Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Some remarks on the strong maximum principle arising in nonlocal operators

by

Pascal Autissier, and Jerome Coville

Preprint no.: 6 2008

SOME REMARKS ON THE STRONG MAXIMUM PRINCIPLE ARISING IN NONLOCAL OPERATORS

PASCAL AUTISSIER AND JÉRÔME COVILLE

ABSTRACT. In this note we make some remarks concerning maximum principles holding for nonlocal diffusion operator of the form

$$\mathcal{M}[u](x) := \int_G J(g)u(x*g^{-1})d\mu(g) - u(x),$$

where G is a group acting continuously on a Hausdorff space X and $u \in C(X)$. We first investigate the existence of a strong maximum principle in the general situation and then focus on the case of homogeneous spaces. Depending on the topology of the homogenerous space, we give contidions on J and $d\mu$ such that $\mathcal{M}$ achieves a strong maximum principle. We also revisit the classical case of convolution operator on $\mathbb{R}^n$.

1. **Introduction and Main results**

This note is devoted to the strong maximum principle and some conditions to obtain the strong maximum principle for operator of the form

$$\mathcal{M}[u] := \int_G J(g)u(x*g^{-1})d\mu - u \tag{1.1}$$

where $(G, *, X, J, d\mu)$ satisfies the following set of assumptions :

(H1) X is an Hausdorff's space,
(H2) $(G, *)$ be a topological group acting continuously on X through the operation $*$,
(H3) $d\mu$ is a Borel measure on G such that for all open set $A \in G$, $d\mu(A) > 0$,
(H4) $J \in C(G, \mathbb{R})$ is a non-negative function of unit mass with respect to $d\mu$.

Such kind of operator has been recently introduced to analyse nonlocal effects in various models ranging from Ising model to cellular growth, see [1, 4, 5, 7, 10, 13]. A first example is the well known nonlocal reaction diffusion equation below,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^n} J(x-y)u(y)\,dy - u + f(u) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n. \tag{1.2}$$

In this case, we have $(G, *) = (\mathbb{R}^n, +)$, $X = \mathbb{R}^n$, $J \in C(\mathbb{R}^n)$ and $d\mu = dy$ is the Lebesgue measure. Such equation appears in particular in ecology and in some Ising models see [1, 5, 6, 13] and their many references. Another example of such model is given by the two following discrete versions of (1.2),

$$\frac{\partial u}{\partial t} = \frac{1}{2}[u(x+1) + u(x-1) - 2u(x)] + f(u) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n, \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 35B50, 47G20; Secondary 35J60 .
Key words and phrases. Nonlocal diffusion operators, maximum principles, Geometric condition.
Support information for the first author.
This project was partially supported by the Ceremade- Université Paris Dauphine, CMM-Universidad de Chile on an Ecos-Conicyt project and the Max Planck Institut for Mathematics in the Sciences.

$$\frac{\partial u}{\partial t} = \frac{1}{2}[u(p+1) + u(p-1) - 2u(p)] + f(u) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{Z}^n. \tag{1.4}$$

In both situation the diffusion operator can be rewrited in term of a nonlocal operator $\mathcal{M}$ defined by (1.1). Indeed, in these two cases, $(G,*) = (\mathbb{Z}^n, +), d\mu = \delta_1 + \delta_{-1}$ where δ_x is the Dirac measure at a point x and $J \in C(\mathbb{Z}^n, \mathbb{R})$ is defined as follows:

$$J(p) := \begin{cases} \frac{1}{2} & \text{if } \{(\pm 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, \pm 1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

and the Hausdorff space is either $X = \mathbb{R}^n$ or $X = \mathbb{Z}^n$. As their continuous version (Equation (1.2)), such two equations appear in particular in some discrete reaction diffusion models describing a wide variety of phenomena, ranging from combustion to ecology, nerve propagation or phase transitions. We point the interested reader to [4, 3, 9] and they many references.

Another example comes from the following size structured population model, recently introduced by Perthame *et ale* in [10, 11]

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \int_0^{+\infty} u(\frac{x}{y})b(y)dy - u(x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.5}$$

In such case, $(G,*) = (\mathbb{R}^+ \setminus \{0\}, \bullet)$, $X = \mathbb{R}^+$ and again $d\mu = dy$ the Lebesgue measure.

In all theses cases, depending on the group and the measure considered, the properties satisfied by the corresponding operator $\mathcal{M}$ show significantly differences. However, as the classical Laplace operator (Δ) see [2], they all satisfy the following positive maximum principle property

Courrège Positive maximum principle :
An operator $A \in \mathcal{L}(C(X))$ is said satisfying the positive maximum principle if for all $f \in C(X)$ and $x \in X$ such that $f(x) = \sup(f)$ we have $A(f)(x) \leq 0$.

For the Laplace operator (Δ), in addition to the above property, it is well known [8, 12], that sub-harmonic function satisfies a strong maximum principle, that is :

Elliptic Strong maximum principle :
Let $u \in C^2(\mathbb{R}^n)$ be such that

$$\Delta u \geq 0 \quad \textit{in} \quad \mathbb{R}^n.$$

Then u cannot achieve a global maximum without being constant.

In this note, we investigate conditions on $(G,*), X, J$ and $d\mu$ in order to achieve such strong maximum principles for the general operator $\mathcal{M}$. More precisely , we are interested in finding simple conditions on $(G,*), X, J$ and $d\mu$ for the strong maximum principle to hold, that is:

Strong maximum principle :
Let $u \in C(X, \mathbb{R})$ be such that

$$\mathcal{M}[u] \geq 0 \quad \textit{in} \quad X.$$

Then u cannot achieve a global maximum without being constant.

Such kind of strong maximum principle plays an important role in the analysis of elliptic non linear equations since it has already proved to be a very efficient tool to provide various *a priory* estimates which is usually a crucial point in the analysis of non linear pde's. It is therefore of great interest to obtain simple conditions to achieve a strong maximum principle for $\mathcal{M}$.

In this direction, we first establish a generic result that we call pre-maximum principle satisfied by operators $\mathcal{M}$. More precisely, we show

Theorem 1.1. *(Pre-maximum principle)*
Let $(G,*,X,J,d\mu)$ satisfying $(H1-H4)$ and let $u\in C(X,\mathbb{R})$ be such that

$$\mathcal{M}[u]\geq 0 \qquad (\textit{resp.} \leq 0).$$

Assume that u achieves a global maximum (resp. minimum) in x_0 and let us denoted F_{x_0}, the smallest closed subset of X such that

- $x_0\in F_{x_0}$,
- $F_{x_0} * \{g^{-1}\in G|J(g)>0\}\subset F_{x_0}$.

Then $u\equiv u(x_0)$ in F_{x_0}.

Our next result is a characterization of the set F_{x_0} defined in the above Theorem 1.1. Namely, we have

Proposition – 1.1. Let $(G,*,X,J,d\mu)$ satisfying $(H1-H4)$ and let F_{x_0} be the set defined in Theorem 1.1, then

$$F_{x_0}=\overline{\bigcup_{n\in\mathbb{N}} F_n},$$

where the F_n are defined by induction as follows

$$\begin{aligned}&F_0=\{x_0\},\\ &\forall n\geq 0 \quad F_{n+1}:=F_n * \{g^{-1}\in G|J(g)>0\}.\end{aligned}$$

In view of the above generic result, in order to get a strong maximum principle for $\mathcal{M}$, we need to find conditions on $(G,*)$, X, $d\mu$ and J which implies that $F_{x_0}=X$. Note that, from the characterization of the set F_{x_0}, the condition $F_{x_0}=X$ implies that $X=F_{x_0}\subset orb(x):=\{x*g^{-1}|g\in G\}\subset X$, which means that $orb(x_0)$ is a dense set in X.

Remark that for the diffusion operator considered in (1.3), the set $orb(x)$ is never dense in X. Therefore, we cannot expect to have a strong maximum principle for such diffusion operator. On the contrary, for the diffusion operator considered in (1.4), the set $orb(x)$ is always dense in X and we can find conditions on J and $d\mu$ to ensure a strong maximum principle.

Considering the above remarks, in what follows, we restrict our investigation to the case of Hausdorff homogeneous spaces X (i.e. $X:=G/H$, where H is a closed subgroup of G). For such Hausdorff's spaces, the set $orb(x)$ is always dense in X and sufficient conditions $(G,*)$, X, J and $d\mu$ for the strong maximum principle to hold reduce to simple condition on J. In this direction, we first give a sufficient condition on J to ensure that $\mathcal{M}$ satisfies the strong maximum principle. Namely, we have the following

Theorem 1.2.
Let X be a connected homogeneous space and $(G,*)$, J, $d\mu$ as in Theorem 1.1. Assume that $J(e)>0$, then $\mathcal{M}$ satisfies the strong maximum principle.

Adding some compactness assumption on X, we can generalize the previous statement to

Theorem 1.3.
Let $(G,*), X, J, d\mu$ as in the above Theorem and assume further that X is compact. Then $\mathcal{M}$ satisfies the strong maximum principle.

Next, we state optimal conditions on J in some special cases. Namely, we first retrieve the Markov necessary and sufficient condition for the convolution operator (i.e. $(G,*) = (\mathbb{R}^n,+), X = \mathbb{R}^n, d\mu = dy$, which is well known among experts in Stochastic processes.

Theorem 1.4. *(Markov condition)*
Assume that the $(G,*) = (\mathbb{R}^n,+), X = \mathbb{R}^n$ and $d\mu = dy$ then $\mathcal{M}$ satisfies the strong maximum principle iff the convex hull of $\{y \in \mathbb{R}^n | \, J(y) > 0\}$ contains 0.

From the above Markov condition, we also derive the following optimal condition when $(G,*) = (\mathbb{R}^+ \setminus \{0\}, \bullet), X = \mathbb{R}^+$ and $d\mu = dy$. Namely, we have the following :

Theorem 1.5.
Assume that the $(G,*) = (\mathbb{R}^+ \setminus \{0\}, \bullet), X = \mathbb{R}^+$ and $d\mu = dy$ then $\mathcal{M}$ satisfies the strong maximum principle iff there exists 2 points x_1 and x_2 such that $J(x_i) > 0$, $x_1 \neq x_2$ and $0 \leq x_1 \leq 1 \leq x_2$).

1.1. **General comments.**
We first note that, provide an extra assumption on the non-negativity (-positivity) of the maximum (minimum), we can easily extend the above results to operators $\mathcal{M}[u] + c(x)u$ with non-positive zero order term (i.e. $c(x) \leq 0$). As for $\mathcal{M}$, the operator $\mathcal{M} + c(x)$ satisfies a Courrèges positive maximum principle [2] which in this case state the following

Positive maximum principle :
An operator $A \in \mathcal{L}(C(X))$ is said satisfying the positive maximum principle if for all $f \in C(X)$ and $x \in X$ such that $f(x) \geq 0$ and $f(x) = \sup(f)$ we have $A(f)(x) \leq 0$.

Along our investigation, we also observe that to obtain a strong maximum principle for $\mathcal{M}$, we only need the inequality $\mathcal{M}[u](x) \geq 0$ at points of global maximum of the function u. As a consequence, for operators satisfying the strong maximum principle and the Courrèges positive maximum principle, we have the following characterization

Proposition :
Assume that $\mathcal{M}$ satisfies the strong maximum principle and the Courrège positive maximum principle then for all $u \in C(X)$ and $x \in X$ such that $u(x) = \sup(u)$ we have the following alternative:

- *Either $\exists\, y \in \{y \in X \,|\, u(y) = u(x)\}$ such that $\mathcal{M}[u](y) < 0$,*
- *Or $\forall\, y \in \{y \in X \,|\, u(y) = u(x)\}, \mathcal{M}[u](y) = 0$ and $u \equiv Cste$.*

We also wanted to point out that although the Markov condition is well known among the expert in Stochastic analysis, we present here a simple analitical proof, which we believe is new. Using such point of view, allows us to relate conditions to have the maximum property to a simple recovering problem.

The outline of this paper is the following. In the two first sections (Section 2 and 3), we recall some basic topological results and prove the pre-maximum principle and the characterization of F_x (Theorems 1.1 and Proposition 1.1). Then in section 4, we establish the strong maximum principle (Theorems 1.2 and 1.3). Finally, in the last section, we deal with the optimal conditions (Theorems 1.4 and 1.5).

2. **Preliminaries**

In this section, we present some definitions that we use along this paper and establish a useful proposition. Let us first define some notations

- $\Sigma := \{g^{-1} \in G | J(g) > 0\}$
- For a function u, we define $\Gamma_y := \{x \in X | u(y) = u(x)\}$

and introduce this two definitions:

Definition 2.1. Let $A \subset X$ and $B \subset G$ be two sets, then we define $A * B \subset X$ as follows

$$A * B := \{a * b \mid a \in A \quad \text{and} \quad b \in B\}.$$

Definition 2.2. Let $A \subset X$ and $B \subset G$ be two sets, then we say that A is $B*$ stable iff

$$A * B \subset A.$$

Next, let us recall the following basic property of $*$ stable sets :

Proposition – 2.1.
Let $A \subset X$ and $B \subset G$ be two sets. If A is $B*$ stable, then $\bar{A}$ is $B*$ stable, where $\bar{A}$ denotes the closure of A.

Proof:

Let $y \in \bar{A} * B$ and $V(y)$ be an open neighbourhood of y. By definition, we have $y := x_1 * b_1$ for some $x_1 \in \bar{A}$ and $b_1 \in B$. Since the operation $*$ is continuous, the following application T is continuous:

$$\begin{array}{rcl} T : X & \to & X \\ z & \mapsto & z * b_1. \end{array}$$

Therefore $T(V(y))^{-1}$ is a open neighbourhood of x_1. Since $\bar{A}$ is a closed set and $x_1 \in \bar{A}$, we have $T(V(y))^{-1} \cap A \neq \emptyset$. By definition of $T(V(y))^{-1}$, using the stability of A, it follows that for all $z \in T(V(y))^{-1} \cap A$,

$$z * b_1 \in A.$$

Therefore,

$$z * b_1 \in V(y) \cap A \quad \text{for all} \quad z \in T(V(y))^{-1} \cap A,$$

and yields to

$$V(y) \cap A \neq \emptyset.$$

The above argumentation being independant of the choice of $V(y)$, it follows that $y \in \bar{A}$. Now, since y is chosen arbitrary, we end up with

$$\bar{A} * B \subset \bar{A}.$$

□

3. **Pre-maximum principle and Characterizations: Proof of Theorems 1.1 and Proposition 1.1**

In this section, we deal with the proof of Theorems 1.1 and the characterization of the set F_x defined in Theorem 1.1. We also give some characterization of the corresponding set Γ_x.

Let us first start with the proof of the pre-maximum principle.

Proof of Theorem 1.1

The proof is rather simple. Let us first recall the definition of Γ_{x_0} :

$$\Gamma_{x_0} := \{x \in X | u(x) = u(x_0)\}. \tag{3.1}$$

Since u is continuous, Γ_{x_0} is a closed subset of X.
Now observe that Γ_{x_0} is $\Sigma*$ stable (i.e. $\Gamma_{x_0} * \Sigma \subset \Gamma_{x_0}$).
Indeed, choose any $\bar{x} \in \Gamma_{x_0}$. At $\bar{x}, u$ satisfies the following

$$0 \leq \mathcal{M}[u](\bar{x}) = \int_G J(g) u(\bar{x} * g^{-1})\, d\mu - u(\bar{x}) = \int_G J(g)[u(\bar{x} * g^{-1}) - u(\bar{x})]\, d\mu \leq 0.$$

Therefore,

$$\int_G J(g)[u(\bar{x} * g^{-1}) - u(\bar{x})]\, d\mu = 0. \tag{3.2}$$

Using that $J \geq 0$ and for all $g \in G$, $[u(\bar{x} * g) - u(\bar{x})] \leq 0$, from (3.2), it follows that

$$u(\bar{x} * g^{-1}) = u(\bar{x}) \quad \text{for all} \quad g \in \Sigma.$$

Thus, we have

$$u(y) = u(x_0) \quad \text{for all} \quad y \in \{\bar{x}\} * \Sigma.$$

Hence,

$$\{\bar{x}\} * \Sigma \subset \Gamma_{x_0}.$$

Since this computation holds for any element of Γ_{x_0}, we then have

$$\Gamma_{x_0} * \Sigma \subset \Gamma_{x_0}.$$

Recall now that F_{x_0} is the smallest closed subset of X such that

- $x_0 \in F_{x_0}$,
- $F_0 * \Sigma \subset F_{x_0}$.

Since Γ_{x_0} satisfies the above conditions, we then have $F_{x_0} \subset \Gamma_{x_0}$.

$\square$

Note that Γ_{x_0} is independent of the choice of the point where u takes its global maximum. Indeed, we easily see that $\Gamma_{x_0} = \Gamma_y$ for any $y \in \Gamma_{x_0}$. On the contrary, the set F_{x_0} strongly depends on x_0 and there is no reason to always have $F_{x_0} = F_y$. Indeed, for $X = G = \mathbb{R}$, if $\Sigma = \mathbb{R}^+$ then for $x_0 < y$, $F_y \subset_{\neq} F_{x_0}$.

Now, we give a characterization of the set F_{x_0} defined in Theorem 1.1 and prove Proposition 1.1. For the sake of clarity, let us first recall Proposition 1.1:

Proposition – 3.1. Let F_{x_0} be the set defined in Theorem 1.1, then

$$F_{x_0} = \overline{\bigcup_{n \in \mathbb{N}} F_n},$$

where the F_n are defined by induction as follows

$$F_0 = \{x_0\},$$
$$\forall n \geq 0 \quad F_{n+1} := F_n * \Sigma.$$

Proof:

Let us define the following set

$$F_\infty := \bigcup_{n\in\mathbb{N}} F_n.$$

Using the definition of F_∞, we easily see that F_∞ is $\Sigma*$ stable. From Proposition 2.1, it follows that $\bar{F}_\infty$ is $\Sigma*$ stable. Therefore, by definition of F_{x_0}, we have $F \subset \bar{F}_\infty$.

Now, since $x_0 \in F_{x_0}$ and F_{x_0} is $\Sigma*$ stable, by induction we easily see that $\forall n \in \mathbb{N}, F_n \subset F_{x_0}$. Thus, $F_\infty \subset F_{x_0}$ and yields to $F_{x_0} \subset \bar{F}_\infty \subset F_{x_0}$.

□

Using Theorem 1.1, we can also give a characterization of the set Γ_{x_0} in term of the F_y. More precisely,

$$\Gamma_{x_0} = \overline{\bigcup_{n\in\mathbb{N}} A_n},$$

where the set A_n are defined by induction as follows:

- $A_0 := F_{x_0}$
- $A_{n+1} := \begin{cases} F_x & \text{for some } x \in \Gamma_{x_0} \setminus A_n \\ \emptyset & \text{otherwise} \end{cases}$

Remark 3.1. As already mentioned in the introduction, in order to get a strong maximum principle for $\mathcal{M}$, we only need to find condition on $X, d\mu$ and J such that $F_{x_0} = \Gamma_{x_0} = X$.

4. Strong maximum principle when X is an homogeneous space

In this section, we deal with the case of homogeneous space (i.e. $X = G/H$, where G is a topological group and H is a closed subgroup of G) and give some sufficient conditions on J (Theorems 1.2 and 1.3) in order to have a strong maximum principle property for $\mathcal{M}$.

For convenience, let us first recall Theorem 1.2,

Theorem – *Let X be connected homogeneous space. Assume that $J(e) > 0$, then $\mathcal{M}$ satisfies the strong maximum principle.*

Proof:

Again the proof is rather simple, we must check that for any $u \in C(X, \mathbb{R})$ such that

$$\mathcal{M}[u] \geq 0 \qquad \text{(resp. } \leq 0)$$

then u cannot achieve a global maximum (resp. minimum) in X without being constant. So consider $u \in C(X, \mathbb{R})$ such that u achieves a maximum at x_0 and satisfies

$$\mathcal{M}[u] \geq 0 \qquad \text{(resp. } \leq 0).$$

By definition of Γ_x, we are reduced to show that $\Gamma_{x_0} = X$. To this end, we will prove that Γ_{x_0} is a closed and open set. By definition of Γ_{x_0}, Γ_{x_0} is a closed set of X. Now, let us show that Γ_{x_0} is open. Choose any $y \in \Gamma_{x_0}$, then at this point we have:

$$0 \leq \mathcal{M}[u](y) = \int_G J(g)u(y * g^{-1})\, d\mu - u(y) = \int_G J(g)[u(y * g^{-1}) - u(y)]\, d\mu(g) \leq 0.$$

Arguing as in the proof of Theorem 1.1, we have $u(y * g^{-1}) = u(y) = u(x_0)$ for all $g \in \Sigma$. Since, $e \in \Sigma$, we have

$$u(y * g^{-1}) = u(x_0) \quad \text{for all} \quad g^{-1} \in B(e).$$

Using that G is a topological group, $y * B(e)$ is then an open neighborhood of y. Thus,

$$B(y) := y * B(e) \subset \Gamma_{x_0}.$$

Hence, $X = \Gamma_{x_0}$ since X is connected.

□

Let us now turn our attention to the case of compact homogeneous space and prove Theorem 1.3 that we recall below.

Theorem – *If X is compact then $\mathcal{M}$ satisfies the strong maximum principle.*

Before going to the proof, let us prove the following practical Lemma

Lemma – 4.1. For any $g \in X$ there exists a sequences of integer $(n_k)_{k\in\mathbb{N}}$, such that $g^{n_k} \to e$, where e is the unit element.

Proof:

Take $g \in X$ and let us consider the following sequence $(g^m)_{m\in\mathbb{N}}$. Since X is compact, $(g_m)_{m\in\mathbb{N}}$ has a convergent sub-sequence $(g_{m_k})_{k\in\mathbb{N}}$. Without any restriction, we can assume that $m_{k+1} \geq m_k+1$. Consider now the following sequence, $w_k := g^{m_{k+1}-m_k}$. By construction, $w_k \to e$ and $m_{k+1} - m_k \in \mathbb{N}$. Hence, with $n_k := m_{k+1} - m_k$, $g^{n_k} \to e$.

□ .

Let us now turn our attention to the proof of Theorem 1.3.

Proof:

As for Theorem 1.2 we have to check that for any $u \in C(X, \mathbb{R})$ such that

$$\mathcal{M}[u] \geq 0 \qquad (\text{resp.} \leq 0)$$

then u cannot achieve a global maximum (resp. minimum) in X without being constant. So consider $u \in C(X, \mathbb{R})$ such that u achieves a maximum at x_0 and satisfies

$$\mathcal{M}[u] \geq 0 \qquad (\text{resp.} \leq 0).$$

By definition of Γ_x, we are reduced to show that $\Gamma_{x_0} = X$. Again, as in the proof of Theorem 1.2, we prove that Γ_{x_0} is an open and closed set and therefore $X = \Gamma_{x_0}$ since X is connected. By definition Γ_{x_0} is closed. Let $y \in \Gamma_{x_0}$ and F_y be the set defined in Theorem 1.1 with y instead of x_0. Using now the description of F_y given in Proposition 1.1, we have

$$F_y := \overline{\bigcup_{n\in\mathbb{N}} F_n} \subset \Gamma_{x_0}, \tag{4.1}$$

where $F_n := \{y\} * \Sigma^n$.

Choose now $g \in \Sigma$, according to Lemma 4.1 there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $g^{n_k} \to e$. By assumption, Σ is an open subset of G, therefore Σ^{n_k} is a sequence of open subset of G. Since $g^{n_k} \to e$, Σ^{n_k} is a open neighborhood of e for k sufficiently large.

Therefore

$$\{y\} * \Sigma^{n_k} \subset F_y \subset \Gamma_{x_0}$$

Since Σ^{n_k} is a open neighborhood of e for k sufficiently large, $\{y\} * \Sigma^{n_k}$ is then an open neighborhood of y. Thus, Γ_{x_0} contains an open neighborhood of y for any y in Γ_{x_0}. Hence, Γ_{x_0} is open.

5. **Some optimal conditions**

In this section we deal with the optimal Markov condition for convolution operator (Theorem 1.4) and prove Theorem 1.5 .

The classical convolution case $(X = G = \mathbb{R}^n)$ **and** $d\mu = dy$**:**

When $(X = G = \mathbb{R}^n)$ the operator $\mathcal{M}$ take the form of the usual convolution, i.e

$$\mathcal{M}[u] := \int_{\mathbb{R}^n} J(y)u(x-y)\,dy - u.$$

For such convolution operator, the optimal condition on J in order that $\mathcal{M}$ satisfies a strong maximum principle is the following:

Theorem – *$\mathcal{M}$ satisfies a strong maximum principle iff the convex hull of $\{y \in \mathbb{R}^n | J(y) > 0\}$ contains 0.*

This condition is known as the Markov condition.

Proof:

Let us start with the necessary condition. Assume the Markov condition fails. We will show that $\mathcal{M}$ does not satisfy the strong maximum principle. To this end, we construct a non constant function u which achieves a global maximum and satisfies

$$\mathcal{M}[u] \geq 0.$$

Let us denote $conv(\{y \in \mathbb{R}^n | J(y) > 0\})$ the convex hull of $\{y \in \mathbb{R}^n | J(y) > 0\}$. By assumption, $0 \notin conv(\{y \in \mathbb{R}^n | J(y) > 0\})$. Using Hahn-Banach Theorem, there exists an Hyperplane H such that $conv(\{y \in \mathbb{R}^n | J(y) > 0\}) \subset H^+$ where $H^+ := \{x \in \mathbb{R}^n | x_n \geq 0\}$ in an orthonormal basis $(e_1; e_2; \ldots; e_n)$. Consider v a non-increasing function which is constant in $\mathbb{R}^-$, and let us compute $\mathcal{M}[u]$ with $u(x) := v(x_n)$. Since the Lebesgue measure is invariant under rotation and $supp(J) \subset H^+$ we have

$$\begin{aligned}
\mathcal{M}[u] &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} J(t, x_n - y_n)[v(y_n) - v(x_n)]\,dx_n\,dt \\
&= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{x_n} J(t, x_n - y_n)[v(y_n) - v(x_n)]\,dx_n\,dt.
\end{aligned}$$

Therefore, since v is non increasing we end up with

$$\mathcal{M}[u] \geq 0.$$

Since u achieves a global maximum without being constant, u is our desired function.

Let now turn our attention to the sufficient condition. Assume that $0 \in conv(\{y \in \mathbb{R}^n | J(y) > 0\})$, then there exists a simplex $S(x_i)$ form by $n+1$ point of $\mathbb{R}^n$ such that $0 \in S$ and $J(x_i) > 0$. By continuity, we can always assume that $(x_1; ...; x_n)$ is a basis of $\mathbb{R}^n$. Let us now rewrite x_0 in the basis $(x_1; ...; x_n)$:

$$x_0 = -a_1 x_1 ... - a_n x_n \quad \text{with} \quad a_i > 0.$$

Observe now that for $\mathbb{R}^n$ equipped with the sup norm associated to the base $(x_1; ...; x_n)$, there exists $r > 0$ so that $B(x_0, r) \subset \{J > 0\}$. Now for all integer $m > 0$, set $y_m = mx_0 + [ma_1]x_1 + ... + [ma_n]x_n$, where [ ] denotes the integer part. Now let u be a continuous function satisfying

$$\mathcal{M}[u] \geq 0$$

which achieves a global maximum at some point $z \in \mathbb{R}$. Without loosing generality, we may always assume that $z = 0$. Indeed, if $z \neq 0$, we consider the function $u_z(x) := u(x - z)$, instead of u. We easily see that u_z achieves a global maximum at 0 and satisfies $\mathcal{M}[u_z] \geq 0$. Using now Theorem 1.1, we see that for all $m \in \mathbb{N}$,

$$\|y_m\| < 1 \qquad \text{and} \qquad B(y_m; mr) \subset F_0.$$

Therefore,

$$\bigcup_{m \in \mathbb{N}} B(y_m; mr) \subset F_0.$$

Hence, $\mathbb{R}^n \subset F$.

□

The above necessary and sufficient condition for the convolution operator, can be weakened depending and the underlying topological structure of the space. In particular, we have in mind the following setting. Since $\mathcal{M}$ is translation invariant, $\mathcal{M}$ is also an operator on the set of periodic functions. On this set of functions, the strong maximum principle always holds. This condition is not so surprising since the additional periodic structure will in some sense compactify the homogeneous space $\mathbb{R}^n$.

Another special case: $X = \mathbb{R}^+, (G, *) = (\mathbb{R}^+ \setminus \{0\}, \bullet)$ **and** $d\mu = dy$**:**

In this situation,

$$\mathcal{M}[u] := \int_{\mathbb{R}^+} J(y) u(\frac{x}{y})\, dy - u,$$

and the above operator has essentially the same property that the usual convolution operator. Indeed, let us make the following change of variables $x := e^t$, then we have

$$\mathcal{M}[u](e^t) = \int_{\mathbb{R}} \widetilde{J}(t - s) u(e^s)\, ds - u(e^t),$$

where $\widetilde{J}(t) := J(e^t)e^t$. Therefore, letting $v(t) = u(e^t)$, we end up with

$$\mathcal{M}[v](t) = \widetilde{J} \star v(t) - v(t) \quad \text{ in } \mathbb{R},$$

with $\int_{\mathbb{R}} \widetilde{J}(t)dt = 1$. Hence, the optimal condition to achieve a strong maximum principle for such kind of operator will be of the same type as the one used for the convolution operator.

Namely, there exists two points $a < 1 < b$ such that $J(a) > 0$ and $J(b) > 0$. This condition corresponds to the one given for the convolution operator which is the existence of two points $a' < 0 < b'$ such that $\widetilde{J}(a') > 0$ and $\widetilde{J}(b') > 0$.

The above observation proves of Theorem 1.5.

□

REFERENCES

[1] Peter W. Bates, Paul C. Fife, Xiaofeng Ren, and Xuefeng Wang. Traveling waves in a convolution model for phase transitions. *Arch. Rational Mech. Anal.*, 138(2):105–136, 1997.

[2] Jean-Michel Bony, Philippe Courrège, and Pierre Priouret. Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 2):369–521 (1969), 1968.

[3] Xinfu Chen and Jong-Sheng Guo. Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. *Math. Ann.*, 326(1):123–146, 2003.

[4] Ricardo Coutinho and Bastien Fernandez. Fronts in extended systems of bistable maps coupled via convolutions. *Nonlinearity*, 17(1):23–47, 2004.

[5] Jérome Coville and Louis Dupaigne. On a nonlocal reaction diffusion equation arising in population dynamics. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(4):727–755, 2007.
[6] A. De Masi, T. Gobron, and E. Presutti. Travelling fronts in non-local evolution equations. *Arch. Rational Mech. Anal.*, 132(2):143–205, 1995.
[7] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo. Uniqueness and global stability of the instanton in nonlocal evolution equations. *Rend. Mat. Appl. (7)*, 14(4):693–723, 1994.
[8] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[9] G. Harris, W. Hudson, and B. Zinner. Traveling wavefronts for the discrete Fisher's equation. *J. Differential Equations*, 105(1):46–62, 1993.
[10] Philippe Michel, Stéphane Mischler, and Benoît Perthame. General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appl. (9)*, 84(9):1235–1260, 2005.
[11] Benoît Perthame and Lenya Ryzhik. Exponential decay for the fragmentation or cell-division equation. *J. Differential Equations*, 210(1):155–177, 2005.
[12] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.
[13] Konrad Schumacher. Travelling-front solutions for integro-differential equations. I. *J. Reine Angew. Math.*, 316:54–70, 1980.

I.R.M.A.R., Université de Rennes I, campus de Beaulieu, 35042 RENNES cedex, France
E-mail address: pascal.autissier@univ-rennes1.fr

Max Planck Institut for mathematical science, Inselstrasse 22, D-04103 Leipzig, Deutschland
E-mail address: coville@mis.mpg.de