_n(t')| \geq C_2 \int_0^{t'} |K_n|$ then $|K_n(t)| \geq C_2 \int_0^t |K_n|$ for all $t \in [t', c_1]$.

To prove Claim #2, denote by $C_3 \in (0, \infty)$ the supremum of $|g_2(s^{\pm}, \kappa)|$ on I_η with η as above. Let $c_1 \in (0, c_1^{\#})$ be such that

$$\sqrt{\frac{\delta}{C_3}} \geq 2|K_n|^{\frac{1}{2}} \text{ on } [0, c_1]. \tag{60}$$

This is possible since $K_n(t) \to 0$ as $t \downarrow 0$. Let $t' \in (0, c_1)$ be such that $|K_n(t')| \geq C_2 \int_0^{t'} |K_n|$. Then Claim #1 implies that $t' \in I_\eta$ and $|\Omega_2(t')| \geq \delta |K_n(t')|$. So by (60),

$$|\kappa_n(t')| = \Big(\frac{|\Omega_2(t')|}{|g_2(s^{\pm}(t'), \kappa(t'))|}\Big)^{\frac{1}{2}}$$
$$\geq \sqrt{\frac{\delta}{C_3}} |K_n(t')|^{\frac{1}{2}}$$
$$\geq 2|K_n(t')|. \tag{61}$$

And $|K_n(t')| \geq C_2 \int_0^{t'} |K_n| \neq 0$. Now, (61) also shows that $|K_n|'(t') = |\kappa_n(t')| > \frac{d}{dt}|_{t=t'}\big(\int_0^t |K_n|\big)$, provided that $\kappa_n(t')$ and $K_n(t')$ have the same sign. So if this is the case, Claim #2 follows.
Let us prove that the signs of $\kappa_n(t')$ and $K_n(t')$ agree. Suppose for contradiction that we had $K_n(t') > 0$ yet $\kappa_n(t') < 0$ (the other case is similar). Let

$t_0 = \sup\{t \in [0,t'] : \kappa_n(t) = 0\}$. Since $\kappa_n(0) = 0 \neq \kappa_n(t')$, continuity of κ_n implies that $t_0 \in [0,t')$ and that $\kappa_n(t_0) = 0$. Moreover, $K_n' = \kappa_n < 0$ on (t_0,t'), whence $K_n(t_0) > K_n(t') \geq C_2 \int_0^{t'} |K_n| \geq C_2 \int_0^{t_0} |K_n|$. In particular, $K_n(t_0) > 0$, so $t_0 \neq 0$. And $t_0 \in I_\eta$ by Claim #1. Thus (15) gives

$$\kappa_n^2(t_0) = \frac{|\Omega_2(t_0)|}{|g_2(s^\pm(t_0), \kappa(t_0))|} \geq c|\Omega_2(t_0)| \geq c\delta |K_n(t_0)| \neq 0$$

by Claim #1. This contradiction finishes the proof of Claim #2.

To complete the proof of the lemma, notice that there exists a sequence $t_j \downarrow 0$ such that $\tilde{C}_j = \frac{|K_n(t_j)|}{\int_0^{t_j} |K_n|} \to \infty$, because otherwise Gronwall's inequality would imply that K_n vanishes identically on a right neighbourhood of zero, a contradiction. Now let C_2 and c_1 be the constants furnished by Claims #1 and #2. There is $J_0 \in \mathbf{N}$ such that for all $j \geq J_0$ we have $\tilde{C}_j \geq C_2$ and $t_j \in (0, c_1)$. Hence Claim #2 implies that $K_n(t) \geq C_2 \int_0^t |K_n|$ for all $t \in \bigcup_{j \geq J_0} (t_j, c_1) = (0, c_1)$. So $(0,c_1) \subset I_\eta$ and $\kappa_n \neq 0$ on $(0,c_1)$ by Claim #1. Finally notice that having $\kappa_n \neq 0$ on some $(0,\eta)$ implies that $|K_n|$ increases monotonically on $(0,\eta)$. And this implies $\int_0^t |K_n| \leq t|K_n(t)|$ on $(0,\eta)$. □

4.6. Proof of Proposition 4.2 (i). By translating we may assume without loss of generality that $t_0 = 0$. If Ω_2 does not vanish identically on any $(0,c_1)$ then the assumptions of Lemma 4.5 are satisfied. So $\lim_{t\downarrow 0} \frac{\int_0^t |K_n|}{|K_n(t)|} = 0$, whence there is $c > 0$ such that $|\Omega_2| \geq c|K_n|$ near zero (by Lemma 4.4 because $m(0) \neq 0$). Also by Lemma 4.5, there are $\eta, c_1 > 0$ such that $\kappa_n \neq 0$ (so also $\Omega_2 \neq 0$) on $(0,c_1)$ and $(0,c_1) \subset I_\eta$. The latter fact implies that $|g_2(s^\pm, \kappa)| \leq C_3 < \infty$ on $(0,c_1)$. Hence by (15)

$$\kappa_n^2 \geq \frac{1}{C_3}|\Omega_2| \geq \frac{c}{C_3}|K_n| \text{ near zero.} \tag{62}$$

Since $\kappa_n \neq 0$ on $(0,c_1)$, by continuity its sign $*$ is constant on $(0,c_1)$, and it clearly agrees with the sign of K_n. So taking square roots in (62) we obtain $*K_n' \geq c\sqrt{*K_n}$. Hence $*K_n(t) \geq ct^2$ for small $t \in (0,c_1)$. By (62) this implies $*\kappa_n(t) \geq ct$.

Finally notice that by $\int_0^t |K_n| \ll |K_n(t)|$ and by Lemma 4.4 we have

$$\lim_{t\downarrow 0} \zeta(t) = \frac{\Lambda_1(0)}{m(0)}. \tag{63}$$

But by (41) and by continuity of $s^\pm$ the left-hand side lies in the interval $[s^-(0), s^+(0)]$ because $\Omega_2 \neq 0$ near zero. □

Next we turn to the case $\frac{\Lambda_1(0)}{m(0)} \in \{s^-(0), s^+(0)\}$. For $* \in \{-,+\}$ we define $D^*(t) = \int_0^t \alpha^*$ ($\equiv -\int_t^0 \alpha^*$ if $t < 0$), with α^* as defined in (44). Notice that by local admissibility $\alpha^* \geq 0$ almost everywhere, so D^* are nondecreasing functions. The following lemma provides the key for the proofs of Proposition 4.2 (ii) and of Proposition 4.3. It states that near points where Λ_2 and $\frac{\Lambda_1}{m} - s^*$ vanish simultaneously, Λ_2 behaves like D^*.

4.7. Lemma. *Suppose that* $\Lambda_2(0) = 0$ *and* $\frac{\Lambda_1(0)}{s^*(0)} = m(0)$. *Then* $m(0) \neq 0 \neq \Lambda_1(0)$ *and there are* $\varepsilon, C > 0$ *such that*

$$|\Lambda_2(t) - \frac{\Lambda_1(t)}{s^*(t)} D^*(t)| \leq C \int_0^t |D^*| \text{ for all } t \in (-\varepsilon, \varepsilon). \tag{64}$$

In particular, there are $C, c > 0$ *such that*

$$c|D^*(t)| \leq |\Lambda_2(t)| \leq C|D^*(t)| \text{ for all } t \in (-\varepsilon, \varepsilon). \tag{65}$$

Proof. We have $m(0) = 0$ if and only if $\Lambda_1(0) = 0$, but since $\Lambda_2(0) = 0$, they cannot be zero because of (35). Thus $m(0) \neq 0 \neq \Lambda_1(0)$.
Since $\kappa \in [\frac{1}{s^-}, \frac{1}{s^+}]$, we have $D^*(t) \geq 0$ if $t \geq 0$ and $D^*(t) \leq 0$ if $t \leq 0$, and D^* is nondecreasing. Without loss of generality let us restrict to the case $t \geq 0$. We have

$$\Lambda_2' = m - \kappa\Lambda_1 = m - \frac{\Lambda_1}{s^*} + \alpha^* \frac{\Lambda_1}{s^*}. \tag{66}$$

Since $\Lambda_2(0) = 0$ and $(D^*)' = \alpha^*$, after a partial integration this implies

$$\Lambda_2(t) = \frac{\Lambda_1(t)}{s^*(t)} D^*(t) - \int_0^t (\frac{\Lambda_1}{s^*})' D^* + \int_0^t (m - \frac{\Lambda_1}{s^*}). \tag{67}$$

But

$$|m - \frac{\Lambda_1}{s^*}| \leq \int_0^t |m'| + |\Big(\frac{\Lambda_1}{s^*}\Big)'| \leq C(|\Lambda_2| + |D^*|)$$

because $m(0) - \frac{\Lambda_1(0)}{s^*(0)} = 0$, by assumption, since $s^*(0) \neq 0$ and because

$$|m'| + |\Lambda_1'| + |(s^*)'| \leq C(|\Lambda_2| + \alpha^*).$$

This latter estimate follows from (32) and from

$$(s^*)' = *\alpha^* \nu_1(\Gamma, *N) \cdot \Gamma' \text{ for } * \in \{-, +\}. \tag{68}$$

Equation (68) is proven in the appendix to [11].
By the triangle inequality we have $|\Lambda_2| \leq |\Lambda_2 - \frac{\Lambda_1}{s^*} D^*| + CD^*$. Thus (67) implies

$$|\Lambda_2(t) - \frac{\Lambda_1(t)}{s^*(t)} D^*(t)| \leq C\Big(\int_0^t D^* + \int_0^t |\Lambda_2 - \frac{\Lambda_1}{s^*} D^*|\Big).$$

Now (64) follows from Gronwall's inequality. The estimates (65) follow from (64) because by monotonicity $\int_0^t D^* \le tD^*(t)$ and because $\Lambda_1(0) \ne 0$. $\square$

4.8. Proof of Proposition 4.2 (ii). Without loss of generality we assume that $t_0 = 0$. We consider only the case when $\frac{\Lambda_1(0)}{m(0)} = s^+(0)$; the other case is similar. We only prove that $(0, \varepsilon) \subset I_0$. By an analogous argument one shows $(-\varepsilon, 0) \subset I_0$. Then (33) and continuity of Λ_2 imply that $(-\varepsilon, \varepsilon) \subset \{\Lambda_2 = 0\}$. Thus (38) and continuity of Λ_1, m and s^* imply that also $0 \in I_0$.
We claim that there is $\varepsilon > 0$ such that $(0, \varepsilon) \subset I_0$. Suppose that this were false. Since $\Omega_2(0) = 0$, by (34) we have $\Lambda_2(0) = 0$. So (65) implies the key fact that there is $\varepsilon > 0$ such that $\Lambda_2 \ne 0$ on $(0, \varepsilon)$. So $\Omega_2 \ne 0$ on $(0, \varepsilon)$ by (34) and $\kappa_n \ne 0$ by (33), and $\kappa, \kappa_n \in C^\infty(0, \varepsilon)$ by Proposition 3.4. Moreover,

$$c|K_n| \le |\Omega_2| \le C|K_n| \text{ near zero.} \tag{69}$$

Since $m(0) \ne 0$, this follows from Lemma 4.4 because $\int_0^t |K_n| \le t|K_n(t)|$ (since by continuity κ_n does not change its sign on $(0, \varepsilon)$, so K_n is monotone).
Claim #1. We have $\lim_{t\downarrow 0} \zeta(t) = \lim_{t\downarrow 0} \frac{1}{\kappa(t)} = \frac{\Lambda_1(0)}{m(0)} = s^+(0)$.
As we have just seen, $\int_0^t |K_n| \le t|K_n(t)|$. So $\lim_{t\downarrow 0} \zeta(t) = \frac{\Lambda_1(0)}{m(0)}$ by Lemma 4.4. Since $\kappa = Q(s^\pm, \zeta)$ on $(0, \varepsilon)$ (by (40)) and since $\frac{\Lambda_1(0)}{m(0)} = s^+(0)$ (and $Q(s^\pm, s^+) = \frac{1}{s^+}$ by Lemma 7.6), we conclude that $\lim_{t\downarrow 0} \kappa(t) = \frac{1}{s^+(0)}$. This proves Claim #1.

As in the proof of Proposition 4.3 we will show that $\alpha^+ \le CD^+$. However, the simple scaling argument used there does not work in the present case since $\Omega_2(0) = 0$. Instead, we estimate the derivative of α^+.

Claim #2. For $t > 0$ small enough we have the estimate $|\kappa'(t)| \le C\alpha^+(t)$.
Recall that $\zeta = Z(s^\pm, \kappa)$ on $(0, \varepsilon)$. By (68) we have $(s^-)' \sim \alpha^- \sim 1$ and $(s^+)' \sim \alpha^+$. By Lemma 7.6 we have $Z_+(s^\pm, \kappa) \sim |\log \alpha^+|$ and $Q_-(s^\pm, \kappa) \sim \alpha^+$. Hence by the Z_3-estimate in Lemma 7.6,

$$\begin{aligned} |\kappa'| &\le \frac{1}{|Z_3(s^\pm, \kappa)|} \Big(|Z_+(s^\pm, \zeta)||(s^+)'| + |Z_-(s^\pm, \zeta)||(s^-)'| + |\zeta'| \Big) \\ &\le C(\alpha^+ + |\log \alpha^+|^{-1} \alpha^+ + |\log \alpha^+|^{-1} |\zeta'|). \end{aligned} \tag{70}$$

So we must show that $|\zeta'| \le C|\log \alpha^+|\alpha^+$. We take derivatives in the definition of ζ and use that by Claim #1 we have $\sigma = s^+$ near zero, to find:

$$\begin{aligned} -\zeta' &= \frac{\Omega_3'}{\Omega_2} + \zeta \frac{\Omega_2'}{\Omega_2} \\ &= \frac{h\sigma\Omega_2 - \kappa_n \Lambda_1 - \kappa_n^2 F_2}{\Omega_2} + \zeta \cdot \frac{-h\Omega_2 + \kappa_n m + \kappa_n^2 F_1}{\Omega_2} \\ &= h(s^+ - \zeta) + \frac{\kappa_n}{\Omega_2}(m\zeta - \Lambda_1) + \frac{\kappa_n^2(F_1\zeta - F_2)}{\Omega_2}. \end{aligned} \tag{71}$$

F_1 and F_2 are uniformly bounded, see (19). Moreover, $\Omega_2 = \kappa_n^2 g_2(s^\pm, \kappa)$ (cf. (15)) and $|g_2(s^\pm,\kappa)|^{-1} \le C\alpha^+$. So the last term in (71) is dominated by α^+. Since $\zeta = Z(s^\pm, \kappa)$, by (127) we have

$$\begin{aligned} |s^+ - \zeta| &\le |s^+ - \frac{1}{\kappa}| + |Z(s^\pm, \kappa) - \frac{1}{\kappa}| \\ &\le C\alpha^+ + \frac{1}{|\kappa|}\Big|\frac{g(s^\pm,\kappa)}{g_2(s^\pm,\kappa)}\Big| \le C\alpha^+ |\log \alpha^+|. \end{aligned} \tag{72}$$

Since $\alpha^+ \le C|\log\alpha^+|\alpha^+$ by Claim #1, we conclude from (71):

$$|\zeta'| \le C(\alpha^+|\log\alpha^+| + \frac{|\kappa_n||m\zeta - \Lambda_1|}{|\Omega_2|}). \tag{73}$$

It remains to prove

$$|m(t)\zeta(t) - \Lambda_1(t)| \le C|\Lambda_2(t)| \text{ for } t > 0 \text{ small enough.} \tag{74}$$

In fact, assume (74) were established. By (14) we have $|\Lambda_2| = 2|\kappa_n g(s^\pm, \kappa)| \le C|\kappa_n||\log\alpha^+|$ and by (15) we have $|\Omega_2| = \kappa_n^2|g_2(s^\pm,\kappa)| \ge c\kappa_n^2(\alpha^+)^{-1}$ Hence (74) implies that $\frac{|\kappa_n||m\zeta-\Lambda_1|}{|\Omega_2|} \le C\alpha^+|\log\alpha^+|$. And the claim would follow from (73) and from (70).
Let us prove (74). By (42) and by (32) we have

$$|(\Lambda_1\Omega_2 + m\Omega_3)'| \le |\Lambda_1'\Omega_2 + m'\Omega_3| + |\Lambda_1\Omega_2' + m\Omega_3'| \le C|\Lambda_2||\Omega_2| + |\Lambda_1\Omega_2' + m\Omega_3'|. \tag{75}$$

Now $s^+(0) = \frac{\Lambda_1(0)}{m(0)}$ implies that

$$|m(t)s^+(t) - \Lambda_1(t)| \le C\int_0^t (|m'| + |(s^+)'| + |\Lambda_1'|) \le C(D^+(t) + \int_0^t |\Lambda_2|) \le C|\Lambda_2(t)|,$$

because of (32, 68). In the last step we used (65) and the monotonicity of D^+. Using this together with (17, 18) we find

$$\begin{aligned} |\Lambda_1\Omega_2' + m\Omega_3'| &= |h(ms^+ - \Lambda_1)\Omega_2 + \kappa_n^2(\Lambda_1 F_1 - mF_2)| \\ &\le C(|\Lambda_2||\Omega_2| + |\kappa_n^2|). \end{aligned} \tag{76}$$

Since by Claim #1 we have $\Lambda_1(0)\Omega_2(0) + m(0)\Omega_3(0) = 0$, the estimates (75, 76) imply

$$|\Lambda_1(t)\Omega_2(t) + m(t)\Omega_3(t)| \le C\int_0^t |\Lambda_2||\Omega_2| + C\int_0^t |\Lambda_2||\kappa_n|. \tag{77}$$

We have used that $|\kappa_n| \le C|\Lambda_2|$ by (14). By (69), the estimate (77) implies

$$\begin{aligned} |m(t)\zeta(t) - \Lambda_1(t)| &\le \frac{C}{|K_n(t)|}\Big(\int_0^t |K_n||\Lambda_2| + \int_0^t |\Lambda_2||\kappa_n|\Big) \\ &\le C\Big(|\Lambda_2(t)|\frac{\int_0^t |K_n|}{|K_n(t)|} + |\Lambda_2(t)|\frac{\int_0^t |\kappa_n|}{|K_n(t)|}\Big) \\ &\le C|\Lambda_2(t)|. \end{aligned}$$

Again we used that by (65) Λ_2 is equivalent to the monotone function D^+, and we used that K_n is monotone and that $\int_0^t |\kappa_n| = |\int_0^t \kappa_n|$ because κ_n does not change its sign. Hence (74) follows. This finishes the proof of Claim #2.

By Claim #1 we have $\lim_{t\downarrow 0} \kappa(t) = \frac{1}{s^+(0)}$. So Claim #2 implies the estimate $|\kappa(t) - \frac{1}{s^+(0)}| \leq C \int_0^t \alpha^+$. Hence by (68)

$$\alpha^+(t) \leq \frac{1}{s^+(0)}|s^+(t) - s^+(0)| + s^+(t)|\kappa(t) - \frac{1}{s^+(0)}| \leq C \int_0^t \alpha^+.$$

So $\alpha^+ = 0$ on $(0, \varepsilon)$ by Gronwall's inequality. This means that $(0, \varepsilon) \subset I_0$, a contradiction. $\square$

4.9. Proof of Proposition 4.3. Without loss of generality we may assume that $t_0 = 0$. First of all recall from (34) that $0 \in I_0$. Proposition 3.4 and continuity of Ω_2 imply that κ is continuous in a neighbourhood of zero. So Λ_2 is continuously differentiable near zero. Hence, if $\Lambda_2'(0) \neq 0$ then there is $\varepsilon > 0$ such that $|\Lambda_2(t)| \geq \varepsilon t$ on $(-\varepsilon, \varepsilon)$. So if $\Lambda_2'(0) \neq 0$ then the proof is finished. It remains to consider the case $\Lambda_2'(0) = 0$.
We claim that in this case $0 \in \operatorname{int} I_0$. Since $0 \in I_0$ we have $\kappa(0) \in \{\frac{1}{s^-(0)}, \frac{1}{s^+(0)}\}$. For definiteness say $\kappa(0) = \frac{1}{s^+(0)}$. The other case is similar. Let us assume for contradiction that there were no $\varepsilon > 0$ such that $(0, \varepsilon) \subset I_0$. The formula $\Lambda_2' = m - \kappa\Lambda_1$ implies that

$$\frac{\Lambda_1(0)}{m(0)} = \frac{1}{\kappa(0)} = s^+(0). \tag{78}$$

So the assumptions of Lemma 4.7 are satisfied with $* = +$. Thus there is $\varepsilon > 0$ such that (65) holds with $* = +$, that is, $\Lambda_2 \sim D^+$. On the other hand, $|g_2(s^\pm, \kappa)| \leq C(\alpha^+)^{-1}$ near zero because $\kappa(0) = \frac{1}{s^+(0)}$ and because κ and s^+ are continuous. So the fact that $\kappa_n^2 |g_2(s^\pm, \kappa)| = |\Omega_2| \geq c > 0$ (by (15)) implies that $|\kappa_n| \geq c\sqrt{\alpha^+}$ near zero. So from (14) we deduce that $|\Lambda_2| \geq |\kappa_n g(s^\pm, \kappa)| \geq c\sqrt{\alpha^+}|\log \alpha^+|$. Since $\lim_{t\to 0} \alpha^+(t) = 0$, we conclude that $\alpha^+ \leq \sqrt{\alpha^+}|\log \alpha^+| \leq C|\Lambda_2|$. By (65) this implies $\alpha^+ \leq CD^+$. Hence by Gronwall's inequality, $\alpha^+ = 0$ in a neighbourhood of zero. This contradiction finishes the proof. $\square$

4.10. Corollary. *The set I_0 has empty interior. If, in addition, we assume (A), then the whole set $\{t \in [-T, T] : \kappa_n(t) = 0\}$ has empty interior.*

Proof. Recall our standing hypothesis that κ_n differs from zero on a set of positive measure. Suppose that the interior of I_0 were nonempty. Let (t_0, t_1) be a maximal interval in $\operatorname{int} I_0$. Since κ_n does not vanish identically, we must have $t_0 > -T$ or $t_1 < T$. For definiteness let us assume $t_1 < T$. If $\Omega_2(t_1) \neq 0$ then $t_1 \in \operatorname{int} I_0$ by Proposition 4.3 because $\Lambda_2 = 0$ in a left neighbourhood of t_1. If $\Omega_2(t_1) = 0$ then we use (38) and continuity of Λ_1, m and $s^\pm$ to conclude that

$$\frac{\Lambda_1(t_1)}{m(t_1)} \in \{s^+(t_1), s^-(t_1)\}.$$

So $t_1 \in \operatorname{int} I_0$ by Proposition 4.2 (ii). But on the other hand $t_1 \notin \operatorname{int} I_0$ by maximality of (t_0, t_1). This contradiction proves that I_0 has empty interior.
Now make the extra assumption (A). Suppose that $\operatorname{int}\{\kappa_n = 0\} \neq \emptyset$. Let $(t_0, t_1) \subset \operatorname{int}\{\kappa_n = 0\}$ be a maximal interval. We claim that then

$$(t_0, t_1) \subset I_0, \tag{79}$$

contradicting the first part of the corollary.
To prove (79) notice that $\mathcal{L}^1((t_0, t_1) \cap I_0) > 0$: Otherwise (A) implies that κ takes at least two different values on (t_0, t_1), but (38) implies that κ is constant on (t_0, t_1), a contradiction.
By (38) m and $\Lambda_1 \neq 0$ are constant on (t_0, t_1) and $\kappa = \frac{m}{\Lambda_1}$ almost everywhere on (t_0, t_1). But $\kappa \in \{\frac{1}{s^+}, \frac{1}{s^-}\}$ almost everywhere on $I_0 \cap (t_0, t_1)$, so by continuity of $s^\pm$, m and Λ_1 (and since $|s^+ - s^-| \geq \inf_{[-T,T]} \operatorname{dist}_{\partial S}(\Gamma) > 0$) there is $* \in \{+, -\}$ such that

$$\kappa(t) = \frac{m(t)}{\Lambda_1(t)} = \frac{1}{s^*(t)} \text{ for almost every } t \in (t_0, t_1) \cap I_0. \tag{80}$$

By constancy of κ we conclude that (80) in fact holds for all $t \in (t_0, t_1)$, so (79) follows. ☐

4.11. Proof of Proposition 4.1. Recall that we assume (A) (or even that $\operatorname{int}\{\kappa_n = 0\} = \emptyset$) and that κ_n does not vanish identically on $[-T, T]$. So by Corollary 4.10 the zero set of κ_n has empty interior.
Claim #1. The set $\{t \in [-T, T] : \Omega_2(t) = 0\}$ is finite.
In fact, assume that the claim were wrong, i.e. there is an accumulation point t_0 of $\{\Omega_2 = 0\}$. For definiteness assume that $(t_0, t_0 + \varepsilon)$ intersects $\{\Omega_2 = 0\}$ for all $\varepsilon > 0$; the other case is similar. By continuity we have $\Omega_2(t_0) = 0$. But there is no $\varepsilon > 0$ such that $\Omega_2(t) \neq 0$ for all $t \in (t_0, t_0 + \varepsilon) \setminus \{t_0\}$. So Proposition 4.2 implies that $\kappa_n = 0$ on $(t_0, t_0 + c)$ for some $c > 0$ (either because $\Omega_2 = 0$ or because $t_0 \in \operatorname{int} I_0$). This contradiction to Corollary 4.10 proves Claim #1.

By Claim #1, Proposition 3.4 implies that

$$\kappa = Q(s^\pm, \zeta) \text{ almost everywhere.} \tag{81}$$

Next notice that Proposition 4.2 (ii) and Corollary 4.10 imply that

$$\left\{ t \in [-T, T] : \Omega_2(t) = 0 \text{ and } \frac{\Lambda_1(t)}{m(t)} \in \{s^-(t), s^+(t)\} \right\} = \emptyset. \tag{82}$$

So the second alternative in Proposition 4.2 (i) applies at both sides of each of the finitely many zeroes of Ω_2. Thus $\zeta \in C^0([-T,T])$. Hence $\kappa \in C^0([-T,T])$ by (81), by continuity of $s^\pm$ and by continuity of Q.

Claim #2. The set $\{t \in [-T,T] : \Lambda_2(t) = 0\}$ is finite.
In fact, otherwise this set would have an accumulation point t_0. By Proposition 4.3 and since $\operatorname{int} I_0 = \emptyset$ we conclude that $\Omega_2(t_0) = 0$. But then Proposition 4.2 (i) shows that $|\kappa_n(t)| \geq c|t - t_0|$ near t_0. By (33) this contradicts the fact that t_0 is an accumulation point of $\{\Lambda_2 = 0\}$.

Next notice that if $\Omega_2(t_0) = 0$ then by Proposition 4.2 (i) we have $|\Lambda_2(t)| \geq c|t - t_0|$ near t_0 (because $|\Lambda_2| = 2|\kappa_n|g(s^\pm, \kappa)$). If $\Lambda_2(t_0) = 0$ and $\Omega_2(t_0) \neq 0$ then Proposition 4.3 implies the same estimate. But Λ_2 is C^1 because κ is continuous. So Λ_2 changes its sign at each of its zeroes. Hence so does κ_n by (14). And if $t_0 \notin I_0$ then $g(s^\pm, \kappa) \leq C$ near t_0 by continuity of κ, so $|\kappa_n(t)| \geq c|\Lambda_2(t)| \geq c|t - t_0|$ for t near t_0. Summarizing, statements (i) and (ii) of Proposition 4.1 are proven.
Next we claim that

$$I_0 = \{t \in [-T,T] : \Omega_2(t) \neq 0 \text{ and } \Lambda_2(t) = 0\}. \tag{83}$$

In fact, the inclusion $\{\Omega_2 \neq 0\} \cap \{\Lambda_2 = 0\} \subset I_0$ was proven in §3.2, see (34). To prove the opposite inclusion recall that $I_0 \subset \{\kappa_n = 0\} = \{\Lambda_2 = 0\}$ by (33). So we must show $\{\Omega_2 = 0\} \cap I_0 = \emptyset$. Let $t_0 \in \{\Omega_2 = 0\}$. By (82) and by Claim #1, Proposition 4.2 (i) implies that there is $\eta > 0$ such that $(t_0 - \eta, t_0 + \eta) \setminus \{t_0\} \subset I_\eta$. So by continuity of κ and $s^\pm$ we conclude that $t_0 \in I_\eta$. This proves (83) and hence part (iii).

Let us prove part (iv). Let $t_0 \in I_0$. By translation we may assume without loss of generality that $t_0 = 0$, and we also assume that $\kappa(0) = \frac{1}{s^+(0)}$; the case $\kappa(0) = \frac{1}{s^-(0)}$ is analogous. By (83) we have $\Omega_2(0) \neq 0$ and $\Lambda_2(0) = 0$. Since $\operatorname{int} I_0 \neq \emptyset$, Proposition 4.3 implies existence of $\varepsilon > 0$ such that $|\Lambda_2(t)| \geq \varepsilon|t|$ on $(-\varepsilon, \varepsilon)$. Hence $\kappa,\ \kappa_n \in C^\infty\big((-\varepsilon, \varepsilon) \setminus \{0\}\big)$ by Proposition 3.4.
We claim that (after possibly making ε smaller) there are $c, C > 0$ such that (45) holds for $t \in (-\varepsilon, \varepsilon)$. In fact, since $\Lambda_2 \neq 0$ on $(-\varepsilon, \varepsilon) \setminus \{0\}$, we have $I_0 \cap (-\varepsilon, \varepsilon) = \{0\}$. Hence $\Omega_2(t) = \kappa_n^2(t) g_2(s^\pm(t), \kappa(t))$ for all $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ by (15). We claim that

$$|\Omega_2| \sim 1 \text{ and } |\kappa_n| \sim \sqrt{\alpha^+} \text{ near zero.} \tag{84}$$

The first estimate follows from continuity because $\Omega_2(0) \in (0, \infty)$. The second one follows from the first one and from (15) because $|g_2(s^\pm, \kappa)| \sim \frac{1}{\alpha^+}$ near zero (recall that by continuity of κ we have $\lim_{t \to 0} \alpha^+(t) = 0$). This proves (84) and thus (49). On the other hand, we have $c|t| \leq |\Lambda_2(t)| \leq C|t|$ near zero, so (14) and the fact that $g(s^\pm, \kappa)$ is of the order $|\log \alpha^+|$ near zero imply that

$$c|t| \leq |\kappa_n(t) \log \alpha^+(t)| \leq C|t| \tag{85}$$

The scalings (84) and (85) imply (45). Now since $\kappa = \frac{1-\alpha^+}{s^+}$, we have

$$\kappa(t) - \kappa(0) = \Big(\frac{1}{s^+(t)} - \frac{1}{s^+(0)}\Big) - \frac{\alpha^+(t)}{s^+(t)}. \tag{86}$$

By (68) and since α^+ is equivalent to an increasing function by (45), the first term on the right hand side of (86) is dominated by $t\alpha^+(t)$. This proves (46). To prove (47) we recall that for t near 0 we have $\zeta = Z(s^\pm, \kappa)$ by §3.2 and because $\Omega_2 \neq 0$. Taking derivatives we obtain

$$Z_3(s^\pm, \kappa)\kappa' = -Z_-(s^\pm, \zeta)(s^-)' - Z_+(s^\pm, \zeta)(s^+)' + \zeta'. \tag{87}$$

The first two terms are estimated using (68) and (131): both are dominated by α^+. To estimate the third term in (87), notice that (71) implies

$$\begin{aligned} |-\zeta' - \frac{\kappa_n}{\Omega_2}(m\zeta - \Lambda_1)| &= |h(s^+ - \frac{1}{\kappa}) + h(\frac{1}{\kappa} - \zeta) + \frac{\kappa_n^2(F_1\zeta - F_2)}{\Omega_2}| \\ &\leq C(\alpha^+ + \alpha^+|\log\alpha^+| + \alpha^+) \leq C\alpha^+|\log\alpha^+| \end{aligned} \tag{88}$$

because $\frac{1}{\kappa} - \zeta = \frac{1}{\kappa} - Z(s^\pm, \kappa)$ (cf. (127)) and because of (19, 84). Near zero $|m\zeta - \Lambda_1|$ is uniformly bounded from below and above by positive constants: In fact, boundedness from above is clear. Boundedness from below follows by continuity from the fact that $m(0)\zeta(0) \neq \Lambda_1(0)$. To prove this, notice that $\Lambda_2 \in C^1$ since $\kappa \in C^0$, and recall that $|\Lambda_2(t)| \geq c|t|$. Therefore,

$$0 \neq |\Lambda_2'(0)| = |m(0) - \kappa(0)\Lambda_1(0)|.$$

But $\zeta(0) = \frac{1}{\kappa(0)}$ e.g. by (16) since $0 \in I_0$. Thus indeed $|m\zeta - \Lambda_1| \sim 1$.
By (84) we therefore conclude from (88) that near zero

$$c\sqrt{\alpha^+} \leq |\zeta'| \leq C\sqrt{\alpha^+} \tag{89}$$

because $\alpha^+|\log\alpha^+| \ll \sqrt{\alpha^+}$. Hence (47) follows from (87) and from the Z_3-estimates in Lemma 7.6.
To prove (50) we take derivatives in (14) to find

$$\kappa_n' = -\frac{\Lambda_2'}{2g(s^\pm, \kappa)} + \kappa_n \cdot \frac{\sum_* g_*(s^\pm, \kappa)(s^*)' + g_3(s^\pm, \kappa)\kappa'}{g(s^\pm, \kappa)}. \tag{90}$$

Now $\Lambda_2' \sim 1$, $g(s^\pm, \kappa) \sim |\log\alpha^+|$, $|g_*(s^\pm, \kappa)(s^*)'| \sim 1$ (since $(s^*)' = *\alpha^*\nu_1(\Gamma, *N) \cdot \Gamma'$ and $|g_*(s^\pm, \kappa)| \sim (\alpha^*)^{-1}$) and $|g_3(s^\pm, \kappa)| \sim (\alpha^+)^{-1}$. So the first term in (90) is of the order $\frac{1}{|\log\alpha^+|}$, whereas by (47) the second term in (90) is of the order $\frac{1}{|\log\alpha^+|^2}$. Since the latter expression is much smaller than the former, (90) implies (50). And (50) implies that κ_n' is continuous at 0 with

$\kappa_n'(0) = 0$.
To prove (48) we take derivatives in (87) (we omit the argument $(s^\pm, \kappa)$)

$$Z_3\kappa'' = \zeta'' - \sum_{*,*'} Z_{**'}(s^*)'(s^{*'})' - \sum_* Z_*(s^*)'' - 2\sum_* Z_{*3}(s^*)'\kappa' - Z_{33}(\kappa')^2. \tag{91}$$

The first term on the right-hand side gives an expression dominated by $|Z_{++}|(\alpha^+)^2 + |Z_{-+}||\alpha^+| + |Z_{--}|$. By Lemma 7.6 this is dominated by $\alpha^+ \log \alpha^+$, which is much smaller than $|\log \alpha^+|^{-1}$. Also, $|Z_{*3}(s^*)'| \le C|\log \alpha^+|$, which by (47) shows that $|Z_{*3}(s^*)'\kappa'| \ll |\log \alpha^+|^{-1}$. Again using (47) we find $|Z_{33}(\kappa')^2| \le C|\log \alpha^+|^{-1}$. Thus dividing by Z_3 and using (130), the estimate (91) gives

$$|\kappa''| \le C\Big(|\log \alpha^+|^{-2} + \alpha^+|\log \alpha^+|^{-1}|(s^-)''| + |(s^+)''| + |\log \alpha^+|^{-1}|\zeta''|\Big). \tag{92}$$

(The leading $|\log \alpha^+|^{-2}$-term comes from $Z_{33}(\kappa')^2$.) Using (68) we estimate further:

$$|(s^*)''| \le C(|\alpha^*| + |(\alpha^*)'|) \le C(|\alpha^*| + |\kappa'|).$$

So by (47) we conclude $|(s^+)''| \le C|\kappa'| \le C\sqrt{\alpha^+}|\log \alpha^+|^{-1}$ and $|(s^-)''| \le C$.
To estimate the last term in (92) let us take derivatives in (71):

$$\begin{aligned}-\zeta'' &= h'(s^+ - \zeta) + h(s^+ - \zeta)' + \frac{\kappa_n'(m\zeta - \Lambda_1) + \kappa_n(m\zeta - \Lambda_1)'}{\Omega_2} - \frac{\kappa_n(m\zeta - \Lambda_1)}{\Omega_2} \cdot \frac{\Omega_2'}{\Omega_2} \\ &+ \frac{2\kappa_n\kappa_n'(F_1\zeta - F_2) + \kappa_n^2(F_1\zeta - F_2)'}{\Omega_2} - \frac{\kappa_n^2(F_1\zeta - F_2)}{\Omega_2} \cdot \frac{\Omega_2'}{\Omega_2}.\end{aligned} \tag{93}$$

Using that Ω_2 is Lipschitz, that h is C^1 near zero because so is κ, that the F_i are bounded, using $|s^+ - \zeta| \le C\alpha^+|\log \alpha^+|$ (by (72)) and using (84, 50), we see that the right-hand side of (93) is dominated by

$$\begin{aligned}&\alpha^+|\log \alpha^+| + |(s^+ - \zeta)'| + \Big(|\log \alpha^+|^{-1} + \sqrt{\alpha^+}|(m\zeta - \Lambda_1)'|\Big) + \sqrt{\alpha^+} \\ &+ \Big(\sqrt{\alpha^+}|\log \alpha^+|^{-1} + \alpha^+|(F_1\zeta - F_2)'|\Big) + \alpha^+.\end{aligned} \tag{94}$$

We will show below that the F_i are C^1 near zero. Moreover, we employ the rough estimates $|(s^+ - \zeta)'| \le |(s^+)'| + |\zeta'| \le C\sqrt{\alpha^+}$ (by (68) and (89)) and $|(m\zeta - \Lambda_1)'| \le C$. Inserting these estimates into (94) and exploiting that $\alpha^+(t) \to 0$ as $t \to 0$, we conclude that

$$|\zeta''| \le C|\log \alpha^+|^{-1}.$$

Together with (92) this yields the estimate (48). To finish the proof of part (iv) it remains to prove that F_1 and F_2 are C^1 near zero. In fact, since κ is continuous and $\kappa(0) = \frac{1}{s^+(0)}$, equation (20) reduces to

$$F_1 = \frac{1}{\alpha^-}\Big(\nu_1(\Gamma, N) + \nu_1(\Gamma, -N)\Big) \cdot \Gamma'.$$

Since $\alpha^-(0) \neq 0$ we have $F_1 \in C^1$ because $\kappa \in C^1$ implies $\alpha^- \in C^1$. A similar proof applies to F_2.
To prove (v) notice that the estimates (47, 48) imply that κ', κ'' are continuous at 0, and that $\kappa'(0) = \kappa''(0) = 0$. If we had $\kappa \in C^{2,\varepsilon}(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then we could Taylor expand κ about 0 to find $|\kappa(t) - \kappa(0)| \leq C|t|^{2+\varepsilon}$. But since $\alpha^+(0) = 0$, this implies $|\alpha^+(t)| \leq C|s^+(t) - s^+(0)| + C|t|^{2+\varepsilon}$. Since $|(s^+)'(t)| \leq \alpha^+(t)$, Gronwall's inequality implies that $\alpha^+(t) \leq C|t|^{2+\varepsilon}$. But this contradicts the lower bound in (45). Therefore, if $\kappa \in C^{2,\varepsilon}(-\varepsilon, \varepsilon)$ then $0 \notin I_0$. Similarly, if $\kappa_n \in C^{1,\varepsilon}(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ then (49, 50) imply that $|\sqrt{\alpha^+(t)}| \leq C|t|^{1+\varepsilon}$, which is easily seen to contradict (45). □

5 Higher Regularity

5.1. In this section we assume, in addition to the standing hypotheses, that κ_n does not vanish identically (since otherwise the surface is trivially smooth). Moreover, we assume that $\{\kappa_n = 0\}$ has empty interior. Proposition 3.4 and Proposition 4.1 provide us the following picture: The functions κ and κ_n are continuous up to the boundary on $[-T, T]$. The set of zeroes of κ_n is finite. Moreover, $\kappa, \kappa_n \in C^\infty\big(\{t \in [-T, T] : \kappa_n(t) \neq 0\}\big)$. In this section we improve this to $\kappa, \kappa_n \in C^\infty\big([-T, T] \setminus I_0\big)$. This cannot be improved further because at I_0 we already have an optimal regularity result, see Proposition 4.1.

5.2. Let $t_0 \in [-T, T]$ be a zero of κ_n. We are interested in the local behaviour at t_0. The cases $t_0 \in \{-T, T\}$ are obtained by applying the arguments below only on one side. So we assume without loss of generality that $t_0 \in (-T, T)$. Since the zero set of κ_n is finite, after translating, rescaling and restricting we may assume that $t_0 = 0$, that $T = 1$ and that 0 is the only zero of κ_n in $[-1, 1]$. We consider the case when $0 \notin I_0$, so $I_0 = \emptyset$. Thus $\kappa, \kappa_n \in C^\infty\big((-1, 1) \setminus \{0\}\big)$. And both are continuous at 0. By continuity of $s^\pm\kappa$ we conclude that there is $\eta > 0$ such that $[-1, 1] \subset I_\eta$.
We define $\tilde{\kappa}_n(t) = \frac{\kappa_n(t)}{t}$. We claim that $\tilde{\kappa}_n$ is continuous at 0. In fact, $\tilde{\kappa}_n(t) = -\frac{\Lambda_2(t)}{2tg(s^\pm(t), \kappa(t))}$ by (14). By continuity of κ, we have $\Lambda_2 \in C^1(-1, 1)$, with $\Lambda_2'(0) = m(0) - \kappa(0)\Lambda_1(0)$ and $\Lambda_2(0) = 0$ by (33). So

$$\lim_{t \to 0} \tilde{\kappa}_n(t) = -\frac{m(0) - \kappa(0)\Lambda_1(0)}{2g(s^\pm(0), \kappa(0))}. \tag{95}$$

It is important that $\tilde{\kappa}_n(0) \neq 0$. This follows from Proposition 4.1 (ii), according to which $m(0) - \kappa(0)\Lambda_1(0) = \Lambda_2'(0) \neq 0$. Notice that $0 \in \{\kappa_n = 0\} \setminus I_0$ implies that $\Lambda_2(0) = \Omega_2(0) = \Omega_3(0) = 0$, see e.g. §3.2. Moreover, we have $\Omega_2 \neq 0$ on $(-1, 1) \setminus \{0\}$ since $\kappa_n \neq 0$ here. So by Proposition 4.2 we have $\frac{\Lambda_1(0)}{m(0)} \in (s^-(0), s^+(0))$ and $\lim_{t \to 0} -\frac{\Omega_3(t)}{\Omega_2(t)} = \lim_{t \to 0} \zeta(t) = \frac{\Lambda_1(0)}{m(0)}$. Since $I_0 = \emptyset$, by

(15) we have $\Omega_2 = \kappa_n^2 g_2(s^\pm, \kappa)$ almost everywhere on $(-1,1)$. So (18) simplifies to (23), i.e.

$$\Omega_3' = -\kappa_n \Lambda_1 - \kappa_n^2 \sum_* \frac{s^* \nu_1(\Gamma, *N) \cdot \Gamma'}{1 - s^* \kappa} \tag{96}$$

5.3. For $t \in (-1,1)$ and $x \in \mathbb{R}^2$ with $x_1 \in (\frac{1}{s^-(t)}, \frac{1}{s^+(t)})$ we define

$$\tilde{P}(t,x) = \begin{pmatrix} -2x_2^2 g_3(s^\pm(t), x_1) - x_2 \Lambda_1(t) - 2t x_2^2 \sum_* \frac{s^*(t)\nu_1(\Gamma(t), *N(t)) \cdot \Gamma'(t)}{1 - s^*(t) x_1} \\ 2x_2 g(s^\pm(t), x_1) + m(t) - x_1 \Lambda_1(t) + 2t x_2 \sum_* \nu_1(\Gamma(t), *N(t)) \cdot \Gamma'(t) \end{pmatrix}. \tag{97}$$

For brevity we define $\vec{\kappa}(t) \in \mathbb{R}^2$ by setting $\vec{\kappa}_1(t) = \kappa(t)$ and $\vec{\kappa}_2(t) = \tilde{\kappa}_n(t)$. On $(-1,1) \setminus \{0\}$ the Euler-Lagrange equation (16) gives

$$2\tilde{\kappa}_n \tilde{\kappa}_n' g_3 + \tilde{\kappa}_n^2 g_{33} \kappa' + \tilde{\kappa}_n^2 \sum_* g_{3*}(s^*)' = (t^{-2}\Omega_3)' = -\frac{2\Omega_3/t^2}{t} + \frac{\Omega_3'}{t^2}. \tag{98}$$

Here and below we omit the argument $(s^\pm(t), x_1)$ of all functions involving g or its derivates. Now (96), the definition of $\tilde{P}$ and (98) imply $2\tilde{\kappa}_n \tilde{\kappa}_n' g_3 + \tilde{\kappa}_n^2 g_{33} \kappa' = \frac{1}{t}\tilde{P}_1(t, \vec{\kappa})$. Here we used that $g_{3*}(s^*)' = \frac{s^*\nu_1(\Gamma, *N) \cdot \Gamma'}{1 - s^*\kappa}$.
Equation (14) gives, on $(-1,1) \setminus \{0\}$:

$$-2\tilde{\kappa}_n' g - 2\tilde{\kappa}_n g_3 \kappa' - 2\tilde{\kappa}_n \sum_* g_*(s^*)' = (t^{-1}\Lambda_2)' = -\frac{\Lambda_2/t}{t} + \frac{\Lambda_2'}{t}. \tag{99}$$

By $\Lambda_2' = m - \kappa \Lambda_1$ and since $g_*(s^*)' = \nu_1(\Gamma, *N) \cdot \Gamma'$ we deduce from (99) that $-2\tilde{\kappa}_n' g - 2\tilde{\kappa}_n g_3 \kappa' = \frac{1}{t}\tilde{P}_2(t, \vec{\kappa})$. We claim that

$$\lim_{t \to 0} \frac{g_3(s^\pm(t), \kappa(t))}{g(s^\pm(t), \kappa(t))} = \frac{\Lambda_1(0)}{m(0) - \kappa(0)\Lambda_1(0)}. \tag{100}$$

In fact, since $Z(s^\pm, \kappa) = \zeta$, Proposition 4.2 implies $\frac{\Lambda_1(0)}{m(0)} = \lim_{t \to 0} Z(s^\pm(t), \kappa(t))$. So if $g_3(s^\pm(0), \kappa(0)) = 0$, so $Z(s^\pm(0), \kappa(0)) = 0$, then $\Lambda_1(0) = 0$ (recall that $m(0) \neq \kappa(0)\Lambda_1(0)$). So it remains to consider the case $g_3(s^\pm(0), \kappa(0)) \neq 0$. But then (100) follows from (127):

$$\frac{g(s^\pm(t), \kappa(t))}{g_3(s^\pm(t), \kappa(t))} = -\kappa(t) + Z(s^\pm(t), \kappa(t)) \to -\kappa(0) + \frac{m(0)}{\Lambda_1(0)}.$$

We have $\tilde{P}_1(t, \vec{\kappa}(t)) = -2\tilde{\kappa}_n^2(t) g_3(s^\pm(t), \kappa(t)) - \tilde{\kappa}_n(t)\Lambda_1(t) + \mathcal{O}(t)$. Recalling (95) we conclude $\lim_{t \to 0} \tilde{P}_1(t, \vec{\kappa}(t)) = 0$ by (100). Also $\tilde{P}_2(t, \vec{\kappa}(t)) = 2\tilde{\kappa}_n(t) g(s^\pm(t), \kappa(t)) + m(t) - \kappa(t)\Lambda_1(t)$ converges to zero as $t \to 0$ by (95).
In vector notation, (98, 99) imply $tM(t, \vec{\kappa}(t))\vec{\kappa}'(t) = \tilde{P}(t, \vec{\kappa}(t))$. Here

$$M(t,x) = \begin{pmatrix} x_2^2 g_{33}(s^\pm(t), x_1) & 2x_2 g_3(s^\pm(t), x_1) \\ -2x_2 g_3(s^\pm(t), x_1) & -2g(s^\pm(t), x_1) \end{pmatrix}. \tag{101}$$

In §5.4 below we prove that $M(t,\vec{\kappa}(t))$ is invertible for each $t \in (-1,1)$. So if we set $P(t,x) = M(t,x)^{-1}\tilde{P}(t,x)$ then $t\vec{\kappa}'(t) = P(t,\vec{\kappa}(t))$. Notice that $P(0,\vec{\kappa}(0)) = 0$ because $\tilde{P}(0,\vec{\kappa}(0)) = 0$.
Let us compute $\nabla P(0,\vec{\kappa}(0))$, where ∇ refers to the x-variable. We have $P_{,i}(0,x) = (M^{-1}(0,x))_{,i}\tilde{P}(0,x) + M^{-1}(0,x)\tilde{P}_{,i}(0,x)$. The first term converges to zero as $x \to \vec{\kappa}(0)$ because so does $\tilde{P}(0,x)$ (clearly $M^{-1}(0,\cdot)$ is C^∞ in a neighbourhood of $\vec{\kappa}(0)$). Now $\tilde{P}_{1,1}(0,x) = -2x_2^2 g_{33}(s^\pm(0),x_1)$, $\tilde{P}_{1,2}(0,x) = -4x_2 g_3(s^\pm(0),x_1) - \Lambda_1(0)$, $\tilde{P}_{2,1}(0,x) = 2x_2 g_3(s^\pm(0),x_1) - \Lambda_1(0)$ and $\tilde{P}_{2,2}(0,x) = 2g(s^\pm(0),x_1)$. Since $\Lambda_1(0) = -2\tilde{\kappa}_n(0)g_3(s^\pm(0),\kappa(0))$ by (95, 100), we conclude that

$$\nabla P(0,\vec{\kappa}(0)) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \tag{102}$$

because $\nabla\tilde{P}(0,\vec{\kappa}(0)) = M(0,\vec{\kappa}(0))A$. Here and below A denotes the right-hand side of (102). We conclude

$$\begin{aligned} t\vec{\kappa}'(t) =& A(\vec{\kappa}(t)-\vec{\kappa}(0)) + \\ &+ [P(t,\vec{\kappa}(t)) - P(0,\vec{\kappa}(t))] + [P(0,\vec{\kappa}(t)) - \nabla P(0,\vec{\kappa}(0))(\vec{\kappa}(t)-\vec{\kappa}(0))]. \end{aligned} \tag{103}$$

5.4. Let us estimate the determinants of the matrices $M(t,x)$ defined in (101). We have $\det M(t,x) = -2x_2^2\Big(g_{33}(s^\pm,x_1)g(s^\pm,x_1) - 2g_3^2(s^\pm,x_1)\Big)$. We claim that for all $\tilde{\eta} > 0$ there is $c > 0$ such that whenever $s^- < -\tilde{\eta} < \tilde{\eta} < s^+$ and $x_1 \in (\frac{1}{s^-},\frac{1}{s^+})$ then

$$\Big(g(s^\pm,x_1)g_{33}(s^\pm,x_1)\Big)^{\frac{1}{2}} - \sqrt{2}|g_3(s^\pm,x_1)| \geq c. \tag{104}$$

To prove (104) first note from §7.5 that $g_{33}(s^\pm,x_1) = \int_{s^-}^{s^+} \frac{2s^2}{(1-sx_1)^3}\,ds > 0$. Next we apply Hölder's inequality to find

$$\sqrt{g(s^\pm,x_1)g_{33}(s^\pm,x_1)} \geq \int_{s^-}^{s^+} \frac{1}{(1-sx_1)^{\frac{1}{2}}}\cdot\frac{\sqrt{2}|s|}{(1-sx_1)^{\frac{3}{2}}}\,ds = \int_{s^-}^{s^+} \frac{\sqrt{2}|s|}{(1-sx_1)^2}\,ds.$$

So

$$\frac{1}{\sqrt{2}}\sqrt{g(s^\pm,x_1)g_{33}(s^\pm,x_1)} - |g_3(s^\pm,x_1)| \geq \int_{s^-}^{s^+} \frac{|s|}{(1-sx_1)^2}\,ds - \Big|\int_{s^-}^{s^+} \frac{s}{(1-sx_1)^2}\,ds\Big|.$$

From this and the assumptions on $s^\pm$ and x_1, estimate (104) follows easily.

5.5. Lemma. *Let $f \in C^\infty((0,1);\mathbb{R}^n)$ be bounded. Assume that $a \in \mathbb{R}^n$, that $B \in \mathbb{R}^{n\times n}$ has only negative eigenvalues and that $\alpha, Y \in L^\infty(0,1)$ with $|Y(t)| \le Ct$ and $|\alpha(t)| \to 0$ as $t \downarrow 0$. If*

$$tf'(t) = Bf(t) + Y(t) + a + \alpha(t)(f(t) + B^{-1}a) \text{ for all } t \in (0,1) \tag{105}$$

then $f \in W^{1,\infty}(0,1)$ with $f(0) = -B^{-1}a$.

Proof. After possibly replacing f by $f + B^{-1}a$ we may assume without loss of generality that $a = 0$. Let $\varepsilon > 0$. By the variation of constants formula we have

$$f(t) = e^{\int_\varepsilon^t \frac{B+\alpha(r)}{r}\, dr} f(\varepsilon) + \int_\varepsilon^t e^{\int_s^t \frac{B+\alpha(r)}{r}\, dr} \frac{Y(s)}{s}\, ds. \tag{106}$$

Since the eigenvalues of B are negative, there is $c > 0$ such that $|e^{\int_\varepsilon^t \frac{B+\alpha(r)}{r}\, dr} f(\varepsilon)| \le e^{-c\log\frac{t}{\varepsilon}}|f(\varepsilon)|$. As $\varepsilon \to 0$ this converges to zero because f is bounded on $(0,1)$. So letting $\varepsilon \downarrow 0$ in (106) we find

$$f(t) = \int_0^t e^{\int_s^t \frac{B+\alpha(r)}{r}\, dr} \frac{Y(s)}{s}\, ds.$$

Hence $|f(t)| \le \int_0^t e^{-c\log\frac{t}{s}} \frac{|Y(s)|}{s}\, ds \le t^{-c}\int_0^t s^c\, ds \le Ct$. Plugging this into (105) (recall that $a = 0$) shows that $f' \in L^\infty(0,1)$. □

5.6. Now we argue by induction. The function $\alpha(t) = \frac{P(0,\vec\kappa(t)) - \nabla P(0,\vec\kappa(0))(\vec\kappa(t)-\vec\kappa(0))}{\vec\kappa(t)-\vec\kappa(0)}$ converges to zero as $t \downarrow 0$ because $P(0,\cdot)$ is C^∞ near $\vec\kappa(0)$ and because $\vec\kappa$ is continuous. Hence (103) is of the form (105) with $f = \vec\kappa - \vec\kappa(0)$, $a = 0$ and $Y(t) = P(t,\vec\kappa(t)) - P(0,\vec\kappa(t))$. Since $\kappa \in C^0$ we have $s^\pm \in C^1$. So M^{-1} and $\tilde P$ are C^1. Therefore, $P \in C^1$ in a neighbourhood of $(0,\vec\kappa(0))$. Thus, $|Y(t)| \le Ct$. So we can apply Lemma 5.5 to conclude that $\vec\kappa \in W^{1,\infty}(0,1)$. This implies $s^\pm \in W^{2,\infty}$, so $P \in W^{2,\infty}$ near $(0,\vec\kappa(0))$.
Taking derivatives in (103) and setting $\beta(t) = \nabla P(t,\vec\kappa(t)) - \nabla P(0,\vec\kappa(0))$ we find

$$t\vec\kappa''(t) = (A-I)\vec\kappa'(t) + P_t(t,\vec\kappa(t)) + \beta(t)\cdot\vec\kappa'(t). \tag{107}$$

But $|\beta(t)| \le Ct$ because $\vec\kappa$ and ∇P are Lipschitz. And P_t is Lipschitz as well. So we can apply Lemma 5.5 to (107) (with $Y(t) = P_t(t,\vec\kappa(t)) - P_t(0,\vec\kappa(0)) + \beta(t)\vec\kappa'(t)$, $\alpha = 0$, $a = P_t(0,\vec\kappa(0))$ $B = A - I$ and $f = \vec\kappa' + B^{-1}a$) to conclude $\vec\kappa \in W^{2,\infty}(0,1)$. And $\lim_{t\downarrow 0}\vec\kappa'(t) = -(A-I)^{-1}P_t(0,\vec\kappa(0))$.
Now assume that $\vec\kappa \in W^{m,\infty}(0,1)$ for some integer $m \ge 2$. Then $s^\pm \in W^{m+1,\infty}$, so $P \in W^{m+1,\infty}$ near $(0,\vec\kappa(0))$. Differentiating (107) $m-1$ times we find

$$t\vec\kappa^{(m+1)} = (A - mI)\vec\kappa^{(m)} + P^m(t) + \beta(t)\cdot\vec\kappa^{(m)}(t). \tag{108}$$

Here $P^m(t) = \frac{d^{m-1}}{dt^{m-1}}(P_t(t,\vec\kappa(t)) + \beta(t)\cdot\vec\kappa'(t)) - \beta(t)\cdot\vec\kappa^{(m)}$. Since $P \in W^{m+1,\infty}$ and $\vec\kappa \in W^{m,\infty}$, we have that $t \mapsto P_t(t,\vec\kappa(t))$ and $t \mapsto \beta(t)$ are in $W^{m,\infty}$. The

highest derivative of $\vec{\kappa}$ occurring in P^m is $\vec{\kappa}^{(m-1)}$. Hence P^m is Lipschitz near zero. So Lemma 5.5 applies to (108) with $\alpha = 0$, $a = P^m(0)$ and $Y(t) = P^m(t) + \beta(t) \cdot \vec{\kappa}^{(m)}(t) - P^m(0)$. Thus $\vec{\kappa} \in W^{m+1,\infty}(0,1)$ and $\lim_{t\downarrow 0} \vec{\kappa}^{(m)}(t) = -(A - mI)^{-1}P^m(0)$. Notice that $P^m(0) = \frac{d^{m-1}}{dt^{m-1}}|_{t=0}\big(P_t(t, \vec{\kappa}(t)) + \beta(t) \cdot \vec{\kappa}'(t)\big)$. Inductively, we conclude that $\vec{\kappa} \in C^\infty([0,1))$.
Of course the value of the m^{th} derivative at zero, $\lim_{t\downarrow 0} \vec{\kappa}^{(m)}(t)$, can be calculated from the values of $\vec{\kappa}(0), \vec{\kappa}'(0), ..., \vec{\kappa}^{(m-1)}(0)$. But $\vec{\kappa}$ is continuous at 0. So inductively $\vec{\kappa}^{(m)}(0)$ depends only on the values of functions which are continuous at 0. In particular, applying the above arguments on $(-1,0)$ gives $\lim_{t\uparrow 0} \vec{\kappa}^{(m)}(t) = \lim_{t\downarrow 0} \vec{\kappa}^{(m)}(t)$ for all $m \in \mathbf{N}$.

Proof of Theorem 2.4. First we apply Proposition 3.4 and Corollary 4.10. If we assume (A) or even that $\{\kappa_n = 0\}$ has empty interior, then we can apply Proposition 4.1. The higher regularity on $[-T,T] \setminus I_0$ then follows by applying §5.1 through §5.6 above to each of the finitely many subintervals of $[-T,T] \setminus I_0$. □

Proof of Theorem 2.7. Let $x \in D_{\nabla u} \setminus (C_{\nabla u} \cup \Sigma_c \cup \Sigma_\tau)$. Then by [11] Theorem 2.10 there are $T > 0$ and a ∇u-integral curve $\Gamma \in W^{2,\infty}([-T,T]; S)$ with $\Gamma(0) = x$ which is transversal on $[-T,T]$, and there is $\kappa_n \in L^2(-T,T)$ such that $u = (\Gamma, \kappa_n)$ on $[\Gamma(-T,T)]$ and (Γ, κ_n) solves the Euler-Lagrange equations. And (6) is satisfied. By Lemma 7.4 we may assume without loss of generality that

$$\text{either } \operatorname{int}\{\kappa_n = 0\} = \emptyset \text{ or } \operatorname{int} I_0 \neq \emptyset. \tag{109}$$

Since $x \notin C_{\nabla u}$, we know that κ_n does not vanish identically in any neighbourhood of zero (a similar argument is given in detail in the proof of Proposition 6.3). Thus $\operatorname{int} I_0 = \emptyset$ by Theorem 2.4. Hence $\{\kappa_n = 0\}$ has empty interior as well by (109). Thus all claimed statements follow from Theorem 2.4 and from the equality

$$\frac{\partial^2}{\partial x_i \partial x_j}(\Gamma, \kappa_n)(\Gamma(t) + sN(t)) = \Big(\frac{\kappa_n(t)}{1 - s\kappa(t)}\Gamma_i'(t)\Gamma_j'(t)\Big)\ n(t),$$

which holds for all $t \in (-T,T)$ and $s \in (s^-(t), s^+(t))$ (cf. Proposition 2.2 in [10]).
If $x \in C_{\nabla u}$ then u is obviously C^∞ in a neighbourhood of x. Hence it remains to prove that the set $C_{\nabla u} \cup \Big(D_{\nabla u} \setminus (\Sigma_\tau \cup \Sigma_c)\Big)$ agrees with $S \setminus (\partial \hat{C}_{\nabla u} \cup \Sigma_\tau \cup \Sigma_c)$. This will proven in Proposition 6.3. □

6 The singular set

Above we proved that minimizers are C^3 on the open set $C_{\nabla u} \cup \Big(D_{\nabla u} \setminus (\Sigma_\tau \cup \Sigma_c) \Big)$, which by definition of Σ_τ and Σ_c agrees with $(C_{\nabla u} \cup D_{\nabla u}) \setminus (\Sigma_\tau \cup \Sigma_c)$. Proposition 6.3 below provides a simple formula for the complement of this set. First, however, we collect some facts that are true for arbitrary $W^{2,2}$ isometric immersions.

6.1. Proposition. *Let $S \subset \mathbb{R}^2$ be a bounded C^1 domain and let $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$. Then the following hold:*

(i) Each of the sets Σ_τ, Σ_c and $S \cap \partial C_{\nabla u}$ consists of straight line segments in S on which ∇u is constant and which intersect ∂S at both endpoints. We have $\Sigma_\tau \cup \Sigma_c \subset S \setminus C_{\nabla u}$. Moreover, the sets $S \cap \partial C_{\nabla u}$ and Σ_τ are relatively closed in S and have empty interior.

(ii) We have

$$S \subset D_{\nabla u} \cup \overline{\hat{C}}_{\nabla u} \tag{110}$$

(iii) We have

$$S \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial\hat{C}_{\nabla u}) \subset \hat{C}_{\nabla u} \cup \Big(D_{\nabla u} \setminus (\Sigma_\tau \cup \Sigma_c) \Big). \tag{111}$$

6.2. Remarks.

(i) By definition of $\hat{C}_{\nabla u}$ and Lemma 7.1 (ii) we have $\partial\hat{C}_{\nabla u} \subset \partial C_{\nabla u}$. Hence $C_{\nabla u} \setminus \hat{C}_{\nabla u} = C_{\nabla u} \setminus \overline{\hat{C}}_{\nabla u}$. So (110) implies that $C_{\nabla u} \setminus \hat{C}_{\nabla u} \subset D_{\nabla u}$.

(ii) The example in the appendix to [10] shows that for arbitrary $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ the set $C_{\nabla u}$ can be dense in S and $S \cap \partial C_{\nabla u}$ can consist of uncountably many line segments. Slightly modifying this example shows that in general also $S \cap \partial\hat{C}_{\nabla u}$ consists of uncountably many segments. For irregular boundary conditions, this situation cannot be exluded for minimizers either. However, Proposition 6.4 below identifies a class of boundary conditions for which $S \cap \partial\hat{C}_{\nabla u}$ consists of only finitely segments when u is a minimizer. More sophisticated results will be given in [12].

Proof. In this proof we omit the index ∇u. To prove (i), notice that the statement about the geometry of Σ_c and Σ_τ is an immediate consequence of

their definitions. That $\Sigma_c \subset S \setminus C$ follows from closedness of the latter set, and $\Sigma_\tau \subset S \setminus C$ by definition. The statement about the geometry of $S \cap \partial C$ follows from Lemma 7.1. And $\operatorname{int} S \cap \partial C = \emptyset$ because ∂C is the boundary of an open set.
To prove (ii), we note that by Proposition 2.2.5 in [9] the mapping ∇u is S-developable on $S \setminus \hat{C}$. Hence

$$\operatorname{int}(S \setminus \hat{C}) \subset D \tag{112}$$

by maximality of D. Since S is open, we have $S \setminus \overline{\hat{C}} = \operatorname{int}(S \setminus \hat{C})$. Thus (110) follows from (112).

To prove (iii), notice that $\hat{C} = \overline{\hat{C}} \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial\hat{C})$ because $\hat{C}$ is open and $\Sigma_\tau \cup \Sigma_c \subset S \setminus C$. Hence from (110) we conclude that

$$S \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial\hat{C}) = \hat{C} \cup \Big(D \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial\hat{C}) \Big).$$

This proves (111).
To prove the statements about Σ_τ in (i), notice that it is relatively closed by continuity of $q_{\nabla u}$ (cf. Proposition 2.2.1 in [9]) and by continuity of the outer unit normal to S. To prove $\operatorname{int}\Sigma_\tau = \emptyset$, we argue by contradiction. Let $x \in \operatorname{int}\Sigma_\tau$, so $x \in S \setminus \overline{C}$ because $\Sigma_\tau \subset S \setminus C$ by definition. Thus $x \in D$ by (110). Hence there are $T > 0$ and a ∇u-integral curve $\Gamma : [-T, T] \to \operatorname{int}\Sigma_\tau$ with $\Gamma(0) = x$. Since S is C^1, it satisfies condition $(*)$ from [9]. Hence we can apply Proposition 3.1.11 (iii) in [9] to conclude that the set

$$J_\Gamma := \{t \in (-T, T) : [\Gamma(t)]_{N(t)} \text{ intersects } \partial S \text{ tangentially}\}$$

is contained in I_0 (up to a null set). But since $\Gamma([-T, T]) \subset \Sigma_\tau$, we have $(-T, T) \subset J_\Gamma$ up to a null set. Thus $\kappa_n = 0$ almost everywhere on $(-T, T)$ because of (6) (which holds because $(\Gamma, \kappa_n) \in W^{2,2}([\Gamma(-T, T)]; \mathbb{R}^3)$). As in the proof of Proposition 6.3 below, this would imply that $x \in C$, a contradiction.
□

6.3. Proposition. *Assume that the hypotheses of Theorem 2.7 are satisfied. Then*

$$C_{\nabla u} \cup \Big(D_{\nabla u} \setminus (\Sigma_\tau \cup \Sigma_c) \Big) = S \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial\hat{C}_{\nabla u}). \tag{113}$$

Moreover, the sets Σ_0 and

$$\Sigma_0' := \{x \in S \setminus (\overline{C}_{\nabla u} \cup \Sigma_\tau \cup \Sigma_c) : \nabla^2 u(x) = 0\}$$

consist of countably many line segments intersecting ∂S at both endpoints. Lines in Σ_0' can only accumulate at $\Sigma_\tau \cup \Sigma_c \cup \partial S$. We have $\Sigma_0 \subset \Sigma_0'$.

Proof. We omit the index ∇u. One inclusion in (113) follows from (111) because $\hat{C} \subset C$. To prove that $C \cup (D \setminus (\Sigma_\tau \cup \Sigma_c)) \subset S \setminus (\Sigma_\tau \cup \Sigma_c \cup \partial\hat{C})$, notice that $D \cup \hat{C} = D \cup C$ because $C \setminus \hat{C} \subset D$ (cf. the remark to Proposition 6.1). So we must prove that $D \cup \hat{C} \subset (S \setminus \partial\hat{C}) \cup \Sigma_\tau \cup \Sigma_c$, But $\hat{C} \subset S \setminus \partial\hat{C}$ by openness. So it remains to show that $D \subset (S \setminus \partial\hat{C}) \cup \Sigma_\tau \cup \Sigma_c$, which is equivalent to $D \cap \partial\hat{C} \subset \Sigma_\tau \cup \Sigma_c$. We claim that even (recall $\partial\hat{C} \subset \partial C$ by the remark to Proposition 6.1)

$$D \cap \partial C \subset \Sigma_\tau \cup \Sigma_c. \tag{114}$$

In fact, since D is open and $\Sigma_\tau \cup \Sigma_c$ is closed, by Lemma 7.1 (ii), it suffices to prove that $D \cap \partial U \subset \Sigma_\tau \cup \Sigma_c$ for each connected component U of C.
To prove this, we argue by contradiction. Suppose there were $x_0 \in (D \cap \partial U) \setminus (\Sigma_\tau \cup \Sigma_c)$. Theorem 2.10 in [11] implies that there is $T > 0$ and $\Gamma \in W^{2,\infty}([-T,T];S)$ transversal and with $\Gamma(0) = x_0$, and there exists $\kappa_n \in L^2(-T,T)$ such that $u|_{[\Gamma(-T,T)]} = (\Gamma, \kappa_n)$ and such that (Γ, κ_n) satisfy the Euler-Lagrange equations in the sense of Definition 2.1. Moreover, (6) is satisfied, and by Lemma 7.4 we may also assume that the alternative (109) holds. By Proposition 2.2.3 in [9], there exists $r \in (0,T)$ such that $B_r(x_0) \setminus [x_0]$ consists of precisely two connected components B^1 and B^2 (of course $B^{1,2}$ are open half-disks) such that $B^1 \subset U$ and $B^2 \subset S \setminus \bar{U}$. Since Γ is parametrized by arclength, we have $\Gamma(-r,r) \subset B_r(x_0)$. Since $\Gamma'(0)$ is perpendicular to $[x_0]$, after possibly changing the orientation of Γ we therefore have $\Gamma(-r,0) \subset B^1$ and $\Gamma(0,r) \subset B^2$. (For a proof of this simple fact see e.g. Remark 3.2.4 in [9].) Hence $\kappa_n = 0$ on $(-r,0)$. Hence $\operatorname{int} I_0 \neq \emptyset$ by (109). Thus $\kappa_n = 0$ on $(-T,T)$ by the first part of Theorem 2.4. This implies that $[\Gamma(-T,T)] \subset C$. But since $\kappa \in L^\infty$, there is $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset [\Gamma(-T,T)]$ (cf. Remark 3.2.4 in [9]). Hence $x_0 \in C$, contradicting the fact that $C \cap \partial U = \emptyset$ by openness of C. This concludes the proof of (114).
By the first part of Theorem 2.7, the mapping u is C^3 on the complement of $\overline{C} \cup \Sigma_\tau \cup \Sigma_c$. Thus the set Σ_0' is well defined. Moreover, by (110) and by Theorem 2.10, for all x_0 in the complement of $\overline{C} \cup \Sigma_\tau \cup \Sigma_c$ there exist (Γ, κ_n) as above. Thus Σ_0 is well defined, and (6) implies that $\Sigma_0 \subset \Sigma_0'$. Moreover, we have

$$\Sigma_0' \cap [\Gamma(-T,T)] = \{[\Gamma(t)] : t \in (-T,T),\ \kappa_n(t) = 0\}. \tag{115}$$

Hence Σ_0' does not accumulate at $x_0 = \Gamma(0)$. In fact, otherwise the second part of Theorem 2.4 would imply that $\operatorname{int}\{\kappa_n = 0\} \neq \emptyset$. As above this would imply $x_0 \in C$, a contradiction. □

Certain regularity assumptions on the boundary conditions allow us to conclude that, for a minimizer u, the set $\hat{C}_{\nabla u}$ consists of only finitely many connected components U and that $S \cap \partial U$ consists of finitely many line segments for each U. In other words, (gradients of) minimizers are finitely developable in the sense of [9]. In particular, the modified version of the example from [10] mentioned in the remarks to Proposition 6.1 is excluded. We end this section by giving a simple example for such regularity assumptions on the boundary conditions. More sophisticated examples can be found in [12].

6.4. Proposition. *Let $S \subset \mathbb{R}^2$ be a convex bounded domain and assume that $\partial_c S$ consists of finitely many connected components $\partial_c^1 S$, ..., $\partial_c^M S$. Let $u_0 \in W^{2,2}_{iso}(S;\mathbb{R}^3)$ be such that ∇u_0 is constant on each $\partial_c^i S$ (in the trace sense). Then, for every minimizer of $\mathcal{E}$ within $\mathcal{A}_{u_0}(S, \partial_c S)$ the sets $\hat{C}_{\nabla u}$ and $S \cap \partial \hat{C}_{\nabla u}$ consist of at most finitely many connected components.*

Proof. We argue quickly since more detailed proofs in a similar spirit can be found in [9, 12]. We assume that $\hat{C}_{\nabla u} \neq \emptyset$. Let U be a connected component of $\hat{C}_{\nabla u}$ and let $x \in \partial U$. Then $[x]$ is well defined. Moreover, $S \setminus [x] = S_x^1 \cup S_x^2$ for two disjoint connected open sets S_x^1, S_x^2, and $\partial S \setminus \{x^+, x^-\} = \partial^1 S \cup \partial^2 S$ for two disjoint open subarcs $\partial^1 S \cup \partial^2 S$ of ∂S. And $\partial S_i = [x] \cup \partial^i S$. (A proof of these facts can be found e.g. in the appendix of [9].) By the regularity of ∂S there exists a constant $c > 0$, depending only on S, such that

$$\mathcal{H}^1([x]) \geq c \min_{j=1,2} \mathcal{H}^1(\partial^j S). \tag{116}$$

On the other hand,

$$\overline{[x]} \cap \operatorname{int} \partial_c S = \emptyset. \tag{117}$$

(Here, the interior is understood relative to ∂S.) In fact, by convexity of S we know that $\overline{[x]}$ intersects ∂S transversally. So if $\overline{[x]}$ intersects $\operatorname{int} \partial_c^i S$ for some i, then by constancy of ∇u on $\partial_c^i S$ (by the hypothesis on ∇u_0) and on $[x]$, we would conclude that ∇u is constant in a neighbourhood of x. This would imply that $x \in C_{\nabla u}$, contradicting the fact that $x \in \partial C_{\nabla u}$.
Now (117) implies that, for each component $\operatorname{int} \partial_c^i S$ of $\operatorname{int} \partial_c S$ either $\operatorname{int} \partial_c^i S \subset \partial^1 S$ or $\operatorname{int} \partial_c^i S \subset \partial^2 S$. But if $\partial^j S$ did not intersect $\partial_c S$ then by minimality u would be affine on S_j. Thus $U = S_j$, and $S \cap \partial U = [x]$, contradicting the fact that $U \subset \hat{C}_{\nabla u}$. Hence both $\partial^1 S$ and $\partial^2 S$ must intersect $\operatorname{int} \partial_c S$, and so both must contain a connected component of $\operatorname{int} \partial_c S$. Thus

$$\min_{j=1,2} \mathcal{H}^1(\partial^j S) \geq \delta_2 := \min_{i=1,\dots,M} \mathcal{H}^1(\partial_c^i S). \tag{118}$$

From (116, 118) and arbitrariness of x we conclude that $\mathcal{H}^1([x]) \geq c\delta_2$ for all $x \in \partial U$ and for all connected components U of $\hat{C}_{\nabla u}$. But by Proposition 2.2.4 in [9] there exists only finitely many components U of $\hat{C}_{\nabla u}$ for which all boundary segments are longer than a given constant. $\square$

7 Appendix

7.1. Lemma. *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $u \in W^{2,2}_{iso}(S;\mathbb{R}^3)$. Then the following hold:*

(i) The boundary of $C_{\nabla u}$ agrees with the closure of

$$\bigcup\{x \in \partial U : U \text{ is a connected component of } C_{\nabla u}\}.$$

(ii) If $y \in S \cap \partial C_{\nabla u}$ then $[y] \subset \partial C_{\nabla u}$.

Proof. We omit the index ∇u. To prove (i), notice that since C is open, it consists of countably many connected components U, each of which is itself open. Now let $x \in \partial C$ and let $x_n \in C$ be such that $x_n \to x$. Denote by U_n the connected component of C that contains x_n. Since $x \notin U_n$ (because by openness $x \notin C$), we have $\text{dist}_{U_n}(x) = \text{dist}_{\partial U_n}(x)$. But $\text{dist}_{U_n}(x) \leq |x - x_n| \to 0$. Hence there exist $y_n \in \partial U_n$ such that $y_n \to x$.
To prove (ii) notice that $[y]$ is well defined because by openness $y \in S \setminus C$. By (i) there exist components U_n of C and $y_n \in \partial U_n$ such that $y_n \to y$. After passing to subsequences (in fact, this is not necessary), the Hausdorff limit Y of the sets $\overline{[y_n]}$ exists. Lemma 2.2.6 in [9] implies that $[y] \subset S \cap Y$. But $Y \subset \partial C$ because $[y_n] \subset \partial U_n$ by Proposition 2.2.3 in [9]. This proves (ii). □

For a given curve $\Gamma \in W^{2,\infty}([-T,T];S)$ we define $\beta^{\pm}_{\Gamma}(t) = \Gamma(t) + s^{\pm}_{\Gamma}(t)N(t)$.

7.2. Lemma. *Assume that $\Gamma \in W^{2,\infty}([-T,T];S)$ is admissible and transversal, and that $\beta^{\pm}_{\Gamma}(-T) \neq \beta^{\pm}_{\Gamma}(T)$. Then Γ is simple in the sense of [10]. (This means that $\beta^{\pm}_{\Gamma}(-T,T)$ are singletons or Jordan arcs, that $s^{\pm}_{\Gamma} \in C^0([-T,T])$ and that $\beta^{+}_{\Gamma}([-T,T]) \cap \beta^{-}_{\Gamma}([-T,T]) = \emptyset$.)*

Proof. We omit the index Γ. Since Γ is transversal, we have $s^{\pm} \in C^0([-T,T])$ (cf. Proposition 3.1.11 in [9]). Since Γ is also admissible, Lemma 7.8 in [11] implies that $\beta^{+}_{\Gamma}([-T,T]) \cap \beta^{-}_{\Gamma}([-T,T]) = \emptyset$.
Since Γ is admissible, we have Lemma 3.1.9 from [9] at our disposal. By the remark following it (i.e. by the nondegeneracy of the lifting), if $\beta^*(-T) \neq \beta^*(T)$ then $\beta^*([-T,T])$ does not agree with the whole connected component of ∂S that contains $\beta^*([-T,T])$. Thus Remark 3.1.7 in [9] implies that Γ is simple. □

7.3. Proposition. *Let* $\Gamma \in C^\infty([-T,T];S)$ *be admissible and transversal, and let* $\kappa_n \in C^\infty([-T,T])$. *Assume, moreover, that* $1 - s_\Gamma^\pm \kappa > 0$ *on* $[-T,T]$ *and that* $\beta_\Gamma^\pm(T) \neq \beta_\Gamma^\pm(-T)$. *Then there exists an open set* $U \subset \mathbb{R}^2$ *that contains the closure of* $[\Gamma(-T,T)]$ *and there exists* $\tilde{u} \in C^\infty_{iso}(U;\mathbb{R}^3)$ *such that* $(\Gamma, \kappa_n) = \tilde{u}$ *on* $[\Gamma(-T,T)]$.

Proof. We omit the index Γ. Since as above by transversality $s^\pm$ are continuous on $[-T,T]$, and since by hypothesis κ is continuous as well, the curve Γ is uniformly admissible. This means that $1 - s^\pm\kappa \geq c > 0$ and that Γ is admissible (which is true by hypothesis). On the other hand, Lemma 7.2 implies that Γ is simple. Now we can argue (with some minor modifications) as in the proofs of Lemma 2.4 in [10] and of Proposition 2.5 in [10] to obtain the desired conclusion.

For the convenience of the reader, we give some details. The only difference to [10] is that in the present context both Γ and κ_n are smooth up to the boundary, so we do not need the condition $\kappa_n = 0$ near $\{-T,T\}$ required in Proposition 2.5 in [10]. Instead, we can extend Γ smoothly to all of $\mathbb{R}$, so $\kappa \in C^\infty(\mathbb{R})$. Define $\tilde{s}_\delta^\pm$ as in Lemma 2.4 in [10]. Then there exists $\delta > 0$ such that $1 - \tilde{s}_\delta\kappa \geq \frac{c}{2}$ on $[-T-\delta, T+\delta]$. After possibly multiplying κ with a smooth cutoff function we may assume that $\operatorname{spt}\kappa \subset (-T-\delta, T+\delta)$. Then $1 - \tilde{s}_\delta^\pm\kappa \geq \frac{c}{2}$ on all of $\mathbb{R}$. Thus we obtain a mapping $\Phi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ as in the proof of Lemma 2.4 in [10]. As there one deduces from simpleness and uniform admissibility of Γ on $[-T-\delta, T+\delta]$ that Φ is globally injective on $\tilde{M}_\delta$. Defininig $U := \Phi(\tilde{M}_\delta)$ and $\tilde{u} := (\Gamma, \kappa_n)$ (with κ_n and Γ extended smoothly to $\mathbb{R}$), we obtain the claim. □

7.4. Lemma. *Let* $\Gamma \in W^{2,\infty}([0,T];S)$ *be admissible and transversal, and let* $\kappa_n \in L^2(0,T)$ *be such that* $\operatorname{int}\{t \in (0,T) : \kappa_n(t) = 0\} \neq \emptyset$.
Then there exist $T' > 0$ *and* $\hat{\Gamma} \in W^{2,\infty}([0,T'];S)$ *(parameterized by arclength as well) admissible and transversal, and there exists* $\hat{\kappa}_n \in L^2(0,T')$ *such that the following hold:*

$$\operatorname{int} I_0^{\hat{\Gamma}} \neq \emptyset \tag{119}$$

$$[\hat{\Gamma}(0,T')] = [\Gamma(0,T)] \tag{120}$$

$$(\hat{\Gamma}, \hat{\kappa}_n) = (\Gamma, \kappa_n) \text{ on } [\Gamma(0,T)]. \tag{121}$$

Moreover, if (6) holds, then the analogue for $(\hat{\Gamma}, \hat{\kappa}_n)$ *holds as well.*

Proof. By the results from [9] outlined in §2.1, the hypotheses imply that (Γ, κ_n) is well defined as an element of $W^{2,2}_{\text{loc, iso}}([\Gamma(0,T)];\mathbb{R}^3)$. Moreover, for all $\varepsilon > 0$ small enough there exist $t_0, t_1 \in (0,T)$ such that $t_1 - t_0 \in (0,\varepsilon)$ and $[t_0,t_1] \subset \operatorname{int}\{\kappa_n = 0\}$. Throughout this proof we tacitly assume that ε is chosen

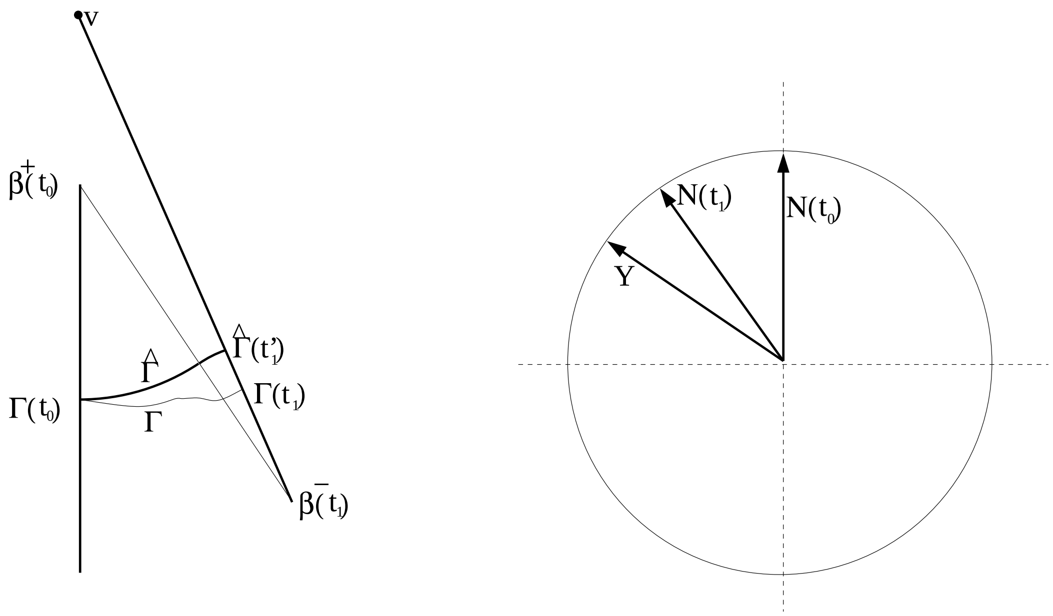

Figure 1: The construction used in the proof of Lemma 7.4.

small enough to justify all steps. If $N(t_0) \neq N(t_1)$ (this is equivalent to $N(t_0) \not\parallel N(t_1)$ because ε is small), then there exists a unique $v \in [\Gamma(t_0)]^{\mathbb{R}^2} \cap [\Gamma(t_1)]^{\mathbb{R}^2}$, where $[\Gamma(t)]^{\mathbb{R}^2} := \Gamma(t) + \operatorname{span} N(t)$. Since by admissibility $[\Gamma(t_0)] \cap [\Gamma(t_1)] = \emptyset$, we have $v \notin [\Gamma(t_0)]$ or $v \notin [\Gamma(t_1)]$. After possibly changing the orientation of Γ, we may assume that $v \notin [\Gamma(t_0)]$. We translate coordinates such that $\Gamma(t_0) = 0$ and rotate coordinates such that $[\Gamma(t_0)]$ is contained in the x_2-axis and $\Gamma'(t_0) = e_1$. (If $N(t_0) = N(t_1)$ the we do the same and directly proceed with the next paragraph.) Since $v \notin [\Gamma(t_0)]$ but $0 = \Gamma(t_0) \in [\Gamma(t_0)]$ and v is contained in the x_2-axis, we have $v \cdot e_2 \neq 0$. We assume that $v \cdot e_2 > 0$. The other case is similar. The situation is depicted in Figure 1 (left).

For $t \in (t_0, t_1)$ we have $N(t) \cdot e_2 = 1 + \int_{t_0}^{t} (-\kappa \Gamma') \geq 1 - \varepsilon \|\kappa\|_{L^\infty(0,T)}$ because $N(t_0) = e_2$. Similarly, $|\Gamma \cdot e_2| \leq \|\kappa\|_{L^\infty(0,T)} \varepsilon^2$ on (t_0, t_1). Thus $N(t_1) \cdot e_2 \geq |\Gamma(t_1) \cdot e_2|$. Also, $\Gamma(t) \cdot e_1 \geq (t - t_0)(1 - \|\kappa\|_{L^\infty(0,T)}(t - t_0))$ for $t > t_0$. So $\Gamma(t_1) \cdot e_1 \geq (1 - \varepsilon \|\kappa\|_{L^\infty(0,T)})\varepsilon$. This is positive for small $\varepsilon > 0$. Since $N(t_1) = \frac{v - \Gamma(t_1)}{|v - \Gamma(t_1)|}$ (unless $N(t_0) = N(t_1)$), we conclude that $N(t_1) \cdot e_1 < 0$ (with equality if $N(t_0) = N(t_1)$). Since also $N(t_1) \cdot e_2 > 0$, we conclude that there exists $\alpha_2 \in [0, \frac{\pi}{2})$ such that, identifying $\mathbb{R}^2$ with $\mathbb{C}$, we have $N(t_1) = ie^{i\alpha_2}$, cf. Figure 1 (right). (The case $\alpha_2 = 0$ occurs if $i = N(t_0) = N(t_1)$.) Similarly, for $Y := \frac{\beta^+(t_0) - \beta^-(t_1)}{|\beta^+(t_0) - \beta^-(t_1)|}$, one shows that there is $\alpha_0 \in [\alpha_2, \frac{\pi}{2})$ such that $Y = ie^{i\alpha_0}$. Thus $e^{-i\alpha_1} Y = N(t_1)$, where $\alpha_1 = \alpha_0 - \alpha_2 \geq 0$. (The case $\alpha_1 = 0$ occurs if $v = \beta^+(t_0)$.)

Set $L = |\beta^+(t_0) - \beta^-(t_1)| - s^+(t_0)$, and set $t' = t_0 + \alpha_0 s^+(t_0)$ and $t_1' = t' + \alpha_1 L$.

Define

$$\hat\kappa(t) = \begin{cases} \kappa(t) & \text{for } t \in (0, t_0) \\ \frac{1}{s^+(t_0)} & \text{for } t \in (t_0, t') \\ -\frac{1}{L} & \text{for } t \in (t', t_1'). \end{cases}$$

Denote by $\hat\Gamma : (0, t_1') \to \mathbb{R}^2$ the unique arclength parametrized curve with $\hat\Gamma(0) = \Gamma(0)$ and $\hat\Gamma'(0) = \Gamma(0)$ and with curvature $\hat\Gamma'' \cdot \hat N = \hat\kappa$.
Obviously $\hat\Gamma = \Gamma$ on $(0, t_0)$. On (t_0, t'), one easily checks that $\hat\Gamma$ is a subarc of the circle with center $\beta^+(t_0)$ and radius $s^+(t_0)$. Moreover, $\beta^+(t_0) = \hat\Gamma(t) + s^+(t_0)\hat N(t)$ for all $t \in (t_0, t')$. (In fact, this is true for $t = t_0$, and $(\hat\Gamma + s^+(t_0)\hat N)' = (1 - s^+(t_0)\hat\kappa)\hat\Gamma' = 0$.) In particular, $\hat\Gamma(t') = \beta^+(t_0) - s^+(t_0)Y$ because $\hat N(t') = Y$. (In fact, defining $\hat K(t) := \int_{t_0}^t \hat\kappa$ we have $\hat N(t') = e^{i\hat K(t')}\hat N(t_0)$. Since $\hat N(t_0) = N(t_0) = i$, the choices of α_0 and t' imply that indeed $\hat N(t') = Y$.)
Similarly, $\hat\Gamma$ is a subarc of the circle with center $\beta^-(t_1)$ and radius L, and $\beta^-(t_1) = \hat\Gamma(t) - L\hat N(t)$ for all $t \in (t', t_1')$. Thus $\hat\Gamma(t_1') = \beta^-(t_1) + L\hat N(t_1')$. But by the choices of α_1 and t_1', a similar calculation as above shows that $\hat N(t_1') = N(t_1)$. We conclude that $\hat\Gamma(t_1') \in [\Gamma(t_1)]$, and therefore $[\hat\Gamma(t_1')] = [\Gamma(t_1)]$. (As usual, $[\hat\Gamma(t)]$ denotes $[\hat\Gamma(t)]_{\hat N(t)}$.) More precisely, we have $\hat\Gamma(t_1') - \Gamma(t_1) = \theta N(t_1)$, since $\Gamma(t_1) = \beta^-(t_1) - s^-(t_1)N(t_1)$. Here we have introduced

$$\theta = L - s^-(t_1) = |\beta^+(t_0) - \beta^-(t_1)| - s^+(t_0) + s^-(t_1). \tag{122}$$

Inserting the definitions of $\beta^\pm$, we can estimate

$$\begin{aligned} |\theta| &\le |\Gamma(t_0) - \Gamma(t_1)| + |s^-(t_1)(N(t_0) - N(t_1))| \\ &\quad + |(s^+(t_0) - s^-(t_1))N(t_0)| - (s^+(t_0) - s^-(t_1)) \\ &\le C\varepsilon, \end{aligned} \tag{123}$$

because the second line equals zero. Define $\tau(t) = t_1' + \int_{t_1}^t (1 - \theta\kappa)$ for all $t \in (t_1, T)$. Since $|\theta| \le C\varepsilon$, we have $|\tau' - 1| \le C\varepsilon$, so τ is Bilipschitz. Hence, writing $T' = \tau(T)$, we can define $\hat\kappa : (t_1', T') \to \mathbb{R}$ by setting $\hat\kappa(\tau(t)) := \frac{\kappa(t)}{\tau'(t)}$ for all $t \in (t_1, T)$. Extend $\hat\Gamma$ to (t_1', T') with curvature $\hat\kappa$. Then $\hat\Gamma(\tau)$ is equivalent to Γ on (t_1, T) in the sense of [10]. This means that $[\hat\Gamma(\tau(t))] = [\Gamma(t)]$ for all $t \in (t_1, T)$. More precisely, $\hat\Gamma(\tau) - \Gamma = \theta N$ and $\hat N(\tau) = N$ on (t_1, T). This is easy to verify, and similar arguments can be found in [10].
By equivalence, $\hat\Gamma$ is transversal on $[0, t_0] \cup [t_1', T']$. Since $|\hat K| \le C\varepsilon$ on (t_0, t_1'), for small ε it is also transversal on (t_0, t_1'). (Notice in passing that this and (123) ensure that $\hat\Gamma([0, T']) \subset S$.)
By construction $\hat\Gamma$ is obviously admissible on $[t_0, t']$ and on $[t', t_1']$. Moreover, $[\hat\Gamma([t_0, t'))]$ and $[\hat\Gamma((t', t_1'])]$ are disjoint because they are contained in different connected components (half-spaces) of $\mathbb{R}^2 \backslash \operatorname{span} Y$. (In fact, $[\hat\Gamma(t)] \cap \operatorname{span} Y = \emptyset$ for all $t \in (t_0, t_1') \setminus \{t'\}$ because $[\hat\Gamma(t)]^{\mathbb{R}^2} \cap \operatorname{span} Y \subset \{\beta^+(t_0), \beta^-(t_1)\}$, and these

two points are in $\mathbb{R}^2 \setminus S$, so none of them is contained in any $[\hat{\Gamma}(t)]$. But both $[\hat{\Gamma}([t_0, t'))]$ and $[\hat{\Gamma}((t', t'_1])]$ are connected. And they intersect different connected components of $\mathbb{R}^2 \setminus \operatorname{span} Y$ because $\hat{\Gamma}'(t')$ is perpendicular to Y.) Thus $\hat{\Gamma}$ is admissible on $[t_0, t'_1]$.
We claim that

$$[\hat{\Gamma}(t_0, t'_1)] = [\Gamma(t_0, t_1)]. \tag{124}$$

In fact, since $\hat{\Gamma}$ is admissible and transversal on $[t_0, t'_1]$, Proposition 3.1.8 in [9] implies that $[\hat{\Gamma}(t_0, t'_1)]$ agrees with a connected component of $S \setminus ([\hat{\Gamma}(t_0)] \cup [\hat{\Gamma}(t'_1)]) = S \setminus ([\Gamma(t_0)] \cup [\Gamma(t_1)])$. The same is true for $[\Gamma(t_0, t_1)]$ because Γ is transversal and admissible on $[t_0, t_1]$. But clearly $[\hat{\Gamma}(t_0, t'_1)] \cap [\Gamma(t_0, t_1)] \neq \emptyset$ (because $\hat{\Gamma}(t_0) = \Gamma(t_0)$ and $\hat{\Gamma}'(t_0) = \Gamma'(t_0)$ is perpendicular to $[\Gamma(t_0)]$). Hence (124) follows.
The analoga to (124) on $(0, t_0)$ and on (t'_1, T') hold trivially by equivalence of $\hat{\Gamma}$ and Γ on the corresponding sets. Hence (120) is proven.
Since Γ is admissible on $[0, T]$, by equivalence $\hat{\Gamma}$ is admissible on $[0, t_0] \cup [t'_1, T']$. And admissibility of Γ implies that $[\Gamma(0, t_0)]$, $[\Gamma(t_0, t_1)]$ and $[\Gamma(t_1, T)]$ are mutually disjoint. Hence by (124) and since, by equivalence, $[\hat{\Gamma}(0, t_0)] = [\Gamma(0, t_0)]$ and $[\hat{\Gamma}(t'_1, T')] = [\Gamma(t_1, T)]$, admissibility of $\hat{\Gamma}$ on $[0, t_0]$, $[t_0, t'_1]$ and on $[t'_1, T']$ implies its admissibility on $[0, T']$.
Define $\hat{\kappa}_n : (0, T') \to \mathbb{R}$ by

$$\hat{\kappa}_n(t) = \begin{cases} \kappa_n(t) & \text{if } t \in (0, t_0) \\ 0 & \text{if } t \in (t_0, t'_1) \end{cases} \tag{125}$$

and $\hat{\kappa}_n(\tau(t)) = \frac{\kappa_n(t)}{\tau'(t)}$ if $t \in (t_1, T)$. We claim that the second fundamental form $\hat{A}$ of $(\hat{\Gamma}, \hat{\kappa}_n)$ and that of (Γ, κ_n), called A, agree on $[\Gamma(0, T)]$. Since the metrics are the same and since obviously $(\hat{\Gamma}, \hat{\kappa}_n) = (\Gamma, \kappa_n)$ on $[\Gamma(0, t_0)]$, this will imply (121) by Bonnet's Theorem (cf. [2]).
By Proposition 2.2 in [10] we have $A(\Gamma(t) + sN(t)) = \frac{\kappa_n(t)}{1 - s\kappa(t)} \Gamma'(t) \otimes \Gamma'(t)$, and a similar expression for $\hat{A}(\hat{\Gamma}(t) + s\hat{N}(t))$. Obviously $A = \hat{A}$ on $[\Gamma(0, t_0)]$ and $A = 0 = \hat{A}$ on $[\Gamma(t_0, t_1)]$ (here we use (124) again). It remains to consider the set $[\Gamma(t_1, T)]$. But $\Gamma(t) + sN(t) = \hat{\Gamma}(\tau(t)) + (s - \theta)\hat{N}(\tau(t))$ for $t \in (t_1, T)$. From this and since

$$\frac{\hat{\kappa}_n(\tau)}{1 - (s - \theta)\hat{\kappa}(\tau)} = \frac{\kappa_n}{\tau' - (s - \theta)\kappa} = \frac{\kappa_n}{1 - s\kappa} \tag{126}$$

on (t_1, T), we deduce that $A = \hat{A}$ also on $[\Gamma(t_1, T)]$. This concludes the proof of (121).
Equation (126) also implies that $(\hat{\Gamma}, \hat{\kappa}_n)$ satisfies the analogue of (6) if (Γ, κ_n) satisfies (6). Finally, notice that (119) is satisfied because $(t_0, t'_1) \subset I_0^{\hat{\Gamma}}$ by construction. □

7.5. Recall the definitions of g, g_*, g_2 and g_3 in §2.1. Let us compute the derivatives of g and g_2 needed in the proof of Lemma 7.6 below. We will slightly abuse notation and write α^* to denote $1 - s^*x$.

$$g_{**}(s^\pm, x) = *\frac{x}{(1-s^*x)^2} \quad \text{and} \quad g_{*\bar{*}}(s^\pm, x) = 0$$

$$g_{23}(s^\pm, x) = -\int_{s^-}^{s^+} \frac{2s}{(1-sx)^3}\,ds \quad \text{and} \quad g_{33}(s^\pm, x) = \int_{s^-}^{s^+} \frac{2s^2}{(1-sx)^3}\,ds$$

$$g_{2*}(s^\pm, x) = - * (\alpha^*)^{-2} \quad \text{and} \quad g_{23*}(s^\pm, x) = - * 2s^*(\alpha^*)^{-3}$$

$$g_{2**}(s^\pm, x) = - * 2x(\alpha^*)^{-3} \quad \text{and} \quad g_{233}(s^\pm, x) = -\int_{s^-}^{s^+} \frac{6s^2}{(1-sx)^4}\,ds.$$

Recall the definition of Z in (31). Using that $g_3(s^\pm, x) = -\frac{1}{x}(g(s^\pm, x) + g_2(s^\pm, x))$ for $x \neq 0$, that $g_2 \leq -c < 0$ and using the definitions of g_2, g_3 at $x = 0$, we have

$$Z(s^\pm, x) = \begin{cases} \frac{1}{x}\left(1 + \frac{g(s^\pm,x)}{g_2(s^\pm,x)}\right) & \text{if } x \neq 0 \\ \frac{s^+ + s^-}{2} & \text{if } x = 0. \end{cases} \tag{127}$$

In Lemma 7.6 below, we denote by Z_* the partial derivative of $Z(s^\pm, x)$ with respect to s^* and by Z_3 the partial derivative of $Z(s^\pm, x)$ with respect to x. Similar definitions apply to Q.

7.6. Lemma. *Let $\tilde{\eta} > 0$ and let*

$$M_0 = \{(s^-, s^+, x) \in \mathbb{R}^3 : |s^\pm| \in (\tilde{\eta}, 2\operatorname{diam} S) \text{ and } x \in (\frac{1}{s^-}, \frac{1}{s^+})\}, \tag{128}$$

$$M_0' = \{(s^-, s^+, z) \in \mathbb{R}^3 : |s^\pm| \in (\tilde{\eta}, 2\operatorname{diam} S) \text{ and } z \in (s^-, s^+)\}. \tag{129}$$

Then $Z \in C^0(\bar{M_0}) \cap C^\infty(M_0)$. In addition, for all $s^- < 0 < s^+$, the function $Z(s^\pm, \cdot)$ is a homeomorphism from $[\frac{1}{s^-}, \frac{1}{s^+}]$ onto $[s^-, s^+]$, and it is an orientation preserving diffeomorphism from $(\frac{1}{s^-}, \frac{1}{s^+})$ onto (s^-, s^+). We have $Z(s^\pm, x) = \frac{1}{x}$ if and only if $x \in \{\frac{1}{s^-}, \frac{1}{s^+}\}$.
There is $\varepsilon > 0$ such that, whenever $\alpha^* \in (0, \varepsilon)$, we have:

$$|Z_3(s^\pm, x)| \sim |\log \alpha^*| \tag{130}$$

$$|Z_*(s^\pm, x)| \sim |\log \alpha^*| \text{ and } |Z_{\bar{*}}(s^\pm, x)| \sim \alpha^*. \tag{131}$$

And:

$$|Z_{33}(s^\pm, x)| + |Z_{3*}(s^\pm, x)| \leq C\frac{|\log \alpha^*|}{\alpha^*} \text{ and } |Z_{3\bar{*}}(s^\pm, x)| \leq C \tag{132}$$

$$|Z_{**}(s^\pm, x)| \leq C\frac{|\log \alpha^*|}{\alpha^*} \text{ and } |Z_{*\bar{*}}(s^\pm, x)| \leq C \text{ and } |Z_{\bar{*}\bar{*}}(s^\pm, x)| \leq C\alpha^*. \tag{133}$$

The function Q, defined for all $s^- < 0 < s^+$ and all $z \in [s^-, s^+]$ by

$$Z(s^\pm, Q(s^\pm, z)) = z, \tag{134}$$

satisfies $Q \in C^0(\bar{M}_0') \cap C^\infty(M_0')$.

Proof. From the definition of Z in (31) and the properties of g_2 and g_3 we deduce that $Z \in C^0(\bar{M}_0) \cap C^\infty(M_0)$. In particular, $Z(s^\pm, \cdot) \in C^\infty((\frac{1}{s^-}, \frac{1}{s^+}))$. Using $g_2(s^\pm, x) = -xg_3(s^\pm, x) - g(s^\pm, x)$ one easily shows that $Z_3 = \frac{gg_{33} - 2g_3^2}{g_2^2}$. By §5.4 this is strictly positive. So $Z(s^\pm, \cdot)$ is increasing on $(\frac{1}{s^-}, \frac{1}{s^+})$. From this one also sees that $Z(s^\pm, \cdot)$ is continuous on $[\frac{1}{s^-}, \frac{1}{s^+}]$. From (127) and since $\frac{|g(s^\pm,x)|}{|g_2(s^\pm,x)|} \sim \frac{|\log(1-s^*x)|}{|1-s^*x|}$ as $x \to \frac{1}{s^*}$, we conclude

$$\lim_{(\frac{1}{s^-}, \frac{1}{s^+}) \ni x \to \frac{1}{s^*}} Z(s^\pm, x) = \frac{1}{x} = s^*.$$

Continuity and monotonicity therefore imply that $Z(s^\pm, \cdot)$ maps $[\frac{1}{s^-}, \frac{1}{s^+}]$ onto $[s^-, s^+]$.

By monotonicity and smoothness, for all $s^\pm$ there clearly exists a smooth inverse $Q(s^\pm, \cdot)$ of $Z(s^\pm, \cdot)$. That $Q \in C^\infty(M_0') \cap C^0(\bar{M}_0')$ can be seen e.g. by applying the implicit function theorem to the function $(s^\pm, z, x) \mapsto Z(s^\pm, x) - z$. We now come to the proof of the scaling estimates for small $\alpha^\pm$. Taking $\varepsilon < \frac{1}{2}$ we may suppose that $x \neq 0$. To avoid heavy notation we prove (130) through (133) only for $* = +$; the case $* = -$ is similar. In what follows $*$ is a dummy variable and not the $*$ from the statement of the lemma (which we just fixed to be $+$). We will omit the argument $(s^\pm, x)$ of Z, g, g_2, g_3 and their derivatives. By (127) we have (again using $-xg_3 = g + g_2$)

$$Z_3 = -\frac{1}{x}\Big(2Z + \frac{g_{23}g}{(g_2)^2}\Big) \tag{135}$$

$$Z_* = \frac{1}{x}\Big(\frac{g_*}{g_2} - \frac{g_{2*}g}{(g_2)^2}\Big). \tag{136}$$

Taking further derivatives we find (again $*, *' \in \{+, -\}$ are dummy variables)

$$Z_{33} = -\frac{3}{x}Z_3 - \frac{1}{x}\Big(\frac{g_{233}g + g_{23}g_3}{g_2^2} - \frac{g_{23}g}{g_2^2} \cdot \frac{2g_{23}}{g_2}\Big) \tag{137}$$

$$Z_{3*} = -\frac{2}{x}Z_* - \frac{1}{x}\Big(\frac{g_{23*}g + g_{23}g_*}{g_2^2} - \frac{g_{23}g}{g_2^2} \cdot \frac{2g_{2*}}{g_2}\Big) \tag{138}$$

$$Z_{**'} = \frac{1}{x}\Big(\Big(\frac{g_{**'}}{g_2} - \frac{g_*}{g_2} \cdot \frac{g_{2*'}}{g_2}\Big) - \Big(\frac{g_{2**'}g + g_{2*}g_{*'}}{g_2^2} - \frac{g_{2*}g}{g_2^2} \cdot \frac{2g_{2*'}}{g_2}\Big)\Big) \tag{139}$$

To derive the scaling of the above derivatives as $\alpha^+ \downarrow 0$, we use the following facts (which are easily verified): We have $|g| \sim |\log \alpha^+|$, $|g_+| \sim (\alpha^+)^{-1}$, $g_- \sim 1$.

For $i = 2, 3$, we have $|g_i| \sim (\alpha^+)^{-1}$, $|g_{i+}| \sim (\alpha^+)^{-2}$, $|g_{i-}| \sim 1$, and similarly $|g_{i++}| \sim (\alpha^+)^{-3}$, $g_{i+-} = 0$, $|g_{i--}| \sim 1$. Also $|g_{23}| \sim (\alpha^+)^{-2}$, $|g_{23+}| \sim (\alpha^+)^{-3}$, $|g_{23-}| \sim 1$, $|g_{++}| \sim (\alpha^+)^{-2}$, $|g_{--}| \sim 1$, and $|g_{233}| \sim (\alpha^+)^{-3}$. Thus (130) through (133) follow from (135) through (139). □

Acknowledgements. Some of the results were obtained in my PhD thesis. I thank my advisor Stefan Müller for proposing this problem and for helpful suggestions. Interesting discussions with Sergio Conti and Reza Pakzad are also gratefully acknowledged.

References

[1] C. Bohle, G. P. Peters and U. Pinkall, Constrained Willmore surfaces, Calc. Var. **32** (2008), 263-277.

[2] M. P. do Carmo, Differential geometry of curves and surfaces, Prentice-Hall (1976).

[3] G. Friesecke, R. D. James and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity, Comm. Pure Appl. Math. **55** (2002), 1461-1506.

[4] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, CRC Press (1992).

[5] P. Hartman and L. Nirenberg, On spherical image maps whose Jacobians do not change sign, Amer. J. Math. **81** (1959), 901-920.

[6] W. Helfrich, Elastic properties of lipid bilayers: theory and possible experiments, Z. Naturforsch. A **C28** (1973), 693-703.

[7] P. Hornung, Approximating $W^{2,2}$ isometric immersions, C. R. Acad. Sci. Paris, Ser. I **346** (2008).

[8] P. Hornung, Minimizers of Kirchhoff's plate functional: Euler-Lagrange equations and regularity, C. R. Acad. Sci. Paris, Ser. I **347** (2009).

[9] P. Hornung, Approximation of $W^{2,2}$ isometric immersions by smooth isometric immersions I: Geometry of countably developable mappings, Preprint, http://www.app-ana2.uni-bonn.de/publications.

[10] P. Hornung, Approximation of $W^{2,2}$ isometric immersions by smooth isometric immersions II: Local construction, Preprint, http://www.app-ana2.uni-bonn.de/publications.

[11] P. Hornung, Flat minimizers of the Willmore functional: Euler-Lagrange equations, Preprint, http://www.app-ana2.uni-bonn.de/publications.

[12] P. Hornung, Regularity and non-regularity of isometric immersions minimizing the Willmore functional, in preparation.

[13] R. Jerrard, Some rigidity results related to Monge-Ampere functions, Preprint (2007).

[14] B. Kirchheim, Geometry and Rigidity of Microstructures, Habilitation thesis, University of Leipzig (2001).

[15] E. Kuwert and R. Schätzle, Removability of point singularities of Willmore surfaces, Ann. of Math. (2) **160** (2004), 315-357.

[16] E. Kuwert and R. Schätzle, Branch points of Willmore surfaces, Duke Math. J. **138** (2007), 179-201.

[17] S. Müller and M. R. Pakzad, Regularity properties of isometric immersions, Math. Z. **251** (2005), 313-331.

[18] M. R. Pakzad, On the Sobolev space of isometric immersions, J. Differential Geom. **66** No. 1 (2004), 47-69.

[19] U. Pinkall and I. Sterling, Willmore surfaces, Math. Intelligencer **9** (1987), 38-43.

[20] A. V. Pogorelov, Extrinsic geometry of convex surfaces, Translations of Mathematical Monographs **35** (1973).

[21] T. Rivière, Analysis aspects of Willmore surfaces, Invent. math. (2008).

[22] L. Simon, Existence of surfaces minimizing the Willmore functional, Comm. Anal. Geom. **1** (1993), 281-326.

[23] E. L. Starostin and G. H. M. van der Heijden, The shape of a Möbius strip, Nature materials **6** (2007), 563-567.

[24] M. Sadowsky, Theorie der elastisch biegsamen undehnbaren Bänder mit Anwendungen auf das Möbius'sche Band, Sitzungsber. Preuss. Akad. Wiss. **22** (1930), 412-415.

[25] T. J. Willmore, Riemannian geometry, The Clarendon Press, Oxford University Press, New York (1993).