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homology with conormal boundary conditions

by

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# The homology of path spaces and Floer homology with conormal boundary conditions

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## Abstract

We define the Floer complex for Hamiltonian orbits on the cotangent bundle of a compact manifold satisfying non-local conormal boundary conditions. We prove that the homology of this chain complex is isomorphic to the singular homology of the natural path space associated to the boundary condition.

## Introduction

Let  $H : [0, 1] \times T^*M \rightarrow \mathbb{R}$  be a smooth time-dependent Hamiltonian on the cotangent bundle of a compact manifold  $M$ , and let  $X_H$  be the Hamiltonian vector field induced by  $H$  and by the standard symplectic structure of  $T^*M$ . The aim of this paper is to define the *Floer complex* for the orbits of  $X_H$  satisfying *non-local conormal boundary value conditions*, and to compute its homology. More precisely, we fix a compact submanifold  $Q$  of  $M^2 = M \times M$  and we look for solutions  $x : [0, 1] \rightarrow T^*M$  of the equation

$$x'(t) = X_H(t, x(t)),$$

such that the pair  $(x(0), -x(1))$  belongs to the *conormal bundle*  $N^*Q$  of  $Q$  in  $T^*M^2$ . We recall that the conormal bundle of a submanifold  $Q$  of the manifold  $N$  (here  $N = M^2$ ) is the set of covectors in  $T^*N$  which are based at points of  $Q$  and vanish on the tangent space of  $Q$ . Conormal bundles are Lagrangian submanifolds of the cotangent bundle, and the Liouville form vanishes identically on them.

When  $Q = Q_0 \times Q_1$  is the product of two submanifolds  $Q_0, Q_1$  of  $M$ , the above boundary condition is a local one, requiring that  $x(0) \in N^*Q_0$  and  $x(1) \in N^*Q_1$ . Extreme cases are given by  $Q_0$  and/or  $Q_1$  equal to a point or equal to  $M$ : since the conormal bundle of a point  $q \in M$  is the fiber  $T_q^*M$ , the first case produces a *Dirichlet boundary condition*, while since  $N^*M$  is the zero section in  $T^*M$ , the second one corresponds to a *Neumann boundary condition*. A non-local example is given by  $Q = \Delta$ , the diagonal in  $M \times M$ , inducing the periodic orbit problem (provided that  $H$  can be extended to a smooth function on  $\mathbb{R} \times T^*M$  which is 1-periodic in time). Another interesting choice is the one producing the figure-eight problem:  $M$  is itself a product  $O \times O$ , and  $Q$  is the subset of  $M^2 = O^4$  consisting of points of the form  $(o, o, o, o)$ ,  $o \in O$ . The Floer complex for the figure-eight problem enters in the factorization of the pair-of-pants product on  $T^*O$  (see [4]).

The set of solutions of the above non-local boundary value Hamiltonian problem is denoted by  $\mathcal{S}^Q(H)$ . If  $H$  is generic, all these solutions are non-degenerate, meaning that the linearized problem has no non-zero solutions, and  $\mathcal{S}^Q(H)$  is at most countable (and in general infinite). The free Abelian group generated by the elements of  $\mathcal{S}^Q(H)$  is denoted by  $F^Q(H)$ . This group can be graded by the Maslov index of the path  $\lambda$  of Lagrangian subspaces of  $T^*(\mathbb{R}^n \times \mathbb{R}^n)$  produced by the

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graph of the differential of the Hamiltonian flow along  $x \in \mathcal{S}^Q(H)$ , with respect to the tangent space of  $N^*Q$ , after a suitable symplectic trivialization of  $x^*(TT^*M) \cong [0, 1] \times T^*\mathbb{R}^n$ ,  $n = \dim M$ . Our first result is that such a definition does not depend on the choice of this trivialization, provided that the trivialization preserves the vertical subbundle and maps the tangent space of  $N^*Q$  at  $(x(0), -x(1))$  into the conormal space  $N^*W$  of some linear subspace  $W \subset \mathbb{R}^n \times \mathbb{R}^n$ . See section 2 for the precise statement.

When the Hamiltonian  $H$  is the Fenchel-dual of a Lagrangian  $L : [0, 1] \times TM \rightarrow \mathbb{R}$  which is fiber-wise strictly convex, the  $M$ -projection of the orbit  $x \in \mathcal{S}^Q(H)$  is an extremal curve  $\gamma$  of the Lagrangian action functional

$$\mathbb{S}_L(\gamma) = \int_0^1 L(t, \gamma(t), \gamma'(t)) dt,$$

subject to the non-local constraint  $(\gamma(0), \gamma(1)) \in Q$ . In this case, a theorem of Duistermaat's [5] can be used to show that the above Maslov index  $\mu(\lambda, N^*W)$  coincides up to a shift with the Morse index  $i^Q(\gamma)$  of  $\gamma$  as a critical point of  $\mathbb{S}_L$  in the space of paths on  $M$  satisfying the above non-local constraint. Indeed, in section 3 we prove the identity

$$i^Q(\gamma) = \mu(\lambda, N^*W) + \frac{1}{2}(\dim Q - \dim M) - \frac{1}{2}\nu^Q(x),$$

where  $\nu^Q(x)$  denotes the nullity of  $x$ , i.e. the dimension of the space of solutions of the linearization at  $x$  of the non-local boundary value problem. This formula suggests to incorporate the shift  $(\dim Q - \dim M)/2$  into the grading of  $F^Q(H)$ , which is then graded by the index

$$\mu^Q(x) := \mu(\lambda, N^*W) + \frac{1}{2}(\dim Q - \dim M),$$

which is indeed an integer if  $x$  is non-degenerate. When the Hamiltonian  $H$  satisfies suitable growth conditions on the fibers of  $T^*M$ , the solutions of the Floer equation

$$\partial_s u + J(\partial_t u - X_H(t, u)) = 0$$

on the strip  $\mathbb{R} \times [0, 1]$  with coordinates  $(s, t)$ , satisfying the boundary condition  $(u(s, 0), -u(s, 1)) \in N^*Q$  for every real number  $s$ , and converging to two given elements of  $\mathcal{S}^Q(H)$  for  $s \rightarrow \pm\infty$ , form a pre-compact space. Here  $J$  is the almost complex structure on  $T^*M$  induced by a Riemannian metric on  $M$ . Assuming also that the elements of  $\mathcal{S}^Q(H)$  are non-degenerate, a standard counting process defines a boundary operator on the graded group  $F_*^Q(H)$ , which then carries the structure of a chain complex, the *Floer complex* of  $(T^*M, Q, H, J)$ . This free chain complex is well-defined up to chain isomorphisms.

Changing the Hamiltonian  $H$  produces chain equivalent Floer complexes, so in order to compute the homology of the Floer complex we can assume that  $H$  is the Fenchel-dual of a strictly convex Lagrangian  $L$ . In this case, we prove that the Floer complex of  $(T^*M, Q, H, J)$  is isomorphic to the *Morse complex* of the Lagrangian action functional  $\mathbb{S}_L$  on the Hilbert manifold consisting of the absolutely continuous paths  $\gamma : [0, 1] \rightarrow M$  with square-integrable derivative and such that the pair  $(\gamma(0), \gamma(1))$  is in  $Q$ . The latter space is homotopically equivalent to the path space

$$C_Q([0, 1], M) = \{\gamma : [0, 1] \rightarrow M \mid \gamma \text{ is continuous and } (\gamma(0), \gamma(1)) \in Q\},$$

so Morse theory for  $\mathbb{S}_L$  implies that the homology of the Floer complex of  $(T^*M, Q, H, J)$  is isomorphic to the singular homology of  $C_Q([0, 1], M)$ . The isomorphism between the Morse and the Floer complexes is constructed by counting the space of solutions of a mixed problem, obtained by coupling the negative gradient flow of  $\mathbb{S}_L$  with respect to a  $W^{1,2}$ -metric with the Floer equation on the half-strip  $[0, +\infty[ \times [0, 1]$ .

These results generalize the case of Dirichlet boundary conditions ( $Q$  is the singleton  $\{(q_0, q_1)\}$  for some pair of points  $q_0, q_1 \in M$ , and  $C_Q([0, 1], M)$  has the homotopy type of the based loop space of  $M$ ) and the case of periodic boundary conditions ( $Q = \Delta$ , and  $C_Q([0, 1], M)$  is the free loop space of  $M$ ), studied by the first and last author in [3]. They also generalize the results by

Oh, concerning the case  $Q = M \times S$ , where  $S$  is a compact submanifold of  $M$  (with such a choice, the path space  $C_Q([0, 1], M)$  is homotopically equivalent to  $S$ , so one gets a finitely generated Floer homology, isomorphic to the singular homology of  $S$ ). See [14] and [13] for previous proofs of the isomorphism between the Floer homology for periodic Hamiltonian orbits on  $T^*M$  and the singular homology of the free loop space of  $M$  (see also the review paper [16]). Most of the arguments from [3] readily extend to the present more general setting, so we just sketch them here, focusing the analysis on the index questions, which constitute the more original part of this paper.

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## 1 Linear preliminaries

Let  $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$  be the cotangent space of the vector space  $\mathbb{R}^n$ . The *Liouville one-form* on  $T^*\mathbb{R}^n$  is the tautological one-form  $\theta_0 := p dq$ , that is

$$\theta_0(q, p)[(u, v)] := p[u], \quad \forall q, u \in \mathbb{R}^n, \forall p, v \in (\mathbb{R}^n)^*.$$

Its differential

$$\omega_0 := d\theta_0 = dp \wedge dq, \quad \omega_0[(q_1, p_1), (q_2, p_2)] := p_1[q_2] - p_2[q_1],$$

is the *standard symplectic form* on  $T^*\mathbb{R}^n$ . The group of linear automorphisms of  $T^*\mathbb{R}^n$  which preserve  $\omega_0$  is the *symplectic group*  $\text{Sp}(T^*\mathbb{R}^n)$ . The *Lagrangian Grassmannian*  $\mathcal{L}(T^*\mathbb{R}^n)$  is the space of all  $n$ -dimensional linear subspaces of  $T^*\mathbb{R}^n$  on which  $\omega_0$  vanishes identically.

If  $\lambda, \nu : [a, b] \rightarrow \mathcal{L}(T^*\mathbb{R}^n)$  are two continuous paths of Lagrangian subspaces, the *relative Maslov index*  $\mu(\lambda, \nu)$  is a half-integer counting the intersections  $\lambda(t) \cap \nu(t)$  algebraically. We refer to [11] for the definition and the main properties of the relative Maslov index. Here we just need to recall the formula for the relative Maslov index  $\mu(\lambda, \lambda_0)$  of a continuously differentiable Lagrangian path  $\lambda$  with respect to a constant one  $\lambda_0$ , in the case of *regular crossings*. Let  $\lambda : [a, b] \rightarrow \mathcal{L}(T^*\mathbb{R}^n)$  be a continuously differentiable curve, and let  $\lambda_0$  be in  $\mathcal{L}(T^*\mathbb{R}^n)$ . Fix  $t \in [a, b]$  and let  $\nu_0 \in \mathcal{L}(T^*\mathbb{R}^n)$  be a Lagrangian complement of  $\lambda(t)$ . If  $s$  belongs to a suitably small neighborhood of  $t$  in  $[a, b]$ , for every  $\xi \in \lambda(t)$  we can find a unique  $\eta(s) \in \nu_0$  such that  $\xi + \eta(s) \in \lambda(s)$ . The *crossing form*  $\Gamma(\lambda, \lambda_0, t)$  at  $t$  is the quadratic form on  $\lambda(t) \cap \lambda_0$  defined by

$$\Gamma(\lambda, \lambda_0, t) : \lambda(t) \cap \lambda_0 \rightarrow \mathbb{R}, \quad \xi \mapsto \left. \frac{d}{ds} \omega_0(\xi, \eta(s)) \right|_{s=t}. \quad (1)$$

The number  $t$  is said to be a *crossing* if  $\lambda(t) \cap \lambda_0 \neq (0)$ , and it is called a *regular crossing* if the above quadratic form is non-degenerate. Regular crossings are isolated, and if  $\lambda$  and  $\lambda_0$  have only regular crossings the relative Maslov index of  $\lambda$  with respect to  $\lambda_0$  is defined as

$$\mu(\lambda, \lambda_0) := \frac{1}{2} \text{sgn} \Gamma(\lambda, \lambda_0, a) + \sum_{a < t < b} \text{sgn} \Gamma(\lambda, \lambda_0, t) + \frac{1}{2} \text{sgn} \Gamma(\lambda, \lambda_0, b), \quad (2)$$

where  $\text{sgn}$  denotes the signature.

If  $V$  is a linear subspace of  $\mathbb{R}^n$ , its *conormal space*  $N^*V$  is the linear subspace of  $T^*\mathbb{R}^n$  defined by

$$N^*V := V \times V^\perp = \{(q, p) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \mid q \in V, p[u] = 0 \forall u \in V\}.$$

Conormal spaces are Lagrangian subspaces of  $T^*\mathbb{R}^n$ . The set of all conormal spaces is denoted by  $\mathcal{N}^*(\mathbb{R}^n)$ ,

$$\mathcal{N}^*(\mathbb{R}^n) := \{N^*V \mid V \in \text{Gr}(\mathbb{R}^n)\},$$

where  $\text{Gr}(\mathbb{R}^n)$  denotes the Grassmannian of all linear subspaces of  $\mathbb{R}^n$ . The conormal space of  $(0)$ ,  $N^*(0) = (0) \times (\mathbb{R}^n)^*$ , is called the *vertical subspace*. Note that if  $\alpha$  is a linear automorphism of  $\mathbb{R}^n$  and  $V \in \text{Gr}(\mathbb{R}^n)$ , then

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^T \end{pmatrix} N^*V = \alpha^{-1}V \times \alpha^T V^\perp = \alpha^{-1}V \times (\alpha^{-1}V)^\perp = N^*(\alpha^{-1}V), \quad (3)$$

where  $\alpha^T \in L((\mathbb{R}^n)^*, (\mathbb{R}^n)^*)$  denotes the transpose of  $\alpha$ .

Let  $C : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$  be the linear involution

$$C(q, p) := (q, -p), \quad \forall q \in \mathbb{R}^n, \forall p \in (\mathbb{R}^n)^*,$$

and note that  $C$  is anti-symplectic, meaning that

$$\omega_0(C\xi, C\eta) = -\omega_0(\xi, \eta), \quad \forall \xi, \eta \in T^*\mathbb{R}^n.$$

In particular,  $C$  maps Lagrangian subspaces into Lagrangian subspaces. Changing the sign of the symplectic structure changes the sign of the Maslov index, so the naturality property of the Maslov index implies that

$$\mu(C\lambda, C\nu) = -\mu(\lambda, \nu), \quad (4)$$

for every pair of continuous paths  $\lambda, \nu : [0, 1] \rightarrow \mathcal{L}(T^*\mathbb{R}^n)$ . Since conormal subspaces of  $T^*\mathbb{R}^n$  are  $C$ -invariant, we deduce the following:

**PROPOSITION 1.1** *If  $V, W : [0, 1] \rightarrow \text{Gr}(\mathbb{R}^n)$  are two continuous paths in the Grassmannian of  $\mathbb{R}^n$ , then  $\mu(N^*V, N^*W) = 0$ .*

The subgroup of the symplectic automorphisms of  $T^*\mathbb{R}^n$  which fix the vertical subspace is denoted by

$$\text{Sp}_v(T^*\mathbb{R}^n) := \{B \in \text{Sp}(T^*\mathbb{R}^n) \mid BN^*(0) = N^*(0)\}.$$

The elements of the above subgroup can be written in matrix form as

$$B = \begin{pmatrix} \alpha^{-1} & 0 \\ \beta & \alpha^T \end{pmatrix},$$

where  $\alpha \in \text{GL}(\mathbb{R}^n)$ ,  $\beta \in L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ , and  $\beta\alpha \in \text{L}_s(\mathbb{R}^n, (\mathbb{R}^n)^*)$ , the space of symmetric linear mappings. Note that every element of  $\text{Sp}_v(T^*\mathbb{R}^n)$  can be decomposed as

$$B = \begin{pmatrix} \alpha^{-1} & 0 \\ \beta & \alpha^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ \beta\alpha & I \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^T \end{pmatrix}. \quad (5)$$

The second result of this section is the following:

**PROPOSITION 1.2** *Let  $V_0, V_1$  be linear subspaces of  $\mathbb{R}^n$ , and let  $B : [0, 1] \rightarrow \text{Sp}_v(T^*\mathbb{R}^n)$  be a continuous path such that  $B(0)N^*V_0 = N^*V_0$  and  $B(1)N^*V_1 = N^*V_1$ . Then*

$$\mu(BN^*V_0, N^*V_1) = 0.$$

*Proof.* By (5), there are continuous paths  $\alpha : [0, 1] \rightarrow \text{GL}(\mathbb{R}^n)$  and  $\gamma : [0, 1] \rightarrow \text{L}_s(\mathbb{R}^n, (\mathbb{R}^n)^*)$  such that  $B = GA$  with

$$G := \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix}, \quad A := \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^T \end{pmatrix}.$$

The assumptions on  $B(0)$  and  $B(1)$  and the special form of  $G$  and  $A$  imply that

$$A(0)N^*V_0 = G(0)N^*V_0 = N^*V_0, \quad A(1)N^*V_1 = G(1)N^*V_1 = N^*V_1. \quad (6)$$

The affine path  $F(t) := tG(1) + (1-t)G(0)$  is homotopic with fixed end-points to the path  $G$  within the symplectic group  $\text{Sp}(T^*\mathbb{R}^n)$ , so by the homotopy property of the Maslov index

$$\mu(BN^*V_0, N^*V_1) = \mu(GAN^*V_0, N^*V_1) = \mu(FAN^*V_0, N^*V_1). \quad (7)$$

We can write  $F(t)$  as  $F_1(t)F_0(t)$ , where  $F_0$  and  $F_1$  are the symplectic paths

$$F_0(t) := \begin{pmatrix} I & 0 \\ (1-t)\gamma(0) & I \end{pmatrix}, \quad F_1(t) := \begin{pmatrix} I & 0 \\ t\gamma(1) & I \end{pmatrix}.$$

We note that  $F_0(t)$  preserves  $N^*V_0$ , while  $F_1(t)$  preserves  $N^*V_1$ , for every  $t \in [0, 1]$ . Then, by the naturality property of the Maslov index

$$\begin{aligned} \mu(FAN^*V_0, N^*V_1) &= \mu(F_1F_0AN^*V_0, N^*V_1) \\ &= \mu(F_0AN^*V_0, F_1^{-1}N^*V_1) = \mu(F_0AN^*V_0, N^*V_1). \end{aligned} \quad (8)$$

By the concatenation property of the Maslov index, (3), (6), and the fact that  $F_0(t)$  preserves  $N^*V_0$  for every  $t$  and is the identity for  $t = 1$ , we have the chain of equalities

$$\begin{aligned} \mu(F_0AN^*V_0, N^*V_1) &= \mu(F_0A(0)N^*V_0, N^*V_1) + \mu(F_0(1)AN^*V_0, N^*V_1) \\ &= \mu(F_0N^*V_0, N^*V_1) + \mu(AN^*V_0, N^*V_1) = \mu(N^*V_0, N^*V_1) + \mu(N^*(\alpha^{-1}V_0), N^*V_1) = 0, \end{aligned} \quad (9)$$

where the latter term vanishes because of Proposition 1.1. The conclusion follows from (7), (8), and (9).  $\square$

We conclude this section by discussing how graphs of symplectic automorphisms of  $T^*\mathbb{R}^n$  can be turned into Lagrangian subspaces of  $T^*\mathbb{R}^{2n}$ .

Let us identify the product  $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$  with  $T^*\mathbb{R}^{2n}$ . Then the graph of the linear involution  $C$  is the conormal space of the diagonal  $\Delta_{\mathbb{R}^n}$  in  $\mathbb{R}^n \times \mathbb{R}^n$ ,

$$\text{graph } C = N^*\Delta_{\mathbb{R}^n}.$$

Moreover, the fact that  $C$  is anti-symplectic easily implies that a linear endomorphism  $A : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$  is symplectic if and only if the graph of  $CA$  is a Lagrangian subspace<sup>1</sup> of  $T^*\mathbb{R}^{2n}$ , if and only if the graph of  $AC$  is a Lagrangian subspace of  $T^*\mathbb{R}^{2n}$ .

Theorem 3.2 in [11] implies that if  $A$  is a path of symplectic automorphisms of  $T^*\mathbb{R}^n$  and  $\lambda, \nu$  are paths of Lagrangian subspaces of  $T^*\mathbb{R}^n$ , then

$$\mu(A\lambda, \nu) = \mu(\text{graph } AC, C\lambda \times \nu) = -\mu(\text{graph } CA, \lambda \times C\nu). \quad (10)$$

## 2 Hamiltonian systems on cotangent bundles with conormal boundary conditions

Let  $M$  be a smooth manifold of dimension  $n$ , and let  $T^*M$  be the cotangent bundle of  $M$  with projection  $\tau^* : T^*M \rightarrow M$ . The cotangent bundle  $T^*M$  carries the following canonical structures: The *Liouville one-form*  $\theta$  and the *Liouville vector field*  $\eta$  which can be defined intrinsically by

$$\theta(x)[\zeta] = x[D\tau^*(x)[\zeta]] = d\theta(x)[\eta, \zeta], \quad \forall x \in T^*M, \forall \zeta \in T_x T^*M,$$

and the symplectic structure  $\omega = d\theta$ . Elements of  $T^*M$  are also denoted as pairs  $(q, p)$ , with  $q \in M, p \in T_q^*M$ .

<sup>1</sup>Here  $T^*\mathbb{R}^{2n}$  is endowed with its standard symplectic structure. In symplectic geometry it is also customary to endow the product of a symplectic vector space  $(V, \omega)$  with itself by the symplectic structure  $\omega \times (-\omega)$ . With the latter convention, the product of two Lagrangian subspaces is Lagrangian, and an endomorphism is symplectic if and only if its graph is Lagrangian. When dealing with cotangent spaces and conormal spaces it seems more convenient to adopt the former convention, even if it involves the appearance of the involution  $C$ .

The *vertical subbundle* is the  $n$ -dimensional vector subbundle of  $TT^*M$  whose fiber at  $x \in T^*M$  is the linear subspace

$$T_x^v T^*M := \ker D\tau^*(x) \subset T_x T^*M.$$

Each vertical subspace  $T_x^v T^*M$  is a Lagrangian subspace of the symplectic vector space  $(T_x T^*M, \omega_x)$ .

If  $Q$  is a smooth submanifold of  $M$ , the *conormal bundle* of  $Q$  is defined by

$$N^*Q := \{x \in T^*M \mid \tau^*(x) \in Q, x[\xi] = 0 \forall \xi \in T_{\tau^*(x)}Q\}.$$

It inherits the structure of a vector bundle over  $Q$  of dimension  $\text{codim } Q$ . Moreover,  $N^*Q$  is a *Lagrangian submanifold* of  $T^*M$ , meaning that its tangent space at every point  $x$  is a Lagrangian subspace of  $(T_x T^*M, \omega_x)$ . The conormal bundle of the whole  $M$  is the zero-section, while the conormal bundle of a point  $Q = \{q\}$  is  $T_q^*M$ .

A smooth Hamiltonian  $H : [0, 1] \times T^*M \rightarrow \mathbb{R}$  induces a time dependent vector field  $X_H$  on  $T^*M$  defined by

$$\omega(X_H(t, x), \zeta) = -D_x H(t, x)[\zeta], \quad \forall \zeta \in T_x T^*M.$$

We denote by  $\phi_t^H$  the non-autonomous flow determined by the ODE

$$x'(t) = X_H(t, x(t)). \quad (11)$$

**Local boundary conditions.** Let  $Q_0$  and  $Q_1$  be submanifolds of  $M$ . We are interested in the set of solutions  $x : [0, 1] \rightarrow T^*M$  of the Hamiltonian system (11) such that

$$x(0) \in N^*Q_0, \quad x(1) \in N^*Q_1. \quad (12)$$

In other words, we are considering Hamiltonian orbits  $t \mapsto (q(t), p(t))$  such that  $q(0) \in Q_0$ ,  $q(1) \in Q_1$ ,  $p(0)$  vanishes on  $T_{q(0)}Q_0$ , and  $p(1)$  vanishes on  $T_{q(1)}Q_1$ . In particular, when  $Q_0 = \{q_0\}$  and  $Q_1 = \{q_1\}$  are points, we find Hamiltonian orbits whose projection onto  $M$  joins  $q_0$  and  $q_1$ , without any other conditions. When  $Q_0 = Q_1 = M$ , (12) reduces to the Neumann boundary conditions  $p(0) = p(1) = 0$ .

The *nullity*  $\nu^{Q_0, Q_1}(x)$  of the solution  $x$  of (11-12) is the non-negative integer

$$\nu^{Q_0, Q_1}(x) = \dim D\phi_1^H(x(0))T_{x(0)}N^*Q_0 \cap T_{x(1)}N^*Q_1,$$

and  $x$  is said to be *non-degenerate* if  $\nu^{Q_0, Q_1}(x) = 0$ , or equivalently if  $\phi_1^H(N^*Q_0)$  is transverse to  $N^*Q_1$  at  $x(1)$ .

We wish to associate a Maslov index to each solution of the boundary problem (11-12). If  $x : [0, 1] \rightarrow T^*M$  is such a solution, let  $\Phi$  be a *vertical preserving symplectic trivialization* of the symplectic bundle  $x^*(TT^*M)$ : for every  $t \in [0, 1]$ ,  $\Phi(t)$  is a symplectic linear isomorphism from  $T_{x(t)}T^*M$  to  $T^*\mathbb{R}^n$ ,

$$\Phi(t) : T_{x(t)}T^*M \rightarrow T^*\mathbb{R}^n,$$

which maps  $T_{x(t)}^v T^*M$  onto the vertical subspace  $N^*(0) = (0) \times (\mathbb{R}^n)^*$ , the dependence on  $t$  being smooth. Moreover, we assume that the tangent spaces of the conormal bundles of  $Q_0$  and  $Q_1$  are mapped into conormal subspaces of  $T^*\mathbb{R}^n$ :

$$\Phi(0)T_{x(0)}N^*Q_0 \in \mathcal{N}^*(\mathbb{R}^n) \quad \text{and} \quad \Phi(1)T_{x(1)}N^*Q_1 \in \mathcal{N}^*(\mathbb{R}^n). \quad (13)$$

Let  $V_0^\Phi$  and  $V_1^\Phi$  be the linear subspaces of  $\mathbb{R}^n$  defined by

$$N^*V_0^\Phi = \Phi(0)T_{x(0)}N^*Q_0, \quad N^*V_1^\Phi = \Phi(1)T_{x(1)}N^*Q_1.$$

The fact that  $\Phi$  maps the vertical subbundle into the vertical subspace implies that  $\dim V_0^\Phi = \dim Q_0$  and  $\dim V_1^\Phi = \dim Q_1$ . Since the flow  $\phi_t^H$  is symplectic, the linear mapping

$$G^\Phi(t) := \Phi(t)D\phi_t^H(x(0))\Phi(0)^{-1} \quad (14)$$

is a symplectic automorphism of  $T^*\mathbb{R}^n$ . Notice that

$$\nu^Q(x) = \dim G^\Phi(1)N^*V_0^\Phi \cap N^*V_1^\Phi.$$



DEFINITION 2.1 *The Maslov index of a solution  $x$  of (11-12) is the half-integer*

$$\mu^{Q_0, Q_1}(x) := \mu(G^\Phi N^* V_0^\Phi, N^* V_1^\Phi) + \frac{1}{2}(\dim Q_0 + \dim Q_1 - n).$$

The shift

$$\frac{1}{2}(\dim Q_0 + \dim Q_1 - n)$$

comes from the fact that in the case of convex Hamiltonians we would like the Maslov index of a non-degenerate solution to coincide with the Morse index of the corresponding extremal curve of the Lagrangian action functional (see section 3 below). The next result shows that the Maslov index of  $x$  is well-defined:

PROPOSITION 2.2 *Assume that  $\Phi$  and  $\Psi$  are two vertical preserving symplectic trivializations of  $x^*(TT^*M)$ , and that they both satisfy (13). Then*

$$\mu(G^\Phi N^* V_0^\Phi, N^* V_1^\Phi) = \mu(G^\Psi N^* V_0^\Psi, N^* V_1^\Psi). \quad (15)$$

*If  $x$  is non-degenerate, the number  $\mu^{Q_0, Q_1}(x)$  is an integer.*

*Proof.* Since both  $\Phi$  and  $\Psi$  are vertical preserving, the path  $B(t) := \Psi(t)\Phi(t)^{-1}$  takes values into the subgroup  $\mathrm{Sp}_v(T^*\mathbb{R}^n)$ . We first prove the identity (15) under the extra assumption

$$V_0^\Phi = V_0^\Psi = V_0, \quad V_1^\Phi = V_1^\Psi = V_1. \quad (16)$$

In this case,

$$B(0)N^*V_0 = N^*V_0, \quad B(1)N^*V_1 = N^*V_1. \quad (17)$$

Consider the homotopy of Lagrangian subspaces

$$\lambda(s, t) := B(s)G^\Phi(st)N^*V_0.$$

By the concatenation and the homotopy property of the Maslov index,

$$\mu(\lambda|_{[0,1] \times \{0\}}, N^*V_1) + \mu(\lambda|_{\{1\} \times [0,1]}, N^*V_1) = \mu(\lambda|_{\{0\} \times [0,1]}, N^*V_1) + \mu(\lambda|_{[0,1] \times \{1\}}, N^*V_1). \quad (18)$$

Since  $\lambda(0, t) = B(0)N^*V_0$  is constant in  $t$ ,

$$\mu(\lambda|_{\{0\} \times [0,1]}, N^*V_1) = 0. \quad (19)$$

By the naturality of the Maslov index and since  $B(1)$  preserves  $N^*V_1$ ,

$$\mu(\lambda|_{\{1\} \times [0,1]}, N^*V_1) = \mu(B(1)G^\Phi N^*V_0, N^*V_1) = \mu(G^\Phi N^*V_0, N^*V_1). \quad (20)$$

Moreover,

$$\mu(\lambda|_{[0,1] \times \{0\}}, N^*V_1) = \mu(BN^*V_0, N^*V_1) = 0, \quad (21)$$

because of (17) and Proposition 1.2. Finally,

$$\mu(\lambda|_{[0,1] \times \{1\}}, N^*V_1) = \mu(BG^\Phi N^*V_0, N^*V_1) = \mu(G^\Psi N^*V_0, N^*V_1). \quad (22)$$

Then (18) together with (19), (20), (21), and (22) imply the identity (15) under the extra assumption (16).

Now we deal with the general case. Let  $\alpha_0, \alpha_1 : [0, 1] \rightarrow \mathrm{GL}(\mathbb{R}^n)$  be continuous paths such that

$$\alpha_0(1) = \alpha_1(0) = I, \quad \alpha_0(0)V_0^\Psi = V_0^\Phi, \quad \alpha_1(1)V_1^\Psi = V_1^\Phi.$$

Consider the paths in  $\mathrm{Sp}_v(T^*\mathbb{R}^n)$

$$A_0 = \begin{pmatrix} \alpha_0^{-1} & 0 \\ 0 & \alpha_0^T \end{pmatrix}, \quad A_1 = \begin{pmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_1^T \end{pmatrix}.$$

Then  $A_0(1) = A_1(0) = I$ , and by (3)

$$A_0(0)N^*V_0^\Phi = N^*V_0^\Psi, \quad A_1(1)N^*V_1^\Phi = N^*V_1^\Psi.$$

The trivialization  $\Theta(t) := A_1(t)A_0(t)\Phi(t)$  is vertical preserving, and

$$\begin{aligned} \Theta(0)T_{x(0)}N^*Q_0 &= A_1(0)A_0(0)\Phi(0)T_{x(0)}N^*Q_0 = A_0(0)N^*V_0^\Phi = N^*V_0^\Psi, \\ \Theta(1)T_{x(1)}N^*Q_1 &= A_1(1)A_0(1)\Phi(1)T_{x(1)}N^*Q_1 = A_1(1)N^*V_1^\Phi = N^*V_1^\Psi. \end{aligned}$$

Therefore,  $\Theta$  is an admissible trivialization with  $V_0^\Theta = V_0^\Psi$  and  $V_1^\Theta = V_1^\Psi$ . By the particular case treated above,

$$\mu(G^\Theta N^*V_0^\Theta, N^*V_1^\Theta) = \mu(G^\Psi N^*V_0^\Psi, N^*V_1^\Psi),$$

so it is enough to prove that the left-hand side coincides with  $\mu(G^\Phi N^*V_0^\Phi, N^*V_1^\Phi)$ . By the naturality property of the Maslov index,

$$\mu(G^\Theta N^*V_0^\Theta, N^*V_1^\Theta) = \mu(A_1A_0G^\Phi N^*V_0^\Phi, A_1(1)N^*V_1^\Phi) = \mu(A_1(1)^{-1}A_1A_0G^\Phi N^*V_0^\Phi, N^*V_1^\Phi).$$

By the concatenation property of the Maslov index the latter quantity coincides with

$$\begin{aligned} &\mu(A_1(1)^{-1}A_1A_0G^\Phi(0)N^*V_0^\Phi, N^*V_1^\Phi) + \mu(A_1(1)^{-1}A_1(1)A_0(1)G^\Phi N^*V_0^\Phi, N^*V_1^\Phi) \\ &= \mu(A_1(1)^{-1}A_1A_0N^*V_0^\Phi, N^*V_1^\Phi) + \mu(G^\Phi N^*V_0^\Phi, N^*V_1^\Phi). \end{aligned}$$

By (3),  $A_1(1)^{-1}A_1(t)A_0(t)N^*V_0^\Phi$  is a conormal subspace of  $T^*\mathbb{R}^n$  for every  $t \in [0, 1]$ , so the first addendum in the latter expression vanishes because of Proposition 1.1. The identity (15) follows.

If  $x$  is non-degenerate,  $G^\Phi(1)V_0^\Phi \cap V_1^\Phi = (0)$ , whereas the intersection  $G^\Phi(0)V_0^\Phi \cap V_1^\Phi = V_0^\Phi \cap V_1^\Phi$  might be non-trivial. By Corollary 4.12 in [11], the relative Maslov index  $\mu(G^\Phi V_0^\Phi, V_1^\Phi)$  differs by an integer from the number  $d/2$ , where

$$d := \dim N^*V_0^\Phi \cap N^*V_1^\Phi.$$

Since

$$N^*V_0^\Phi \cap N^*V_1^\Phi = (V_0^\Phi \cap V_1^\Phi) \times (V_0^{\Phi\perp} \cap V_1^{\Phi\perp}) = (V_0^\Phi \cap V_1^\Phi) \times (V_0^\Phi + V_1^\Phi)^\perp,$$

the number

$$d = \dim V_0^\Phi \cap V_1^\Phi + n - \dim(V_0^\Phi + V_1^\Phi) = \dim V_0^\Phi + \dim V_1^\Phi + n - 2\dim(V_0^\Phi + V_1^\Phi)$$

has the parity of

$$\dim Q_0 + \dim Q_1 - n = \dim V_0^\Phi + \dim V_1^\Phi - n.$$

It follows that

$$\mu^{Q_0, Q_1}(x) = \mu(G^\Phi V_0^\Phi, V_1^\Phi) + \frac{1}{2}(\dim Q_0 + \dim Q_1 - n)$$

is an integer, as claimed.  $\square$

**Non local boundary conditions.** The smooth involution

$$\mathcal{C} : T^*M \rightarrow T^*M, \quad \mathcal{C}(x) = -x,$$

is anti-symplectic, meaning that  $\mathcal{C}^*\omega = -\omega$ . Its graph in  $T^*M \times T^*M = T^*M^2$  is the conormal bundle of the diagonal  $\Delta_M$  of  $M \times M$ . Note also that conormal subbundles in  $T^*M$  are  $\mathcal{C}$ -invariant.

Given a smooth submanifold  $Q \subset M \times M$ , we are interested in the set of all solutions  $x : [0, 1] \rightarrow T^*M$  of (11) satisfying the nonlocal boundary condition

$$(x(0), -x(1)) \in N^*Q. \tag{23}$$

Equivalently, we are looking at the Lagrangian intersection problem

$$(\text{graph } \mathcal{C} \circ \phi_1^H) \cap N^*Q$$

in  $T^*M^2$ . A solution  $x$  of (11-23) is called *non-degenerate* if the above intersection is transverse at  $(x(0), -x(1))$ , or equivalently if the *nullity* of  $x$ , defined as

$$\nu^Q(x) := \dim(T_{(x(0), -x(1))} \text{graph } \mathcal{C} \circ \phi_1^H) \cap T_{(x(0), -x(1))} N^*Q,$$

is zero.

When  $Q = Q_0 \times Q_1$  is the product of two submanifolds  $Q_0, Q_1$  of  $M$ , the boundary condition (23) reduces to the local boundary condition (12). A common choice for  $Q$  is the diagonal  $\Delta_M$  in  $M \times M$ : this choice produces 1-periodic Hamiltonian orbits (provided that  $H$  can be extended to  $\mathbb{R} \times T^*M$  as a 1-periodic function). Other choices are also interesting: for instance in [4] it is shown that the pair-of-pants product on Floer homology for periodic orbits on the cotangent bundle of  $M$  factors through a Floer homology for Hamiltonian orbits on  $T^*(M \times M)$  with nonlocal boundary condition (23) given by the submanifold  $Q$  of  $M \times M \times M \times M$  consisting of all 4-uples  $(q, q, q, q)$ .

The nonlocal boundary value problem (11-23) on  $T^*M$  can be turned into a local boundary value problem on  $T^*M^2 = T^*M \times T^*M$ . Indeed, a curve  $x : [0, 1] \rightarrow T^*M$  is an orbit for the Hamiltonian vector field  $X_H$  on  $T^*M$  if and only if the curve

$$y : [0, 1] \rightarrow T^*M^2, \quad y(t) = (x(t/2), -x(1-t/2)),$$

is an orbit for the Hamiltonian vector field  $X_K$  on  $T^*M^2$ , where  $K \in C^\infty([0, 1] \times T^*M^2)$  is the Hamiltonian

$$K(t, y_1, y_2) := \frac{1}{2}H(t/2, y_1) + \frac{1}{2}H(1-t/2, -y_2).$$

By construction,

$$y(1) = (x(1/2), -x(1/2)) \in \text{graph } \mathcal{C} = N^*\Delta_M,$$

and the curve  $x$  satisfies the nonlocal boundary condition (23) if and only if

$$y(0) = (x(0), -x(1)) \in N^*Q.$$

Therefore, the nonlocal boundary value problem (11-23) for  $x : [0, 1] \rightarrow T^*M$  is equivalent to the following local boundary value problem for  $y : [0, 1] \rightarrow T^*M^2$ :

$$y'(t) = X_K(t, y(t)), \tag{24}$$

$$y(0) \in N^*Q, \quad y(1) \in N^*\Delta_M. \tag{25}$$

Using the identity (30) below, it is easy to show that

$$\nu^{Q, \Delta_M}(y) = \nu^Q(x).$$

In particular,  $x$  is a non-degenerate solution of (11- 23) if and only if  $y$  is non-degenerate solution of (24-25). We define the Maslov index of the solution  $x$  of (11-23) as the Maslov index of the solution  $y$  of (24-25):

$$\mu^Q(x) := \mu^{Q, \Delta_M}(y).$$

It is also convenient to have a formula for the latter Maslov index which avoids the above local reformulation.

**PROPOSITION 2.3** *Assume that  $\Phi$  is a vertical preserving symplectic trivialization of  $x^*(TT^*M)$ , and that the linear subspace*

$$(\Phi(0) \times C\Phi(1)D\mathcal{C}(-x(1)))T_{(x(0), -x(1))}N^*Q \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n = T^*\mathbb{R}^{2n}$$

*is a conormal subspace of  $T^*\mathbb{R}^{2n}$ , that we denote by  $N^*W^\Phi$ , with  $W^\Phi \in \text{Gr}(\mathbb{R}^{2n})$ . Then*

$$\mu^Q(x) = \mu(\text{graph } G^\Phi C, N^*W^\Phi) + \frac{1}{2}(\dim Q - n), \tag{26}$$

*where  $G^\Phi$  is defined by (14). In particular, if  $Q = Q_0 \times Q_1$  with  $Q_0$  and  $Q_1$  smooth submanifolds of  $M$ , then  $\mu^Q(x) = \mu^{Q_0, Q_1}(x)$ .*

*Proof.* The isomorphisms

$$\Psi(t) := \Phi(t/2) \times C\Phi(1-t/2)D\mathcal{C}(-x(1-t/2)) : T_{y(t)}T^*M^2 \rightarrow T^*\mathbb{R}^{2n}$$

provide us with a vertical preserving symplectic trivialization of  $y^*(TT^*M^2)$ . By assumption,

$$\Psi(0)T_{y(0)}N^*Q = (\Phi(0) \times C\Phi(1)D\mathcal{C}(-x(1)))T_{(x(0),-x(1))}N^*Q = N^*W^\Phi.$$

Moreover, since  $\mathcal{C}$  is an involution,

$$\begin{aligned} \Psi(1)T_{y(1)}N^*\Delta_M &= \Psi(1)T_{(x(1/2),-x(1/2))}\text{graph } \mathcal{C} = \Psi(1)\text{graph } D\mathcal{C}(x(1/2)) \\ &= (\Phi(1/2) \times C\Phi(1/2)D\mathcal{C}(-x(1/2)))\text{graph } D\mathcal{C}(x(1/2)) \\ &= \{(\Phi(1/2)\xi, C\Phi(1/2)\xi) \mid \xi \in T_{x(1/2)}T^*M\} = \text{graph } C = N^*\Delta_{\mathbb{R}^n}. \end{aligned} \quad (27)$$

Therefore,  $\Psi$  is an admissible trivialization of  $y^*(TT^*M^2)$ , and

$$\mu^Q(x) = \mu^{Q,\Delta_M}(y) = \mu(G^\Psi N^*W^\Phi, N^*\Delta_{\mathbb{R}^n}) + \frac{1}{2}(\dim Q + \dim \Delta_M - 2n), \quad (28)$$

where  $G^\Psi$  is defined as

$$G^\Psi(t) := \Psi(t)D\phi_t^K(y(0))\Psi(0)^{-1},$$

see (14). We denote by  $\phi_{t,s}^H$  the solution of

$$\begin{cases} \phi_{s,s}^H(z) &= z, \\ \partial_t \phi_{t,s}^H(z) &= X_H(t, \phi_{t,s}^H(z)), \end{cases} \quad \forall z \in T^*M, \forall s, t \in [0, 1],$$

and we omit the second subscript  $s$  when  $s = 0$ . By differentiating the identity

$$\phi_{r,t}(\phi_{t,s}(z)) = \phi_{r,s}(z) \quad \forall z \in T^*M, \forall r, s, t \in [0, 1],$$

we find

$$D\phi_{r,t}(\phi_{t,s}(z))D\phi_{t,s}(z) = D\phi_{r,s}(z) \quad \forall z \in T^*M, \forall r, s, t \in [0, 1]. \quad (29)$$

By construction, the flow of  $X^K$  is related to the flow of  $X^H$  by the formula

$$\phi_t^K(y_1, y_2) = (\phi_{t/2}^H(y_1), -\phi_{1-t/2,1}^H(-y_2)).$$

It follows that

$$D\phi_t^K(y(0)) = D\phi_{t/2}^H(x(0)) \times D\mathcal{C}(x(1-t/2))D\phi_{1-t/2,1}^H(x(1))D\mathcal{C}(-x(1)), \quad (30)$$

and

$$G^\Psi(t) = \Phi(t/2)D\phi_{t/2}^H(x(0))\Phi(0)^{-1} \times C\Phi(1-t/2)D\phi_{1-t/2,1}^H(x(1))\Phi(1)^{-1}C.$$

By (29), the inverse of this isomorphism can be written as

$$G^\Psi(t)^{-1} = \Phi(0)D\phi_{t/2}^H(x(0))^{-1}\Phi(t/2)^{-1} \times C\Phi(1)D\phi_{1,1-t/2}^H(x(1-t/2))\Phi(1-t/2)^{-1}C.$$

Then

$$G^\Psi(t)^{-1}N^*\Delta_{\mathbb{R}^n} = G^\Psi(t)^{-1}\text{graph } C = \text{graph } CA(t),$$

where  $A$  is the symplectic path

$$A(t) := \Phi(1)D\phi_{1,1-t/2}^H(x(1-t/2))\Phi(1-t/2)^{-1}\Phi(t/2)D\phi_{t/2}^H(x(0))\Phi(0)^{-1}.$$

Note that

$$A(0) = I, \quad A(1) = \Phi(1)D\phi_1^H(x(0))\Phi(0)^{-1},$$

and that this path is homotopic with fixed end-points to the path  $G^\Phi$  by the symplectic homotopy mapping  $(s, t)$  into

$$\Phi(1)D\phi_{1,1-\frac{1}{2}-s\frac{1-t}{2}}^H\left(x\left(1-\frac{t}{2}-s\frac{1-t}{2}\right)\right)\Phi\left(1-\frac{t}{2}-s\frac{1-t}{2}\right)^{-1}\Phi\left(\frac{t}{2}+s\frac{1-t}{2}\right)D\phi_{\frac{1}{2}+s\frac{1-t}{2}}^H(x(0))\Phi(0)^{-1}.$$

Therefore, by the naturality and the homotopy properties of the Maslov index,

$$\begin{aligned}\mu(G^\Psi N^*W^\Phi, N^*\Delta_{\mathbb{R}^n}) &= \mu(N^*W^\Phi, G^{\Psi^{-1}}N^*\Delta_{\mathbb{R}^n}) = \mu(N^*W^\Phi, \text{graph } CA) \\ &= \mu(N^*W^\Phi, \text{graph } CG^\Phi) = -\mu(\text{graph } CG^\Phi, N^*W^\Phi).\end{aligned}\quad (31)$$

The conormal subspace  $N^*W^\Phi$  is invariant with respect to the anti-symplectic involution  $C \times C$ , while  $(C \times C)\text{graph } CG^\Phi = \text{graph } G^\Phi C$ . Then the identity (4) implies that

$$\mu(\text{graph } CG^\Phi, N^*W^\Phi) = -\mu(\text{graph } G^\Phi C, N^*W^\Phi).\quad (32)$$

Formulas (31), and (32) imply

$$\mu(G^\Psi N^*W^\Phi, N^*\Delta_{\mathbb{R}^n}) = \mu(\text{graph } G^\Phi C, N^*W^\Phi).\quad (33)$$

Identities (28) and (33) imply (26).

If now  $Q = Q_0 \times Q_1$ , we have  $N^*Q = N^*Q_0 \times N^*Q_1$ , and we can choose a vertical preserving symplectic trivialization  $\Phi$  of  $x^*(TT^*M)$  such that

$$\Phi(0)T_{x(0)}N^*Q_0 = N^*V_0^\Phi, \quad \Phi(1)T_{x(1)}N^*Q_1 = N^*V_1^\Phi,$$

with  $V_0^\Phi$  and  $V_1^\Phi$  in  $\text{Gr}(\mathbb{R}^n)$ . It follows that  $W^\Phi = V_0^\Phi \times V_1^\Phi$ , and by the identity (10) we have

$$\mu(\text{graph } G^\Phi C, N^*W^\Phi) = \mu(\text{graph } G^\Phi C, N^*V_0^\Phi \times N^*V_1^\Phi) = \mu(G^\Phi N^*V_0^\Phi, N^*V_1^\Phi).\quad (34)$$

By (26) and (34) we deduce that

$$\mu^Q(x) = \mu(G^\Phi N^*V_0^\Phi, N^*V_1^\Phi) + \frac{1}{2}(\dim Q_0 + \dim Q_1 - n),$$

which is precisely  $\mu^{Q_0, Q_1}(x)$ . This concludes the proof.  $\square$

REMARK 2.4 (Periodic boundary conditions) *In the particular case  $Q = \Delta_M$ , we have*

$$(\Phi(0) \times C\Phi(1)D\mathcal{C}(-x(1)))T_{(x(0), -x(1))}N^*\Delta_M = N^*\Delta_{\mathbb{R}^n},$$

see (27). So any vertical preserving trivialization  $\Phi$  of  $x^*(TT^*M)$  satisfies the assumption of Proposition 2.3, and the Maslov index of the periodic orbit  $x : \mathbb{T} \rightarrow T^*M$ , where  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ , is

$$\mu^{\Delta_M}(x) = \mu(\text{graph } G^\Phi C, N^*\Delta_{\mathbb{R}^n}),$$

which is precisely the Conley-Zehnder index  $\mu_{CZ}(G^\Phi)$  of the symplectic path  $G^\Phi$ . Note that the trivialization  $\Phi$  need not be periodic: if  $M$  is not orientable and  $x = (q, p)$  is a closed loop such that the vector bundle  $q^*(TM)$  over  $\mathbb{T}$  is not orientable, then there are no vertical preserving periodic trivializations of  $x^*(TT^*M)$ . In this case, one can identify suitable classes of non-vertical-preserving periodic trivializations for which the formula relating  $\mu^{\Delta_M}(x)$  to the Conley-Zehnder index of the corresponding symplectic path involves just a correction term  $+1$ , see [15].

### 3 The dual Lagrangian formulation and the index theorem

In this section we assume that the Hamiltonian  $H \in C^\infty([0, 1] \times T^*M)$  satisfies the classical Tonelli assumptions: It is  $C^2$ -strictly convex and superlinear, that is

$$D_{pp}H(t, q, p) > 0 \quad \forall (t, q, p) \in [0, 1] \times T^*M, \quad (35)$$

$$\lim_{|p| \rightarrow \infty} \frac{H(t, q, p)}{|p|} = +\infty \quad \text{uniformly in } (t, q) \in [0, 1] \times M. \quad (36)$$

Here the norm  $|p|$  of the covector  $p \in T_q^*M$  is induced by some fixed Riemannian metric on  $M$ . If  $M$  is compact, the superlinearity condition does not depend on the choice of such a metric.

Under these assumptions, the Fenchel transform defines a smooth time-dependent Lagrangian on  $TM$ ,

$$L(t, q, v) := \max_{p \in T_q^*M} (\langle p, v \rangle - H(t, q, p)), \quad \forall (t, q, v) \in [0, 1] \times TM,$$

which is also  $C^2$ -strictly convex and superlinear,

$$D_{vv}L(t, q, v) > 0 \quad \forall (t, q, v) \in [0, 1] \times TM,$$

$$\lim_{|v| \rightarrow \infty} \frac{L(t, q, v)}{|v|} = +\infty \quad \text{uniformly in } (t, q) \in [0, 1] \times M.$$

Since the Fenchel transform is an involution, we also have

$$H(t, q, p) = \max_{v \in T_qM} (\langle p, v \rangle - L(t, q, v)), \quad \forall (t, q, p) \in [0, 1] \times T^*M. \quad (37)$$

Furthermore, the Legendre duality defines a diffeomorphism

$$\mathcal{L} : [0, 1] \times TM \rightarrow [0, 1] \times T^*M, \quad (t, q, v) \rightarrow (t, q, D_vL(t, q, v)),$$

such that

$$L(t, q, v) = \langle p, v \rangle - H(t, q, p) \iff (t, q, p) = \mathcal{L}(t, q, v). \quad (38)$$

A smooth curve  $x : [0, 1] \rightarrow T^*M$  is an orbit of the Hamiltonian vector field  $X_H$  if and only if the curve  $\gamma := \pi \circ x : [0, 1] \rightarrow M$  is an absolutely continuous extremal of the Lagrangian action functional

$$\mathbb{S}_L(\gamma) := \int_0^1 L(t, \gamma(t), \gamma'(t)) dt.$$

The corresponding Euler-Lagrange equation can be written in local coordinates as

$$\frac{d}{dt} \partial_v L(t, \gamma(t), \gamma'(t)) = \partial_q L(t, \gamma(t), \gamma'(t)). \quad (39)$$

If  $Q$  is a non-empty submanifold of  $M \times M$ , the non-local boundary condition (23) is translated into the conditions

$$(\gamma(0), \gamma(1)) \in Q, \quad (40)$$

$$D_v L(0, \gamma(0), \gamma'(0))[\xi_0] = D_v L(1, \gamma(1), \gamma'(1))[\xi_1] \quad \forall (\xi_0, \xi_1) \in T_{(\gamma(0), \gamma(1))}Q. \quad (41)$$

The second condition is the natural boundary condition induced by the first one, meaning that every curve which is an extremal curve of  $\mathbb{S}_L$  among all curves satisfying (40) necessarily satisfies (41).

In order to study the second variation of  $\mathbb{S}_L$  at the extremal curve  $\gamma$ , it is convenient to localize the problem in  $\mathbb{R}^n$ . This can be done by choosing a smooth local coordinate system

$$[0, 1] \times \mathbb{R}^n \rightarrow [0, 1] \times M, \quad (t, q) \mapsto (t, \varphi_t(q)),$$

such that  $\gamma(t) \in \varphi_t(\mathbb{R}^n)$  for every  $t \in [0, 1]$ . Such a diffeomorphism induces the tangent bundle and cotangent bundles coordinate systems

$$[0, 1] \times T\mathbb{R}^n \rightarrow [0, 1] \times TM, \quad (t, q, v) \mapsto (t, \varphi_t(q), D\varphi_t(q)[v]), \quad (42)$$

$$[0, 1] \times T^*\mathbb{R}^n \rightarrow [0, 1] \times T^*M \quad (t, q, p) \mapsto (t, \varphi_t(q), (D\varphi_t(q))^*{}^{-1}[p]). \quad (43)$$

If we pull back the Lagrangian  $L$  and the Hamiltonian  $H$  by the above diffeomorphisms, we obtain a smooth Lagrangian on  $[0, 1] \times T\mathbb{R}^n$  - that we still denote by  $L$  - and a smooth Hamiltonian on  $[0, 1] \times T^*\mathbb{R}^n$  - that we still denote by  $H$ . These new functions are still related by Fenchel duality. The submanifold  $Q \subset M \times M$  can also be pulled back in  $\mathbb{R}^n \times \mathbb{R}^n$  by the map  $\varphi_0 \times \varphi_1$ . The resulting submanifold of  $\mathbb{R}^n \times \mathbb{R}^n$  is still denoted by  $Q$ . The cotangent bundle coordinate system (43) induces a symplectic trivialization of  $x^*(TT^*M)$  which preserves the vertical subspaces and maps conormal subbundles into conormal subbundles. In particular, this trivialization satisfies the assumptions of Proposition 2.3.

The solution  $\gamma$  of (39-40-41) is now a curve  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ . Let  $i^Q(\gamma)$  be its Morse index, that is the dimension of a maximal subspace of the Hilbert space

$$W_W^{1,2}([0, 1[, \mathbb{R}^n) := \{u \in W^{1,2}([0, 1[, \mathbb{R}^n) \mid (u(0), u(1)) \in W\}, \text{ where } W := T_{(\gamma(0), \gamma(1))}Q,$$

on which the second variation

$$\begin{aligned} d^2\mathbb{S}_L(\gamma)[u, v] := & \int_0^1 \left( D_{vv}L(t, \gamma, \gamma')[u', u'] + D_{qv}L(t, \gamma, \gamma')[u', v] \right. \\ & \left. + D_{vq}L(t, \gamma, \gamma')[u, v'] + D_{qq}L(t, \gamma, \gamma')[u, v] \right) dt \end{aligned}$$

is negative definite. The nullity of such a quadratic form is denoted by  $\nu^Q(\gamma)$ ,

$$\nu^Q(\gamma) := \dim \left\{ u \in W_W^{1,2}([0, 1[, \mathbb{R}^n) \mid d^2\mathbb{S}_L(\gamma)[u, v] = 0 \text{ for every } v \in W_W^{1,2}([0, 1[, \mathbb{R}^n) \right\}.$$

The following index theorem relates the Morse index and nullity of  $\gamma$  to the relative Maslov index and nullity of the corresponding Hamiltonian orbit:

**THEOREM 3.1** *Let  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  be a solution of (39-40-41), and let  $x : [0, 1] \rightarrow T^*\mathbb{R}^n$  be the corresponding Hamiltonian orbit. Let  $\lambda$  be the path of Lagrangian subspaces of  $T^*\mathbb{R}^n \times T^*\mathbb{R}^n = T^*\mathbb{R}^{2n}$  defined by*

$$\lambda(t) := \text{graph } D\phi_t^H(x(0))C, \quad t \in [0, 1],$$

where  $\phi_t^H$  denotes the Hamiltonian flow and  $C$  is the anti-symplectic involution  $C(q, p) = (q, -p)$ . Let  $W = T_{(\gamma(0), \gamma(1))}Q \in \text{Gr}(\mathbb{R}^n \times \mathbb{R}^n)$ . Then

$$\begin{aligned} \nu^Q(\gamma) &= \dim \lambda(1) \cap N^*W, \\ i^Q(\gamma) &= \mu(\lambda, N^*W) + \frac{1}{2}(\dim Q - n) - \frac{1}{2}\nu^Q(\gamma). \end{aligned}$$

This theorem is essentially due to Duistermaat, see Theorem 4.3 in [5]. However, in Duistermaat's formulation the Morse index of  $\gamma$  is related to an absolute Maslov-type index  $i(\lambda)$  of the Lagrangian path  $\lambda$  (see Definition 2.3 in [5]). This choice makes the index formula more complicated. The use of the relative Maslov index  $\mu(\lambda, \cdot)$  introduced by Robbin and Salamon in [11] simplifies such a formula. Rather than deducing Theorem 3.1 from Duistermaat's statement, we prefer to go over his proof, using the relative Maslov index  $\mu$  instead of the absolute Maslov-type index  $i$ .

*Proof.* Let  $c$  be a real number, chosen to be so large that the bilinear form  $d^2\mathbb{S}_{L+c|q|^2}(\gamma)$  is positive definite, hence a Hilbert product on  $W_W^{1,2}([0, 1], \mathbb{R}^n)$ . We denote by  $\mathcal{E}$  the bounded self-adjoint operator on  $W_W^{1,2}([0, 1], \mathbb{R}^n)$  which represents the symmetric bilinear form  $d^2\mathbb{S}_L(\gamma)$  with respect to such a Hilbert product. It is a compact perturbation of the identity, and  $i^Q(\gamma)$  is the

number of its negative eigenvalues, counted with multiplicity (see Lemma 1.1 in [5]), while  $\nu^Q(\gamma)$  is the dimension of its kernel. The eigenvalue equation  $\mathcal{E}u = \lambda u$  corresponds to a second order Sturm-Liouville boundary value problem in  $\mathbb{R}^n$ . Legendre duality shows that such a linear second order problem is equivalent to the following first order linear Hamiltonian boundary value problem on  $T^*\mathbb{R}^n$ :

$$\xi'(t) = A(\mu, t)\xi(t), \quad (\xi(0), C\xi(1)) \in N^*W. \quad (44)$$

Here

$$A(\mu, t) := \begin{pmatrix} D_{qp}H(t, x(t)) & D_{pp}H(t, x(t)) \\ -\mu cT - D_{qq}H(t, x(t)) & -D_{pq}H(t, x(t)) \end{pmatrix},$$

where  $\mu = \lambda/(1 - \lambda)$  and  $T : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*$  is the isomorphism induced by the Euclidean inner product. The fact that  $d^2\mathbb{S}_{L+c|q|^2}(\gamma)$  is positive definite implies that problem (44) has only the zero solution when  $\mu \leq -1$ . Let  $\Phi(\mu, t)$  be the solution of

$$\frac{\partial \Phi}{\partial t}(\mu, t) = A(\mu, t)\Phi(\mu, t), \quad \Phi(\mu, 0) = I.$$

When  $\mu = 0$ ,  $\Phi(0, \cdot)$  is the differential of the Hamiltonian flow, so

$$\lambda(t) = \text{graph } \Phi(0, t)C.$$

In particular, using also the fact that  $N^*W$  is invariant with respect to the involution  $C \times C$ , we find

$$\nu^Q(\gamma) = \dim \ker \mathcal{E} = \dim(\text{graph } C\Phi(0, 1)) \cap N^*W = \dim \lambda(1) \cap N^*W,$$

as claimed. The eigenvalue  $\lambda$  is negative if and only if  $\mu$  belongs to the interval  $(-1, 0)$ , so

$$i^Q(\gamma) = \sum_{-1 < \mu < 0} \dim(\text{graph } \Phi(\mu, 1)C) \cap N^*W, \quad (45)$$

see equation (1.23) in [5]. By Proposition 4.1 in [5], the Lagrangian path

$$[-1, 0] \mapsto \mathcal{L}(T^*\mathbb{R}^{2n}), \quad \mu \mapsto \text{graph } \Phi(\mu, 1)C,$$

has non-trivial intersection with the Lagrangian subspace  $N^*W$  for finitely many  $\mu \in ]-1, 0]$ , and the corresponding crossing forms (see (1)) are positive definite. Then (45) and formula (2) for the relative Maslov index in the case of regular crossings imply that if  $\epsilon > 0$  is so small that there are no non-trivial intersections for  $\mu \in [-\epsilon, 0)$ , there holds

$$i^Q(\gamma) = \mu(\text{graph } \Phi(\cdot, 1)C|_{[-1, -\epsilon]}, N^*W), \quad (46)$$

$$\mu(\text{graph } \Phi(\cdot, 1)C|_{[-\epsilon, 0]}, N^*W) = \frac{1}{2} \dim(\text{graph } \Phi(0, 1)C) \cap N^*W = \frac{1}{2} \dim \nu^Q(\gamma). \quad (47)$$

By considering the homotopy

$$[-1, 0] \times [0, 1] \rightarrow \mathcal{L}(T^*\mathbb{R}^{2n}), \quad (\mu, t) \mapsto \text{graph } \Phi(\mu, t)C,$$

and by using the homotopy and concatenation properties of the relative Maslov index, we obtain from (46) the identity

$$\begin{aligned} i^Q(\gamma) &= -\mu(\text{graph } \Phi(-1, \cdot)C|_{[0, 1]}, N^*W) + \mu(\text{graph } \Phi(\cdot, 0)C|_{[0, 1]}, N^*W) \\ &\quad + \mu(\text{graph } \Phi(0, \cdot)C|_{[0, 1]}, N^*W) - \mu(\text{graph } \Phi(\cdot, 1)C|_{[-\epsilon, 0]}, N^*W). \end{aligned} \quad (48)$$

The path  $t \mapsto \text{graph } \Phi(-1, t)C$  appearing in the first term can intersect  $N^*W$  only for  $t = 0$ , where it coincides with  $\text{graph } C = N^*\Delta$ , where  $\Delta = \Delta_{\mathbb{R}^n}$  is the diagonal in  $\mathbb{R}^n \times \mathbb{R}^n$ . By Lemma 4.2 in [5], the corresponding crossing form is non-degenerate and has Morse index equal to

$$\dim \tau^*(N^*W \cap N^*\Delta),$$



where  $\tau^* : T^*\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is the standard projection. Since  $\tau^*(N^*W \cap N^*\Delta) = W \cap \Delta$ , we deduce that this crossing form has signature

$$\begin{aligned} \dim N^*W \cap N^*\Delta - 2 \dim W \cap \Delta &= \dim W^\perp \cap \Delta^\perp - \dim W \cap \Delta = \dim(W + \Delta)^\perp - \dim W \cap \Delta \\ &= 2n - \dim(W + \Delta) - \dim W \cap \Delta = 2n - \dim W - \dim \Delta = n - \dim W. \end{aligned}$$

So by (2),

$$\mu(\text{graph } \Phi(-1, \cdot)C|_{[0,1]}, N^*W) = \frac{1}{2}(n - \dim W). \quad (49)$$

Since  $\text{graph } \Phi(\mu, 0)C = \text{graph } C = N^*\Delta$  does not depend on  $\mu$ , the second term in (48) vanishes,

$$\mu(\text{graph } \Phi(\cdot, 0)C|_{[0,1]}, N^*W) = 0. \quad (50)$$

The third term in (48) is precisely

$$\mu(\text{graph } \Phi(0, \cdot)C|_{[0,1]}, N^*W) = \mu(\lambda, N^*W), \quad (51)$$

and the last one is computed in (47). Formulas (47), (48), (49), (50), and (51) imply

$$i^Q(\gamma) = \mu(\lambda, N^*W) + \frac{1}{2}(\dim W - n) - \frac{1}{2}\nu^Q(\gamma),$$

concluding the proof.  $\square$

We conclude this section by reformulating the above result in terms of the Maslov index for solutions of non-local conormal boundary value Hamiltonian problems introduced in section 2.

**COROLLARY 3.2** *Assume that the Hamiltonian  $H \in C^\infty([0,1] \times T^*M)$  satisfies (35-36), and let  $L \in C^\infty([0,1] \times TM)$  be its Fenchel dual Lagrangian. Let  $x : [0,1] \rightarrow T^*M$  be a solution of the non-local conormal boundary value Hamiltonian problem (11-23), and let  $\gamma = \tau^* \circ x : [0,1] \rightarrow M$  be the corresponding solution of (39-40-41). Then*

$$\nu^Q(x) = \nu^Q(\gamma), \quad \mu^Q(x) = i^Q(\gamma) + \frac{1}{2}\nu^Q(x).$$

**REMARK 3.3** *The Tonelli assumptions (35-36) are needed in order to have a globally defined Lagrangian  $L$ . Since the Maslov and Morse indexes are local invariants, the above result holds if we just assume the Legendre positivity condition, that is*

$$D_{pp}H(t, q(t), p(t)) > 0 \quad \forall t \in [0,1],$$

along the Hamiltonian orbit  $x(t) = (q(t), p(t))$ .

## 4 The Floer complex

Let us fix a metric  $\langle \cdot, \cdot \rangle$  on  $M$ , with associated norm  $|\cdot|$ . We denote by the same symbol the induced metric on  $TM$  and on  $T^*M$ . This metric determines an isometry  $TM \rightarrow T^*M$  and a direct summand of the vertical tangent bundle  $T^vT^*M$ , the horizontal bundle  $T^hT^*M$ . It also induces a preferred  $\omega$ -compatible almost complex structure  $J$  on  $T^*M$ , which in the splitting  $TT^*M = T^hT^*M \oplus T^vT^*M$  takes the form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

In order to have a well-defined Floer complex, we assume that  $M$  is compact, that the submanifold  $Q$  of  $M \times M$  is also compact, and that the smooth Hamiltonian  $H : [0,1] \times T^*M \rightarrow \mathbb{R}$  satisfies the following conditions:

(H0) every solution  $x$  of the non-local boundary value Hamiltonian problem (11-23) is non-degenerate, meaning that  $\nu^Q(x) = 0$ ;

(H1) there exist  $h_0 > 0$  and  $h_1 \geq 0$  such that

$$DH(t, q, p)[\eta] - H(t, q, p) \geq h_0|p|^2 - h_1,$$

for every  $(t, q, p) \in [0, 1] \times T^*M$  ( $\eta$  denotes the Liouville vector field);

(H2) there exists  $h_2 \geq 0$  such that

$$|\nabla_q H(t, q, p)| \leq h_2(1 + |p|^2), \quad |\nabla_p H(t, q, p)| \leq h_2(1 + |p|),$$

for every  $(t, q, p) \in [0, 1] \times T^*M$  ( $\nabla_q$  and  $\nabla_p$  denote the horizontal and the vertical components of the gradient).

Condition (H0) holds for a generic choice of  $H$ , in basically every reasonable space. Since  $M$  is compact, it is easy to show that conditions (H1) and (H2) do not depend on the choice of the metric on  $M$  (it is important here that the exponent of  $|p|$  in the second inequality of (H2) is one unit less than the corresponding exponent in the first inequality).

We denote by  $\mathcal{S}^Q(H)$  the set of solutions of (11-23), which by (H0) is at most countable. The first variation of the Hamiltonian action functional

$$\mathbb{A}_H(x) := \int x^*(\theta - H dt) = \int_0^1 (\theta(x'(t)) - H(t, x(t))) dt$$

on the space of free paths on  $T^*M$  is

$$d\mathbb{A}_H(x)[\zeta] = \int_0^1 (\omega(\zeta, x'(t)) - dH(t, x)[\zeta]) dt + \theta(x(1))[\zeta(1)] - \theta(x(0))[\zeta(0)], \quad (52)$$

where  $\zeta$  is a section of  $x^*(TT^*M)$ . Since the Liouville one-form  $\theta \times \theta$  of  $T^*M^2$  vanishes on the conormal bundle of every submanifold of  $M^2$ , the extremal curves of  $\mathbb{A}_H$  on the space of paths satisfying (23) are precisely the elements of  $\mathcal{S}^Q(H)$ . A first consequence of conditions (H0), (H1), (H2) is that the set of solutions  $x \in \mathcal{S}^Q(H)$  with an upper bound on the action,  $\mathbb{A}_H(x) \leq A$ , is finite (see Lemma 1.10 in [3]).

Let us consider the Floer equation

$$\partial_s u + J(u)(\partial_t u - X_H(t, u)) = 0 \quad (53)$$

where  $u : \mathbb{R} \times [0, 1] \rightarrow T^*M$ , and  $(s, t)$  are the coordinates on the strip  $\mathbb{R} \times [0, 1]$ . It is a nonlinear first order elliptic PDE, a perturbation of order zero of the equation for  $J$ -holomorphic strips on the almost-complex manifold  $(T^*M, J)$ . The solutions of (53) which do not depend on  $s$  are the orbits of the Hamiltonian vector field  $X_H$ . If  $u$  is a solution of (53), an integration by parts and formula (52) imply the identity

$$\int_a^b \int_0^1 |\partial_s u|^2 ds dt = \mathbb{A}_H(u(a, \cdot)) - \mathbb{A}_H(u(b, \cdot)) + \int_a^b (\theta(u(s, 1))[\partial_s u(s, 1)] - \theta(u(s, 0))[\partial_s u(s, 0)]) ds.$$

In particular, if  $u$  satisfies also the non-local boundary condition

$$(u(s, 0), -u(s, 1)) \in N^*Q, \quad \forall s \in \mathbb{R}, \quad (54)$$

the fact that the Liouville form vanishes on conormal bundles implies that

$$\int_a^b \int_0^1 |\partial_s u|^2 ds dt = \mathbb{A}_H(u(a, \cdot)) - \mathbb{A}_H(u(b, \cdot)). \quad (55)$$

Given  $x^-, x^+ \in \mathcal{S}^Q(H)$ , we denote by  $\mathcal{M}(x^-, x^+)$  the set of all solutions of (53-54) such that

$$\lim_{s \rightarrow \pm\infty} u(s, t) = x^\pm(t), \quad \forall t \in [0, 1].$$

By elliptic regularity, such solutions are smooth up to the boundary. Moreover, conditions (H0) implies that the above convergence of  $u(s, t)$  to  $x^\pm(t)$  is exponentially fast in  $s$ , uniformly with respect to  $t$ . Furthermore, (55) implies that the elements  $u$  of  $\mathcal{M}(x^-, x^+)$  satisfy the energy identity

$$E(u) := \int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|^2 ds dt = \mathbb{A}_H(x^-) - \mathbb{A}_H(x^+). \quad (56)$$

In particular,  $\mathcal{M}(x^-, x^+)$  is empty whenever  $\mathbb{A}_H(x^-) \leq \mathbb{A}_H(x^+)$  and  $x^- \neq x^+$ , and it consists of the only element  $u(s, t) = x(t)$  when  $x^- = x^+ = x$ .

A standard transversality argument (see [8]) shows that we can perturb the Hamiltonian  $H$  in order to ensure that the linear problem obtained by linearizing (53-54) along every solution in  $\mathcal{M}(x^-, x^+)$  has no non-zero solutions, for every pair  $x^-, x^+ \in \mathcal{S}^Q(H)$ . The same task could also be achieved by keeping  $H$  fixed and by perturbing the almost complex structure  $J$ . The Levi-Civita almost complex structure  $J$  has some special features which turn out to be useful when dealing with compactness questions for  $\mathcal{M}(x^-, x^+)$ , so we prefer to perturb the Hamiltonian. The support and the size of this perturbation can be chosen in such a way to keep the set  $\mathcal{S}^Q(H)$  and the linearization of (11) along its elements unaffected.

It follows that  $\mathcal{M}(x^-, x^+)$  has the structure of a smooth manifold. Then Theorem 7.42 in [12] implies that the dimension of  $\mathcal{M}(x^-, x^+)$  equals the difference of the Maslov indices of the Hamiltonian orbits  $x^-, x^+$ :

$$\dim \mathcal{M}(x^-, x^+) = \mu^Q(x^-) - \mu^Q(x^+).$$

The manifolds  $\mathcal{M}(x^-, x^+)$  can be oriented in a way which is *coherent with gluing*. This fact is true for more general Lagrangian intersection problems on symplectic manifolds (see [7] for periodic orbits and [9] for Lagrangian intersections), but the special situation of conormal boundary conditions on cotangent bundles allows simpler proofs (see section 1.4 in [3], where the meaning of coherence is also explained; see also section 5.2 in [10] and section 5.9 in [4]).

By conditions (H1) and (H2), the solution spaces  $\mathcal{M}(x^-, x^+)$  are pre-compact in the  $C_{\text{loc}}^\infty$  topology. In fact by the energy identity (55), Lemma 1.12 in [3] implies that setting  $u = (q, p)$ ,  $p$  has a uniform bound in  $W^{1,2}([s, s+1] \times [0, 1])$ . From this fact, Theorem 1.14 in [3] produces an  $L^\infty$  bound for the elements of  $\mathcal{M}(x^-, x^+)$ . A  $C^1$  bound is then a consequence of the fact that bubbling off of  $J$ -holomorphic spheres cannot occur (because the symplectic form  $\omega$  of  $T^*M$  is exact), nor can occur the bubbling off of  $J$ -holomorphic disks (because the Liouville form - a primitive of  $\omega$  - vanishes on conormal bundles). Finally,  $C^k$  bounds for all positive integers  $k$  follow from elliptic bootstrap.

When  $\mu^Q(x^-) - \mu^Q(x^+) = 1$ ,  $\mathcal{M}(x^-, x^+)$  is an oriented one-dimensional manifold. Since the translation of the  $s$  variables defines a free  $\mathbb{R}$ -action on it,  $\mathcal{M}(x^-, x^+)$  consists of lines. Compactness and transversality imply that the number of these lines is finite. Denoting by  $[u]$  the equivalence class of  $u$  in the compact zero-dimensional manifold  $\mathcal{M}(x^-, x^+)/\mathbb{R}$ , we define  $\epsilon([u]) \in \{+1, -1\}$  to be  $+1$  if the  $\mathbb{R}$ -action is orientation preserving on the connected component of  $\mathcal{M}(x^-, x^+)$  containing  $u$ , and  $-1$  in the opposite case. The integer  $n_F(x^-, x^+)$  is defined as

$$n_F(x^-, x^+) := \sum_{[u] \in \mathcal{M}(x^-, x^+)/\mathbb{R}} \epsilon([u]),$$

If  $k$  is an integer, we denote by  $F_k^Q(H)$  be the free Abelian group generated by the elements  $x \in \mathcal{S}^Q(H)$  with Maslov index  $\mu^Q(x) = k$ . These groups need not be finitely generated. The homomorphism

$$\partial_k : F_k^Q(H) \rightarrow F_{k-1}^Q(H)$$

is defined in terms of the generators by

$$\partial_k x^- := \sum_{\substack{x^+ \in \mathcal{S}(H) \\ \mu^Q(x^+) = k-1}} n_F(x^-, x^+) x^+, \quad \forall x^- \in \mathcal{S}^Q(H), \mu^Q(x^-) = k.$$

The above sum is finite because, as already observed, the set of elements of  $\mathcal{S}^Q(H)$  with an upper action bound is finite. A standard gluing argument shows that  $\partial_{k-1} \circ \partial_k = 0$ , so  $\{F_*^Q(H), \partial_*\}$  is a complex of free Abelian groups, called the *Floer complex* of  $(T^*M, Q, H, J)$ . The homology of such a complex is called the *Floer homology* of  $(T^*M, Q, H, J)$ :

$$HF_k^Q(H, J) := \frac{\ker(\partial_k : F_k^Q(H) \rightarrow F_{k-1}^Q(H))}{\text{ran}(\partial_{k+1} : F_{k+1}^Q(H) \rightarrow F_k^Q(H))}.$$

The Floer complex has an  $\mathbb{R}$ -filtration defined by the action functional: if  $F_k^{Q,A}(H)$  denotes the subgroup of  $F_k^Q(H)$  generated by the  $x \in \mathcal{S}^Q(H)$  such that  $\mathbb{A}_H(x) < A$ , the boundary operator  $\partial_k$  maps  $F_k^{Q,A}(H)$  into  $F_{k-1}^{Q,A}(H)$ , so  $\{F_*^{Q,A}(H), \partial_*\}$  is a subcomplex (which is finitely generated).

By changing the orientation data, we obtain isomorphic chain complexes. Moreover, a different choice of the (small) perturbation of the original Hamiltonian or a different choice of the metric on  $M$  - hence of the almost complex structure  $J$  - produces isomorphic chain complexes. Therefore, if we do not assume transversality, the Floer complex of  $(T^*M, Q, H)$  is well-defined up to isomorphisms.

On the other hand, a different choice of the Hamiltonian (still satisfying (H0), (H1), (H2)) produces chain equivalent complexes. These facts can be proven by standard homotopy argument in Floer theory, but the Hamiltonians to be joined have to be chosen close enough, in order to guarantee compactness (see Theorems 1.12 and 1.13 in [3]).

**REMARK 4.1** *Conditions (H1) and (H2) do not require  $H$  to be convex in  $p$ , not even for  $|p|$  large. They are used to get compactness of both the set of Hamiltonian orbits below a certain action and the set of solutions of the Floer equation connecting them. They could be replaced by suitable convexity and super-linearity assumptions on  $H$ . This approach is taken in the context of generalized Floer homology in [6]. Since this class has a non-empty intersection with the class of Hamiltonians satisfying (H1) and (H2), the homotopy type of the Floer complex is the same in both classes.*

## 5 The Morse complex

In order to define the Morse complex of the Lagrangian action functional, we need the smooth Lagrangian  $L : [0, 1] \times TM \rightarrow \mathbb{R}$  to satisfy the following conditions:

(L1) there exists  $l_0 > 0$  such that

$$\nabla_{vv}L(t, q, v) \geq l_0 I$$

for every  $(t, q, v) \in [0, 1] \times TM$ ;

(L2) there exists  $l_1 \geq 0$  such that

$$|\nabla_{vv}L(t, q, v)| \leq l_1, \quad |\nabla_{qv}L(t, q, v)| \leq l_1(1 + |v|), \quad |\nabla_{qq}L(t, q, v)| \leq l_1(1 + |v|^2)$$

for every  $(t, q, v) \in [0, 1] \times TM$ .

These conditions are expressed in terms of the Riemannian metric of  $M$ , but the compactness of  $M$  easily implies that they do not depend on the choice of the metric. Again, it is important here that every derivative with respect to  $v$  in (L2) lowers the exponent of  $|v|$  by one unit. Assumption (L1) implies that  $L$  is strictly convex and grows at least quadratically in  $v$ , while (L2) implies that

$L$  grows at most quadratically in  $v$ . In particular,  $L$  satisfies the classical Tonelli assumptions. As recalled in section 3, these assumptions imply the equivalence between the Euler-Lagrange equation (39) associated to  $L$  and the Hamiltonian equation (11) associated to its Fenchel dual  $H : [0, 1] \times T^*M \rightarrow \mathbb{R}$ . It is also easy to show that if  $L$  satisfies (L1) and (L2) then  $H$  satisfies (H1) and (H2).

By the Legendre transform, the elements  $x$  of  $\mathcal{S}^Q(H)$  are in one-to-one correspondence with the solutions  $\gamma$  of (39) satisfying the boundary conditions (40) and (41). Let  $\mathcal{S}^Q(L)$  denote the set of these  $M$ -valued curves. By (L2), the Lagrangian action functional  $\mathbb{S}_L$  is twice continuously differentiable on the Hilbert manifold  $W_Q^{1,2}([0, 1[, M)$  consisting of the absolutely continuous curves  $\gamma : [0, 1] \rightarrow M$  whose derivative is square integrable and such that  $(\gamma(0), \gamma(1)) \in Q$  (see e.g. Appendix A in [1]). The elements of  $\mathcal{S}^Q(L)$  are precisely the critical points of  $\mathbb{S}_L$  on such a manifold, and condition (H0) is equivalent to:

(L0) all the critical points  $\gamma \in \mathcal{S}^Q(L)$  of  $\mathbb{S}_L$  on  $W_Q^{1,2}([0, 1[, M)$  are non-degenerate,

meaning that the bounded self-adjoint operator on  $T_\gamma W_Q^{1,2}([0, 1[, M)$  representing the second differential of  $\mathbb{S}_L$  at  $\gamma$  with respect to a  $W^{1,2}$  inner product is an isomorphism. Under this assumption, Corollary 3.2 implies that the Morse index  $i^Q(\gamma)$  of  $\gamma \in \mathcal{S}^Q(L)$  as a critical point of  $\mathbb{S}_L$  on  $W_Q^{1,2}([0, 1[, M)$  coincides with the Maslov index  $\mu^Q(x)$  of the corresponding element  $x \in \mathcal{S}^Q(H)$ .

By (L1),  $L$  is bounded from below and so is the action functional  $\mathbb{S}_L$ . The metric of the compact manifold  $M$  induces a complete Riemannian structure on the Hilbert manifold  $W_Q^{1,2}([0, 1[, M)$ , namely

$$\langle\langle \xi, \zeta \rangle\rangle := \int_0^1 (\langle \nabla_t \xi, \nabla_t \zeta \rangle_{\gamma(t)} + \langle \xi, \zeta \rangle_{\gamma(t)}) dt, \quad \forall \gamma \in W_Q^{1,2}([0, 1[, M), \forall \xi, \zeta \in T_\gamma W_Q^{1,2}([0, 1[, M),$$

where  $\nabla_t$  denotes the Levi-Civita covariant derivative along  $\gamma$ . Conditions (L1) and (L2) imply that  $\mathbb{S}_L$  satisfies the *Palais-Smale condition* on the Riemannian manifold  $W_Q^{1,2}([0, 1[, M)$ , that is every sequence  $(\gamma_h) \subset W_Q^{1,2}([0, 1[, M)$  such that  $\mathbb{S}_L(\gamma_h)$  is bounded and  $\|\nabla \mathbb{S}_L(\gamma_h)\|$  is infinitesimal has a subsequence which converges in the  $W^{1,2}$  topology (see e.g. Appendix A in [1]).

Therefore, the functional  $\mathbb{S}_L$  is twice continuously differentiable, bounded from below, has non-degenerate critical points with finite Morse index, and satisfies the Palais-Smale condition on the complete Riemannian manifold  $W_Q^{1,2}([0, 1[, M)$ . Under these assumptions, the Morse complex of  $\mathbb{S}_L$  on  $W_Q^{1,2}([0, 1[, M)$  is well-defined (up to chain isomorphisms) and its homology is isomorphic to the singular homology of  $W_Q^{1,2}([0, 1[, M)$ . The details of the construction are contained in [2]. Here we just state the results and fix the notation.

Let  $M_k^Q(\mathbb{S}_L)$  be the free Abelian group generated by the elements  $\gamma$  of  $\mathcal{S}^Q(L)$  of Morse index  $i^Q(\gamma) = k$ . Up to perturbing the Riemannian metric of  $W_Q^{1,2}([0, 1[, M)$ , the unstable and stable manifolds  $W^u(\gamma^-)$  and  $W^s(\gamma^+)$  of  $\gamma^-$  and  $\gamma^+$  with respect to the negative gradient flow of  $\mathbb{S}_L$  on  $W_Q^{1,2}([0, 1[, M)$  have transverse intersections of dimension  $i^Q(\gamma^-) - i^Q(\gamma^+)$ , for every pair of critical points  $\gamma^-, \gamma^+$ . An arbitrary choice of an orientation for the (finite-dimensional) unstable manifold of each critical point induces an orientation of all these intersections. When  $i^Q(\gamma^-) - i^Q(\gamma^+) = 1$ , such an intersection consists of finitely many oriented lines. The integer  $n_M(\gamma^-, \gamma^+)$  is defined to be the number of those lines where the orientation agrees with the direction of the negative gradient flow minus the number of the other lines. Such integers are the coefficients of the homomorphisms

$$\partial_k : M_k^Q(\mathbb{S}_L) \rightarrow M_{k-1}^Q(\mathbb{S}_L), \quad \partial_k \gamma^- = \sum_{\substack{\gamma^+ \in \mathcal{S}^Q(L) \\ i^Q(\gamma^+) = k-1}} n_M(\gamma^-, \gamma^+) \gamma^+,$$

defined in terms of the generators  $\gamma^- \in \mathcal{S}^Q(L)$ ,  $i^Q(\gamma^-) = k$ . This sequence of homomorphisms can be identified with the boundary operator associated to a cellular filtration of  $W_Q^{1,2}([0, 1[, M)$  induced by the negative gradient flow of  $\mathbb{S}_L$ . Therefore,  $\{M_*^Q(\mathbb{S}_L), \partial_*\}$  is a chain complex of free Abelian groups, called the *Morse complex* of  $\mathbb{S}_L$  on  $W_Q^{1,2}([0, 1[, M)$ , and its homology is isomorphic

to the singular homology of  $W_Q^{1,2}(]0, 1[, M)$ . Changing the (complete) Riemannian metric on  $W_Q^{1,2}(]0, 1[, M)$  produces a chain isomorphic Morse complex. The Morse complex is filtered by the action level, and the homology of the subcomplex generated by all elements  $\gamma \in \mathcal{S}^Q(L)$  with  $\mathbb{S}_L(\gamma) < A$  is isomorphic to the singular homology of the sublevel

$$\left\{ \gamma \in W_Q^{1,2}(]0, 1[, M) \mid \mathbb{S}_L(\gamma) < A \right\}.$$

The embedding of  $W_Q^{1,2}(]0, 1[, M)$  into the space  $C_Q([0, 1], M)$  of *continuous* curves  $\gamma : [0, 1] \rightarrow M$  such that  $(\gamma(0), \gamma(1)) \in Q$  is a homotopy equivalence. Therefore, the homology of the above Morse complex is isomorphic to the singular homology of the path space  $C_Q([0, 1], M)$ .

## 6 The isomorphism between the Morse and the Floer complex

We are now ready to state and prove the main result of this paper. Here  $M$  is a compact manifold and  $Q$  is a compact submanifold of  $M \times M$ .

**THEOREM 6.1** *Let  $L \in C^\infty([0, 1] \times TM)$  be a time-dependent Lagrangian satisfying conditions (L0), (L1) and (L2). Let  $H \in C^\infty([0, 1] \times T^*M)$  be its Fenchel-dual Hamiltonian. Then there is a chain complex isomorphism*

$$\Theta : \{M_*^Q(\mathbb{S}_L), \partial_*\} \longrightarrow \{F_*^Q(H), \partial_*\}$$

*uniquely determined up to chain homotopy, having the form*

$$\Theta\gamma = \sum_{\substack{x \in \mathcal{S}^Q(H) \\ \mu^Q(x) = i^Q(\gamma)}} n_\Theta(\gamma, x) x, \quad \forall \gamma \in \mathcal{S}^Q(L),$$

*where  $n_\Theta(\gamma, x) = 0$  if  $\mathbb{S}_L(\gamma) \leq \mathbb{A}_H(x)$ , unless  $\gamma$  and  $x$  correspond to the same solution, in which case  $n_\Theta(\gamma, x) = \pm 1$ . In particular,  $\Theta$  respects the action filtrations of the Morse and the Floer complexes.*

*Proof.* Let  $\gamma \in \mathcal{S}^Q(L)$  and  $x \in \mathcal{S}^Q(H)$ . Let  $\mathcal{M}(\gamma, x)$  be the space of all  $T^*M$ -valued maps on the half-strip  $[0, +\infty[ \times ]0, 1]$  solving the Floer equation (53) with the asymptotic condition

$$\lim_{s \rightarrow +\infty} u(s, t) = x(t), \quad (57)$$

and the boundary conditions

$$(u(s, 0), -u(s, 1)) \in N^*Q, \quad \forall s \geq 0, \quad (58)$$

$$\tau^* \circ u(0, \cdot) \in W^u(\gamma), \quad (59)$$

where  $\tau^* : T^*M \rightarrow M$  is the standard projection and  $W^u(\gamma)$  denotes the unstable manifold of  $\gamma$  with respect to the negative gradient flow of  $\mathbb{S}_L$  on  $W_Q^{1,2}(]0, 1[, M)$ . By elliptic regularity, these maps are smooth on  $]0, +\infty[ \times ]0, 1]$  and continuous on  $[0, 1] \times [0, +\infty[$  (actually, the fact that  $\tau^* \circ u(0, \cdot)$  is in  $W^{1,2}(]0, 1[)$  implies that  $u \in W^{3/2,2}(]0, S[ \times ]0, 1[)$  for every  $S > 0$ , in particular  $u$  is Hölder continuous up to the boundary).

The proof of the energy estimate for the elements of  $\mathcal{M}(\gamma, x)$  is based on the following immediate consequence of the Fenchel formula (37) and of (38):

**LEMMA 6.2** *If  $x = (q, p) : [0, 1] \rightarrow T^*M$  is continuous, with  $q$  of class  $W^{1,2}$ , then*

$$\mathbb{A}_H(x) \leq \mathbb{S}_L(q),$$

*the equality holding if and only if the curves  $(q, q')$  and  $(q, p)$  are related by the Legendre transform, that is  $(t, q(t), q'(t)) = \mathcal{L}(t, q(t), q'(t))$  for every  $t \in [0, 1]$ . In particular, the Hamiltonian and the Lagrangian action coincide on corresponding solutions of the two systems.*

In fact, if  $u \in \mathcal{M}(\gamma, x)$  the above Lemma together with (55) (which holds because of (58)), (57), and (59) imply that

$$\begin{aligned} E(u) &:= \int_0^{+\infty} \int_0^1 |\partial_s u|^2 ds dt = \mathbb{A}_H(u(0, \cdot)) - \mathbb{A}_H(x) \\ &\leq \mathbb{S}_L(\tau^* \circ u(0, \cdot)) - \mathbb{A}_H(x) \leq \mathbb{S}_L(\gamma) - \mathbb{A}_H(x). \end{aligned} \quad (60)$$

This energy estimate allows to prove that  $\mathcal{M}(\gamma, x)$  is pre-compact in  $C_{\text{loc}}^\infty$ , as in section 1.5 of [3]. It also implies that:

- (E1)  $\mathcal{M}(\gamma, x)$  is empty if either  $\mathbb{S}_L(\gamma) < \mathbb{A}_H(x)$ , or  $\mathbb{S}_L(\gamma) = \mathbb{A}_H(x)$  but  $\gamma$  and  $x$  do not correspond to the same solution;
- (E2)  $\mathcal{M}(\gamma, x)$  consists of the only element  $u(s, t) = x(t)$  if  $\gamma$  and  $x$  correspond to the same solution.

The computation of the dimension of  $\mathcal{M}(\gamma, x)$  is based on the following linear result, which is a particular case of Theorem 5.24 in [4]:

**PROPOSITION 6.3** *Let  $A : [0, +\infty] \times [0, 1] \rightarrow \text{L}_s(\mathbb{R}^{2n})$  be a continuous map into the space of symmetric linear endomorphisms of  $\mathbb{R}^{2n}$ . Let  $V$  and  $W$  be linear subspaces of  $\mathbb{R}^n$  and  $\mathbb{R}^n \times \mathbb{R}^n$ , respectively. We assume that  $W$  and  $V \times V$  are partially orthogonal, meaning that their quotients by the common intersection  $W \cap (V \times V)$  are orthogonal in the quotient space. We assume that the path  $G$  of symplectic automorphisms of  $\mathbb{R}^{2n}$  defined by*

$$G'(t) = J_0 A(+\infty, t) G(t), \quad G(0) = I, \quad \text{where } J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

satisfies

$$\text{graph } G(1)C \cap N^*W = (0).$$

Then for every  $p \in ]1, +\infty[$  the bounded linear operator

$$v \mapsto \partial_s v + J_0 \partial_t v + A(s, t)v$$

from the Banach space

$$\{v \in W^{1,p}([0, +\infty[ \times ]0, 1[, \mathbb{R}^{2n}) \mid v(0, t) \in N^*V \forall t \in [0, 1], (v(s, 0), -v(s, 1)) \in N^*W \forall s \geq 0\}$$

to the Banach space  $L^p([0, +\infty[ \times ]0, 1[, \mathbb{R}^{2n})$  is Fredholm of index

$$\frac{n}{2} - \mu(\text{graph } G(\cdot)C, N^*W) - \frac{1}{2}(\dim W + 2 \dim V - 2 \dim W \cap (V \times V)). \quad (61)$$

If we linearize the problem given by (53-57-58) and (59) replaced by the condition that  $\tau^* \circ u(0, \cdot)$  should be a given curve on  $M$ , we obtain an operator of the kind introduced in the above Proposition, where  $V = (0)$ ,  $\dim W = \dim Q$ , and  $G$  is the linearization of the Hamiltonian flow along  $x$ . By Proposition 2.3,

$$\mu(\text{graph } G(\cdot)C, N^*W) = \mu^Q(x) - \frac{1}{2}(\dim Q - n),$$

so by (61) this operator has index  $-\mu^Q(x)$ . Since (59) requires that the curve  $\tau^* \circ u(0, \cdot)$  varies within a manifold of dimension  $i^Q(\gamma)$ , the linearization of the full problem (53-57-58-59) produces an operator of index  $i^Q(\gamma) - \mu^Q(x)$ . By perturbing the Lagrangian  $L$  (hence the Hamiltonian  $H$ ) and the metric on  $W_Q^{1,2}([0, 1[, M)$ , we may assume that this operator is onto, for every  $u \in \mathcal{M}(\gamma, x)$ , and every  $\gamma \in \mathcal{S}^Q(L)$ ,  $x \in \mathcal{S}^Q(H)$ , except for the case in which  $\gamma$  and  $x$  correspond to the same solution. In the latter case,  $\mathcal{M}(\gamma, x)$  consists of the only map  $u(s, t) = x(t)$ , see (E2), and the corresponding linear operator is not affected by the above perturbations. However, in this case this operator is automatically onto. The proof of this fact is based on the following consequence of Lemma 6.2, and is analogous to the proof of Proposition 3.7 in [3].

LEMMA 6.4 *If  $x \in \mathcal{S}^Q(H)$  and  $\gamma = \tau^* \circ x$ , then*

$$d^2 \mathbb{A}_H(x)[\xi, \xi] \leq d^2 \mathbb{S}_L(\gamma)[D\tau^*(x)[\xi], D\tau^*(x)[\xi]],$$

*for every section  $\xi$  of  $x^*(TT^*M)$ .*

We conclude that whenever  $\mathcal{M}(\gamma, x)$  is non-empty it is a manifold of dimension

$$\dim \mathcal{M}(\gamma, x) = i^Q(\gamma) - \mu^Q(x).$$

See section 3.1 in [3] for more details on the arguments just sketched

When  $i^Q(\gamma) = \mu^Q(x)$ , compactness and transversality imply that  $\mathcal{M}(\gamma, x)$  is a finite set. Each of its points carries an orientation-sign  $\pm 1$ , as explained in section 3.2 of [3]. The sum of these contributions defines the integer  $n_\Theta(\gamma, x)$ . A standard gluing argument shows that the homomorphism

$$\Theta : \{M_*^Q(\mathbb{S}_L), \partial_*\} \longrightarrow \{F_*^Q(H), \partial_*\}, \quad \Theta\gamma = \sum_{\substack{x \in \mathcal{S}^Q(H) \\ \mu^Q(x) = i^Q(\gamma)}} n_\Theta(\gamma, x) x, \quad \forall \gamma \in \mathcal{S}^Q(L),$$

is a chain map. By (E1) such a chain map preserves the action filtration. In other words, if we order the elements of  $\mathcal{S}^Q(L)$  and  $\mathcal{S}^Q(H)$  - that is the generators of the Morse and the Floer complex - by increasing action, the homomorphism  $\Theta$  is upper-triangular with respect to these ordered sets of generators. Moreover, by (E2) the diagonal elements of  $\Theta$  are  $\pm 1$ . These facts imply that  $\Theta$  is an isomorphism and concludes the proof.  $\square$

COROLLARY 6.5 *Let  $H : [0, 1] \times T^*M \rightarrow \mathbb{R}$  be a Hamiltonian satisfying (H0), (H1), (H2). Then the homology of the Floer complex of  $(T^*M, Q, H, J)$  is isomorphic to the singular homology of the path space  $C_Q([0, 1], M)$ .*

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