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A Morse theoretic description of string topology

by

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This paper is dedicated to Yasha Eliashberg on the occasion of his 60th birthday.

Abstract

Let M be a closed, oriented, n-dimensional manifold. In this paper we give a Morse theoretic description of the string topology operations introduced by Chas and Sullivan, and extended by the first author, Jones, Godin, and others. We do this by studying maps from surfaces with cylindrical ends to M, such that on the cylinders, they satisfy the gradient flow equation of a Morse function on the loop space, LM. We then give Morse theoretic descriptions of related constructions, such as the Thom and Euler classes of a vector bundle, as well as the shriek, or umkehr homomorphism.

Contents

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Introduction

String topology operations were first defined by Chas and Sullivan in [7]. Their basic loop product is an algebra structure on the homology of the free loop space of a closed, oriented manifold, $H_*(LM)$, which comes from studying maps from a "pair of pants" surface to M. Since then generalizations and applications of these operations have been widely studied. In particular the work of V. Godin [17] describes operations based on families of Riemann surfaces, varying in moduli space.

More specifically, let $\mathcal{M}_{g,p+q}$ be the space of oriented, connected surfaces embedded in $\mathbb{R}^\infty$ having genus g and $p+q$ parameterized boundary components. We think of these surfaces as cobordisms between p parameterized circles, thought of as "incoming", and q parameterized circles, thought of as "outgoing". This space is homotopy equivalent to the moduli space of bordered Riemann surfaces, and is a model for the classifying space, $BDiff^+(\Sigma_{g,p+q};\partial\Sigma)$, where $Diff^+(\Sigma_{g,p+q};\partial\Sigma)$ is the group of orientation preserving diffeomorphisms of a surface Σ that are fixed pointwise on the boundary.

For a closed, oriented n-manifold M, let $\mathcal{M}_{g,p+q}(M)$ denote the space of pairs,

$$\mathcal{M}_{g,p+q}(M) = \{(\Sigma, f) : \Sigma \in \mathcal{M}_{g,p+q} \text{ and } f : \Sigma \to M \text{ is a smooth map}\}.$$

$\mathcal{M}_{g,p+q}(M)$ is a model of the homotopy orbit space,

$$\mathcal{M}_{g,p+q}(M) \simeq EDiff^+(\Sigma_{g,p+q};\partial\Sigma) \times_{Diff} C^\infty(\Sigma, M)$$

where the subscript $Diff$ refers to taking the orbit space by the diagonal $Diff^+(\Sigma_{g,p+q};\partial\Sigma)$-action.

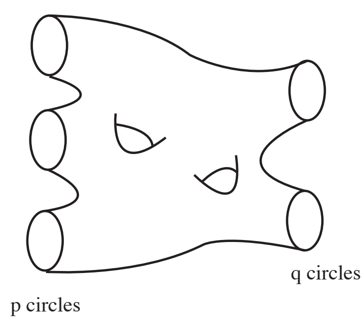

Figure 1: The surface Σ

Using the restriction maps to the incoming and outgoing parameterized boundary components as well as the projection map $\mathcal{M}_{g,p+q}(M) \to \mathcal{M}_{g,p+q}$, one has a diagram,

$$(LM)^q \xleftarrow{\rho_{out}} \mathcal{M}_{g,p+q}(M) \xrightarrow{\rho_{in}} \mathcal{M}_{g,p+q} \times (LM)^p. \tag{1}$$

In [17], this correspondence diagram lead to higher string topology operations in a (generalized) homology theory h_*, by first constructing an "umkehr map"

$$(\rho_{in})_! : h_*(\mathcal{M}_{g,p+q} \times (LM)^p) \to h_{*+\chi(F)\cdot d}(\mathcal{M}_{g,p+q}(M))$$

where $\chi(F)$ is the Euler characteristic of the surface Σ and $d = dim(M)$. For this one needs that the generalized homology theory supports an orientation of M. The higher string topology operations of [17] were then defined via the composition

$$\mu_{g,p+q} : h_*(\mathcal{M}_{g,p+q} \times (LM)^p) \xrightarrow{(\rho_{in})_!} h_{*+\chi(\Sigma)\cdot d}(\mathcal{M}_{g,p+q}(M)) \xrightarrow{(\rho_{out})_*} h_{*+\chi(\Sigma)\cdot d}((LM)^q). \tag{2}$$

In the case of an operation based on a fixed surface (i.e a point in $\mathcal{M}_{g,p+q}$), and in the case of families of genus zero surfaces when $q = 1$, the umkehr map has been given several descriptions. In [7] it was constructed on the chain level, and in [10] and [9] it was constructed as a generalized Pontrjagin-Thom construction on the homotopy theoretic level. Pontrjagin-Thom constructions are also the basis of the umkehr map defined by Godin in [17] for surfaces varying in higher genus moduli space.

In this paper we will show how string topology operations can be constructed using Morse theory on the loop space LM. In section one we show how to construct the umkehr map, and therefore the resulting string topology operations on the level of the Morse chain complex of appropriate energy functions on LM. We will prove that the operations defined this way are equal to the original string topology operations on the level of homology. We then indicate how this construction can be generalized to families, using the work of Godin [17]. In section 2 we will show that under the appropriate transversality conditions, operations obtained by explicitly counting the "gradient flow surfaces" whose boundaries lie in appropriate stable and unstable manifolds of critical points, define the same string topology operations. We discuss these transversality conditions in some detail. The operations constructed this way were defined and studied by the second author with Abbondandolo in [5]. In that paper the authors described an isomorphism of rings between the Floer homology of the cotangent bundle, $HF_*(T^*M)$ with the "pair of pants" product, to this Morse loop product in $H_*(LM)$. The arguments in this paper verify that this product agrees with the string topology product as constructed by Chas and Sullivan [7]. This verification uses related Morse theoretic constructions of the Thom and Euler classes of any oriented vector bundle, as well as the "umkehr" map, which is done in section 3. These constructions may be of independent interest.

1 Fat graphs and the space of gradient surfaces in a manifold

We begin by describing the types of Morse functions on the loop space that we will consider. We refer the reader to [4] for more details.

Endow M with a Riemannian metric. Consider a smooth Lagrangian

$$L : \mathbb{R}/\mathbb{Z} \times TM \to \mathbb{R}$$

that satisfies the following convexity property, bounds on its second derivatives, as well as nondegeneracy properties.

(L1) There exists $\ell_0 > 0$ such that

$$\nabla_{v,v} L(t,(q,v)) \geq \ell_0 I$$

for every $(t,(q,v)) \in \mathbb{R}/\mathbb{Z} \times TM$. (Here $q \in M$ and $v \in T_qM$.)

(L2) There exists $\ell_1 \geq 0$ such that

$$|\nabla_{v,v} L(t,(q,v))| \leq \ell_1 \quad |\nabla_{q,v} L(t,(q,v))| \leq \ell_1(1+|v|), \quad |\nabla_{q,q} L(t,(q,v))| \leq \ell_1(1+|v|^2)$$

for every $(t,(q,v)) \in \mathbb{R}/\mathbb{Z} \times TM$.

We explain a bit of this notation. The Riemannian metric on M induces a splitting of the tangent bundle $T(TM)$ into a vertical and horizontal part, via the Levi-Civita connection. Then $\nabla_{v,v}$, $\nabla_{q,v}$, and $\nabla_{q,q}$ denote the components of the Hessian in this splitting.
With such a Lagrangian one can define an energy function,

$$\mathcal{E} : LM \longrightarrow \mathbb{R}$$
$$\mathcal{E}(\gamma) = \int_0^1 L(t, \gamma(t), \frac{d\gamma}{ds}(t))dt$$

which is C^2 on the space LM, which we take to be those loops of Sobolev class $W^{1,2}$.
However, if we want the energy function to be smooth on this Hilbert manifold of loops, we have to assume stronger conditions on the Lagrangian L, namely similar bounds on all partial derivatives of L. For example, for any Riemannian metric g on M and a time-dependent potential $V(t,q)$ on M we can take $L(t,(q,v)) = \frac{1}{2}|v|_g^2 + V(t,q)$, and $\mathcal{E}$ will be smooth. In Section 2, it will be necessary to assume such smoothness of $\mathcal{E}$ for reasons of transversality. Therefore, we will from now on only consider such Lagrangians with a fibrewise quadratic kinetic term.
We then also assume

(L0) The critical points of $\mathcal{E}$ are all nondegenerate.

We denote the set of critical points of $\mathcal{E}$ by $\mathcal{P}(L)$.
In this context, the energy functional, $\mathcal{E} : LM \longrightarrow \mathbb{R}$ is a Morse function that is bounded below, its critical points, $\mathcal{P}_L$, have finite Morse indices, and it satisfies the Palais-Smale condition. Again, we refer the reader to [4] for details.

When the Lagrangian L satisfies these assumptions, standard Morse theory applies, and one can construct a space $LM_{\mathcal{E}}$ which is defined to be the union of the unstable manifolds of $\mathcal{E}$, and is topologized as a subspace of the loop space, LM. $LM_{\mathcal{E}}$ has one cell for each critical point in $\mathcal{P}(L)$, and the inclusion $LM_{\mathcal{E}} \hookrightarrow LM$ is a homotopy equivalence. The cellular chain complex of $LM_{\mathcal{E}}$ is the Morse complex,

$$\longrightarrow \cdots \xrightarrow{\partial_{p+1}} C_p^{\mathcal{E}}(LM) \xrightarrow{\partial_p} C_{p-1}^{\mathcal{E}}(LM) \to \cdots \tag{3}$$

where $C_p^{\mathcal{E}}(LM)$ is the free abelian group generated by those $a \in \mathcal{P}(L)$ of Morse index p, and

$$\partial_p([a]) = \sum_{\substack{b \in \mathcal{P}(L) \\ ind(b)=p-1}} \#\mathcal{M}(a,b)[b]$$

where $\mathcal{M}(a,b)$ is the space of gradient flow lines connecting a to b, which is a compact, zero dimensional, oriented manifold in this setting. The number $\#\mathcal{M}(a,b)$ refers to the oriented count of the points in this moduli space. We note that the stable attaching maps of the cells of the $LM_{\mathcal{E}}$ can be described by the framed bordism types of the higher dimensional compact spaces of piecewise flows, $\bar{\mathcal{M}}(a,b)$ [11][8].

The homotopy theoretic string topology operations were defined using "fat graph" models for surfaces [9] [17]. We will likewise use these graphs to define our Morse theoretic operations.

We recall the definition (see [19], [23]).

Definition 1. *A fat graph is a finite graph with the following properties:*

1. *Each vertex is at least trivalent*
2. *Each vertex comes equipped with a cyclic order of the half edges emanating from it.*

The cyclic order of the half edges is quite important in this structure. It allows for the graph to be "thickened" to a surface with boundary. As a way of describing this thickening, recall that the cyclic orderings of the half edges at each vertex define a partition of the set of oriented edges, that identify boundary components of the thickened surface. More explicitly, let $E(\Gamma)$ be the set of edges, and let $\tilde{E}(\Gamma)$ be the set of oriented edges. $\tilde{E}(\Gamma)$ is a 2-fold cover of $E(\Gamma)$. It has an involution $e \to \bar{e}$ which represents changing the orientation. The partition of $\tilde{E}(\Gamma)$ is best illustrated by the following example.

The cyclic orderings at the vertices are determined by the counterclockwise orientation of the plane. To obtain the partition, notice that an oriented edge has well defined source and target vertices. Start with an oriented edge, and follow it to its target vertex. The next edge in the partition is the next oriented edge in the cyclic ordering at that vertex. Continue in this way until one is back at the original oriented edge. This will be the first cycle in the partition. Then continue with this process until one exhausts all the oriented edges. The resulting cycles in the partition will be called *"boundary cycles"* as they reflect the boundary circles of the thickened surface. In the case of Γ_2 illustrated in figure 2, the partition into boundary cycles are given by:

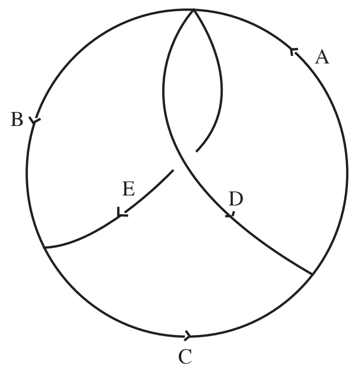

Figure 2: The fat graph Γ

$$\text{Boundary cycles of } \Gamma_2\text{:} \quad (A, B, C)\,(\bar{A}, \bar{D}, E, \bar{B}, D, \bar{C}, \bar{E}).$$

So one can compute combinatorially the number of boundary components in the thickened surface of a fat graph. Furthermore the graph and the surface have the same homotopy type, so one can compute the Euler characteristic of the surface directly from the graph. Then using the formula $\chi(F) = 2 - 2g - n$, where n is the number of boundary components, we can solve for the genus directly in terms of the graph. The space of *metric fat graphs* (i.e fat graphs with lengths assigned to each edge) of topological type (g, n) gives a model for the homotopy type moduli space of Riemann surfaces of genus g and n-marked points [19], [23].

Notice that the boundary cycles of a metric fat graph Γ nearly determines a parameterization of the boundary of the thickened surface. For example, the boundary cycle (A, B, C) of the graph Γ above can be represented by a map $S^1 \to \Gamma_2$ where the circle is of circumference equal to the sum of the lengths of sides A, B, and C. The ambiguity of the parameterization is the choice of where to send the basepoint $1 \in S^1$. By choosing a marked point in each boundary cycle, one can describe such a parameterization. This was carried out by Godin [16] [17] in which she used marked fat graphs to give models of the homotopy type of moduli spaces of bordered Riemann surfaces. In this paper, however, we will be mostly concerned about individual graphs. So for our purposes we can define a marking on a metric fat graph to simply be a choice of basepoint in each boundary cycle. Also, as part of our data in a marking of a metric fat graph, we assume that the n boundary cycles are partitioned into p incoming, and $q = n - p$ outgoing cycles.

Let Γ be a metric marked fat graph. In particular this means that the boundary cycles of Γ are partitioned into p incoming and q outgoing cycles, and there are parameterizations determined by the markings,

$$\alpha^- : \coprod_p S^1 \longrightarrow \Gamma, \quad \alpha^+ : \coprod_q S^1 \longrightarrow \Gamma. \tag{4}$$

By taking the circles to have circumference equal to the sum of the lengths of the edges making up the boundary cycle it parameterizes, each component of α^+ and α^- is a local isometry.
Define the surface Σ_Γ to be the mapping cylinder of these parameterizations,

$$\Sigma_\Gamma = \left(\coprod_p S^1 \times (-\infty, 0]\right) \sqcup \left(\coprod_q S^1 \times [0, +\infty)\right) \bigcup \Gamma / \sim \tag{5}$$

where $(t, 0) \in S^1 \times (-\infty, 0] \sim \alpha^-(t) \in \Gamma$, and $(t, 0) \in S^1 \times [0, +\infty) \sim \alpha^+(t) \in \Gamma$

Notice that the graph in figure 3 is a fat graph representing a surface of genus $g = 0$ and 3 boundary components. This graph has two edges, say A and B, and has boundary cycles $(A), (B), (\bar{A}, \bar{B})$. If we let (A) and (B) be the incoming cycles and $(\bar{A}, \bar{B})$ the outgoing cycle, then figure 3 is a picture of the surface Σ_Γ, for Γ equal to the figure 8.

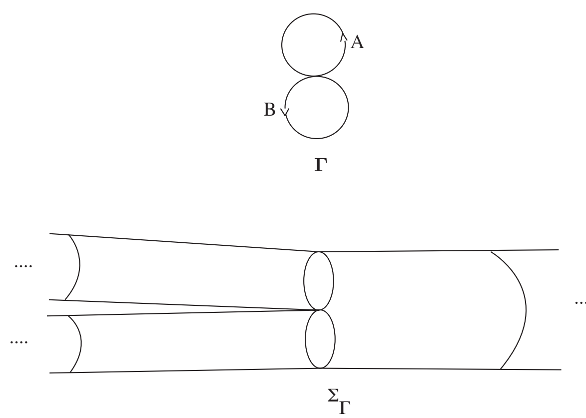

Figure 3: Σ_Γ

Notice that a map $\phi : \Sigma_\Gamma \to M$ is a collection of p maps from half cylinders, $\phi_i : (-\infty, 0] \times S^1 \to M$ and q- half cylinders, $\phi_j : [0, +\infty) \times S^1 \to M$, that have an intersection property at $t = 0$ determined by the combinatorics of the fat graph Γ. In the definition of $\phi_i : (-\infty, 0) \times S^1 \to M$, the circle factor is rescaled in a canonical way so as to have radius one.
We now define a "gradient flow surface" to be a map $\phi : \Sigma_\Gamma \to M$ so that when restricted to each half cylinder satisfies a gradient flow equation. Now, similar to what was done in constructing

cohomology operations on closed manifolds using Morse theory (see [6], [15], [14]), we will need to allow our gradient flow equations to be perturbed on each cylinder. More specifically, define a "Lagrangian labeling" of a marked fat graph Γ to be a labeling $\mathcal{E}(\Gamma)$ of each of the boundary cycles of Γ by a Lagrangian, L_i, and therefore by an energy functional $\mathcal{E}_i : LM \to \mathbb{R}$. We write $\mathcal{E}(\Gamma) = (\mathcal{E}_1, \cdots \mathcal{E}_{p+q})$ where $\mathcal{E}_i$ is the energy functional labeling the i^{th} boundary cycle.

Definition 2. *Let Γ be a marked fat graph with Lagrangian labeling $\mathcal{E} = \mathcal{E}(\Gamma)$. Define the moduli space of "gradient flow surfaces", $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$ to be the space of maps*

$$\phi : \Sigma_\Gamma \to M$$

that are smooth in the interiors of the cylinders, and that the restrictions to the incoming cylinders define maps $\phi_i : (-\infty, 0) \times S^1 \to M$, $i = 1, \cdots, p$ satisfying the gradient flow equation

$$\frac{d\phi_i(t,s)}{dt} + \nabla \mathcal{E}_i = 0 \tag{6}$$

such that $\lim_{t\to-\infty} \phi_i(t,\cdot) : S^1 \to M$ converges uniformly to a critical point in $\mathcal{P}(L_i)$. Similarly on outgoing cylinders ϕ defines maps $\phi_j : (0, +\infty) \times S^1 \to M$, $j = 1, \cdots, q$ which satisfy the gradient flow equation $\frac{d\phi_j(t,s)}{dt} + \nabla \mathcal{E}_j = 0$ and $\lim_{t\to+\infty} \phi_j(t,\cdot) : S^1 \to M$ converges uniformly to a critical point in $\in \mathcal{P}(L_j)$.
The space $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$ is topologized as a subspace of the space of continuous maps $\Sigma_\Gamma \to M$ in the compact-open topology.

The spaces $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$ will be essential in our definition of the Morse theoretic string topology. For example, we now describe a correspondence diagram analogous to (1).
Let $\phi \in \mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$. For $i = 1, \cdots, p$, let $\phi_{i,-1} : S^1 \to M$ be the restriction of $\phi_i : S^1 \times (-\infty, 0] \to M$ to $S^1 \times \{-1\}$. Notice that by definition, each $\phi_{i,-1}$ lies in an unstable manifold of some critical point in $\mathcal{P}(\mathcal{E})$. Therefore $\phi_{i,-1} \in LM_{\mathcal{E}_i}$.
Similarly, for $j = 1, \cdots, q$, let $\phi_{j,1} : S^1 \to M$ be the restriction of ϕ_j to $S^1 \times \{1\}$. These restrictions define the following maps.

$$\prod_{j=1}^{q} LM_{\mathcal{E}_j} \xleftarrow{\rho_{out}} \mathcal{M}^{\mathcal{E}}_{\Gamma}(LM) \xrightarrow{\rho_{in}} \prod_{i=1}^{p} LM_{\mathcal{E}_i}. \tag{7}$$

Our goal is to construct an umkehr map on the level of chains,

$$(\rho_{in})_! : \bigotimes_{i=1}^{p} C^{\mathcal{E}_i}_*(LM) \to C_*(\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM))$$

so that our string topology operation on the level of Morse homology will be induced by the composition on the level of Morse chains,

$$\mu_\Gamma : (\rho_{out})_* \circ (\rho_{in})_! : \bigotimes_{i=1}^{p} C^{\mathcal{E}_i}_*(LM) \longrightarrow \bigotimes_{j=1}^{q} C^{\mathcal{E}_j}_*(LM).$$

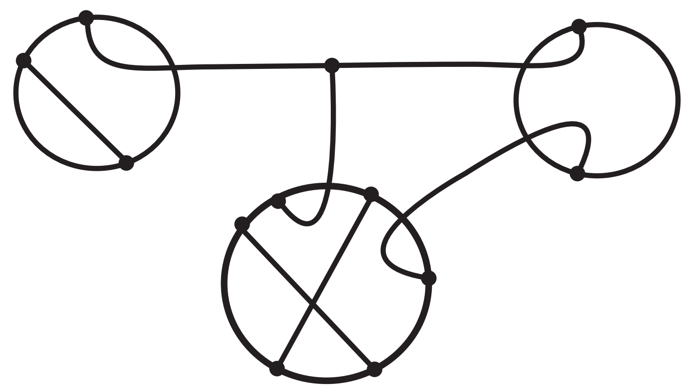

Figure 4: Sullivan chord diagram of type (1;3,3)

To do this, we will find it convenient, as was the case in [9], to consider a particular type of marked fat graph, known as a "Sullivan chord diagram".

Definition 3. *A "Sullivan chord diagram" of type $(g; p, q)$ is a fat graph representing a surface of genus g with $p+q$ boundary components, that consists of a disjoint union of p disjoint closed circles together with the disjoint union of connected trees whose endpoints lie on the circles. The cyclic orderings of the edges at the vertices must be such that each of the p disjoint circles is a boundary cycle. These p circles are referred to as the incoming boundary cycles, and the other q boundary cycles are referred to as the outgoing boundary cycles.*

The ordering at the vertices in the diagrams that follow are indicated by the clockwise cyclic ordering of the plane. Also in a Sullivan chord diagram, the vertices and edges that lie on one of the p disjoint circles will be referred to as circular vertices and circular edges respectively. The others will be referred to as ghost vertices and edges.
Let Γ be a marked chord diagram. The marking defines a parameterization of the incoming and outgoing boundary circles, and hence if $\phi : \Gamma \to M$, we can identify the restriction to these boundary circles with loops, $\phi_{i,0} : S^1 \to M$.
We now go about studying the topology of $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$. The following is our main result.

Theorem 1. *Let Γ be a marked chord diagram. The natural map from the space of gradient flow surfaces to the continuous mapping space,*

$$\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM) \to Map(\Sigma_{\Gamma}, M)$$

is a homotopy equivalence.

Proof. Let Γ be a chord diagram, and let $v(\Gamma)$ be the collection of circular vertices (i.e vertices that lie on the incoming boundary circles). There is a natural evaluation map

$$ev_\Gamma : \prod_{i=1}^{p} LM_{\mathcal{E}_i} \to M^{v(\Gamma)}$$

that evaluates the i^{th} loop on the vertices lying on the i^{th} boundary circle of Γ. Put an equivalence relation on the set of circular vertices $v(\Gamma)$ by saying that two vertices v_2 and v_2 are equivalent if there is a ghost subtree of Γ that contains both v_1 and v_2. Let $\sigma(\Gamma) = v(\Gamma)/\sim$ be the set of equivalence classes of these circular vertices. The projection map $\pi : v(\Gamma) \to \sigma(\Gamma)$ defines a diagonal embedding

$$\Delta_\Gamma : M^{\sigma(\Gamma)} \hookrightarrow M^{v(\Gamma)}. \tag{8}$$

Lemma 2. *$\mathcal{M}^{\mathcal{E}}_\Gamma(LM)$ is homotopy equivalent to the homotopy pullback of the map $ev_\Gamma : \prod_{i=1}^{p} LM_{\mathcal{E}_i} \to M^{v(\Gamma)}$ along the diagonal embedding $\Delta_\Gamma : M^{\sigma(\Gamma)} \hookrightarrow M^{v(\Gamma)}$.*

Proof. We now describe a locally trivial fiber bundle,

$$e\tilde{v}_\Gamma : (LM_{\mathcal{E}})^p_\Gamma \to M^{v(\Gamma)}$$

of the same homotopy type as the evaluation map, $ev_\Gamma : \prod_{i=1}^{p} LM_{\mathcal{E}_i} \to M^{v(\Gamma)}$.
We define $(LM_{\mathcal{E}})^p_\Gamma$ to be the space of "hairy loops" defined by the graph Γ. Namely, let $C_1, \cdots C_p$ be the p -incoming circles of chord diagram Γ. Let $v_{i,1}, \cdots v_{i,n_i} \subset C_i$ be the set of circular vertices lying on C_i. We define "hairy incoming circles" by attaching intervals at these vertices: Let $C_i^h = C_i \cup \bigcup_{n_i} [0,1]$ where the j^{th} interval is attached at $t = 0$ to the j^{th} vertex $v_{i,j} \in C_i$. We now define the space of hairy loops as follows. We let $(LM_{\mathcal{E}})^p_\Gamma = \{\theta \in Map(\cup_{i=1}^p C_i^h, M) \, : \, \theta_{|C_i} : C_i \cong S^1 \to M \text{ lies in } LM_{\mathcal{E}_i}\}$. We have an inclusion $\iota : \prod_{i=1}^{p} LM_{\mathcal{E}_i} \hookrightarrow (LM_{\mathcal{E}})^p_\Gamma$ which are maps that are defined to be constant on the intervals ("hairs"). Clearly this map is a homotopy equivalence.
Notice that another way to describe the space hairy loops is as follows:

$$(LM_{\mathcal{E}})^p_\Gamma = \{(\gamma, \alpha) \in \prod_{i=1}^{p} LM_{\mathcal{E}_i} \times (M^{v(\Gamma)})^I \, : \, ev_\Gamma(\gamma) = \alpha(0)\}$$

where X^I denotes the space of paths $\alpha : I = [0,1] \to X$. This implies that there is a Serre fibration,

$$\begin{aligned} \tilde{ev}_\Gamma : (LM_{\mathcal{E}})^p_\Gamma &\longrightarrow M^{v(\Gamma)} \\ (\gamma, \alpha) &\longrightarrow \alpha(1) \end{aligned} \tag{9}$$

Indeed this fibration has the structure of a locally trivial fiber bundle. (See [18] for descriptions of local trivializations.)
Let P_Γ be the pullback (restriction) of the bundle $\tilde{ev}_\Gamma : (LM_{\mathcal{E}})^p_\Gamma \longrightarrow M^{v(\Gamma)}$ to the image of the embedding, $\Delta_\Gamma(M^{\sigma(\Gamma)}) \subset M^{v(\Gamma)}$

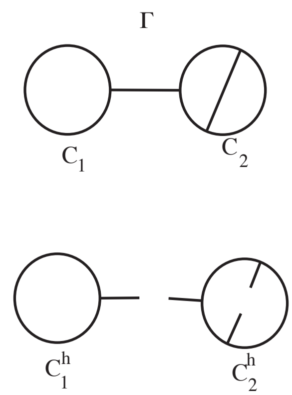

Figure 5: The hairy incoming circles of a chord diagram Γ

$$\begin{array}{ccc}
P_\Gamma & \xrightarrow{\hookrightarrow} & (LM_{\mathcal{E}})^p_\Gamma \\
{\scriptstyle \tilde{ev}}\downarrow & & \downarrow{\scriptstyle \tilde{ev}} \\
M^{\sigma(\Gamma)} & \xrightarrow[\Delta_\Gamma]{\hookrightarrow} & M^{v(\Gamma)}
\end{array} \tag{10}$$

Now by definition, the P_Γ is defined to be the space of pairs $((\gamma, \alpha) \in (LM_{\mathcal{E}})^p_\Gamma$ such that $\alpha(1) \in \Delta_\Gamma(M^{\sigma(\Gamma)})$. This space can be described alternatively as follows.
Let $\tilde{\Gamma}$ be the graph constructed from the union of the hairy incoming circles,

$$\tilde{\Gamma} = \bigcup_{i=1}^{p} C_i^h / \sim$$

where we make the following identifications: We identify the endpoint of the "hair" (i.e $t = 1$ in the interval) emanating from vertex v_1 with the endpoint of the hair emanating from vertex v_2 if and only if v_1 and v_2 are vertices of the same ghost subtree in Γ. That is, they are identified if and only if these vertices are in the same equivalence relation defined in (8) above. We initially put a vertex at each of these identification points, but then remove those new vertices that are only bivalent.
Notice then that P_Γ consists of maps $\theta : \tilde{\Gamma} \to M$ whose restriction to the i^{th} incoming circles, C_i lies in in $LM_{\mathcal{E}_i}$, $i = 1, \cdots, p$. Notice furthermore, that there is a natural map of graphs

$$p : \Gamma \to \tilde{\Gamma}$$

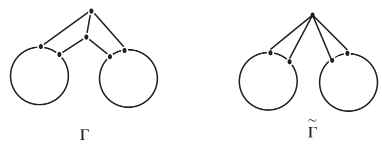

Figure 6: The chord diagrams Γ and $\tilde{\Gamma}$

that is defined by collapsing various ghost trees in Γ to the new chord vertices of $\tilde{\Gamma}$ defined above. In particular, $p : \Gamma \to \tilde{\Gamma}$ is a homotopy equivalence. Moreover, the cyclic ordering of the half edges emanating at the vertices in Γ define a cyclic orderings of the half edges emanating at the vertices of $\tilde{\Gamma}$, and so $\tilde{\Gamma}$ has the structure of a fat graph, and indeed a Sullivan chord diagram with the same marked incoming circles as Γ.

Let $Map_{\mathcal{E}}(\Gamma, M)$ denote the space of maps $\beta : \Gamma \to M$ whose restriction to the i^{th} incoming circle lies in $LM_{\mathcal{E}_i}$. The map $p : \Gamma \to \tilde{\Gamma}$ defines a homotopy equivalence $P_\Gamma \simeq Map_{\mathcal{E}}(G, M)$. But this latter space is homeomorphic to $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$. This can be seen as follows. Since any map $\theta : \Gamma \to M$ whose restrictions to the incoming circles are loops in $\theta_i \in LM_{\mathcal{E}_i}$, each of these loops extends in a unique way to a map of a half cylinder $\bar{\theta}_i : (-\infty, 0] \times S^1 \to M$ satisfying the gradient flow equation. (6) Moreover, since $\mathcal{E}$ satisfies the Palais-Smale criterion and is bounded below, the restrictions of θ to the outgoing boundary cycles, $\theta_j : S^1 \to M$ also extend to a gradient flow cylinder, $\bar{\theta}_j : [0, +\infty) \times S^1 \to M$. These gradient flow cylinders patch together to give an element in $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$. Thus $P_\Gamma \simeq \mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$, which proves the lemma. $\square$

We now complete the proof of Theorem 1. Since $\mathcal{E} : LM \to \mathbb{R}$ is Palais-Smale and bounded below, the inclusion $LM_{\mathcal{E}} \hookrightarrow LM$ is a homotopy equivalence. Thus homotopy pullback of $ev_\Gamma : \prod_{i=1}^p LM_{\mathcal{E}_i} \to M^{v(\Gamma)}$ along the diagonal embedding $\Delta_\Gamma : M^{\sigma(\Gamma)} \hookrightarrow M^{v(\Gamma)}$, is homotopy equivalent to the pullback of the fibration $ev_\Gamma : LM^p \to M^{v(\Gamma)}$ along the diagonal Δ_G. Now as described in [9], this pullback is the mapping space, $Map(r(\Gamma), M)$, where $r(\Gamma)$ is the "reduced" chord diagram obtained from Γ

by collapsing each chord edge to a point. By the lemma, we then have a homotopy equivalence, $\mathcal{M}^{\mathcal{E}}_\Gamma(LM) \simeq Map(r(\Gamma), M)$. Now since the collapse map $\Gamma \to r(\Gamma)$ is a homotopy equivalence, we have an equivalence of mapping spaces, $Map(r(\Gamma), M) \simeq Map(\Gamma, M)$. But this last mapping space is homotopy equivalent to $Map(\Sigma_\Gamma, M)$, since the surface Σ_Γ retracts onto the graph Γ. This completes the proof of the theorem. $\square$

Notice that this argument yields a commutative diagram,

$$\begin{array}{ccc} P_\Gamma & \xrightarrow{\hookrightarrow} & \prod_{i=1}^p LM_{\mathcal{E}_i} \\ \downarrow & & \downarrow \\ Map(r(\Gamma), M) & \xrightarrow{\hookrightarrow} & (LM)^p \end{array}$$

where the two vertical maps are homotopy equivalences. In particular $P_\Gamma \hookrightarrow \prod_{i=1}^p LM_{\mathcal{E}_i}$ and $Map(r(\Gamma), M) \hookrightarrow (LM)^p$ are both topological embeddings with open neighborhoods given by the inverse image of a tubular neighborhood $\eta(\Delta_\Gamma)$ of the embedding $\Delta_\Gamma : M^{\sigma(\Gamma)} \hookrightarrow> M^{v(\Gamma)}$ of compact manifolds. Even though P_Γ is not smooth, we think of these neighborhoods as "tubular neighborhoods", since they are homeomorphic to the total spaces of the pullbacks of the normal bundle, $\nu(\Delta_\Gamma) \to M^{\sigma(\Gamma)}$. Therefore we have Thom collapse maps,

$$\begin{aligned} \tau : \prod_{i=1}^p LM_{\mathcal{E}_i} &\xrightarrow{project} \prod_{i=1}^p LM_{\mathcal{E}_i} / \left((\prod_{i=1}^p LM_{\mathcal{E}_i}) - \tilde{ev}^{-1}(\eta(\Delta_\Gamma)) \right) \cong (P_\Gamma)^{(\tilde{ev})^*\nu(\Delta_\Gamma)} \quad \text{and} \\ \tau : (LM)^p &\xrightarrow{project} (LM)^p / \left((LM)^p - ev^{-1}(\eta(\Delta_\Gamma)) \right) \cong Map(r(\Gamma), M)^{(ev)^*\nu(\Delta_\Gamma)} \end{aligned} \tag{11}$$

where the targets of these maps are the Thom spaces. Moreover these maps are compatible in the sense that the following diagram commutes:

$$\begin{array}{ccc} \prod_{i=1}^p LM_{\mathcal{E}_i} & \xrightarrow{\tau} & (P_\Gamma)^{(\tilde{ev})^*\nu(\Delta_\Gamma)} \\ \downarrow & & \downarrow \\ (LM)^p & \xrightarrow{\tau} & Map(r(\Gamma), M)^{(ev)^*\nu(\Delta_\Gamma)} \end{array}$$

Notice that in this diagram, we again have that the vertical maps are homotopy equivalences.
The bottom horizontal map, together with the Thom isomorphism, was what defined the umkehr map on the chain level in [9]

$$(\rho_{in})_! : C_*(LM)^{\otimes p} \to C_{*+\chi(\Gamma)n}(Map(r(\Gamma), M).$$

This is then compatible, up to chain homotopy, with the Morse theoretic umkehr map,

$$(\rho_{in}^{morse})_! : \bigotimes_{i=1}^p C_*^{\mathcal{E}_i}(LM) \to C_{*+\chi(\Gamma)n}(\mathcal{M}^{\mathcal{E}}_\Gamma(LM))$$

defined to be the composition defined to be the composition,

$$(\rho_{in}^{morse})_! : \bigotimes_{i=1}^p C_*^{\mathcal{E}_i}(LM) \xrightarrow[\cong]{h_*} C_*(\prod_{i=1}^p LM_{\mathcal{E}_i}) \xrightarrow{(\tau_\Gamma)_*} C_*(P_\Gamma)^{\nu(\Delta_\Gamma)}) \cong C_*(\mathcal{M}_\Gamma^{\mathcal{E}}(LM)^{\tilde{\nu}(\Gamma)}) \xrightarrow[\cong]{\cap u_\Gamma} C_{*+\chi(\Gamma)n}(\mathcal{M}_\Gamma^{\mathcal{E}}(LM)) \tag{12}$$

where the first map h_* identifies the Morse chain complex of the energy functional $\mathcal{E}_i$ with the cellular complex of $LM_{\mathcal{E}_i}$ which then sends it in a canonical way to the singular chain complex $C_*(LM_{\mathcal{E}_i})$. In this diagram the symbol " $\cong$ " denotes chain homotopy equivalence.

We can then define the following Morse-string topology operation on the chain level, analogous to the construction in ([9]).

$$\mu_\Gamma : \bigotimes_{i=1}^p C_*^{\mathcal{E}_i}(LM) \xrightarrow{(\rho_{in}^{morse})_!} C_*(\mathcal{M}_\Gamma^{\mathcal{E}}(LM)^{\tilde{\nu}(\Gamma)}) \xrightarrow{(\rho_{out})_*} C_*(\prod_{j=1}^q LM_{\mathcal{E}_j}). \tag{13}$$

Notice that the compatibility of the umkehr map defined in [9] with the Morse umkehr map implies the following diagram of homology groups commutes:

$$\begin{tikzcd}
H_*((LM^p) \arrow[r, "(\rho_{in}^{morse})_!"] \arrow[d, "="'] & H_{*+\chi(\Gamma)n}(\mathcal{M}_\Gamma^{\mathcal{E}}(LM)) \arrow[r, "(\rho_{out})_*"] \arrow[d] & H_{*+\chi(\Gamma)n}((LM)^q) \arrow[d, "="] \\
H_*((LM^p) \arrow[r, "(\rho_{in})_!"'] & H_{*+\chi(\Gamma)n}(Map(r(\Gamma), M) \arrow[r, "(\rho_{out})_*"'] & H_{*+\chi(\Gamma)n}((LM)^q)
\end{tikzcd}$$

In other words, we've proven the following theorem.

Theorem 3. *For a connected surface Σ of genus g, with p-incoming, and q-outgoing boundary circles, let*

$$\mu_{g,p+q}^{top} : H_*((LM^p) \to H_{*+\chi(\Sigma)n}((LM)^q)$$

be the string topology operation defined in [7] and [9]. Then this operation is equal to the Morse theoretic operation

$$\mu_{g,p+q}^{top} = \mu_\Gamma$$

for any connected Sullivan chord diagram Γ of topological type $(g, p+q)$.

1.1 Morse theoretic string topology operations coming from families of graphs

In [17], V. Godin described "higher" string topology operations that are indexed by the homology of the moduli spaces of bordered Riemann surfaces. In this subsection we indicate how the Morse theoretic approach to string topology described above can be adapted, using Godin's work, to yield these higher order operations. A key ingredient in Godin's work was the generalization of the notion of a Sullivan chord diagram to a more general type of fat graph, that she called "admissible", that had two main features: 1. The space of admissible, marked metric fat graphs are homotopy equivalent to moduli space, and 2. These types of graphs are sufficiently explicit so that they can be used to define the necessary umkehr maps for the definition of (higher) string topology operations. These graphs were defined as follows.

Definition 4. *An "admissible" marked fat graph is one with the property that for every oriented edge E that is part of an incoming boundary cycle, its conjugate $\bar{E}$ (i.e the same edge with the opposite orientation) is part of an outgoing* boundary cycle.

In [17] it was proved that the space of admissible, marked fat graphs of topological type $(g, p+q)$, $\mathcal{G}_{g,p+q}$ is homotopy equivalent to the moduli space of bordered surfaces, $\mathcal{M}_{g,p+q}$. Furthermore, If one lets $\mathcal{G}_{g,p+q}(LM)$ be the space of pairs,

$$\mathcal{G}_{g,p+q}(M) = \{(\Gamma, \phi) \, : \, \Gamma \in \mathcal{G}_{g,p+q}, \, \text{and} \, \phi : \Gamma \to M \, \text{is a continuous map}\}, \tag{14}$$

then $\mathcal{G}^{\mathcal{E}}_{g,p+q}(M)$ is homotopy equivalent to the space $\mathcal{M}_{g,p+q}(M)$ defined in the introduction. Furthermore the following correspondence diagram is homotopy equivalent to diagram 1 of the introduction, and extends diagram 7:

$$(LM)^q \xleftarrow{\rho_{out}} \mathcal{G}_{g,p+q}(M) \xrightarrow{\rho_{in}} \mathcal{G}_{g,p+q} \times (LM)^p. \tag{15}$$

In [17] Godin defined a generalized Pontrjagin-Thom map, which in turn defined an umkehr map

$$(\rho_{in})_! : H_*(\mathcal{G}_{g,p+q} \times (LM)^p) \to H_{*+(2-2g+p+q)n}(\mathcal{G}_{g,p+q}(M)).$$

The higher string topollogy operations were defined as the composition

$$\mu_{g,p+q} = (\rho_{out})_* \circ (\rho_{in})_! : H_*(\mathcal{M}_{g,p+q}) \otimes H_*(LM)^{\otimes p} \cong H_*(\mathcal{G}_{g,p+q} \times (LM)^p) \to H_*((LM)^q) \cong H_*(LM)^{\otimes q}.$$

Here we are taking homology with coefficients in an arbitrary field k, and the tensor products are taken over k.

A particular difficulty in generalizing the above Morse theoretic construction to this setting, is the technical problem that, unlike for a Sullivan chord diagram, for a general admissible graph Γ, the inclusion of the incoming circles,

$$\alpha^- : \coprod S^1 \to \Gamma$$

(4) is not an inclusion of a subcomplex (cofibration). This will mean that our proof of the analogue of Theorem 1 will not go through in this more general setting. We get around this by considering the following larger moduli space of gradient flow surfaces defined as follows.

Definition 5. *Let Γ be a marked admissible fat graph with Lagrangian labeling $\mathcal{E} = \mathcal{E}(\Gamma)$. Define the space of $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1$ to be the space of maps*

$$\phi : \Sigma_\Gamma \to M$$

so that the restrictions to the incoming cylinders $(-\infty,-1)\times S^1$ *and the outgoing cylinders,* $(+1,+\infty)\times S^1$ *satisfy the gradient flow equations determined by the Lagrangian labelling.*

Notice that the difference between the space $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)$ and $\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1$ is that for $\phi \in \mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1$ the gradient flow equations need only be satisfied on the cylinders $(-\infty,-1)\times S^1$ and $(+1,+\infty)\times S^1$ rather than on entire cylinders $(-\infty,0)\times S^1$ and $(0,+\infty)\times S^1$ respectively. This seemingly arbitrary distinction is important because the inclusion of each of the incoming circles $\{-1\}\times S^1 \hookrightarrow (-\infty,0)\times S^1 \hookrightarrow \Sigma_\Gamma$ are cofibrations for all admissible fat graphs Γ, but the inclusions of the incoming boundary circles, $\alpha_i^- : \{0\}\times S^1 \to \Gamma \hookrightarrow \Sigma_\Gamma$ may not be cofibrations (however they would be if Γ were a chord diagram).
With this technical distinction, we can now prove the following analogue of Theorem 1.

Theorem 4. *Let* Γ *be a marked admissible fat graph. Then the natural map from the space of gradient flow surfaces to the continuous mapping space,*

$$\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1 \to Map(\Sigma_\Gamma, M)$$

is a homotopy equivalence.

Proof. Since the inclusion

$$\coprod_{i=1}^{p}\{-1\}\times S^1 \hookrightarrow \coprod_{i=1}^{p}(-\infty,0)\times S^1 \xrightarrow{\alpha^-} \Sigma_\Gamma$$

is a cofibration, the induced adjoint restriction map

$$Map(\Sigma_\Gamma, M) \to \prod_{i=1}^{p} LM$$

is a fibration. One then sees that the following commutative square is a pullback square of fibrations,

$$\begin{array}{ccc} \mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1 & \longrightarrow & Map(\Sigma_\Gamma, M) \\ \downarrow & & \downarrow \\ \prod_{i=1}^{p} LM_{\mathcal{E}_i} & \longrightarrow & (LM)^p. \end{array}$$

Since, by the Palais-Smale condition, the bottom horizontal map is a homotopy equivalence, we may conclude that the top horizontal map is a homotopy equivalence. □

We now let $\mathcal{M}^{\mathcal{E}}_{g,p+q}(LM)_1$ be the space of pairs,

$$\mathcal{M}^{\mathcal{E}}_{g,p+q}(LM)_1 = \{(\Gamma,\phi) : \Gamma \in \mathcal{G}_{g,p+q}, \text{ and } \phi \in \mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1\}.$$

Then Theorem 4 implies that the fibration sequence

$$\mathcal{M}^{\mathcal{E}}_{\Gamma}(LM)_1 \to \mathcal{M}^{\mathcal{E}}_{g,p+q}(LM)_1 \to \mathcal{G}_{g,p+q}$$

is homotopic to the fibration sequence

$$Map(\Sigma_G, M) \to \mathcal{G}_{g,p+q}(M) \to \mathcal{G}_{g,p+q},$$

which, by Godin's result [17] is in turn homotopic to the fibration sequence,

$$Map(\Sigma_G, M) \to \mathcal{M}_{g,p+q}(M) \to \mathcal{M}_{g,p+q}.$$

We therefore have a commutative diagram

$$\begin{array}{ccccc}
\prod_{i=1}^{q} LM_{\mathcal{E}_j} & \xleftarrow{\rho_{out}} & \mathcal{M}^{\mathcal{E}}_{g,p+q}(LM)_1 & \xrightarrow{\rho_{in}} & \mathcal{G}_{g,p+q} \times \prod_{i=1}^{p} LM_{\mathcal{E}_i} \\
\simeq\big\downarrow & & \big\downarrow\simeq & & \big\downarrow \\
(LM)^q & \xleftarrow[\rho_{out}]{} & \mathcal{G}_{g,p+q}(M) & \xrightarrow[\rho_{in}]{} & \mathcal{G}_{g,p+q} \times (LM)^p
\end{array}$$

where the horizontal maps are homotopy equivalences. Using these equivalences and Godin's construction one obtains higher order operations,

$$\mu^{morse} : H_*(\mathcal{M}_{g,p+q}) \otimes \bigotimes_{i=1}^{p} H_*(LM_{\mathcal{E}_i}) \to H_*(\mathcal{M}^{\mathcal{E}}_{g,p+q}(LM)_1 \to \bigotimes_{i=1}^{p} H_*(LM_{\mathcal{E}_i}).$$

2 String operations by counting gradient flow lines

In this section we give a more analytical description of the Morse theoretic string topology operations, via the counting of zero dimensional moduli spaces of gradient trajectories. Such an operation was constructed in [5] corresponding to the figure 8 graph. Our comparison of these operations with the ones constructed in section 1, will imply that this figure 8 product on the Morse homology of LM indeed corresponds to the Chas-Sullivan loop product. Combining this with thei theorem of [5] giving a ring isomorphism between the Floer homology of the cotangent bundle $HF_*(T^*M)$ with $H_*(LM)$ implies that the pair of pants product in Floer homology corresponds to the Chas-Sullivan product in $H_*(LM)$.

We continue to consider a Lagrangian, energy functional, and metric satisfying the conditions described in section 1.

Let Γ be a Sullivan chord diagram of type $(g; p+q)$ and $r(\Gamma)$ the corresponding reduced chord diagram with parametrizations $\alpha^- : \coprod_q S^1 \longrightarrow r(\Gamma)$ for the incoming cycles and $\alpha^+ : \coprod_p S^1 \longrightarrow r(\Gamma)$ for the outgoing cycles, where we reparametrize $\alpha^\pm$ such that $S^1 = \mathbb{R}/\mathbb{Z}$ is the standard circle.

The space $Map\big(r(\Gamma), M\big)$ is endowed with the structure of a Hilbert manifold, the topology given by edgewise Sobolev $W^{1,2}$-maps. The parametrizations $\alpha^\pm$ induce embeddings

$$(LM)^q \overset{r_{out}}{\hookleftarrow} Map\big(r(\Gamma), M\big) \overset{r_{in}}{\hookrightarrow} (LM)^p$$

by $r_{in}(c) = c \circ \alpha^-$ and $r_{out}(c) = c \circ \alpha^+$.

Recall the following pullback square of fibre bundles:

$$
\begin{array}{ccc}
Map(r(\Gamma), M) & \xrightarrow{r_{in}} & (LM)^p \\
{\scriptstyle ev_\Gamma}\downarrow & & \downarrow{\scriptstyle ev_\Gamma} \\
M^{\sigma(\Gamma)} & \xrightarrow[\Delta_\Gamma]{} & M^{v(\Gamma)}
\end{array}
\tag{16}
$$

Since this is a pullback of smooth bundles, $Map(r(\Gamma), M)$ has the structure of a codimension $(-\chi(\Gamma) \cdot n)$-submanifold of $(LM)^p$ with coorientation induced by the embedding Δ_Γ, as M is assumed oriented. Recall that in the definition of the loop product, one uses the figure 8 graph for Γ, which is already a reduced chord diagram. Similarly, the "little cacti" diagrams (see [10], [13], [25]) are also reduced Sullivan chord diagrams. These are the diagrams used to describe the "BV"-structure in string topology.

For our purposes it is not necessary to describe the analogous structure of a cooriented embedding for r_{out}. However, we require that the following transversality conditions hold.

Transversality Condition 1. *For any collection of critical points $a_i \in \mathcal{P}(L_i)$, $i = 1, \dots, p$ the embedding $r_{in} \colon Map\big(r(\Gamma), M\big) \hookrightarrow (LM)^p$ is transverse to $W^u(a_1) \times \cdots \times W^u(a_p) \hookrightarrow LM \times \cdots \times LM$.*

Transversality Condition 2. *For any collection of critical points $a_{p+i} \in \mathcal{P}(L_{p+i})$, $i = 1, \dots, q$, the embedding $r_{out} \colon Map\big(r(\Gamma), M\big) \hookrightarrow (LM)^q$ is transverse to $W^s(a_{p+1}) \times \cdots \times W^s(a_{p+q}) \hookrightarrow LM \times \cdots \times LM$.*

The first transversality condition implies that the intersection

$$\mathcal{M}_{r(\Gamma)}(LM; a_1, \cdot, a_p) \,=\, r_{in}\big(Map(r(\Gamma), M)\big) \pitchfork W^u(a_1) \times \ldots \times W^u(a_p)$$

is an oriented, finite-dimensional submanifold of $(LM)^p$ of dimension $\sum_{i=1}^p Ind(a_i) + \chi(\Gamma) \cdot n$.

However, in order to construct the Morse-theoretical description of μ_Γ, we need a stronger condition than Transversality Condition 2. Condition 2 will be used later.

Transversality Condition 3. *For any collection of $p+q$ critical points $a_i \in \mathcal{P}(L_i)$, $i = 1, \dots, p+q$, the restriction*

$$r_{out|\mathcal{M}_{r(\Gamma)}} \colon \mathcal{M}_{r(\Gamma)}(LM; a_1, \cdots, a_p) \hookrightarrow (LM)^q$$

is transverse to $W^s(a_{p+1}) \times \cdots \times W^s(a_{p+q}) \hookrightarrow LM \times \cdots \times LM$.

We will discuss these tranversality conditions below. Assuming them for now, we have for $\vec{a} = (a_1, \cdots, a_p, a_{p+1}, \cdots, a_{p+q}) \in \prod_{i=1}^{p+q} \mathcal{P}(L_i)$ the following immediate result:

Proposition 5. *The space*

$$\mathcal{M}_{r(\Gamma)}(LM; \vec{a}) := r_{out}\big(\mathcal{M}_{r(\Gamma)}(LM; a_1, \cdots, a_p)\big) \pitchfork W^s(a_{p+1}) \times \cdots \times W^s(a_{p+q}) \,\subset\, (LM)^q$$

is a smooth orientable manifold of dimension

$$\dim \mathcal{M}_{r(\Gamma)}(LM; \vec{a}) \,=\, \sum_{i=1}^{p} Ind(a_i) - \sum_{j=p+1}^{p+q} Ind(a_j) + \chi(\Gamma) \cdot n$$

homeomorphic to

$$\left\{ \phi \in \mathcal{M}^{\mathcal{E}}_{r(\Gamma)}(LM) \,\middle|\, \lim_{t\to-\infty} \phi_i(t,\cdot) = a_i,\ i = 1,\cdots,p,\ \lim_{t\to+\infty} \phi_j(t,\cdot) = a_j,\ j = p+1,\cdots,p+q, \right\}.$$

An orientation of this manifold is induced by orientations of the tangent spaces of the unstable manifolds $T_{a_i}W^u(a_i)$ of the critical points $a_1,\cdots,a_{p+q}$.

In the case when $\vec{a}$ is a collection of critical points so that the dimension of the manifold $\mathcal{M}_{r(\Gamma)}(LM;\vec{a})$ is zero, then standard considerations imply that it is compact (see for example [5]). We may then define an operation on the Morse chain complex (3).

Definition 6. *Define the operation*

$$\nu_\Gamma : C^{\mathcal{E}}_*((LM^p) \to C^{\mathcal{E}}_{*+\chi(\Gamma)n}((LM)^q)$$

by

$$\nu_\Gamma([a_1]\otimes\cdots\otimes[a_p]) = \sum_{dim\,\mathcal{M}_{r(\Gamma)}(LM;a_1,\cdots a_p,a_{p+1},\cdots a_{p+q})=0} \#\mathcal{M}_{r(\Gamma)}(LM;a_1,\cdots a_p,a_{p+1},\cdots a_{p+q})\,[a_{p+1}]\otimes\cdots\otimes[a_{p+q}]$$

Here $\#\mathcal{M}_{r(\Gamma)}(LM;\vec{a})$ is the oriented count of the number of points in this zero dimensional compact manifold.

Concerning the previous transversality conditions, they are fulfilled for generic choices of Riemannian metrics on the Hilbert manifold LM, provided that the Lagrangians L_i have been chosen suitably. Namely, the solution sets of gradient flow trajectories corresponding to the intersections in question should not contain constant solutions. For example, in Condition 1, the p-tuple $(a_1,\ldots,a_p) \in \mathrm{Crit}\,\mathcal{E}_{L_1}\times\ldots\times\mathrm{Crit}\,\mathcal{E}_{L_p}$ should not be contained in $Map\big(r(\Gamma),M\big)$. Moreover, the proof of the generic existence of such metrics uses the theorem of Sard-Smale. All of the above intersection problems are Fredholm problems, but in general we need to consider not only Fredholm indices up to 1, but also higher, as e.g. in Transversality Condition 1. This is the place where we have to assume accordingly high differentiability of our energy functions $\mathcal{E}_{L_i}$, $i = 1,\ldots,p+q$.
Assume for example that $\mathcal{B}$ is a sufficiently defined separable Banach manifold consisting of admissible Riemannian metrics on LM. Then we consider the infinite-dimensional Banach manifold

$$\mathcal{W}^u(a_1,\ldots,a_p) = \left\{ (g,c_1,\ldots,c_p) \in \mathcal{B}\times LM^p \,|\, c_i \in W^u_g(a_i),\ i = 1,\ldots,p \right\}$$

where $W^u_g(a_i)$ is the unstable manifold for the negative gradient flow of $\mathcal{E}_{L_i}$ determined by the Riemannian metric g. It not hard to show that for a sufficiently rich set $\mathcal{B}$ of variations of the Riemannian metric on LM, each projection $\mathcal{W}^u(a_1,\ldots,a_p) - \{(a_1,\ldots,a_p)\} \to LM$, $(g,\vec{c}) \mapsto c_i$ is a submersion onto its image away from the critical point. This, together with the assumed smoothness is the main ingredient in the application of the Sard-Smale theorem in order to prove that the Transversality Condition 1 is generically fulfilled. Similarly, we also obtain the other transversality conditions generically satisfied. For more details on this transversality analysis we refer to [3, 4]

The following is now the main theorem of this section.

Theorem 6. *Under the above assumptions on the Lagrangians and the metric, and assuming transversality conditions 1 and 3, the operation ν_Γ is a chain map, and in homology it gives the string topology operation*

$$\nu_\Gamma = \mu_\Gamma : H_*((LM)^p) \to H_{*+\chi(\Gamma)n}((LM)^q).$$

For the proof, we factorize ν_Γ in close analogy to μ_Γ above, into

$$\begin{aligned} &\nu_\Gamma \;=\; (r_{out})_* \circ (r_{in})_!, \quad \text{where} \\ &(r_{in})_!\colon H_*^{\mathcal{E}}\big((LM)^p\big) \longrightarrow H^{\mathcal{F}}_{*+\chi(\Gamma)\cdot n}\big(Map(r(\Gamma),M)\big), \quad \text{and} \\ &(r_{out})_*\colon H_*^{\mathcal{F}}\big(Map(r(\Gamma),M)\big) \longrightarrow H_*^{\mathcal{E}}\big((LM)^q\big) \end{aligned} \tag{17}$$

are naturally isomorphic to $(\rho_{in})_!$ and $(\rho_{out})_*$ when we compare Morse homology with standard homology.

Here we consider an auxilary smooth Morse function $\mathcal{F}$ on the Hilbert manifold $Map(r(\Gamma),M)$. That is, $\mathcal{F}$ satisfies the Palais-Smale property with a complete negative gradient flow for a complete Riemannian metric, it is bounded below, and all critical points $b \in \mathrm{Crit}\,\mathcal{F}$ are non-degenerate and of finite Morse index. For simplicity, we also assume each set of critical points of Morse index k, $k \in \mathbb{N}$, to be finite. Choosing a generic Riemannian metric on $Map(r(\Gamma),M)$ satisfying Morse-Smale transversality with respect to $\mathcal{F}$ for relative Morse index up to 2, the Morse complex $\big(C_*^{\mathcal{F}}(Map(r(\Gamma),M)),\partial\big)$ is well-defined and its Morse homology naturally isomorphic to standard homology $H_*\big(Map(r(\Gamma),M)\big)$, see e.g. [1, 5, 21].

Essentially, up to a small perturbation, we can take for $\mathcal{F}$ the restricted energy functional $r_{in}^*\mathcal{E}^{\otimes p} = \mathcal{E}^{\otimes p}|r_{in}(Map(r(\Gamma),M))$, for $\mathcal{E}^{\otimes p}\colon (LM)^p \to \mathbb{R}$ given by $\mathcal{E}^{\otimes p}(c_1,\ldots,c_p) = \mathcal{E}_{L_1}(c_1)+\ldots+\mathcal{E}_{L_p}(c_p)$.

We now focus on the embedding $r_{out}\colon Map(r(\Gamma),M) \hookrightarrow (LM)^q$. In addition to Transversality Condition 2, we assume

Generic Condition 4. *r_{out} maps no critical point of $\mathcal{F}$ to a critical point of $\mathcal{E}$.*

Given $b \in \mathrm{Crit}\,\mathcal{F}$ and $a_{p+i} \in \mathcal{P}(L_{p+i})$, $i = 1,\ldots,q$, we define

$$\mathcal{M}^{out}_{r(\Gamma)}(b, a_{p+1},\cdots,a_{p+q}) \;=\; r_{out}\big(W^u_{\mathcal{F}}(b)\big) \cap \big(W^s_{\mathcal{E}}(a_{p+1}) \times \cdots \times W^s_{\mathcal{E}}(a_{p+q})\big),$$

where $W^u_{\mathcal{F}}(b)$ is the unstable manifold for the negative gradient flow of $\mathcal{F}$. Condition 4 allows us to find a generic metric on $Map(r(\Gamma),M)$ such that $W^u_{\mathcal{F}}(b)$ intersects the submanifold $r_{out}^{-1}\big(W^s(a_{p+1})\times \cdots \times W^s(a_{p+q})\big)$ transversely. Altogether we obtain $\mathcal{M}^{out}_{r(\Gamma)}(b,a_{p+1},\cdots,a_{p+q})$ as a manifold of dimension $ind(b) - \sum_{j=1}^q ind(a_{p+j})$ with orientation induced by the orientations of the unstable manifolds $W^u(b)$, $W^u(a_{p+1}),\cdots,W^u(a_{p+q})$.

We define

$$\begin{aligned} &(r_{out})_*\colon C_*^{\mathcal{F}}\big(Map(r(\Gamma),M)\big) \longrightarrow C_*^{\mathcal{E}}\big((LM)^q\big), \\ &(r_{out})_*([b]) \;=\; \sum_{\dim \mathcal{M}^{out}_{r(\Gamma)}(b,a_{p+1},\cdots,a_{p+q})=0} \#\mathcal{M}^{out}_{r(\Gamma)}(b,a_{p+1},\cdots,a_{p+q})\,[a_{p+1}]\otimes\cdots\otimes[a_{p+q}] \end{aligned} \tag{18}$$

and have the following result.

Proposition 7. $(r_{out})_*\colon C_*^{\mathcal{F}}\big(Map(r(\Gamma),M)\big) \longrightarrow C_*^{\mathcal{E}}\big((LM)^q\big)$ *is a chain map and the natural isomorphism to standard homology intertwines* $(r_{out})_*$ *with* $(\rho_{out})_*$*. That is, the following diagram commutes:*

$$\begin{array}{ccc} H_*^{\mathcal{F}}\big(Map(r(\Gamma),M)\big) & \xrightarrow{(r_{out})_*} & H_*^{\mathcal{E}}\big((LM)^q\big) \\ \big\downarrow\cong & & \big\downarrow\cong \\ H_*\big(Map(r(\Gamma),M)\big) & \xrightarrow{(\rho_{out})_*} & H_*\big((LM)^q\big) \end{array}$$

commutes.

This construction describes the functoriality for Morse homology. More details can be found in [5] and [21].

We now construct the Morse-theoretical version $(r_{in})_!$ of the umkehr map. Recall that

$$r_{in}\colon Map(r(\Gamma),M) \hookrightarrow (LM)^p$$

is a proper embedding of Hilbert manifolds, with finite codimension and cooriented. In addition to Transversality Condition 1, we assume

Generic Condition 5. $r_{in}(\mathrm{Crit}\,\mathcal{F})$ *is disjoint from* $\prod_{i=1}^p \mathcal{P}(L_i)$ *within* $(LM)^p$*.*

Hence, we find a generic metric on $Map(r(\Gamma),M)$ such that for any $b \in \mathrm{Crit}\,\mathcal{F}$ its stable manifold $W^s_{\mathcal{F}}(b)$ is transverse to $\big(W^u_{\mathcal{E}}(a_1)\times\cdots\times W^u_{\mathcal{E}}(a_p)\big)\cap r_{in}\big(Map(r(\Gamma),M\big)$ for all $a_i \in \mathcal{P}(L_i)$, $i=1,\ldots,p$. We obtain the manifold

$$\mathcal{M}^{in}_{r(\Gamma)}(a_1,\cdots,a_p,b) \;=\; \big(W^u_{\mathcal{E}}(a_1)\times\cdots\times W^u_{\mathcal{E}}(a_p)\big)\cap W^s_{\mathcal{F}}(b)$$

of dimension $\sum_{i=1}^p ind(a_i) - ind(b) - \chi(\Gamma)\cdot n$ and induced orientation from the orientations of $W^u_{\mathcal{E}}(a_1),\cdots,W^u_{\mathcal{E}}(a_p)$, coorientation of $W^s_{\mathcal{F}}(b)$ within $Map(r(\Gamma),M)$ by the orientation of $W^u_{\mathcal{F}}(b)$, and the coorientation of $r_{in}\big(Map(r(\Gamma),M)\big)$ within $(LM)^p$.
We define

$$\begin{aligned} &(r_{in})_!\colon C_*^{\mathcal{E}}\big((LM)^p\big) \longrightarrow C^{\mathcal{F}}_{*+\chi(\Gamma)\cdot n}\big(Map(r(\Gamma),M)\big), \\ &(r_{in})_!\big([a_1]\otimes\cdots\otimes[a_p]\big) \;=\; \sum_{\dim \mathcal{M}^{in}_{r(\Gamma)}(a_1,\cdots,a_p,b)=0} \#\mathcal{M}^{in}_{r(\Gamma)}(a_1,\cdots,a_p,b)\,[b]\,. \end{aligned} \tag{19}$$

By the usual arguments of Morse homology, this is a chain map. Note that the compactness of $\mathcal{M}^{in}_{r(\Gamma)}$ also requires the properness of the embedding r_{in}, such that $r_{in}^{-1}\big(W^u(a_1)\times\cdots\times W^u(a_p)\big)$ is relatively compact in $Map(r(\Gamma),M)$.
We have

Proposition 8. *The chain map* $(r_{in})_!\colon C_*^{\mathcal{E}}\big((LM)^p\big) \longrightarrow C^{\mathcal{F}}_{*+\chi(\Gamma)\cdot n}\big(Map(r(\Gamma),M)\big)$ *induces an umkehr map on Morse homology compatible with the umkehr map* $(\rho_{in})_!$ *under the natural isomorphism, i.e.*

$$\begin{array}{ccc} H_*^{\mathcal{E}}\big((LM)^p\big) & \xrightarrow{(r_{in})_!} & H^{\mathcal{F}}_{*+\chi(\Gamma)\cdot n}\big(Map(r(\Gamma),M)\big) \\ \big\downarrow\cong & & \big\downarrow\cong \\ H_*\big((LM)^p\big) & \xrightarrow{(\rho_{in})_!} & H_{*+\chi(\Gamma)\cdot n}\big(Map(r(\Gamma),M)\big) \end{array}$$

commutes.

Before we give a proof of this by a Morse-theoretical description of the Thom isomorphism, we conclude the proof of Theorem 6. Via a standard gluing argument, $(r_{out})_*\circ(r_{in})_!$ on Morse chain level is equal to counting

$$\begin{aligned} &\Theta \in r_{in}^{-1}\big(W^u(a_1)\times\cdots\times W^u(a_p)\big)\;\subset\; Map(r(\Gamma),M), \quad \text{such that} \\ &\phi^R_{\mathcal{F}}(\Theta)\in r_{out}^{-1}\big(W^s(a_{p+1})\times\cdots\times W^s(a_{p+q})\big), \end{aligned}$$

for $R>0$ fixed and sufficiently large, where $t\mapsto\phi^t_{\mathcal{F}}(\Theta)$ is a flow line for the negative gradient flow of $\mathcal{F}$ on $Map(r(\Gamma),M)$.

By homotoping R to 0 we establish a cobordism to the previous solution space $\mathcal{M}_{r(\Gamma)}(LM;\vec{a})$ which gives rise to a chain homotopy operator, proving

$$\nu_\Gamma\;\simeq\;(r_{out})_*\circ(r_{in})_!$$

on chain level. Hence, via the natural functor to standard homology, we have on homology level

$$\nu_\Gamma\;=\;(r_{out})_*\circ(r_{in})_!\;=\;(\rho_{out})_*\circ(\rho_{in})_!\;=\;\mu_\Gamma,$$

proving Theorem 6.

3 Thom-Isomorphism, Euler-Class and the Umkehr Map via Morse Homology

We will now give a Morse-theoretical construction of the umkehr map which is based on a Morse-theoretical construction of the Thom isomorphism. We need to give this construction in the infinite-dimensional setting, at least sufficient for the case of the loop space LM with its $W^{1,2}$-Hilbert manifold structure and the energy functional $\mathcal{E}$. In fact, Morse homology can be defined for a much larger class of infinite-dimensional settings. For more details on the difference between the finite and the infinite dimensional case and for the more general setting of the latter we refer to [1, 2].

3.1 Preliminaries

Let X be a smooth paracompact Hilbert manifold with a complete Riemannian metric, and let $f\colon X \to \mathbb{R}$ be a C^2 Morse function satisfying the Palais-Smale condition. Moreover, for our purpose ($X = LM,\ f = \mathcal{E}$) we assume that f is bounded below and all critical points are of finite Morse index.

Pick a generic Riemannian metric g, s.t. (f,g) is a Morse-Smale-pair [1] with a complete negative gradient flow. For $x, y \in \operatorname{Crit} f$ with $i(x) - i(y) = 1$ set

$$\langle x, y\rangle \;=\; \#_{\mathsf{alg}}\big\{\, \gamma\colon \mathbb{R} \to X \mid \dot\gamma + \nabla f(\gamma) = 0,\ \gamma(-\infty) = x,\ \gamma(\infty) = y \,\big\}\big/\mathbb{R}\,.$$

Here, $\#_{\mathsf{alg}}$ refers to counting with orientations obtained from the concept of coherent orientations, that is arbitary orientations of all unstable manifolds and induced coorientations of all stable ones. We obtain the Morse homology of f from

$$\begin{aligned} &C_*(f) \;= \mathbb{Z} \otimes \operatorname{Crit}_* f, \qquad \text{(finitely generated)}\\ &\partial\colon C_*(f) \to C_{*-1}(f), \quad \partial x = \sum_{i(y)=i(x)-1} \langle x, y\rangle y, \end{aligned}$$

and the Morse cohomology from

$$\begin{aligned} &C^*(f) = \mathbb{Z}^{\operatorname{Crit}_* f}, \qquad \text{(not necessarily finitely generated !)}\\ &\delta\colon C^*(f) \to C^{*+1}(f), \quad (\delta\phi)(x) = \sum_{i(y)=i(x)-1} \langle x, y\rangle \phi(y),\\ &\text{i.e.} \quad \big(C^*(f), \delta\big) \;=\; \operatorname{Hom}\big(C_*(f), \partial\big)\,. \end{aligned}$$

By identifying $\delta_x \in \mathbb{Z}^{\operatorname{Crit} f}$ with $x \in \operatorname{Crit} f$ we see that δ is alternatively defined by counting positive gradient flow lines for f.

Note that from the lower boundedness of f, the completeness of the negative gradient flow and the Palais-Smale property, we obtain that $H_0(f) \cong \mathbb{Z} \cong H^0(f)$ if X is connected. Hence we have a generator $1 \in H_0(f)$ represented by a single critical point of index 0 and $1 \in H^0(f)$ represented by $\phi \in \mathbb{Z}^{\operatorname{Crit}_0 f}$, $\phi \equiv 1$.

3.2 Relative Cohomology

We now recall the Morse-theoretical definition of relative homology and cohomology, see e.g. [21]. Let $A \subset X$ be an open submanifold with smooth boundary ∂A, and we assume that the above Morse function f on X is in addition such that $\nabla f \pitchfork \partial A$, and the gradient ∇f is pointing out of A. This implies that

$$\big(C_*(f_{|A}), \partial\big) \quad \text{is a subcomplex of} \quad \big(C_*(f), \partial\big),$$

[1] Transversality is sufficient up to index difference 2. See 0.5 in [2] for the precise details for genericity here.

and we have the exact sequence of chain complexes

$$0 \to C_*(f_{|A}) \xrightarrow{i} C_*(f) \xrightarrow{j} C_*\big(f; X, A\big) := C_*(f)/C_*(f_{|A}) \to 0,$$

inducing the long exact sequence of homology.
For Morse cohomology we have dually

$$0 \to C^*(f; X, A) \xrightarrow{j^*} C^*(f; X) \xrightarrow{i^*} C^*(f_{|A}; A) \to 0\,.$$

Namely, we have

$$i^*\Big(\sum_{x \in \mathrm{Crit}\, f} a_x x\Big) \;=\; \sum_{x \in \mathrm{Crit}\, f \cap A} a_x x,$$

since $x \in \mathrm{Crit}\, f \cap (X \setminus A)$ implies $\delta x \in \mathbb{Z}^{\mathrm{Crit}\, f \cap (X \setminus A)}$, hence i^* is a cochain complex morphism. Also, we set

$$C^*(f; X, A) \;=\; \mathbb{Z}^{\mathrm{Crit}\, f \cap (X \setminus A)}$$

which by the same argument of ∇f pointing outwards along ∂A turns j^* into a sub-cochain-complex inclusion. Note that the here obvious excision principle is used in the notation

$$C^*(f; X, A) \;=\; C^*(f; X \setminus A, \partial A)\,.$$

Example

Let us consider the following simple illustrating example, which describes the main feature used for the following Thom isomorphism. Let $q\colon \mathbb{R}^n \to \mathbb{R}$ be the standard positive quadratic form with its unique critical point of index 0 in the origin. Choose another coercive Morse function $\tilde q\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\tilde q(x) = \begin{cases} -q(x), & |x| \le 1, \\ q(x), & |x| \ge 2\,. \end{cases}$$

Obviously we have for the unit disk D_1 and its boundary sphere S_1

$$H^*(-q; D_1, S_1) \;=\; H^*(\tilde q; D_1, S_1)\,.$$

The above long exact cohomology sequence and obvious identifications and homotopy invariance give

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^{n-1}(\tilde q; \mathbb{R}^n \setminus D_1) & \xrightarrow{\delta^*} & H^n(\tilde q; D_1, S_1) & \xrightarrow{j^*} & H^n(\tilde q; \mathbb{R}^n) \\
 & & & & \Big\| & & \cong\big\downarrow \\
 & & & & H^n(-q; D_1, S_1) & & H^n(q; \mathbb{R}^n) \\
 & & & & \cong\big\downarrow & & \Big\| \\
 & & & & \mathbb{Z} & & \{0\}
\end{array}$$

This leads to $H^{n-1}(\tilde q; \mathbb{R}^n \setminus D_1) \cong \mathbb{Z}$. In fact $\tilde q$ encodes the Morse cohomology of the $(n-1)$- sphere.

Functoriality for proper embeddings

Consider now a proper embedding of finite and positive codimension of a submanifold $e\colon P \hookrightarrow X$, and let $k\colon P \to \mathbb{R}$ be a Morse function on P of the same type as f, i.e. satisfying the Palais-Smale property, bounded below and all critical points of finite index. We define $e_*\colon C_*(k) \to C_*(f)$ as follows. Consider $p \in \operatorname{Crit} k \subset P$ and $x \in \operatorname{Crit} f \subset X$. We require generic metrics on P and X such that $W^u(p;k)$ as a submanifold of X and $W^s(x;f)$ intersect transversely in X. This can always generically be achieved for sufficiently bounded relative index if $\operatorname{Crit} f \cap P = \emptyset$, which sometimes turns out to be a bit too restrictive. In such a transverse situation we define

$$n(p,x) \;=\; \#_{\mathsf{alg}}\big(W^u(p,k) \pitchfork_X W^s(x;f)\big)$$

if $i(p) = i(x)$. Note that, for this relative Morse index, the intersection is 0-dimensional. We then define

$$e_*(p) = \sum_{i(x)=i(p)} n(p,x)x\,,$$

and we easily see that $\partial_f \circ e_* = e_* \circ \partial_k$. Analogously, we define the pull-back homomorphism

$$e^*\colon C^*(f;X) \to C^*(k;P)\,.$$

Consider now the special case where $\dim P \;=\; \operatorname{ind}(x) \;=\; l$ and $m \;\in\; \operatorname{Crit} f \cap P$, $\operatorname{ind}(p) \;=\; l$ for $p \in \operatorname{Crit} k$. If $T_x W^s(x,f) \;=\; \operatorname{Eig}^+ D^2 f(x)$ is transverse to P, then the above transversality for $n(p,x)$ is automatically satisfied.

Construction of Thom isomorphism

Let us now consider a smooth vector bundle $\pi\colon E \to X$ of finite rank r endowed with an arbitrary Riemannian metric. Let $q\colon E \to [0,\infty)$ be the associated positive quadratic form, and consider the disk and sphere bundle

$$\begin{aligned} D(E) \;&=\; \big\{\, (x,v) \in E_x \,|\, q(v) \le 1 \,\big\}, \\ S(E) \;&=\; \big\{\, (x,v) \in E_x \,|\, q(v) = 1 \,\big\} \;=\; \partial D(E)\,. \end{aligned}$$

Thus $f_{-q} := \pi^* f - q$ is an admissible relative Morse function for the pair $(E, E \setminus D(E))$. If we extend $f_{-q}|D(E)$ to $\tilde f_q$ outside of $D(E)$ such that $\tilde f_q$ is a Morse function and

$$\tilde f_q(x,v) = f(x) + q(v) \quad \text{for} \;\; q(v) \ge 2,$$

we have the canonical identification as in the above example

$$H^*(f_{-q}; D(E), S(E)) \;=\; H^*(\tilde f_q; D(E), S(E)) \;=\; H^*(\tilde f_q; E, E \setminus D(E))$$

and the exact sequence

$$\ldots \to H^{*-1}(\tilde f_q; E \setminus D(E)) \;\to\; H^*(\tilde f_q; D(E), S(E)) \xrightarrow{j^*} H^*(\tilde f_q; E) \;\cong\; H^*(f_q; E), \tag{20}$$

with $f_q = \pi^* f + q$. Moreover, there is the canonical isomorphism

$$\pi^* \colon H^*(f;X) \xrightarrow{\cong} H^*(f_q;E) \tag{21}$$

induced from

$$\mathrm{Crit}_* f \;=\; \mathrm{Crit}_* f_q$$

by identifying X with the zero section of E.

Proposition 3.1. *If E is an orientable bundle, then $\mathrm{Crit}_* f \;=\; \mathrm{Crit}_{*+r} f_{-q}$ induces an isomorphism*

$$T^* \colon H^*(f;X) \xrightarrow{\cong} H^{*+r}(f_{-q}; D(E), S(E))$$

and the element $u = T^*(1) \in H^r(f_{-q})$ satisfies $\varphi_x^*(u) = u_o$ where $\varphi_x \colon (D^r, S^{r-1}) \hookrightarrow (D(E), S(E))$ is the fibre inclusion over any $x \in X$ and $u_o \in H^r(D^r, S^{r-1})$ is the generator compatible with the orientation of E. Moreover, the same identification of critical points induces the dual isomorphism $T_* \colon H_*(f_{-q}; D(E), S(E)) \xrightarrow{\cong} H_{*-r}(f;X)$.

Corollary 3.2. *$u \in H^r(f_{-q}; D(E), S(E)) = H^r(\tilde{f}_q; D(E), S(E))$ is the Thom class of E, and $(\pi^*)^{-1} \circ j^*(u) =: e(E) \in H^r(X)$ is the Euler class of E.*

This follows from the proposition using (20) and (21).
We now prove the proposition.

Proof. Without loss of generality, f on X has a unique minimum in x_o and we can choose $x = x_o$ for the fibre inclusion. Hence $T^*(1)$ is represented by $\{(x_0, 0)\} \in H^r(f_{-q}; D(E), S(E))$. Obviously, $\mathrm{Eig}^+\big((x_o,0); f_{-q}\big) \cong \mathrm{Eig}^+(x_o; f)$ is transverse to the fibre E_{x_o}. Moreover, we have $W^s\big((x_o,0); f_{-q}|E_{x_o}\big) = \{0\}$ and $W^u(0;-q) = T_0 E_{x_o}$. Hence, we have transverse intersection within E and we see $j^*(x_o, 0) = 0$. Obviously, $\{0\} = u_o \in H^r(-q; D^r, S^{r-1})$ is the generator.
Using the canonical identification $\mathrm{Crit}_* f = \mathrm{Crit}_{*+r} f_{-q}$ via $x = (x,0)$, $W^s(x;f) = W^s((x,0); f_{-q})$ for all $x \in \mathrm{Crit}_* f$, and using the fact that each oriented unstable manifold $W^u(x;f)$ together with the orientation of E gives an orientation for $W^u((x,0); f_{-q})$, implies that we have

$$\#_{\mathsf{alg}}\big(W^u(x;f) \pitchfork W^s(x';f)\big) \;=\; \#_{\mathsf{alg}}\big(W^u((x,0); f_{-q}) \pitchfork W^s((x',0); f_{-q})\big)\,.$$

Since there are no critical points for f_q off the zero section we have canonical chain and cochain complex isomorphisms

$$\begin{aligned} T_\bullet &\colon C_{*+r}\big(f_{-q}; D(E), S(E)\big) \xrightarrow{\cong} C_*(f;X), \\ T^\bullet &\colon C^*(f;X) \xrightarrow{\cong} C^{*+r}\big(f_{-q}; D(E), S(E)\big), \end{aligned}$$

inducing the Thom isomorphisms T_* and T^* on homology respectively cohomology level. $\square$

3.3 Umkehr map for proper embeddings

We will now give another, Morse-theoretical description for the umkehr map.
Let $e\colon P \hookrightarrow X$ be again a proper embedding of finite positive codimension with the additional assumption of coorientation, i.e. the normal bundle ν_e is oriented. Consider again Morse functions as above, $k\colon P \to \mathbb{R}$ and $f\colon X \to \mathbb{R}$ with $\operatorname{Crit} f \cap P = \emptyset$. Then, for generic metrics on P and X we have transverse intersections of the unstable manifold of $m \in \operatorname{Crit} f$ in X and the stable manifold of $p \in \operatorname{Crit} k$ in P,

$$W^u(m; f, X) \pitchfork W^s(p; k, P)\,.$$

Note that, this in particular requires the tranverse intersection of $W^u(m; f, X)$ with P. The coherent orientation condition of Morse homology is guaranteed by the assumption of coorientation of $e(P)$ in X. Hence, we obtain a well-defined integer

$$n(m,p) \;=\; \#_{\mathsf{alg}}\big(W^u(m;f) \pitchfork W^s(p;k)\big)$$

if $i(m) = i(p) + r$, where r is the codimension of P.

Proposition 3.3. *The associated Morse chain morphism*

$$e_\bullet\colon C_*(f;X) \;\to\; C_{*-r}(k;P), \quad m \mapsto \sum_{i(p)=i(m)-r} n(m,p)p,$$

induces the umkehr map

$$e_!\colon H_*(f;X) \;\to\; H_{*-r}(k;P)$$

on the level of Morse homology.

Proof. Again, it is a standard Morse homology argument to see that the above chain level map $e_\bullet$ commutes with the respective boundary operators, $\partial_k \circ e_\bullet = e_\bullet \circ \partial_f$.
Consider the tubular neighbourhood η_e of P and let $\tilde{k}_q \in C^\infty(X, \mathbb{R})$ be a Morse function on X whose restriction to η_e equals $k_{-q} = \pi^* k - q$ as above, after identifying η_e with an open subset of the normal bundle ν_e. Using the orientation of ν_e, we can identify

$$C_*(\tilde{k}_q; X, X \setminus \eta_e) \;\cong\; C_{*-r}(k;P)$$

precisely as in the proof of Proposition 3.1. Moreover, from the long exact sequence we have on chain level

$$C_*(\tilde{k}_q; X) \xrightarrow{\;j\;} C_*(\tilde{k}_q; X, X \setminus \eta_e) \;\cong\; C_{*-r}(k;P)\,. \tag{22}$$

Let us now consider on chain level the following definition of the canonical isomorphism $H_*(f;X) \cong H_*(\tilde{k}_q; X)$, known in Floer theory as the continuation isomorphism. Namely, given generic Riemannian metrics both for the negative gradient flow of f and of $\tilde{k}_q$, we have the chain complex morphism, which is a chain homotopy equivalence,

$$\Phi\colon C_*(f;X) \;\to\; C_*(\tilde{k}_q; X), \quad m \mapsto \sum_{i(m')=i(m)} n(m,m')m',$$

with

$$n(m,m') \;=\; \#_{\mathsf{alg}}\big(W^u(m;f) \pitchfork W^s(m';\tilde{k}_q)\big)\,.$$

Composing Φ with the chain morphism from (22), we see that we obtain up to chain homotopy equivalence exactly the chain morphism $e_\bullet\colon C_*(f;X) \to C_{*-r}(k;P)$.

We conclude the proof by comparing this construction in Morse homology with the definition of the umkehr map in standard homology. We have the commutative diagram

$$\begin{tikzcd}
H_*(f;X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_*(\tilde{k}_q;X) \arrow[r, "j_*"] & H_*(\tilde{k}_q;X,X\setminus\eta_e) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_{*-r}(k;P) \arrow[d, "\cong"'] \\
H_*(X) \arrow[rr, "j_*"] \arrow[d, "\mathrm{id}"'] & & H_*(D(\nu_e),S(\nu_e)) \arrow[r, "\cdot\cap u_{\nu_e}"] \arrow[d, "\cong"'] & H_{*-r}(P) \arrow[d, "\mathrm{id}"'] \\
H_*(X) \arrow[rr, "(\tau_e)_*"] & & H_*(P^{\nu_e}) \arrow[r, "\cong"] & H_{*-r}(P)
\end{tikzcd}$$

where the upper row gives $e_!$ and the lower row is by definition the umkehr map. $\square$

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