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A Non-Self-Adjoint Quadratic Eigenvalue Problem Describing a Fluid-Solid Interaction. Part II: Analysis of Convergence

by

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This paper is dedicated to Philippe G. Ciarlet on the occasion of his seventieth birthday

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Abstract

This paper is the second part of a two-part paper treating a non-self-adjoint quadratic eigenvalue problem for the linear stability of solutions to the Taylor-Couette problem for flow of a viscous liquid in a deformable cylinder, with the cylinder modelled as a membrane. The first part formulated the problem, analyzed it, and presented computations. In this second part, we first give a weak formulation of the problem, carefully contrived so that the pressure boundary terms are eliminated from the equations. We prove that the bilinear forms appearing in the weak formulation satisfy continuous inf-sup conditions. We combine a Fourier expansion with the finite element method to produce a discrete problem satisfying discrete inf-sup conditions. Finally, the Galerkin approximation theory for polynomial eigenvalue problems is applied to prove convergence of the spectrum.

1 Introduction

In this paper, the second of a two-part paper, we develop a numerical method for solving a non-self-adjoint quadratic eigenvalue problem arising in a fluid-solid interaction problem, and we prove convergence of this method. The reader interested primarily in the numerical method and the convergence analysis can read this part independently from Part I, which formulated the problem, analyzed it, and presented computations.

The underlying physical problem is to determine the motion of a viscous incompressible liquid occupying the region between a rigid inner cylinder rotating about its axis at a prescribed angular velocity ω and a nonlinearly viscoelastic outer membrane, whose motion is driven by that of the liquid. Here we study motions independent of the coordinate along the axis of rotation. The governing equations, consisting of the Navier-Stokes equations for the liquid and geometrically exact equations for the membrane, admit a solution, called the Couette solution, in which the liquid and the membrane rotate rigidly with the inner cylinder. Normal modes of the linearization of the governing equations about the Couette solution are the eigenfunctions of the non-self-adjoint quadratic eigenvalue problem presented in Section 2.

To numerically solve the quadratic eigenvalue problem we first derive a suitable weak formulation of it. A careful choice of test functions must be made in order to eliminate the pressure boundary terms from the equations. See Section 3. We prove that the bilinear forms appearing in the weak formulation satisfy continuous inf-sup conditions.

From the weak formulation of the quadratic eigenvalue problem we compute the eigenvalues using a Fourier-finite element method: Fourier series are used to reduce the partial differential equations for the fluid on an annulus to ordinary differential equations in the radial variable r, which are discretized using the 1-dimensional finite element method with Taylor-Hood elements. These equations are coupled to algebraic equations for the membrane (coming from the Fourier series). The resulting discrete problem satisfies discrete inf-sup conditions. See

Section 5. The Galerkin approximation theory for polynomial eigenvalue problems is then applied to prove the convergence of the spectrum (Theorem 5.10).

Galerkin methods are a natural choice for solving fluid-solid interaction problems because the boundary terms for the fluid can be treated exactly. Some other methods, such as certain finite difference methods, introduce an artificial pressure boundary condition, which can lead to a numerical boundary layer. This should be avoided for an accurate description of the fluid-solid interface.

An extensive list of references to other work on the numerical approximation of quadratic eigenvalue problems arising in fluid-solid interaction problems is given in Part I of the paper.

We use Gibbs notation for vectors and tensors in which the value of a tensor (linear transformation) $\boldsymbol{A}$ acting on a vector $\boldsymbol{u}$ is denoted $\boldsymbol{A} \cdot \boldsymbol{u}$ and in which in which $\boldsymbol{ab} \cdot \boldsymbol{c} := (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$ for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

2 The Quadratic Eigenvalue Problem

In this section we recall the quadratic eigenvalue problem that was derived in Part I of the paper: Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a right-handed orthonormal basis for Euclidean 3-space. For any angle χ we define the vectors

$$\boldsymbol{e}_1(\chi) := \cos\chi\, \boldsymbol{i} + \sin\chi\, \boldsymbol{j}, \quad \boldsymbol{e}_2(\chi) := -\sin\chi\, \boldsymbol{i} + \cos\chi\, \boldsymbol{j} \equiv \boldsymbol{k} \times \boldsymbol{e}_1(\chi). \tag{2.1}$$

We consider the motion of a viscous incompressible liquid in the region between a rigid circular cylinder of radius $a < 1$ rotating at a prescribed angular velocity ω about its axis, and a nonlinearly viscoelastic membrane whose natural state is a circular cylinder of radius 1. Neither the liquid nor the membrane has any component of velocity along the axis of the cylinder. This motion is governed by the Navier-Stokes equations for the fluid coupled by adherence conditions to a nonlinear parabolic-hyperbolic equation for the viscoelastic membrane. These equations admit a Couette solution: The fluid and the membrane rotate rigidly with the rigid cylinder at the same angular velocity ω, with the membrane having radius $R > 1$.

We introduce constants

ϱ	the density of the fluid,
μ	the dynamic viscosity of the fluid,
γ	the kinematic viscosity of the fluid, $\gamma = \mu/\varrho$,
ρA	the mass density of the membrane per reference length in the azimuthal direction.

Let $\boldsymbol{v}^1$, p^1, $\boldsymbol{r}^1$ be perturbations about the Couette solution of the fluid velocity, fluid pressure, and membrane position. We introduce normal modes $\boldsymbol{v}$, p, $\boldsymbol{r}$ and the perturbation growth rate λ by

$$\boldsymbol{v}^1(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x})\, e^{\lambda t}, \quad p^1(\boldsymbol{x}, t) = p(\boldsymbol{x})\, e^{\lambda t}, \quad \boldsymbol{r}^1(s, t) = \boldsymbol{r}(s)\, e^{\lambda t}. \tag{2.2}$$

Here s is the coordinate of the membrane in the azimuthal direction. The functions $\boldsymbol{v}$ and p have domain $U = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < R\}$. The domain of $\boldsymbol{r}$ is the set of material points $\{s \in [0, 2\pi]\}$. Since the membrane is cylindrical, $\boldsymbol{r}$ satisfies the periodicity condition $\boldsymbol{r}(0) = \boldsymbol{r}(2\pi)$.

Let $\boldsymbol{D}(\boldsymbol{v})$ be the symmetric part of the velocity gradient of the fluid:

$$\boldsymbol{D}(\boldsymbol{v}) := \frac{1}{2}\left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)^*\right] \tag{2.3}$$

where the asterisk denotes the transpose. Let $\boldsymbol{\Sigma}$ be the Cauchy stress tensor for the fluid:

$$\boldsymbol{\Sigma}(\boldsymbol{v}, p) = -\varrho p \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{v}). \tag{2.4}$$

Let $N^\circ > 0$ be the constant tension, $N_\nu^\circ > 0$ the constant elastic tangent modulus, and $N_{\dot\nu}^\circ \geq 0$ the constant viscosity for the membrane for the Couette solution. They depend on the parameter R. (See Part I.) The constant

$$P(R, \omega^2) = \frac{N^\circ}{\varrho R} - \frac{\rho A \omega^2}{\varrho} \tag{2.5}$$

is the pressure of the fluid at its interface with the deformable solid for the Couette solution.

In Part I, we showed that $\lambda \in \mathbb{C}$ and $(\boldsymbol{v}, \boldsymbol{r}, p) \neq (\boldsymbol{0}, \boldsymbol{0}, 0)$ are to satisfy the eigenvalue problem

$$\begin{aligned} \lambda^2 \rho A \boldsymbol{r} = \lambda[N_{\dot\nu}^\circ(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\rho A \omega \boldsymbol{r} \times \boldsymbol{k}] + \tfrac{1}{R} N^\circ \boldsymbol{r}_{ss} \\ + (N_\nu^\circ - \tfrac{1}{R} N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s - \varrho P(R, \omega^2) \boldsymbol{k} \times \boldsymbol{r}_s + \rho A \omega^2 \boldsymbol{r} \\ + \varrho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 - R\, \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1, \end{aligned} \tag{2.6}$$

$$\lambda \boldsymbol{v} = \tfrac{1}{\varrho} \operatorname{div} \boldsymbol{\Sigma}(p, \boldsymbol{v}) - 2\omega\, \boldsymbol{k} \times \boldsymbol{v}, \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{2.7}$$

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{for } |\boldsymbol{x}| = a, \tag{2.8}$$

$$\boldsymbol{v}(R\boldsymbol{e}_1(s)) = \lambda \boldsymbol{r}(s), \tag{2.9}$$

$$\int_0^{2\pi} \boldsymbol{r}(s) \cdot \boldsymbol{e}_1(s)\, ds = 0. \tag{2.10}$$

Equations (2.6) and (2.9) have domain $s \in [0, 2\pi)$ and equation (2.7) has domain $a < |\boldsymbol{x}| < R$. Equation (2.7) is a Stokes-type eigenvalue equation with complicated boundary conditions; the eigenvalue parameter λ appears in the boundary condition (2.9) for the fluid and the boundary values of the Cauchy stress tensor appear in the membrane equation (2.6).

We take the angular velocity ω to be the bifurcation parameter for the problem. The stability of the Couette solution is determined by the eigenvalue trajectories $\omega \mapsto \lambda(\omega)$. In Part I of the paper we showed that there are no nonzero eigenvalues on the imaginary axis, i.e., that any eigenvalue λ that crosses the imaginary axis must cross through the origin. In the rest of this paper we present a numerical method for computing the eigenvalues λ and prove that this method converges.

3 Weak Formulation of the Quadratic Eigenvalue Problem

In this section we derive a mixed weak formulation of the quadratic eigenvalue problem (2.6)–(2.10). The mixed formulation enables us to avoid the inconvenience of using incompressible shape functions. Unless there is a statement to the contrary all the functions $\boldsymbol{v}, p, \boldsymbol{r}$ and their corresponding test functions that appear in the rest of this paper are complex-valued. Complex conjugates are denoted by superposed bars. We adopt the convention that if H is a traditional notation for a space of scalar-valued functions, then the notation $\boldsymbol{q} \in H$ means that each component of $\boldsymbol{q}$ belongs to H.

Recall that $U = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < R\}$. Let $\boldsymbol{\nu}$ be the unit outer normal to ∂U. Let $\boldsymbol{w} \in H^1(U)$ satisfy $\boldsymbol{w} = \boldsymbol{0}$ on $\{|\boldsymbol{x}| = a\}$. Integrate the dot product of $(2.7)_1$ with $\bar{\boldsymbol{w}}$ over U to obtain

(3.1)

$$\begin{aligned}
\lambda \int_U \boldsymbol{v} \cdot \bar{\boldsymbol{w}}\, d\boldsymbol{x} &= \int_U \left\{ \tfrac{1}{\varrho} \operatorname{div} \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{w}} - 2\omega\, (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \right\} d\boldsymbol{x} \\
&= - \int_U \left\{ \tfrac{1}{\varrho}\, \boldsymbol{\Sigma}(\boldsymbol{v}, p) : \frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{x}} + 2\omega\, (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \right\} d\boldsymbol{x} \\
&\quad + \tfrac{1}{\varrho} \int_{|\boldsymbol{x}|=R} \boldsymbol{\nu} \cdot \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{w}}\, dS \\
&= -2 \int_U \{ \gamma\, \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) + \omega\, (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \}\, d\boldsymbol{x} \;+\; \int_U p \operatorname{div} \bar{\boldsymbol{w}}\, d\boldsymbol{x} \\
&\quad + \tfrac{1}{\varrho} \int_0^{2\pi} \boldsymbol{e}_1(s) \cdot \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{w}}\; R\, ds,
\end{aligned}$$

where $\boldsymbol{v}$, p, and $\bar{\boldsymbol{w}}$ are evaluated at $\boldsymbol{x} = R\boldsymbol{e}_1(s)$ in the boundary term in the last line. Define

$$\varPi := \left\{ p \in L^2(U) : \int_U p\, d\boldsymbol{x} = 0 \right\}. \tag{3.2}$$

Let $q \in \varPi$. Multiply $(2.7)_2$ by $\bar{q}$ and integrate the product over U to obtain

$$\int_U \bar{q} \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} = 0. \tag{3.3}$$

Let $\mathbb{T}$ be the circle $\mathbb{R}/2\pi\mathbb{Z}$. Define

$$H^1_{\mathrm{S}} := \left\{ \boldsymbol{r} \in H^1(\mathbb{T}) : \int_0^{2\pi} \boldsymbol{r} \cdot \boldsymbol{e}_1\, ds = 0 \right\}. \tag{3.4}$$

($H^1(\mathbb{T})$ consists of functions of period 2π on the real line that have square-integrable derivatives. Let $\boldsymbol{r} \in H^1(\mathbb{T})$. Since Lusin's Theorem ensures that that $\boldsymbol{r}_s$ is continuous on any interval of $\mathbb{R}$ except for a subset of arbitrarily small measure, there is an $s_0 \in \mathbb{R}$ such that $\boldsymbol{r}_s(s_0 + 2\pi) = \boldsymbol{r}_s(s_0)$. We tacitly exploit this fact in integrating by parts to kill boundary terms.)

Let $\boldsymbol{q} \in H^1_{\mathrm{S}}$. We integrate the dot product of (2.6) with $\bar{\boldsymbol{q}}$ by parts over $\mathbb{T}$ to obtain

(3.5)

$$
\begin{aligned}
&\lambda^2 \rho A \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}}\, ds + \lambda \int_0^{2\pi} \{N_\nu^\circ\, (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) - 2\rho A \omega (\boldsymbol{r} \times \boldsymbol{k}) \cdot \bar{\boldsymbol{q}}\}\, ds \\
&= -\int_0^{2\pi} \{R^{-1} N^\circ \boldsymbol{r}_s \cdot \bar{\boldsymbol{q}}_s + (N_\nu^\circ - R^{-1} N^\circ)\, (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) \\
&\qquad + \varrho P(R, \omega^2)(\boldsymbol{k} \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{q}} - \rho A \omega^2 \boldsymbol{r} \cdot \bar{\boldsymbol{q}} - \varrho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)(\bar{\boldsymbol{q}} \cdot \boldsymbol{e}_1)\}\, ds \\
&\quad - R \int_0^{2\pi} \boldsymbol{e}_1(s) \cdot \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{q}}\, ds.
\end{aligned}
$$

We define the inner product of $\boldsymbol{r}$ and $\boldsymbol{t}$ on the space $H^{1/2}(\mathbb{T})$ of vector-valued functions thus: We decompose $\boldsymbol{r} = r^1(s)\boldsymbol{e}_1(s) + r^2(s)\boldsymbol{e}_2(s)$ and $\boldsymbol{t} = t^1(s)\boldsymbol{e}_1(s) + t^2(s)\boldsymbol{e}_2(s)$. If r^j and t^j have Fourier coefficients $\{r^j_k\}_{k\in\mathbb{Z}}$ and $\{t^j_k\}_{k\in\mathbb{Z}}$, for $j \in \{1, 2\}$, then the $H^{1/2}$-inner product can be defined by

$$
\langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}(\mathbb{T})} := \sum_{k=-\infty}^{\infty} (1 + |k|)\, r^1_k\, \overline{t^1_k} + \sum_{k=-\infty}^{\infty} (1 + |k|)\, r^2_k\, \overline{t^2_k}. \tag{3.6}
$$

Let $\gamma_R \boldsymbol{v}$ denote the trace of $\boldsymbol{v}$ restricted to $\{|\boldsymbol{x}| = R\}$. Taking the $H^{1/2}(\mathbb{T})$-inner product of the adherence condition $(2.9)_2$ with $\boldsymbol{t} \in H^{1/2}(\mathbb{T})$ gives

$$
\lambda\, \langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}(\mathbb{T})} = \langle \gamma_R \boldsymbol{v}, \boldsymbol{t} \rangle_{H^{1/2}(\mathbb{T})}. \tag{3.7}
$$

Observe that if we choose test functions $\boldsymbol{w}$ and $\boldsymbol{q}$ with $\boldsymbol{q} = \gamma_R \boldsymbol{w}$, i.e., with $\boldsymbol{q}(s) = \boldsymbol{w}(R\boldsymbol{e}_1(s))$, then multiplying equation (3.1) by ϱ and adding it to equation (3.5) eliminates the boundary terms involving $\boldsymbol{\Sigma}(\boldsymbol{v}, p)$. The sum of equation (3.5), equation (3.7), and ϱ times equation (3.1) is

(3.8)

$$
\begin{aligned}
&\lambda^2 \rho A \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}}\, ds \\
&+ \lambda \left[\int_0^{2\pi} \{N_\nu^\circ\, (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) - 2\rho A \omega (\boldsymbol{r} \times \boldsymbol{k}) \cdot \bar{\boldsymbol{q}}\}\, ds + \varrho \int_U \boldsymbol{v} \cdot \bar{\boldsymbol{w}}\, d\boldsymbol{x} - \langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}} \right] \\
&= -\int_0^{2\pi} \{R^{-1} N^\circ \boldsymbol{r}_s \cdot \bar{\boldsymbol{q}}_s + (N_\nu^\circ - R^{-1} N^\circ)\, (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) \\
&\qquad + \varrho P(R, \omega^2)(\boldsymbol{r}_s \times \bar{\boldsymbol{q}}) \cdot \boldsymbol{k} - \rho A \omega^2 \boldsymbol{r} \cdot \bar{\boldsymbol{q}} - \varrho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)(\bar{\boldsymbol{q}} \cdot \boldsymbol{e}_1)\}\, ds \\
&\quad - 2 \int_U \{\mu \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) + \varrho \omega (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}}\}\, d\boldsymbol{x} + \varrho \int_U p \operatorname{div} \bar{\boldsymbol{w}}\, d\boldsymbol{x} - \langle \gamma_R \boldsymbol{v}, \boldsymbol{t} \rangle_{H^{1/2}}.
\end{aligned}
$$

Define the complex Hilbert spaces

$$
\begin{aligned}
H^1_a(U) &:= \{\boldsymbol{v} \in H^1(U) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \{|\boldsymbol{x}| = a\}\}, \\
\mathcal{V}_1 &:= \{(\boldsymbol{v}, \boldsymbol{r}) \in H^1_a(U) \times H^1_{\mathrm{S}}\}, \\
\mathcal{V}_2 &:= \left\{(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) \in H^1_a(U) \times H^1_{\mathrm{S}} \times H^{1/2}(\mathbb{T}) : \gamma_R \boldsymbol{w}(R\boldsymbol{e}_1(s)) = \boldsymbol{q}(s)\right\}
\end{aligned} \tag{3.9}
$$

by

$$
\begin{aligned}
\langle (\boldsymbol{v}_1, \boldsymbol{r}_1), (\boldsymbol{v}_2, \boldsymbol{r}_2) \rangle_{\mathcal{V}_1} &:= \langle \boldsymbol{D}(\boldsymbol{v}_1), \boldsymbol{D}(\boldsymbol{v}_2) \rangle_{L^2(U)} + \langle \boldsymbol{r}_1, \boldsymbol{r}_2 \rangle_{H^1(\mathbb{T})}, \\
\langle (\boldsymbol{w}_1, \boldsymbol{q}_1, \boldsymbol{t}_1), (\boldsymbol{w}_2, \boldsymbol{q}_2, \boldsymbol{t}_2) \rangle_{\mathcal{V}_2} &:= \langle \boldsymbol{D}(\boldsymbol{w}_1), \boldsymbol{D}(\boldsymbol{w}_2) \rangle_{L^2(U)} + \langle \boldsymbol{q}_1, \boldsymbol{q}_2 \rangle_{H^1(\mathbb{T})} \\
&\quad + \langle \boldsymbol{t}_1, \boldsymbol{t}_2 \rangle_{H^{1/2}(\mathbb{T})}.
\end{aligned} \tag{3.10}
$$

Equations (3.3) and (3.8) constitute a

Weak formulation of the quadratic eigenvalue problem: Find $\lambda \in \mathbb{C}$ and nonzero $(\boldsymbol{v}, \boldsymbol{r}, p) \in \mathcal{V}_1 \times \mathit{\Pi}$ such that

$$
\begin{gathered}
\lambda^2 a_2(\boldsymbol{r}, \boldsymbol{q}) + \lambda a_1((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) + a_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) + b(\boldsymbol{w}, p) = 0, \\
b(\boldsymbol{v}, q) = 0,
\end{gathered} \tag{3.11}
$$

for all $(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}, q) \in \mathcal{V}_2 \times \mathit{\Pi}$ where

$$
\begin{aligned}
& a_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) \\
&\quad := \int_0^{2\pi} \{ R^{-1} N^\circ \boldsymbol{r}_s \cdot \bar{\boldsymbol{q}}_s + (N^\circ_\nu - R^{-1} N^\circ)(\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) \\
&\qquad + \varrho P(R, \omega^2)(\boldsymbol{k} \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{q}} \\
&\qquad - \rho A \omega^2 \boldsymbol{r} \cdot \bar{\boldsymbol{q}} - \varrho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)(\bar{\boldsymbol{q}} \cdot \boldsymbol{e}_1) \} \, ds \\
&\quad + 2 \int_U \{ \mu \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) + \varrho \omega (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \} \, d\boldsymbol{x} + \langle \gamma_R \boldsymbol{v}, \boldsymbol{t} \rangle_{H^{1/2}},
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
& a_1((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) \\
&\quad := \int_0^{2\pi} \{ N^\circ_\nu (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) - 2\rho A \omega (\boldsymbol{r} \times \boldsymbol{k}) \cdot \bar{\boldsymbol{q}} \} \, ds \\
&\quad + \varrho \int_U \boldsymbol{v} \cdot \bar{\boldsymbol{w}} \, d\boldsymbol{x} - \langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}},
\end{aligned} \tag{3.13}
$$

$$
a_2(\boldsymbol{r}, \boldsymbol{q}) := \rho A \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}} \, ds, \qquad b(\boldsymbol{w}, p) := -\varrho \int_U p \operatorname{div} \bar{\boldsymbol{w}} \, d\boldsymbol{x}. \tag{3.14}
$$

When a fluid interacts with a deformable body, the pressure of the fluid is determined uniquely, not just up to a constant:

Lemma 3.15. *Let* $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p))$ *be a smooth solution of the weak problem* (3.11). *Then there exists a unique constant* Q *such that* $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p + Q))$ *satisfies the classical problem* (2.6)–(2.10).

Proof. We choose $\boldsymbol{w} = \boldsymbol{0}$, $\boldsymbol{q} = \boldsymbol{0}$, and $\boldsymbol{t} = \gamma_R \boldsymbol{v} - \lambda \boldsymbol{r}$ in $(3.11)_1$ to obtain $||\gamma_R \boldsymbol{v} - \lambda \boldsymbol{r}||^2_{H^{1/2}} = 0$, which implies that $\gamma_R \boldsymbol{v} = \lambda \boldsymbol{r}$, i.e., $\boldsymbol{v} = \lambda \boldsymbol{r}$ on $\{|\boldsymbol{x}| = R\}$. By the Divergence Theorem and (2.9),

$$
\int_U \operatorname{div} \boldsymbol{v} \, d\boldsymbol{x} = \int_{\partial U} \boldsymbol{v} \cdot \boldsymbol{n} \, d\boldsymbol{x} = \lambda \int_0^{2\pi} \boldsymbol{r} \cdot \boldsymbol{e}_1 R \, ds = 0 \tag{3.16}
$$

since $\boldsymbol{r} \in H^1_{\mathrm{S}}$. Therefore we can substitute $q = \operatorname{div} \boldsymbol{v}$ into $(3.11)_2$ to obtain $||\operatorname{div} \boldsymbol{v}||^2_{L^2(U)} = 0$. Thus $\operatorname{div} \boldsymbol{v} = 0$.

We set $\boldsymbol{q} = \boldsymbol{t} = \boldsymbol{0}$ in $(3.11)_1$ (and note that $\boldsymbol{q} = \boldsymbol{0}$ implies that $\boldsymbol{w} = \boldsymbol{0}$ on ∂U) to obtain

$$
\begin{aligned}
\lambda \varrho \int_U \boldsymbol{v} \cdot \bar{\boldsymbol{w}} \, d\boldsymbol{x} &= \int_U \{-2\mu \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) - 2\varrho\omega(\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}}\} \, d\boldsymbol{x} \\
&\quad + \int_U \varrho \, p \operatorname{div} \bar{\boldsymbol{w}} \, d\boldsymbol{x} \\
&= \int_U \{\operatorname{div} \boldsymbol{\Sigma}(\boldsymbol{v}, p) - 2\varrho\omega(\boldsymbol{k} \times \boldsymbol{v})\} \cdot \bar{\boldsymbol{w}} \, d\boldsymbol{x}
\end{aligned} \tag{3.17}
$$

for all $\boldsymbol{w} \in H^1_0(U)$. Therefore equation (3.17) implies that the Stokes-like equation $(2.7)_1$ is satisfied.

We integrate $(3.11)_1$ with $\boldsymbol{t} = \boldsymbol{0}$ by parts and then use $(2.7)_1$ and $\gamma_R \boldsymbol{w} = \boldsymbol{q}$ to obtain

$$
\begin{aligned}
0 = \int_0^{2\pi} & \big\{\lambda^2 \rho A \boldsymbol{r} - \lambda[N^\circ_\nu(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\rho A\omega \boldsymbol{r} \times \boldsymbol{k}] - R^{-1} N^\circ \boldsymbol{r}_{ss} \\
& -(N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + \varrho P(R, \omega^2)\boldsymbol{k} \times \boldsymbol{r}_s - \rho A \omega^2 \boldsymbol{r} \\
& -\varrho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + R\,\boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1 \big\} \cdot \bar{\boldsymbol{q}} \, ds
\end{aligned} \tag{3.18}
$$

for all $\boldsymbol{q} \in H^1_{\mathrm{S}}$. Since $\int_0^{2\pi} \boldsymbol{q} \cdot \boldsymbol{e}_1 \, ds = 0$, there exists a constant Q_1 such that

$$
\begin{aligned}
& Q_1 \boldsymbol{e}_1 - R\,\boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1 \equiv -R\,\boldsymbol{\Sigma}(\boldsymbol{v}, p + Q_1/\varrho R) \cdot \boldsymbol{e}_1 \\
& = (\lambda^2 - \omega^2)\rho A \boldsymbol{r} - \lambda[N^\circ_\nu(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\rho A \omega \boldsymbol{r} \times \boldsymbol{k}] - R^{-1}N^\circ \boldsymbol{r}_{ss} \\
& \quad - (N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + \varrho P(R, \omega^2)\boldsymbol{k} \times \boldsymbol{r}_s - \varrho R^2\omega^2(\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1.
\end{aligned} \tag{3.19}
$$

The choice $Q := Q_1/\varrho R$ shows that that $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p+Q))$ satisfies the membrane equation (2.6). □

From the weak formulation (3.11) we could use the 2-dimensional finite element method to compute the eigenvalues. A more efficient method, however, is to write equation (3.11) in polar coordinates and then use Fourier series in the angle variables ϕ, s to reduce the partial differential equations to ordinary differential equations in the radial variable r. Then the 1-dimensional finite element method can be used. Before doing this we check that the bilinear forms a_0 and b satisfy the inf-sup conditions.

Proposition 3.20. *The bilinear form b defined in* (3.14) *satisfies the inf-sup condition*

$$
\inf_{p \in \Pi} \sup_{(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t}) \in \mathcal{V}_2} \frac{|b(\boldsymbol{w}, p)|}{||p||_\Pi ||(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})||_{\mathcal{V}_2}} =: \beta > 0.
$$

This follows immediately from the well-known inf-sup condition for the Stokes equation (see, for example, Girault & Raviart (1986) or Brenner & Scott (2002, Chap. 12, Sec. 2)).

Define

$$\begin{aligned} \mathcal{Z}_1 &:= \{(\boldsymbol{v}, \boldsymbol{r}) \in \mathcal{V}_1 : b(\boldsymbol{v}, p) = 0 \text{ for all } p \in \mathit{\Pi}\}, \\ \mathcal{Z}_2 &:= \{(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) \in \mathcal{V}_2 : b(\boldsymbol{w}, p) = 0 \text{ for all } p \in \mathit{\Pi}\}. \end{aligned} \tag{3.21}$$

Proposition 3.22. *There exists a constant* $C_{\mathrm{g}} > 0$ *such that the bilinear form* $\hat{a}_0 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{C}$ *defined by*

$$\hat{a}_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) := a_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) + C_{\mathrm{g}} \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}} \, ds \tag{3.23}$$

satisfies the inf-sup conditions

$$\inf_{\substack{(\boldsymbol{v}, \boldsymbol{r}) \in \mathcal{Z}_1 \\ ||(\boldsymbol{v}, \boldsymbol{r})||_{\mathcal{V}_1} = 1}} \; \sup_{\substack{(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) \in \mathcal{Z}_2 \\ ||(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})||_{\mathcal{V}_2} = 1}} |\hat{a}_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}))| = \alpha > 0, \tag{3.24}$$

$$(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}) \;\; \textit{if} \;\; \hat{a}_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) = 0 \;\; \textit{for all} \;\; (\boldsymbol{v}, \boldsymbol{r}) \in \mathcal{Z}_1. \tag{3.25}$$

Proof. Recall that the inf-sup conditions (3.24) and (3.25) are equivalent to the well-posedness of the following problem: Given $F \in \mathcal{Z}_2^*$, the space of bounded linear functionals on $\mathcal{Z}_2$, find $(\boldsymbol{v}, \boldsymbol{r}) \in \mathcal{Z}_1$ such that

$$\hat{a}_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) = F(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) \text{ for all } (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) \in \mathcal{Z}_2. \tag{3.26}$$

See, for example, Ern & Guermond (2004, p. 85, Th. 2.6). To prove Proposition 3.22 we prove that equation (3.26) is well-posed.

Decompose the bilinear form a_0 defined in (3.12) into three bilinear forms:

$$a_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) =: d_1(\boldsymbol{r}, \boldsymbol{q}) + d_2(\boldsymbol{v}, \boldsymbol{w}) + \langle \gamma_R \boldsymbol{v}, \boldsymbol{t} \rangle_{H^{1/2}}, \tag{3.27}$$

where d_1 and d_2 correspond to the membrane terms and the fluid terms of a_0. It is easy to check that d_1 satisfies a Gårding inequality: There exists a constant $C_{\mathrm{g}} > 0$ such that

$$d_1(\boldsymbol{r}, \boldsymbol{r}) + C_{\mathrm{g}} ||\boldsymbol{r}||^2_{L^2(0,2\pi)} \geq \alpha_1 ||\boldsymbol{r}||^2_{H^1(\mathbb{T})} \;\; \text{for all} \;\; \boldsymbol{r} \in H^1_{\mathrm{s}}. \tag{3.28}$$

It is also easy to check that d_2 is coercive:

$$\mathrm{Re}[d_2(\boldsymbol{v}, \boldsymbol{v})] = 2\mu ||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(U)} \;\; \text{for all} \;\; \boldsymbol{v} \in H^1_a(U). \tag{3.29}$$

We now construct a solution $(\boldsymbol{v}, \boldsymbol{r})$ to (3.26). Substitute $\boldsymbol{w} = \mathbf{0}$, $\boldsymbol{q} = \mathbf{0}$ into (3.26) to obtain the problem: Find $\boldsymbol{\varphi}(s) := \gamma_R \boldsymbol{v}(R\boldsymbol{e}_1(s)) \in H^{1/2}(\mathbb{T})$ such that

$$\langle \boldsymbol{\varphi}, \boldsymbol{t} \rangle_{H^{1/2}(\mathbb{T})} = F(\mathbf{0}, \mathbf{0}, \boldsymbol{t}) \;\; \text{for all} \;\; \boldsymbol{t} \in H^{1/2}(\mathbb{T}). \tag{3.30}$$

By the Riesz Representation Theorem there exists a unique solution $\boldsymbol{\varphi}$ to (3.30). Note that $||\boldsymbol{\varphi}||_{H^{1/2}(\mathbb{T})} = ||F(\mathbf{0}, \mathbf{0}, \cdot)||$, where $||F(\mathbf{0}, \mathbf{0}, \cdot)||$ denotes the norm of the bounded linear operator $F(\mathbf{0}, \mathbf{0}, \cdot)$.

Let

$$
\begin{aligned}
\mathcal{Z} &= \{\boldsymbol{w} \in H^1(U) : b(\boldsymbol{w}, p) = 0 \text{ for all } p \in \Pi\}, \\
\mathcal{Z}_0 &= H_0^1(U) \cap \mathcal{Z}.
\end{aligned} \tag{3.31}
$$

Substitute $\boldsymbol{q} = \boldsymbol{0}$, $\boldsymbol{t} = \boldsymbol{0}$ in (3.26) to obtain the problem: Find $\boldsymbol{v} \in \mathcal{Z}$ with $\boldsymbol{v} = \boldsymbol{0}$ on $\{|\boldsymbol{x}| = a\}$ and $\gamma_R \boldsymbol{v} = \boldsymbol{\varphi}$ such that

$$
d_2(\boldsymbol{v}, \boldsymbol{w}) = F(\boldsymbol{w}, \boldsymbol{0}, \boldsymbol{0}) \quad \text{for all} \quad \boldsymbol{w} \in \mathcal{Z}_0. \tag{3.32}
$$

There exists a $\boldsymbol{g} \in \mathcal{Z}$ such that $\boldsymbol{g} = \boldsymbol{0}$ on $\{|\boldsymbol{x}| = a\}$, $\gamma_R \boldsymbol{g} = \boldsymbol{\varphi}$, and $||\boldsymbol{g}||_{H^1} \leq C||\boldsymbol{\varphi}||_{H^{1/2}}$. (See, for example, Girault & Raviart (1986) or Temam (1977).) Define $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{g}$. Then problem (3.32) is equivalent to: Find $\boldsymbol{u} \in \mathcal{Z}_0$ such that

$$
d_2(\boldsymbol{u}, \boldsymbol{w}) = F(\boldsymbol{w}, \boldsymbol{0}, \boldsymbol{0}) - d_2(\boldsymbol{g}, \boldsymbol{w}) \text{ for all } \boldsymbol{w} \in \mathcal{Z}_0. \tag{3.33}
$$

By (3.29), Korn's inequality, and the Lax-Milgram Theorem, there exists a unique $\boldsymbol{u} \in \mathcal{Z}_0$ satisfying (3.33). This determines $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{g}$. By substituting $\boldsymbol{w} = \boldsymbol{u}$ into (3.33) and using (3.29) we obtain the estimate

$$
||\boldsymbol{D}(\boldsymbol{u})||_{L^2} \leq C(||F(\cdot, \boldsymbol{0}, \boldsymbol{0})|| + ||\boldsymbol{\varphi}||_{H^{1/2}}) = C(||F(\cdot, \boldsymbol{0}, \boldsymbol{0})|| + ||F(\boldsymbol{0}, \boldsymbol{0}, \cdot)||) \tag{3.34}
$$

and so

$$
||\boldsymbol{D}(\boldsymbol{v})||_{L^2} \leq C(||F(\cdot, \boldsymbol{0}, \boldsymbol{0})|| + ||F(\boldsymbol{0}, \boldsymbol{0}, \cdot)||). \tag{3.35}
$$

Set $\boldsymbol{t} = \boldsymbol{0}$ in (3.26) to obtain the problem: Find $\boldsymbol{r} \in H^1_{\mathrm{S}}$ such that

$$
d_1(\boldsymbol{r}, \boldsymbol{q}) + C_{\mathrm{g}} \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}} \, ds = F(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{0}) - d_2(\boldsymbol{v}, \boldsymbol{w}) \tag{3.36}
$$

for all $(\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{0}) \in \mathcal{Z}_2$. For given $\boldsymbol{q} \in H^1_{\mathrm{S}}$, equation (3.32) implies that the right-hand side of (3.36) is independent of our choice of $\boldsymbol{w}$ for all $\boldsymbol{w}$ satisfying $\gamma_R \boldsymbol{w} = \boldsymbol{q}$. So fix $\boldsymbol{w} = \boldsymbol{h}$, where $\boldsymbol{h} \in H^1(U)$ satisfies $\boldsymbol{h} = \boldsymbol{0}$ on $\{|\boldsymbol{x}| = a\}$, $\boldsymbol{h} = \boldsymbol{q}$ on $\{|\boldsymbol{x}| = R\}$, and $||\boldsymbol{h}||_{H^1} \leq ||\boldsymbol{q}||_{H^{1/2}}$. Substitute $\boldsymbol{w} = \boldsymbol{h}$ into (3.36) to obtain the problem: Find $\boldsymbol{r} \in H^1_{\mathrm{S}}$ such that

$$
d_1(\boldsymbol{r}, \boldsymbol{q}) + C_{\mathrm{g}} \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}} \, ds = F(\boldsymbol{h}, \boldsymbol{q}, \boldsymbol{0}) - d_2(\boldsymbol{v}, \boldsymbol{h}) \tag{3.37}
$$

for all $\boldsymbol{q} \in H^1_{\mathrm{S}}$. By the Gårding inequality (3.28), the bilinear form on the left-hand side of (3.37) is coercive, and so there exists a unique solution $\boldsymbol{r} \in H^1_{\mathrm{S}}$. Moreover $\boldsymbol{r}$ satisfies (3.36). Equations (3.36), (3.28), and (3.35) imply that

$$
||\boldsymbol{r}||_{H^1} \leq C(||F(\cdot, \cdot, \boldsymbol{0})|| + ||F(\cdot, \boldsymbol{0}, \boldsymbol{0})|| + ||F(\boldsymbol{0}, \boldsymbol{0}, \cdot)||) \leq C'||F||. \tag{3.38}
$$

It follows from (3.30) and (3.36) that $(\boldsymbol{v}, \boldsymbol{r})$ satisfies (3.26).

If we have two solutions $(\boldsymbol{v}^1, \boldsymbol{r}^1)$, $(\boldsymbol{v}^2, \boldsymbol{r}^2)$ of (3.26), then their difference $(\boldsymbol{v}^1 - \boldsymbol{v}^2, \boldsymbol{r}^1 - \boldsymbol{r}^2)$ satisfies (3.26) with $F = 0$. By substituting into (3.26) $(\boldsymbol{w}, \boldsymbol{q}) = (\boldsymbol{0}, \boldsymbol{0})$, then $(\boldsymbol{q}, \boldsymbol{t}) = (\boldsymbol{0}, \boldsymbol{0})$, and then $\boldsymbol{t} = \boldsymbol{0}$, as above, we see that $(\boldsymbol{v}^1 - \boldsymbol{v}^2, \boldsymbol{r}^1 - \boldsymbol{r}^2) = (\boldsymbol{0}, \boldsymbol{0})$, and so (3.26) has a unique solution.

Finally, the continuous dependence of $(\boldsymbol{v}, \boldsymbol{r})$ on the data F follows from the estimates (3.35) and (3.38). □

The weak formulation in polar coordinates. To compute the eigenvalues numerically it is convenient to introduce polar coordinates: Decompose the functions in $\mathcal{V}_1 \times \mathit{\Pi}$ as

$$\boldsymbol{v}(r\boldsymbol{e}_1(\phi)) = v^1(r,\phi)\boldsymbol{e}_1(\phi) + v^2(r,\phi)\boldsymbol{e}_2(\phi), \tag{3.39}$$

$$\boldsymbol{r}(s) = r^1(s)\boldsymbol{e}_1(s) + r^2\boldsymbol{e}_2(s), \quad p(r\boldsymbol{e}_1(\phi)) = \tilde{p}(r,\phi). \tag{3.40}$$

Decompose the functions in $\mathcal{V}_2 \times \mathit{\Pi}$ as

$$\boldsymbol{w}(r\boldsymbol{e}_1(\phi)) = w^1(r,\phi)\boldsymbol{e}_1(\phi) + w^2(r,\phi)\boldsymbol{e}_2(\phi), \quad \boldsymbol{q}(s) = q^1(s)\boldsymbol{e}_1(s) + q^2\boldsymbol{e}_2(s), \tag{3.41}$$

$$\boldsymbol{t}(s) = t^1(s)\boldsymbol{e}_1(s) + t^2\boldsymbol{e}_2(s), \quad q(r\boldsymbol{e}_1(\phi)) = \tilde{q}(r,\phi). \tag{3.42}$$

Now drop the tilde from $\tilde{p}$ and $\tilde{q}$. Define

$$(\mathbf{v},\mathbf{r}) := (v^1, v^2, r^1, r^2), \quad (\mathbf{w},\mathbf{q},\mathbf{t}) := (w^1, w^2, q^1, q^2, t^1, t^2). \tag{3.43}$$

We obtain new function spaces V_1, V_2, and $\tilde{\mathit{\Pi}}$ by substituting the polar coordinates for $(\boldsymbol{v},\boldsymbol{r})$, $(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})$, and p into $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathit{\Pi}$:

$$V_1 := \Big\{(\mathbf{v},\mathbf{r}) \in H^1([a,R]\times\mathbb{T}) \times H^1(\mathbb{T}) : \\ v^i(a,\phi) = 0\ \forall\phi,\ \int_0^{2\pi} r^1\, ds = 0\Big\}, \tag{3.44}$$

$$V_2 := \Big\{(\mathbf{w},\mathbf{q},\mathbf{t}) \in H^1([a,R]\times\mathbb{T})) \times H^1(\mathbb{T}) \times H^{1/2}(\mathbb{T}) : \\ w^i(a,\phi) = 0\ \forall\phi,\ w^i(R,s) = q^i(s)\ \forall s,\ \int_0^{2\pi} q^1\, ds = 0\Big\}, \tag{3.45}$$

$$\tilde{\mathit{\Pi}} := \Big\{p \in L^2([a,R]\times\mathbb{T}) : \int_0^{2\pi}\int_a^R p(r,\phi)\, r\, dr\, d\phi = 0\Big\}. \tag{3.46}$$

Now drop the tilde from $\tilde{\mathit{\Pi}}$. If we substitute (3.39)–(3.42) into (3.11), we obtain

The weak formulation of the quadratic eigenvalue problem in polar coordinates: Find $\lambda \in \mathbb{C}$ and nonzero $(\mathbf{v},\mathbf{r},p) \in V_1 \times \mathit{\Pi}$ such that

$$\begin{aligned} \lambda^2\tilde{a}_2(\mathbf{r},\mathbf{q}) + \lambda\tilde{a}_1((\mathbf{v},\mathbf{r}),(\mathbf{w},\mathbf{q},\mathbf{t})) + \tilde{a}_0((\mathbf{v},\mathbf{r}),(\mathbf{w},\mathbf{q},\mathbf{t})) + \tilde{b}(\mathbf{w},p) &= 0, \\ \tilde{b}(\mathbf{v},q) &= 0, \end{aligned} \tag{3.47}$$

for all $(\mathbf{w},\mathbf{q},\mathbf{t},q) \in V_2 \times \mathit{\Pi}$ where

$$\tilde{a}_0((\mathbf{v},\mathbf{r}),(\mathbf{w},\mathbf{q},\mathbf{t})) := a_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})), \tag{3.48}$$

$$\tilde{a}_1((\mathbf{v},\mathbf{r}),(\mathbf{w},\mathbf{q},\mathbf{t})) := a_1((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})), \tag{3.49}$$

$$\tilde{a}_2(\mathbf{r},\mathbf{q}) := a_2(\boldsymbol{r},\boldsymbol{q}), \quad \tilde{b}(\mathbf{w},p) := b(\boldsymbol{w},p). \tag{3.50}$$

Fourier decomposition and a family of weak problems. We expand the functions in V_1, V_2, and Π as Fourier series in the angle variable (ϕ or s) and use this to generate a family of weak problems indexed by the Fourier wave number.

For $j \in \{1,2\}$ decompose

$$v^j(r,\phi) = \sum_{k=-\infty}^{\infty} v_k^j(r) e^{ik\phi}, \quad r^j(s) = \sum_{k=-\infty}^{\infty} r_k^j e^{iks}, \tag{3.51}$$

$$w^j(r,\phi) = \sum_{k=-\infty}^{\infty} w_k^j(r) e^{ik\phi}, \quad q^j(s) = \sum_{k=-\infty}^{\infty} q_k^j e^{iks}, \quad t^j(s) = \sum_{k=-\infty}^{\infty} t_k^j e^{iks}, \tag{3.52}$$

$$p(r,\phi) = \sum_{k=-\infty}^{\infty} p_k(r) e^{ik\phi}. \tag{3.53}$$

Define

$$(\mathbf{v}_k, \mathbf{r}_k) := (v_k^1, v_k^2, r_k^1, r_k^2), \quad (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) := (w_k^1, w_k^2, q_k^1, q_k^2, t_k^1, t_k^2).$$

We define a family of function spaces indexed by the Fourier wave number $k \in \mathbb{Z}$. For $k \neq 0$,

$$\begin{aligned} V_1^k &:= \left\{ (\mathbf{v}_k, \mathbf{r}_k) \in H^1(a,R) \times \mathbb{C}^2 : v_k^j(a) = 0 \right\}, \\ V_2^k &:= \left\{ (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in H^1(a,R) \times \mathbb{C}^2 \times \mathbb{C}^2 : w_k^j(a) = 0,\ w_k^j(R) = q_k^j \right\}, \\ \Pi^k &:= L^2(a,R). \end{aligned} \tag{3.54}$$

(Note that these spaces are independent of k.) For $k = 0$,

$$\begin{aligned} V_1^0 &:= \left\{ (\mathbf{v}_0, \mathbf{r}_0) \in H^1(a,R) \times \mathbb{C}^2 : v_0^j(a) = 0,\ r_0^1 = 0 \right\}, \\ V_2^0 &:= \left\{ (\mathbf{w}_0, \mathbf{q}_0, \mathbf{t}_0) \in H^1(a,R) \times \mathbb{C}^2 \times \mathbb{C}^2 : w_0^j(a) = 0,\ w_0^j(R) = q_0^j,\ q_0^1 = 0 \right\}, \\ \Pi^0 &:= \left\{ p_0 \in L^2(a,R) : \int_0^{2\pi} p_0(r)\, r dr = 0 \right\}. \end{aligned} \tag{3.55}$$

The condition $r_0^1 = 0$ in $(3.55)_1$ corresponds to side condition (2.10). We equip V_1^k, V_2^k, and Π^k with the norms

$$\begin{aligned} ||(\mathbf{v}_k, \mathbf{r}_k)||_{V_1^k}^2 &= ||\mathbf{v}_k||_{H^1(a,R)}^2 + |\mathbf{r}_k|^2, \\ ||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_1^k}^2 &= ||\mathbf{w}_k||_{H^1(a,R)}^2 + |\mathbf{q}_k|^2 + |\mathbf{t}_k|^2, \\ ||p_k||_{\Pi^k}^2 &= \int_a^R |p_k|^2\, r\, dr. \end{aligned} \tag{3.56}$$

Let $k \in \mathbb{Z}$, $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k$, $q_k \in \Pi^k$. Into (3.47) we substitute the Fourier decompositions (3.51) and (3.53) and

$$w^j(r,\phi) = w_k^j(r) e^{ik\phi}, \quad q^j(s) = q_k^j e^{iks}, \quad t^j(s) = t_k^j e^{iks}, \quad q(r,\phi) = q_k(r) e^{ik\phi}$$

to obtain

A family of weak problems indexed by the Fourier wave number: For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and nonzero $(\mathbf{v}_k, \mathbf{r}_k, p_k) \in V_1^k \times \Pi^k$ such that

$$\begin{aligned} &\lambda^2 a_2^k(\mathbf{r}_k, \mathbf{q}_k) + \lambda a_1^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &\qquad + a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k) = 0, \\ &b^k(\mathbf{v}_k, q_k) = 0 \end{aligned} \tag{3.57}$$

for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k) \in V_2^k \times \Pi^k$ where

$$\begin{aligned} &a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &\quad := R^{-1} N^\circ [(ikr_k^1 - r_k^2)(-ik\overline{q_k^1} - \overline{q_k^2}) + (r_k^1 + ikr_k^2)(\overline{q_k^1} - ik\overline{q_k^2})] \\ &\qquad + (N_\nu^\circ - R^{-1}N^\circ)(r_k^1 + ikr_k^2)(\overline{q_k^1} - ik\overline{q_k^2}) \\ &\qquad + \varrho P(R, \omega^2)[\overline{q_k^2}(ikr_k^1 - r_k^2) - \overline{q_k^1}(r_k^1 + ikr_k^2)] \\ &\qquad - \rho A \omega^2 (r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2}) - \varrho R^2 \omega^2 r_k^1 \overline{q_k^1} \\ &\qquad + 2 \int_a^R \Big\{ \mu[(v_k^1)_r (\overline{w_k^1})_r + \tfrac{1}{r^2}(ikv_k^2 + v_k^1)(-ik\overline{w_k^2} + \overline{w_k^1}) \\ &\qquad\qquad + \tfrac{1}{2}(\tfrac{ik}{r} v_k^1 - \tfrac{1}{r} v_k^2 + (v_k^2)_r)(-\tfrac{ik}{r}\overline{w_k^1} - \tfrac{1}{r}\overline{w_k^2} + (\overline{w_k^2})_r)] \\ &\qquad\qquad + \omega \varrho (v_k^1 \overline{w_k^2} - v_k^2 \overline{w_k^1}) \Big\} \, r\,dr \\ &\qquad + (1 + |k|)(v_k^1(R)\overline{t_k^1} + v_k^2(R)\overline{t_k^2}), \end{aligned} \tag{3.58}$$

$$\begin{aligned} &a_1^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &\quad := N_\nu^\circ (ikr_k^2 + r_k^1)(-ik\overline{q_k^2} + \overline{q_k^1}) + 2\rho A \omega (r_k^1 \overline{q_k^2} - r_k^2 \overline{q_k^1}) \\ &\qquad + \varrho \int_a^R \Big\{ v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2} \Big\} \, r\,dr - (1 + |k|)(r_k^1 \overline{t_k^1} + r_k^2 \overline{t_k^2}), \end{aligned} \tag{3.59}$$

$$a_2^k(\mathbf{r}_k, \mathbf{q}_k) := \rho A (r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2}), \tag{3.60}$$

$$b^k(\mathbf{w}_k, p_k) := -\varrho \int_a^R p_k [(\overline{w_k^1})_r + \tfrac{1}{r}\overline{w_k^1} - \tfrac{ik}{r}\overline{w_k^2}] \, rdr. \tag{3.61}$$

Proposition 3.62. *Let $k \in \mathbb{Z}$. The bilinear form b^k in* (3.61) *satisfies the inf-sup condition*

$$\inf_{p_k \in \Pi^k} \sup_{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k} \frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k} ||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_2^k}} =: \beta > 0.$$

This follows from the inf-sup condition (3.20) for the vectorial equations. See Bernardi et al. (1999, Chap. IX). An alternative proof is given in Bourne (2007).

For $k \in \mathbb{Z}$, define

$$Z_1^k := \{(\mathbf{v}_k, \mathbf{r}_k) \in V_1^k : b^k(\mathbf{v}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k\},$$
$$Z_2^k := \{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k : b^k(\mathbf{w}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k\}. \tag{3.63}$$

Proposition 3.64. *Let $k \in \mathbb{Z}$. There exists a constant $C_{\mathrm{g}}^k > 0$ such that the bilinear form $\hat{a}_0^k : V_1^k \times V_2^k \to \mathbb{C}$ defined by*

$$\hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) := a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + C_{\mathrm{g}}^k \left(\int_a^R \{v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2}\}\, dr + r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2} \right)$$

satisfies the inf-sup conditions

$$\inf_{\substack{(\mathbf{v}_k, \mathbf{r}_k) \in Z_1^k \\ \|(\mathbf{v}_k, \mathbf{r}_k)\|_{V_1^k} = 1}} \sup_{\substack{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in Z_2^k \\ \|(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)\|_{V_2^k} = 1}} |\hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k))| = \alpha > 0,$$

$$(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) = 0 \;\; \textit{if} \;\; \hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) = 0 \;\; \textit{for all} \;\; (\mathbf{v}_k, \mathbf{r}_k) \in Z_1^k.$$

The proof is analogous to the proof of Proposition 3.22. See Bourne (2007) for more details.

4 Galerkin Approximation of Polynomial Eigenvalue Problems

This section summarizes the Galerkin approximation theory for polynomial eigenvalue problems of the form

$$\text{Find } \lambda \in \mathbb{C} \text{ and } 0 \neq u \in V_1 \text{ such that for all } v \in V_2$$
$$A(u, v) = \lambda^N B_N(u, v) + \lambda^{N-1} B_{N-1}(u, v) + \cdots + \lambda B_1(u, v) + B_0(u, v) \tag{4.1}$$

where V_1 and V_2 are Hilbert spaces and A, $B_0, \ldots, B_N$ are bilinear forms. We apply this theory in the following section to design a convergent numerical scheme for computing the eigenvalues of (3.57). The spectral approximation theory of standard eigenvalue problems (the case $N = 1$ in (4.1)), which was developed in the 1970s, is described by Babuška & Osborn (1991). Kolata (1976) showed how to extend this theory to polynomial eigenvalue problems. We give a slightly different presentation and weaken one of the hypotheses (see the paragraph preceding Theorem 4.9).

The main idea is to reduce the polynomial eigenvalue problem to a standard eigenvalue problem by introducing new variables, the same way that any ordinary differential equation can be reduced to a system of first-order ordinary differential equations. This reduction process introduces identity operators and so compactness is lost. The resulting eigenvalue problem, however, is polynomially compact, which is sufficient for applying the standard approximation theory.

Let V_1, V_2, and W be complex Hilbert spaces with V_1 compactly embedded in W. Let $A : V_1 \times V_2 \to \mathbb{C}$, $B_0 : W \times V_2 \to \mathbb{C}$, $\dots$, $B_N : W \times V_2 \to \mathbb{C}$ be continuous bilinear forms satisfying

$$
\begin{aligned}
|A(u,v)| &\le C||u||_{V_1}||v||_{V_2} \quad \text{for all } u \in V_1,\ v \in V_2, \\
|B_j(u,v)| &\le C_j||u||_W||v||_{V_2} \quad \text{for all } u \in W,\ v \in V_2, \quad \text{for } j \in \{0 \dots N\}.
\end{aligned} \tag{4.2}
$$

We assume that A satisfies the inf-sup conditions

$$
\inf_{\substack{u \in V_1 \\ ||u||_{V_1} = 1}} \ \sup_{\substack{v \in V_2 \\ ||v||_{V_2} = 1}} |A(u,v)| = \alpha > 0, \tag{4.3}
$$

$$
v = 0 \text{ if } A(u,v) = 0 \text{ for all } u \in V_1. \tag{4.4}
$$

We consider the spectral approximation of

The continuous problem: Find $\lambda \in \mathbb{C}$ and $0 \ne u \in V_1$ such that

$$
A(u,v) = \lambda^N B_N(u,v) + \lambda^{N-1} B_{N-1}(u,v) + \dots + \lambda B_1(u,v) + B_0(u,v) \tag{4.5}
$$

for all $v \in V_2$. If (λ, u) satisfies (4.5), then we call λ an eigenvalue and u an eigenvector of (4.5).

The continuity and inf-sup conditions (4.2)–(4.4) imply that there exist unique bounded linear operators $T_0 : V_1 \to V_1$, $\dots$, $T_N : V_1 \to V_1$ satisfying

$$
A(T_j u, v) = -B_j(u,v) \quad \text{for all } v \in V_2, \quad \text{for } j \in \{0, \dots, N\}.
$$

See, for example, Ern & Guermond (2004, p. 85, Th. 2.6). Moreover, T_0, $\dots$, T_N are compact since V_1 is compactly embedded in W.

For $\lambda \in \mathbb{C}$ define $T(\lambda) : V_1 \to V_1$ by

$$
T(\lambda) := \lambda^N T_N + \dots + \lambda T_1 + T_0 + I, \tag{4.6}
$$

where $I : V_1 \to V_1$ is the identity operator on V_1. It is easy to check that the pair $(\lambda, u) \in \mathbb{C} \times (V_1 \setminus \{0\})$ satisfies problem (4.5) if and only if it is an eigenpair of T, i.e., if and only if

$$
T(\lambda)\, u \equiv \lambda^N T_N u + \dots + \lambda T_1 u + T_0 u + u = 0. \tag{4.7}
$$

We make the additional hypothesis that

(4.8) There exists a $\xi \in \mathbb{C}$ such that $T(\xi) : V_1 \to V_1$ has a bounded inverse.

Note that hypothesis (4.8) holds if and only if there exists a $\xi \in \mathbb{C}$ that is not an eigenvalue of (4.5). Kolata (1976) made the stronger hypothesis that $T(0)$ has a bounded inverse, which is true if and only if -1 is not an eigenvalue of T_0.

Theorem 4.9. *Let (4.2)–(4.4) and (4.8) hold. Then problem (4.5) has a countable set of eigenvalues with infinity as its only possible accumulation point.*

Proof. Since the set of eigenvalues of (4.5) equals the set of eigenvalues of the operator T, Theorem 4.9 follows immediately from the Spectral Theorem for Compact Polynomial Operator Pencils, stated by Markus (1988). We will go through the steps of the proof explicitly, however, since we will need to refer to one of these steps in the proof of Theorem 4.21.

By hypothesis (4.8), there exists a $\xi \in \mathbb{C}$ such that $T(\xi)$ has a bounded inverse. There are clearly compact operators $T_1', \ldots, T_N'$ such that

$$T(\lambda+\xi) = (\lambda+\xi)^N T_N + \cdots + (\lambda+\xi)T_1 + T_0 + I = \lambda^N T_N' + \cdots + \lambda T_1' + T(\xi).$$

Recall that (λ, u) is an eigenpair of (4.5) if and only if $T(\lambda)u = 0$. Define $\mu = \lambda - \xi$. Then

$$\begin{aligned} T(\lambda)u = 0 &\iff T(\mu+\xi)u = 0 \\ &\iff (\mu^N T_N' + \cdots + \mu T_1' + T(\xi))u = 0 \\ &\iff (\mu^N T(\xi)^{-1}T_N' + \cdots + \mu T(\xi)^{-1}T_1' + I)u = 0. \end{aligned} \tag{4.10}$$

Define $(u_1, u_2, \ldots, u_N) := (u, \mu u, \ldots, \mu^{N-1}u)$. Then (4.10) holds if and only if
(4.11)
$$B\boldsymbol{u} := \left[\begin{array}{ccc|c} -T(\xi)^{-1}T_1' & \cdots & -T(\xi)^{-1}T_{N-1}' & -T(\xi)^{-1}T_N' \\ \hline I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right] \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \frac{1}{\mu} \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}.$$

The operator B on the left-hand side of equation (4.11) is not compact since the identity operator is not compact on infinite-dimensional spaces, but B^N is compact, and so B has a countable set of eigenvalues with zero as its only possible accumulation point. See Dunford & Schwartz (1957, Sec. VII.4, Th. 6). This completes the proof. □

Let $V_{1,h}$ and $V_{2,h}$ be finite-dimensional subspaces of V_1 and V_2 parametrized by $h > 0$. We assume that A satisfies the discrete inf-sup conditions

$$\inf_{\substack{u \in V_{1,h} \\ ||u||_{V_1} = 1}} \sup_{\substack{v \in V_{2,h} \\ ||v||_{V_2} = 1}} |A(u,v)| = \alpha(h) > 0, \tag{4.12}$$

$$v = 0 \text{ if } A(u,v) = 0 \text{ for all } u \in V_{1,h}. \tag{4.13}$$

We make the approximability assumption that

$$\lim_{h\to 0} \alpha(h)^{-1} \inf_{\chi \in V_{1,h}} ||u-\chi||_{V_1} = 0 \quad \text{for all } u \in V_1. \tag{4.14}$$

The discrete problem: Find $\lambda \in \mathbb{C}$ and $0 \neq u_h \in V_{1,h}$ such that
(4.15)
$$A(u_h, v) = \lambda^N B_N(u_h, v) + \lambda^{N-1} B_{N-1}(u_h, v) + \cdots + \lambda B_1(u_h, v) + B_0(u_h, v)$$

for all $v \in V_{2,h}$.

We approximate the eigenvalues of the continuous problem (4.5) by the eigenvalues of the discrete problem (4.15). The continuity and inf-sup conditions (4.2), (4.12), and (4.13) imply that there exists unique bounded linear operators $T_{0,h} : V_1 \to V_{1,h}, \ldots, T_{N,h} : V_1 \to V_{1,h}$ satisfying

$$A(T_{j,h}u, v) = -B_j(u, v) \quad \text{for all } v \in V_{2,h}, \quad \text{for } j \in \{0, \ldots, N\}. \tag{4.16}$$

Let $P_h : V_1 \to V_{1,h}$ be the projection defined by

$$A(P_h u, v) = A(u, v) \quad \text{for all } v \in V_{2,h}. \tag{4.17}$$

(P_h is well-defined by the Babuška-Brezzi Theorem; see Brezzi & Fortin (1991), Ciarlet (1978, p. 414 ff.), or Ern & Guermond (2004, p. 85, Th. 2.6).) Observe that $T_{j,h} = P_h T_j$ for all $j \in \{0, \ldots, N\}$. It is well-known that $(4.2)_1$ and (4.12) imply the quasi-optimality estimate

$$||u - P_h u||_{V_1} \leq \left(1 + \frac{C}{\alpha(h)}\right) \inf_{\chi \in V_{1,h}} ||u - \chi||_{V_1}. \tag{4.18}$$

(This is Céa's Lemma. See, for example, Ern & Guermond (2004, Lemma 2.28).) Therefore $P_h \to I$ pointwise by (4.14) and (4.18). Thus, for all $j \in \{0, \ldots, N\}$, $T_{j,h} = P_h T_j \to T_j$ in norm since T_j is compact.

For $\lambda \in \mathbb{C}$ define $T_h(\lambda) : V_1 \to V_1$ by

$$T_h(\lambda) := \lambda^N T_{N,h} + \cdots + \lambda T_{1,h} + T_{0,h} + I. \tag{4.19}$$

It is easy to check that the pair $(\lambda, u_h) \in \mathbb{C} \times (V_1 \setminus \{0\})$ satisfies problem (4.15) if and only if it is an eigenpair of the operator $T_h : V_1 \to V_{1,h}$, i.e.,

$$T_h(\lambda) u_h \equiv \lambda^N T_{N,h} u_h + \cdots + \lambda T_{1,h} u_h + T_{0,h} u_h + u_h = 0. \tag{4.20}$$

Observe that $T_h(\lambda) \to T(\lambda)$ in norm for all λ since $T_{j,h} \to T_j$ in norm for $j \in \{0, \ldots, N\}$. Therefore, for h sufficiently small, $T_h(\xi)$ has a bounded inverse (because $T(\xi)$ has a bounded inverse), and $T_h(\xi)^{-1} \to T(\xi)^{-1}$ in norm. See Kato (1976).

Theorem 4.21. *Let* (4.2)–(4.4), (4.8), (4.12)–(4.14) *hold. Then the eigenvalues of problem* (4.15) *converge to those of problem* (4.5) *as* $h \to 0$.

Proof. Recall from the proof of Theorem 4.9 that (λ, u) is an eigenpair of (4.5) if and only if $(1/\mu, u)$ is an eigenpair of the operator B, where $\mu = \lambda - \xi$. The same method as in the proof of Theorem 4.9 shows that (λ, u_h) is an eigenpair of (4.15) if and only if $(1/\mu, u_h)$ is an eigenpair of the operator

$$B_h = \left[\begin{array}{ccc|c} -T_h(\xi)^{-1}T'_{1,h} & \cdots & -T_h(\xi)^{-1}T'_{N-1,h} & -T_h(\xi)^{-1}T'_{N,h} \\ \hline I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right]. \tag{4.22}$$

But $B_h \to B$ in norm and so the eigenvalues of B_h converge to the eigenvalues of B. (Recall that if noncompact operators $L_h \to L$ in norm, then the isolated points of the spectrum of L_h converge. See Descloux et al. (1978a, 1978b). In our case $L_h = B_h$ are polynomially compact, i.e., B_h^N are compact, and so every point of the spectrum is isolated.) $\square$

Rate of convergence estimates. Kolata (1976) applied the spectral theory for compact operators from Osborn (1975) to obtain rate of convergence estimates for polynomial eigenvalue problems. We do no repeat these estimates here, but will specialize them to problem (3.57) in the remark following Theorem 5.10.

5 Finite Element Discretization and Convergence

In this section we discretize the eigenvalue problem (3.57) using finite elements and prove the convergence of the numerical method.

Let $a = r_0 < r_1 < \cdots < r_N = R$ be a uniform partition of $[a, R]$ with $R - a = Nh$, so that $r_n = a + nh$, $n \in \{0, \ldots, N\}$. Let $V_{1,h}^k$, $V_{2,h}^k$, and Π_h^k be the Taylor-Hood finite-dimensional subspaces of V_1^k, V_2^k, and Π^k defined by

$$\begin{aligned}
V_{1,h}^k &:= \{(\mathbf{v}_k, \mathbf{r}_k) \in V_1^k : \text{for } j \in \{1,2\}, v_k^j \text{ is continuous}, \\
&\qquad\qquad v_k^j|_{[r_n, r_{n+1}]} \text{ is quadratic}\}, \\
V_{2,h}^k &:= \{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k : \text{for } j \in \{1,2\}, w_k^j \text{ is continuous }, \\
&\qquad\qquad w_k^j|_{[r_n, r_{n+1}]} \text{ is quadratic}\}, \\
\Pi_h^k &:= \left\{p_k \in \Pi^k : p_k \text{ is continuous }, p_k|_{[r_n, r_{n+1}]} \text{ is linear}\right\},
\end{aligned} \tag{5.1}$$

for all $k \in \mathbb{Z}$. We will approximate the eigenpairs $(\lambda, (\mathbf{v}_k, \mathbf{r}_k, p_k)) \in \mathbb{C} \times V_1^k \times \Pi^k$ of problem (3.57) by the eigenpairs $(\lambda, (\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h)) \in \mathbb{C} \times V_{1,h}^k \times \Pi_h^k$ of

The discrete eigenvalue problem: For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and nonzero $(\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h) \in V_{1,h}^k \times \Pi_h^k$ such that

$$\begin{aligned}
&\lambda^2 a_2^k(\mathbf{r}_k^h, \mathbf{q}_k) + \lambda a_1^k((\mathbf{v}_k^h, \mathbf{r}_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\
&\qquad + a_0^k((\mathbf{v}_k^h, \mathbf{r}_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k^h) = 0, \\
&b^k(\mathbf{v}_k^h, q_k) = 0
\end{aligned} \tag{5.2}$$

for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_{2,h}^k$, $q_k \in \Pi_h^k$.

Proposition 5.3. *Let $k \in \mathbb{Z}$, $h > 0$. The bilinear form b^k defined in* (3.61) *satisfies the discrete inf-sup condition*

$$\inf_{p_k \in \Pi_h^k} \sup_{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_{2,h}^k} \frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k} ||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_2^k}} = \beta > 0$$

where β is independent of h.

This follows immediately from the discrete inf-sup condition for the Fourier-finite element discretization of the Stokes equations in axisymmetric domains, which was proved by Belhachmi et al. (2006a, 2006b) for Taylor-Hood elements and $\mathbb{P}_2$-bubble/$\mathbb{P}_1$-discontinuous elements in 3-dimensional axisymmetric domains. Bourne (2007) specializes their proof to the 2-dimensional annular region considered here. Since the spatial dimension is reduced by 1 and since there is no polar singularity, many of the technical difficulties in Belhachmi et al. (2006a, 2006b) are not present here.

For $k \in \mathbb{Z}$, define

$$\begin{aligned} Z^k_{1,h} &:= \{(\mathbf{v}_k, \mathbf{r}_k) \in V^k_{1,h} : b^k(\mathbf{v}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k_h\}, \\ Z^k_{2,h} &:= \{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V^k_{2,h} : b^k(\mathbf{w}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k_h\}. \end{aligned} \tag{5.4}$$

Proposition 5.5. *Let $k \in \mathbb{Z}$. The bilinear form $\hat{a}^k_0$ defined in Theorem 3.64 satisfies the discrete inf-sup conditions*

$$\begin{gathered} \inf_{\substack{(\mathbf{v}_k,\mathbf{r}_k) \in Z^k_{1,h} \\ \|(\mathbf{v}_k,\mathbf{r}_k)\|_{V^k_1} = 1}} \sup_{\substack{(\mathbf{w}_k,\mathbf{q}_k,\mathbf{t}_k) \in Z^k_{2,h} \\ \|(\mathbf{w}_k,\mathbf{q}_k,\mathbf{t}_k)\|_{V^k_2} = 1}} |\hat{a}^k_0((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k))| = \alpha > 0, \\ (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) = (\mathbf{0}, \mathbf{0}, \mathbf{0}) \textit{ if } \hat{a}^k_0((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) = 0 \\ \textit{for all } (\mathbf{v}_k, \mathbf{r}_k) \in Z^k_{1,h}. \end{gathered} \tag{5.6}$$

The proof is similar to the proof of Propositions 3.22, 3.64.

Convergence of the numerical method. We apply the abstract spectral approximation theory of Section 4 to eigenvalue problem (3.57) and its discretization (5.2) to show that the finite element approximation of the eigenvalues converges.

Define the bilinear form

$$\begin{aligned} &c^k((\mathbf{v}_k, \mathbf{r}_k, p_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k)) \\ &\qquad := \hat{a}^k_0((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k) + b^k(\mathbf{v}_k, q_k), \end{aligned} \tag{5.7}$$

where $\hat{a}^k_0$ and b^k were defined in equations (3.64) and (3.61). Let C^k_{g} be the constant introduced in Theorem 3.64. Then the weak formulations (3.57) and (5.2) can be given equivalent formulations:

Equivalent formulation of the continuous eigenvalue problem (3.57)**:** For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and nonzero $(\mathbf{v}_k, \mathbf{r}_k, p_k) \in V^k_1 \times \Pi^k$ such that

$$\begin{aligned} &c^k((\mathbf{v}_k, \mathbf{r}_k, p_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k)) \\ &\qquad = -\lambda^2 a^k_2(\mathbf{r}_k, \mathbf{q}_k) - \lambda a^k_1((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &\qquad\quad + C^k_{\mathrm{g}} \left[\int_a^R \{v^1_k \overline{w^1_k} + v^2_k \overline{w^2_k}\}\, dr + r^1_k \overline{q^1_k} + r^2_k \overline{q^2_k} \right] \end{aligned} \tag{5.8}$$

for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k) \in V^k_2 \times \Pi^k$.

Equivalent formulation of the discrete eigenvalue problem (5.2)**:** For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and nonzero $(\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h) \in V_{1,h}^k \times \Pi_h^k$ such that

$$
\begin{aligned}
&c^k((\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k)) \\
&\quad = -\lambda^2 a_2^k(\mathbf{r}_k^h, \mathbf{q}_k) - \lambda a_1^k((\mathbf{v}_k^h, \mathbf{r}_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\
&\qquad + C_{\mathrm{g}}^k \left[\int_a^R \{ v_k^{1,h} \overline{w_k^1} + v_k^{2,h} \overline{w_k^2} \} \, dr + r_k^{1,h} \overline{q_k^1} + r_k^{2,h} \overline{q_k^2} \right]
\end{aligned} \tag{5.9}
$$

for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k) \in V_{2,h}^k \times \Pi_h^k$.

For each $k \in \mathbb{Z}$, we apply the abstract spectral approximation theory from Section 4 to the eigenvalue problem (5.8) and its discretization (5.9) with

$$
A = c^k, \qquad B_2 = -a_2^k, \qquad B_1 = -a_1^k,
$$
$$
B_0 = C_{\mathrm{g}}^k \left[\int_a^R \{ v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2} \} \, dr + r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2} \right].
$$

It is well-known that the continuous and discrete inf-sup conditions for c^k follow from those for $\hat{a}_0^k$ and b^k, which we proved in Propositions 3.62, 3.64, 5.3 and 5.5. See, for example, Brezzi & Fortin (1991) or Ern & Guermond (2004, p. 101, Prop. 2.36). In Part I of the paper we showed that not every complex number is an eigenvalue of (5.8). Therefore hypotheses (4.3), (4.4), (4.8), (4.12), and (4.13) of Theorem 4.21 are satisfied and we have proved

Theorem 5.10. *Problem* (5.8) *has a countable set of eigenvalues with infinity as its only possible accumulation point. The eigenvalues of problem* (5.9) *converge to the eigenvalues of problem* (5.8) *as* $h \to 0$.

Rate of convergence estimates. Since the eigenfunctions of (5.8) are smooth and we are using $\mathbb{P}_2/\mathbb{P}_1$ elements, applying the results of Osborn (1975) and Kolata (1976) yields the rate of convergence estimate $|\lambda - \lambda_h| \leq Ch^4$ for simple eigenvalues.

6 Comments

In this paper we used a membrane theory to model the deformable outer cylinder. The numerical method presented here can easily be applied to the more general case where the deformable body is modelled by a geometrically exact shell theory. The convergence theorem 5.10 can be extended without any technical difficulty; only the equations are more complicated.

Bourne & Antman (2009) consider the related Taylor-Couette problem of axisymmetric flow in a deformable axisymmetric shell. The spectrum of the associated quadratic eigenvalue problem, governing the stability of Couette flow, is computed using a numerical method similar to the one developed in this paper, but the behavior of solutions is very different.

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